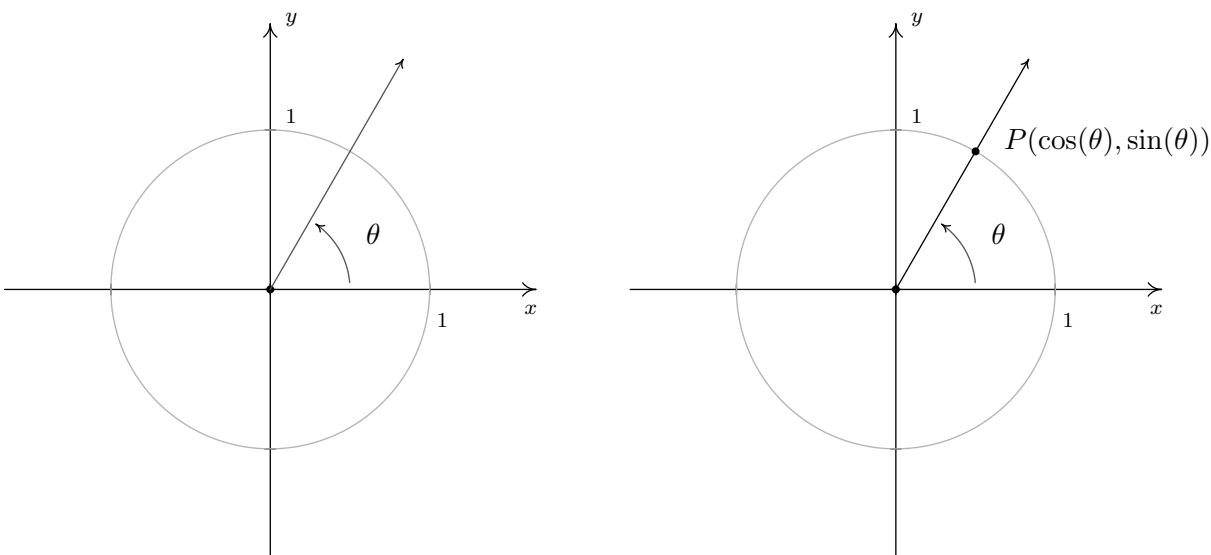


10.2 THE UNIT CIRCLE: COSINE AND SINE

In Section 10.1.1, we introduced circular motion and derived a formula which describes the linear velocity of an object moving on a circular path at a constant angular velocity. One of the goals of this section is describe the *position* of such an object. To that end, consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle. By associating a point P with an angle θ , we are assigning a position P on the Unit Circle to each angle θ . The x -coordinate of P is called the **cosine** of θ , written $\cos(\theta)$, while the y -coordinate of P is called the **sine** of θ , written $\sin(\theta)$.¹ The reader is encouraged to verify that the rules by which we match an angle with its cosine and sine do, in fact, satisfy the definition of function. That is, for each angle θ , there is only one associated value of $\cos(\theta)$ and only one associated value of $\sin(\theta)$.



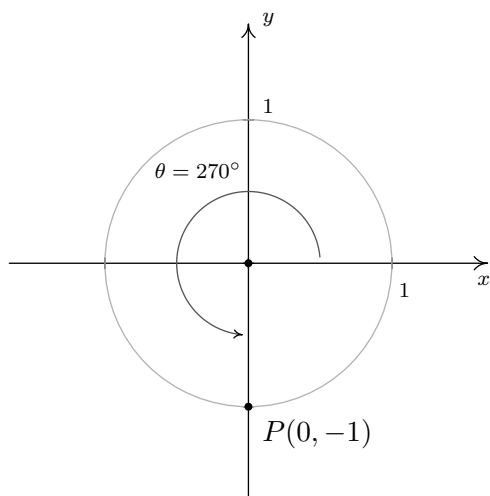
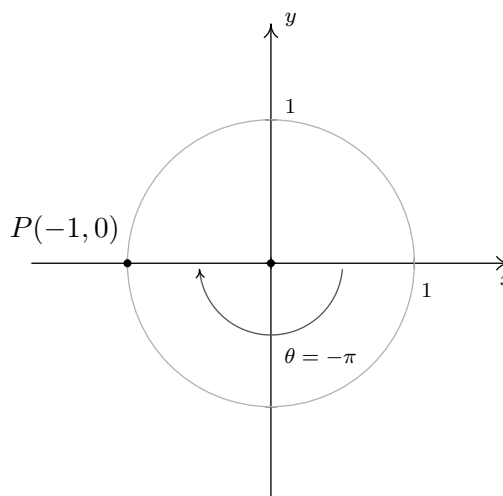
EXAMPLE 10.2.1. Find the cosine and sine of the following angles.

1. $\theta = 270^\circ$
2. $\theta = -\pi$
3. $\theta = 45^\circ$
4. $\theta = \frac{\pi}{6}$
5. $\theta = 60^\circ$

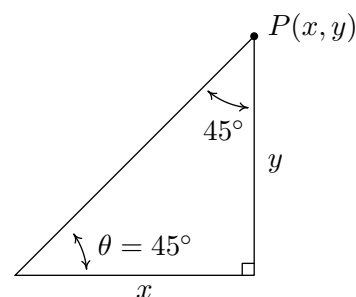
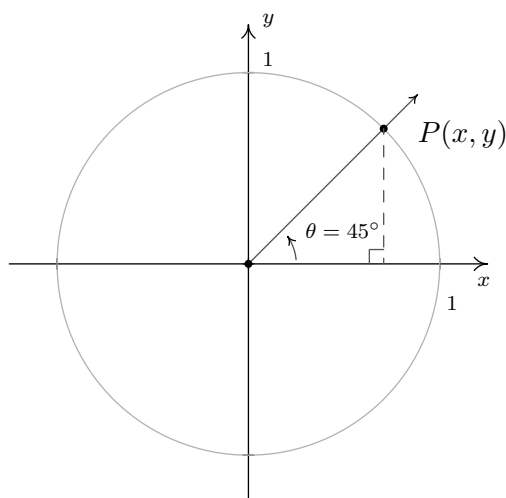
SOLUTION.

1. To find $\cos(270^\circ)$ and $\sin(270^\circ)$, we plot the angle $\theta = 270^\circ$ in standard position and find the point on the terminal side of θ which lies on the Unit Circle. Since 270° represents $\frac{3}{4}$ of a counter-clockwise revolution, the terminal side of θ lies along the negative y -axis. Hence, the point we seek is $(0, -1)$ so that $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$.
2. The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative x -axis. The point on the Unit Circle which lies on the negative x -axis is $(-1, 0)$ which means $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.

¹The etymology of the name 'sine' is quite colorful, and the interested reader is invited to research it; the 'co' in 'cosine' is explained in Section 10.4.

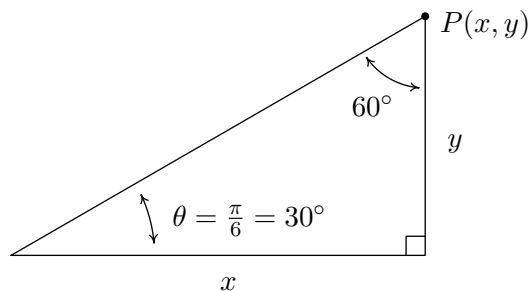
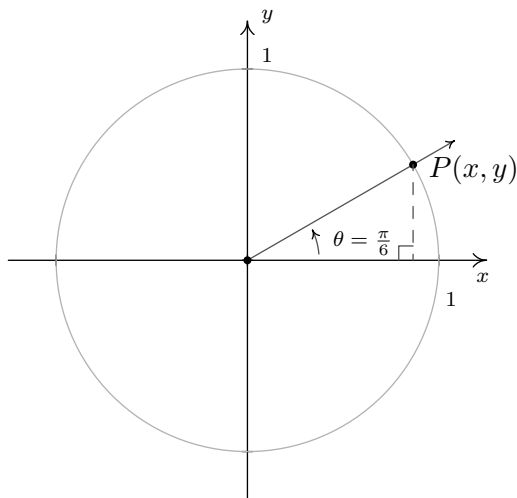
Finding $\cos(270^\circ)$ and $\sin(270^\circ)$ Finding $\cos(-\pi)$ and $\sin(-\pi)$

3. When we sketch $\theta = 45^\circ$ in standard position, we see that its terminal does not lie along any of the coordinate axes which makes our job of finding the cosine and sine values a bit more difficult. Let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. By definition, $x = \cos(45^\circ)$ and $y = \sin(45^\circ)$. If we drop a perpendicular line segment from P to the x -axis, we obtain a $45^\circ - 45^\circ - 90^\circ$ right triangle whose legs have lengths x and y units. From Geometry, we get $y = x$.² Since $P(x, y)$ lies on the Unit Circle, we have $x^2 + y^2 = 1$. Substituting $y = x$ into this equation yields $2x^2 = 1$, or $x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Since $P(x, y)$ lies in the first quadrant, $x > 0$, so $x = \cos(45^\circ) = \frac{\sqrt{2}}{2}$ and with $y = x$ we have $y = \sin(45^\circ) = \frac{\sqrt{2}}{2}$.

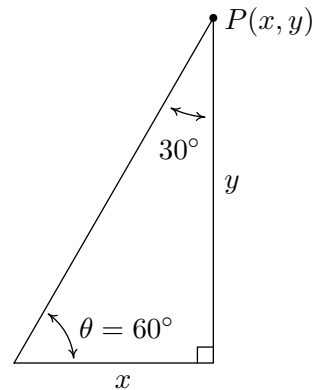
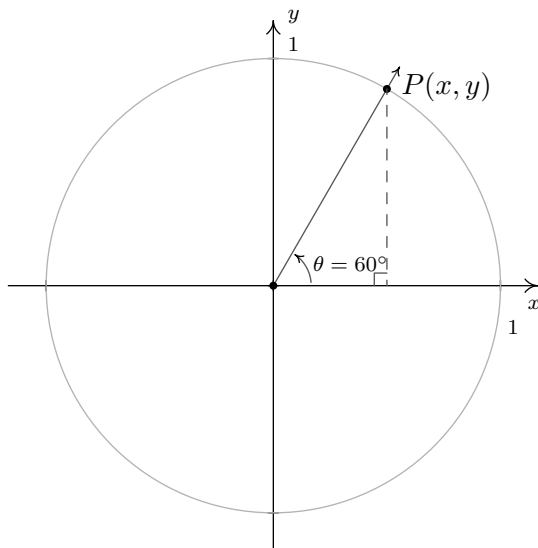


²Can you show this?

4. As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle, we drop a perpendicular line segment from P to the x -axis to form a $30^\circ - 60^\circ - 90^\circ$ right triangle. After a bit of Geometry³ we find $x = y\sqrt{3}$. Since $P(x, y)$ lies on the Unit Circle, we substitute $x = y\sqrt{3}$ into $x^2 + y^2 = 1$ to get $4y^2 = 1$, or $y = \pm\frac{1}{2}$. Here, $y > 0$, so $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and since $x = y\sqrt{3}$, $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.



5. Plotting $\theta = 60^\circ$ in standard position, we find it is not a quadrantal angle and set about using a triangle approach. Once again, we get a $30^\circ - 60^\circ - 90^\circ$ right triangle and, after the usual computations, find $x = \cos(60^\circ) = \frac{1}{2}$ and $y = \sin(60^\circ) = \frac{\sqrt{3}}{2}$.



□

³Again, can you show this?

In Example 10.2.1, it was quite easy to find the cosine and sine of the quadrantal angles, but for non-quadrantal angles, the task was much more involved. In these latter cases, we made good use of the fact that the point $P(x, y) = (\cos(\theta), \sin(\theta))$ lies on the Unit Circle, $x^2 + y^2 = 1$. If we substitute $x = \cos(\theta)$ and $y = \sin(\theta)$ into $x^2 + y^2 = 1$, we get $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. An unfortunate⁴ convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

THEOREM 10.1. The Pythagorean Identity: For any angle θ , $\cos^2(\theta) + \sin^2(\theta) = 1$.

The moniker ‘Pythagorean’ brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived.⁵ The word ‘Identity’ reminds us that, regardless of the angle θ , the equation in Theorem 10.1 is always true. If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 10.1 can be used to determine the other, up to a sign, (\pm) . If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the (\pm) and completely determine the missing value as the next example illustrates.

EXAMPLE 10.2.2. Using the given information about θ , find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.
2. If θ is a Quadrant III angle with $\cos(\theta) = -\frac{\sqrt{5}}{5}$, find $\sin(\theta)$.
3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

SOLUTION.

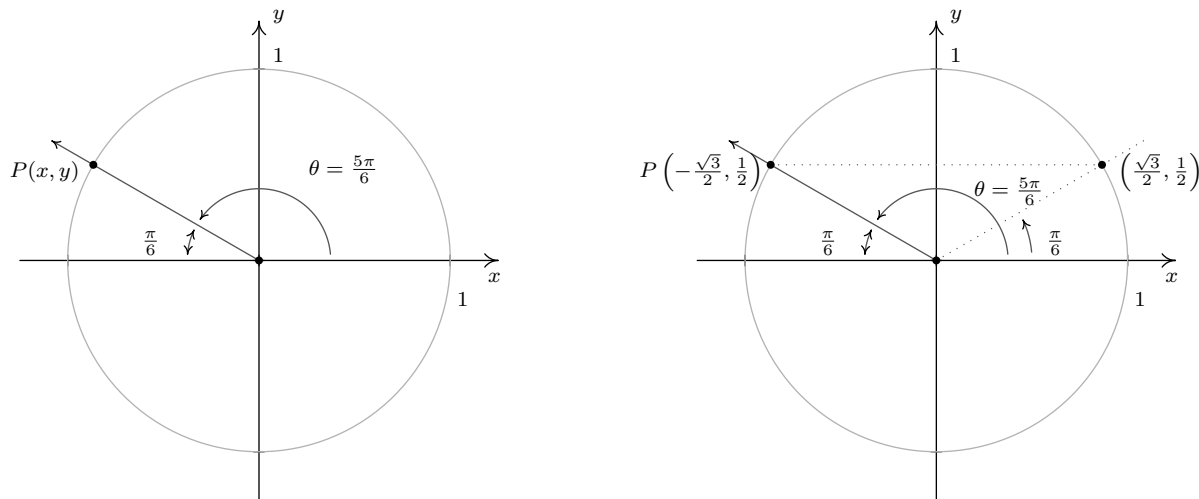
1. When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Since θ is a Quadrant II angle, its terminal side, when plotted in standard position, lies in Quadrant II. Since the x -coordinates are negative in Quadrant II, $\cos(\theta)$ is too. Hence, $\cos(\theta) = -\frac{4}{5}$.
2. Substituting $\cos(\theta) = -\frac{\sqrt{5}}{5}$ into $\cos^2(\theta) + \sin^2(\theta) = 1$ gives $\sin(\theta) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Since θ is a Quadrant III angle, both its sine and cosine are negative (Can you see why?) so we conclude $\sin(\theta) = -\frac{2\sqrt{5}}{5}$.
3. When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$. □

Another tool which helps immensely in determining cosines and sines of angles is the symmetry inherent in the Unit Circle. Suppose, for instance, we wish to know the cosine and sine of $\theta = \frac{5\pi}{6}$. We plot θ in standard position below and, as usual, let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. Note that the terminal side of θ lies $\frac{\pi}{6}$ radians short of one half revolution. In Example 10.2.1, we determined that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\sin(\frac{\pi}{6}) = \frac{1}{2}$. This means

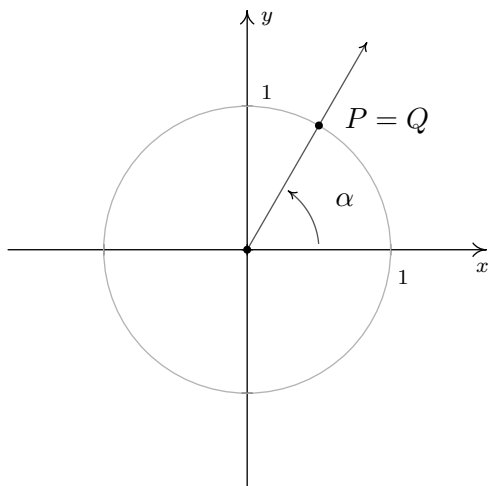
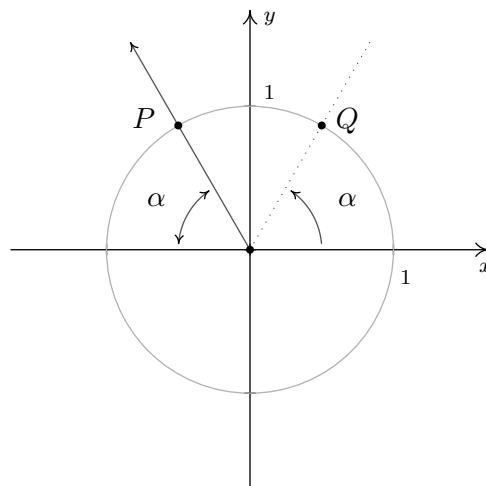
⁴This is unfortunate from a ‘function notation’ perspective. See Section 10.6.

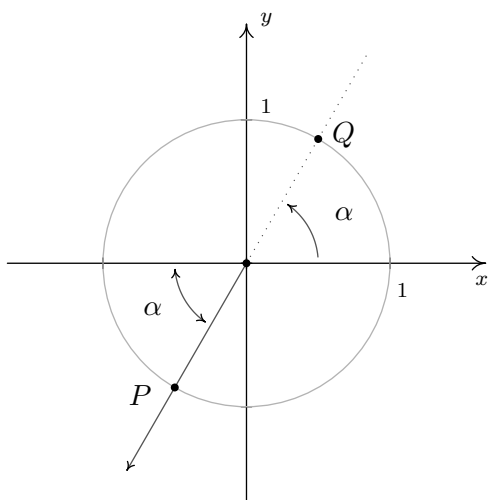
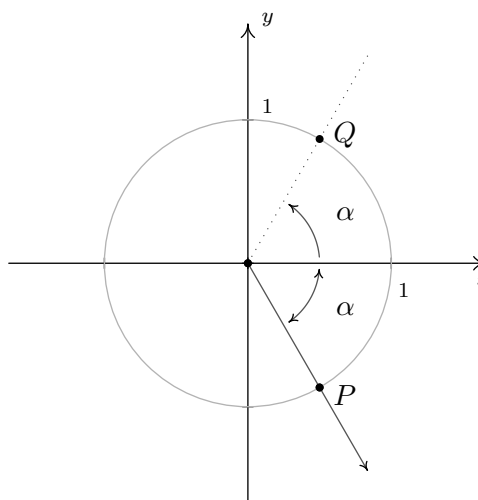
⁵See Sections 1.1 and 7.2 for details.

that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. From the figure below, it is clear that the point $P(x, y)$ we seek can be obtained by reflecting that point about the y -axis. Hence, $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$ and $\sin(\frac{5\pi}{6}) = \frac{1}{2}$.



In the above scenario, the angle $\frac{\pi}{6}$ is called the **reference angle** for the angle $\frac{5\pi}{6}$. In general, for a non-quadrantal angle θ , the reference angle for θ (usually denoted α) is the *acute* angle made between the terminal side of θ and the x -axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* x -axis; if θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* x -axis. If we let P denote the point $(\cos(\theta), \sin(\theta))$, then P lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the x -axis, y -axis and origin, regardless of where the terminal side of θ lies, there is a point Q symmetric with P which determines θ 's reference angle, α as seen below.

Reference angle α for a Quadrant I angleReference angle α for a Quadrant II angle

Reference angle α for a Quadrant III angleReference angle α for a Quadrant IV angle

We have just outlined the proof of the following theorem.

THEOREM 10.2. Reference Angle Theorem. Suppose α is the reference angle for θ . Then $\cos(\theta) = \pm \cos(\alpha)$ and $\sin(\theta) = \pm \sin(\alpha)$, where the choice of the (\pm) depends on the quadrant in which the terminal side of θ lies.

In light of Theorem 10.2, it pays to know the cosine and sine values for certain common angles. In the table below, we summarize the values which we consider essential and must be memorized.

Cosine and Sine Values of Common Angles

θ (degrees)	θ (radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1

EXAMPLE 10.2.3. Find the cosine and sine of the following angles.

1. $\theta = 225^\circ$

2. $\theta = \frac{11\pi}{6}$

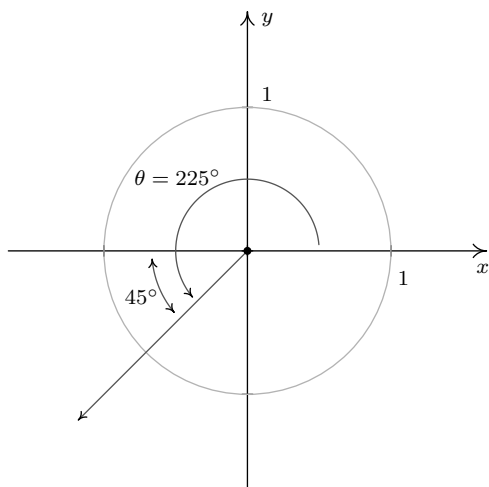
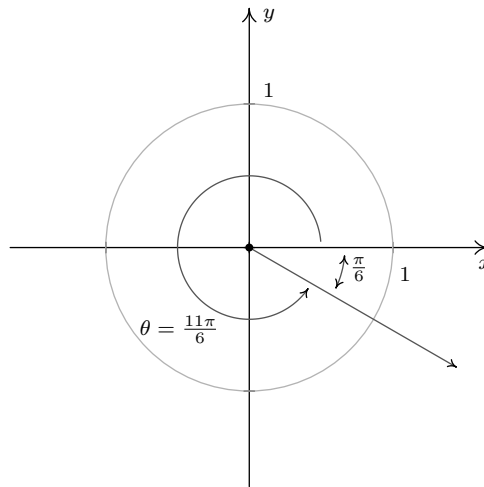
3. $\theta = -\frac{5\pi}{4}$

4. $\theta = \frac{7\pi}{3}$

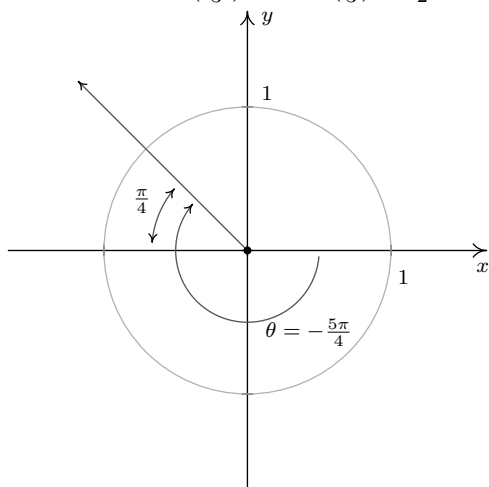
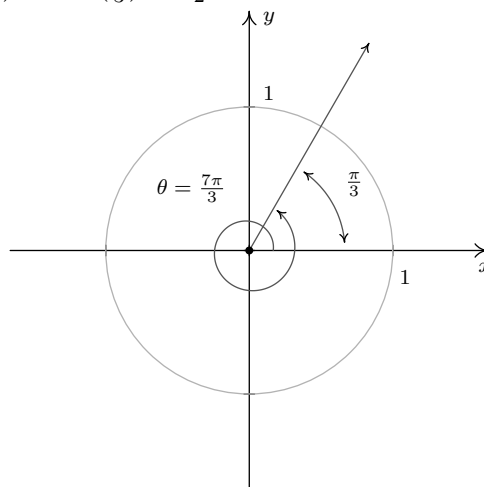
SOLUTION.

- We begin by plotting $\theta = 225^\circ$ in standard position and find its terminal side overshoots the negative x -axis to land in Quadrant III. Hence, we obtain θ 's reference angle α by subtracting: $\alpha = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$. Since θ is a Quadrant III angle, both $\cos(\theta) < 0$ and $\sin(\theta) < 0$. Coupling this with the Reference Angle Theorem, we obtain: $\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$ and $\sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$.

2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive x -axis. To find θ 's reference angle α , we subtract: $\alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

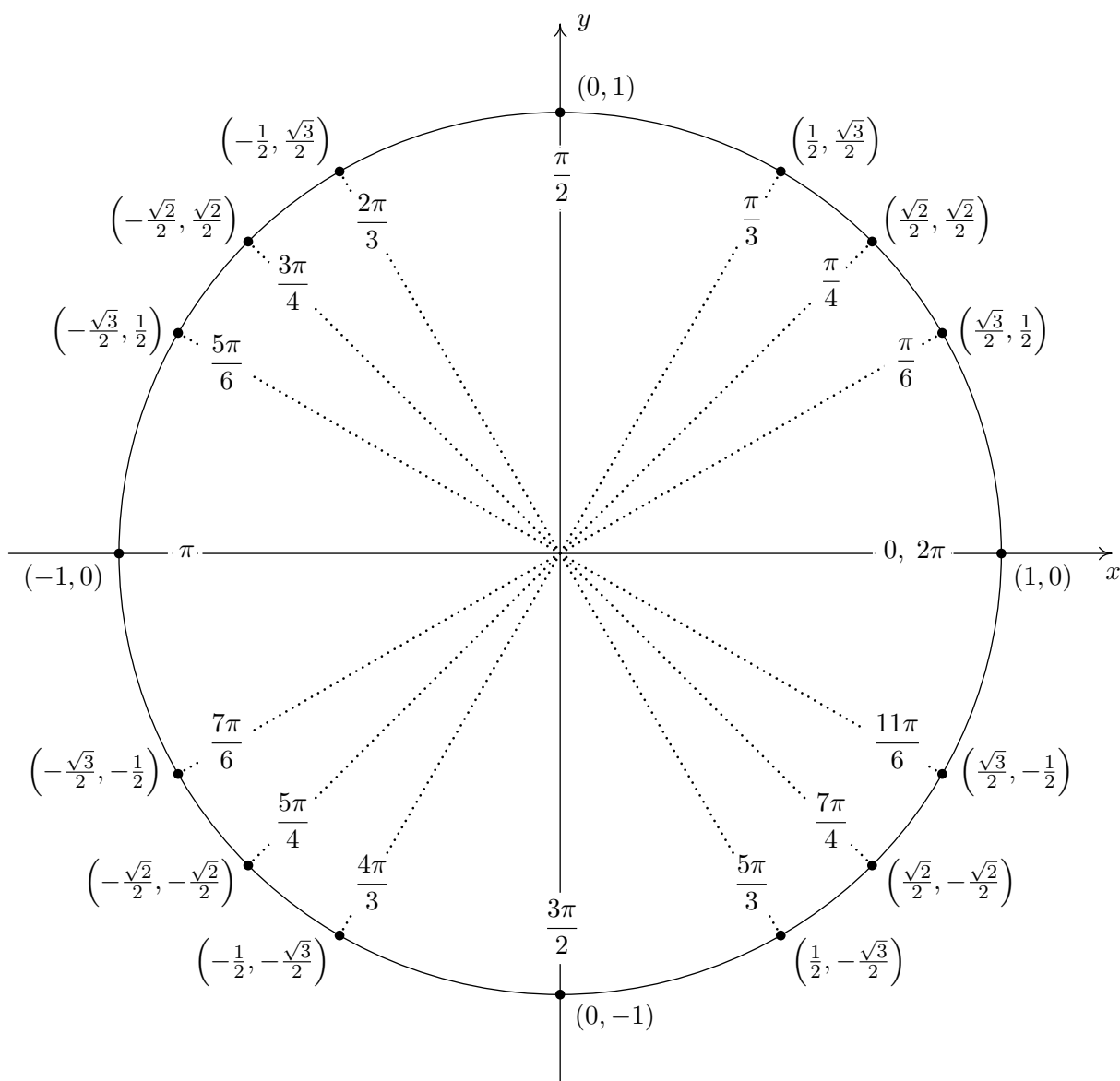
Finding $\cos(225^\circ)$ and $\sin(225^\circ)$ Finding $\cos\left(\frac{11\pi}{6}\right)$ and $\sin\left(\frac{11\pi}{6}\right)$

3. To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive x -axis. The terminal side of θ , therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative x -axis. Since θ is a Quadrant II angle, the Reference Angle Theorem gives: $\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.
4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Since θ and α are coterminal, $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

Finding $\cos\left(-\frac{5\pi}{4}\right)$ and $\sin\left(-\frac{5\pi}{4}\right)$ Finding $\cos\left(\frac{7\pi}{3}\right)$ and $\sin\left(\frac{7\pi}{3}\right)$

□

The reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.⁶ The Reference Angle Theorem in conjunction with the table of cosine and sine values on Page 617 can be used to generate the following figure, which the authors feel should be committed to memory.



Important Points on the Unit Circle

⁶For once, we have something convenient about using radian measure in contrast to the abstract theoretical nonsense about using them as a 'natural' way to match oriented angles with real numbers!

The next example summarizes all of the important ideas discussed thus far in the section.

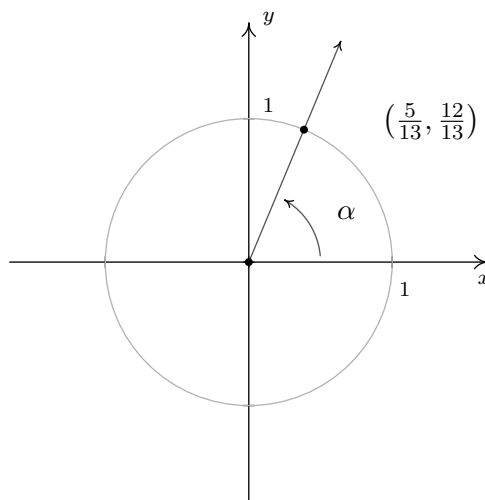
EXAMPLE 10.2.4. Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

1. Find $\sin(\alpha)$ and use this to plot α in standard position.
2. Find the sine and cosine of the following angles:

$$(a) \theta = \pi + \alpha \qquad (b) \theta = 2\pi - \alpha \qquad (c) \theta = 3\pi - \alpha \qquad (d) \theta = \frac{\pi}{2} + \alpha$$

SOLUTION.

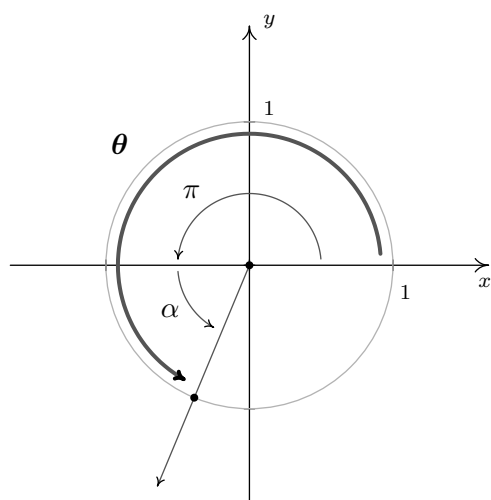
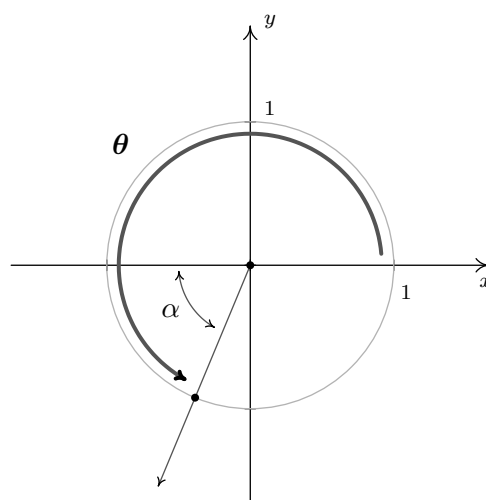
1. Proceeding as in Example 10.2.2, we substitute $\cos(\alpha) = \frac{5}{13}$ into $\cos^2(\alpha) + \sin^2(\alpha) = 1$ and find $\sin(\alpha) = \pm \frac{12}{13}$. Since α is an acute (and therefore Quadrant I) angle, $\sin(\alpha)$ is positive. Hence, $\sin(\alpha) = \frac{12}{13}$. To plot α in standard position, we begin our rotation on the positive x -axis to the ray which contains the point $(\cos(\alpha), \sin(\alpha)) = (\frac{5}{13}, \frac{12}{13})$.



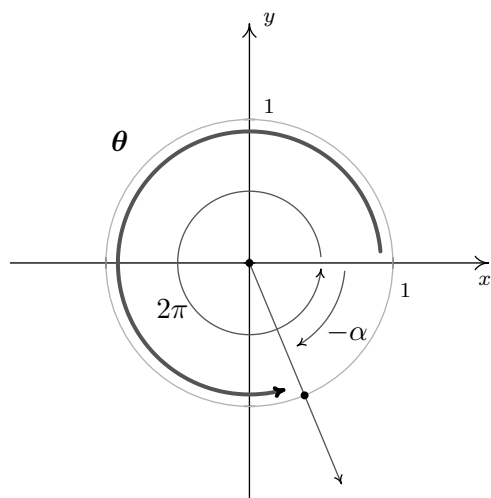
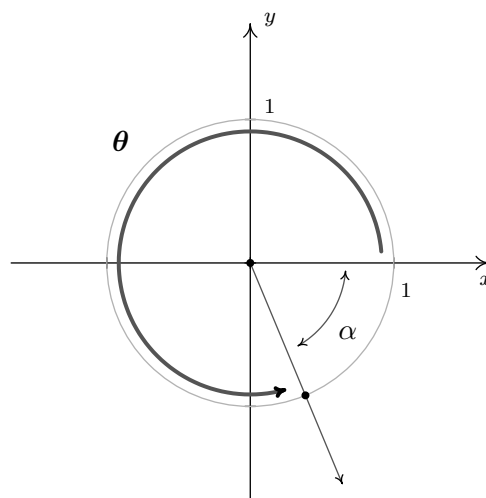
Sketching α

2. (a) To find the cosine and sine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of π radians followed by a rotation of α radians.⁷ We see that α is the reference angle for θ , so by The Reference Angle Theorem, $\cos(\theta) = \pm \cos(\alpha) = \pm \frac{5}{13}$ and $\sin(\theta) = \pm \sin(\alpha) = \pm \frac{12}{13}$. Since the terminal side of θ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative, hence, $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

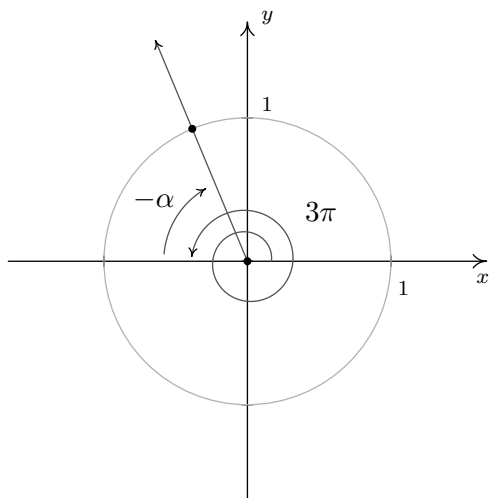
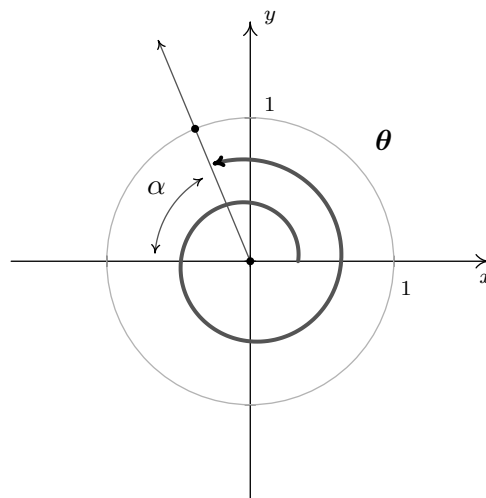
⁷Since $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations given here. You should do this.

Visualizing $\theta = \pi + \alpha$  θ has reference angle α

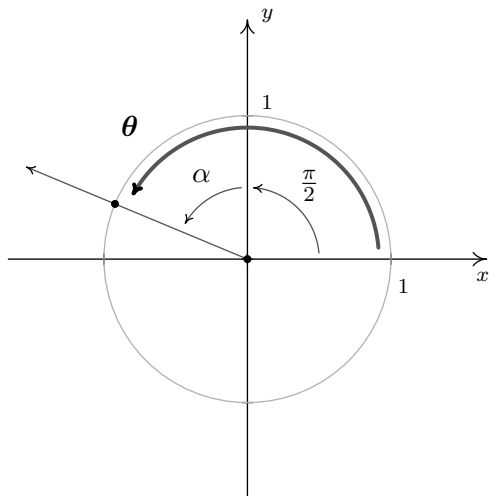
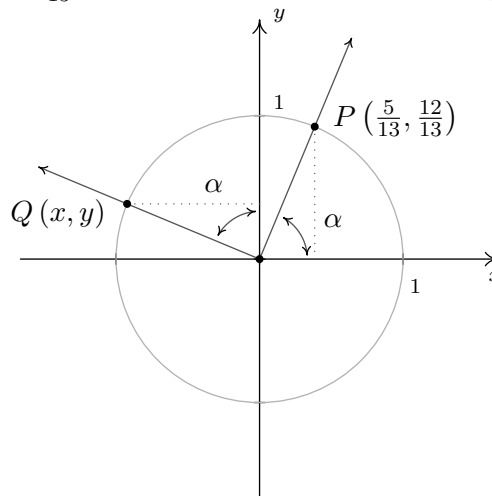
- (b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a *clockwise* revolution, or ‘backing up,’ of α radians. We see that α is θ ’s reference angle, and since θ is a Quadrant IV angle, the Reference Angle Theorem gives: $\cos(\theta) = \frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

Visualizing $\theta = 2\pi - \alpha$  θ has reference angle α

- (c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so that when we ‘back up’ α radians, we end up in Quadrant II. Using the Reference Angle Theorem, we get $\cos(\alpha) = -\frac{5}{13}$ and $\sin(\alpha) = \frac{12}{13}$.

Visualizing $3\pi - \alpha$  θ has reference angle α

- (d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is *not* α , so The Reference Angle Theorem is not immediately applicable. (It's important that you see why this is the case. Take a moment to think about this before reading on.) Let $Q(x, y)$ be the point on the terminal side of θ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph α in standard position, we use the fact that equal angles subtend equal chords to show that the dotted lines in the figure below are equal. Hence, $x = \cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$.

Visualizing $\theta = \frac{\pi}{2} + \alpha$ Using symmetry to determine $Q(x, y)$

□

Our next example asks us to solve some very basic trigonometric equations.⁸

⁸We will more formally study of trigonometric equations in Section 10.7. Enjoy these relatively straightforward exercises while they last!

EXAMPLE 10.2.5. Find all of the angles which satisfy the given equation.

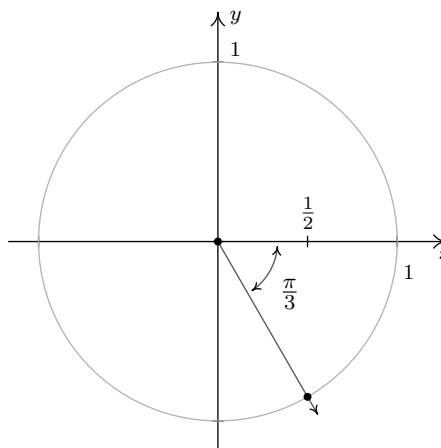
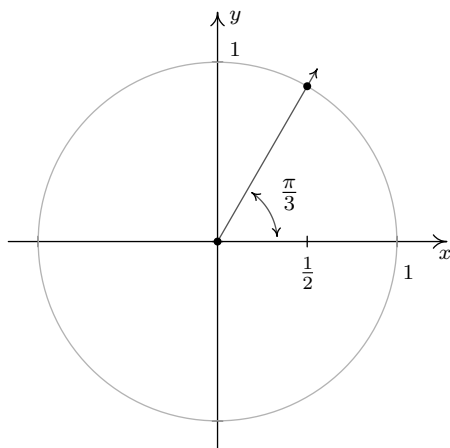
1. $\cos(\theta) = \frac{1}{2}$

2. $\sin(\theta) = -\frac{1}{2}$

3. $\cos(\theta) = 0$.

SOLUTION. Since there is no context in the problem to indicate whether to use degrees or radians, we will default to using radian measure in our answers to each of these problems. This choice will be justified later in the text when we study what is known as Analytic Trigonometry. In those sections to come, radian measure will be the *only* appropriate angle measure so it is worth the time to become “fluent in radians” now.

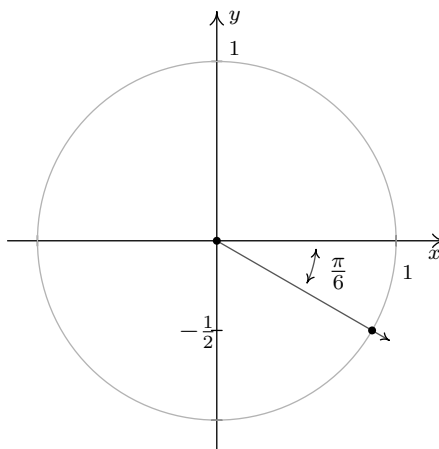
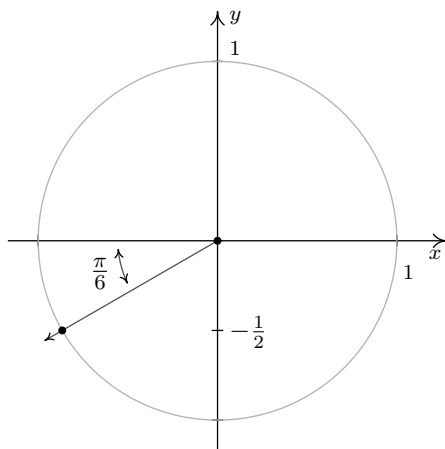
1. If $\cos(\theta) = \frac{1}{2}$, then the terminal side of θ , when plotted in standard position, intersects the Unit Circle at $x = \frac{1}{2}$. This means θ is a Quadrant I or IV angle with reference angle $\frac{\pi}{3}$.



One solution in Quadrant I is $\theta = \frac{\pi}{3}$, and since all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$, we find $\theta = \frac{\pi}{3} + 2\pi k$ for integers k .⁹ Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ here is $\frac{5\pi}{3}$, so our answer in this Quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k .

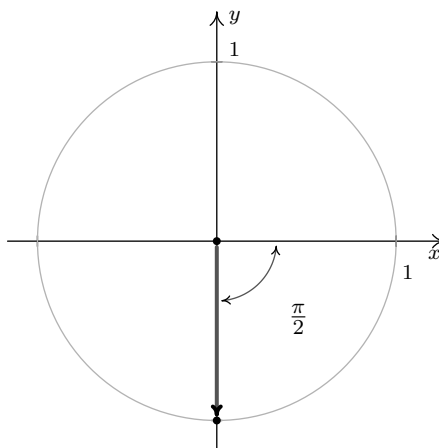
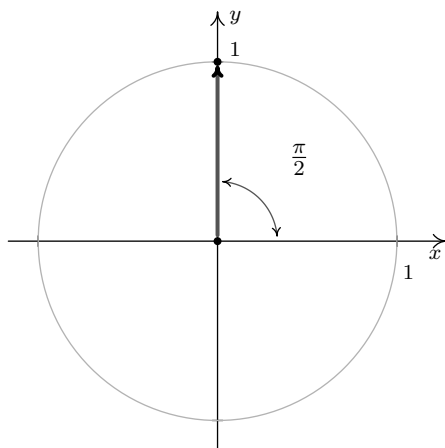
2. If $\sin(\theta) = -\frac{1}{2}$, then when θ is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine θ is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$.

⁹Recall in Section 10.1, two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π . Hence to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k = 0, \pm 1, \pm 2, \dots$



In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π : $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k .

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the y -axis.



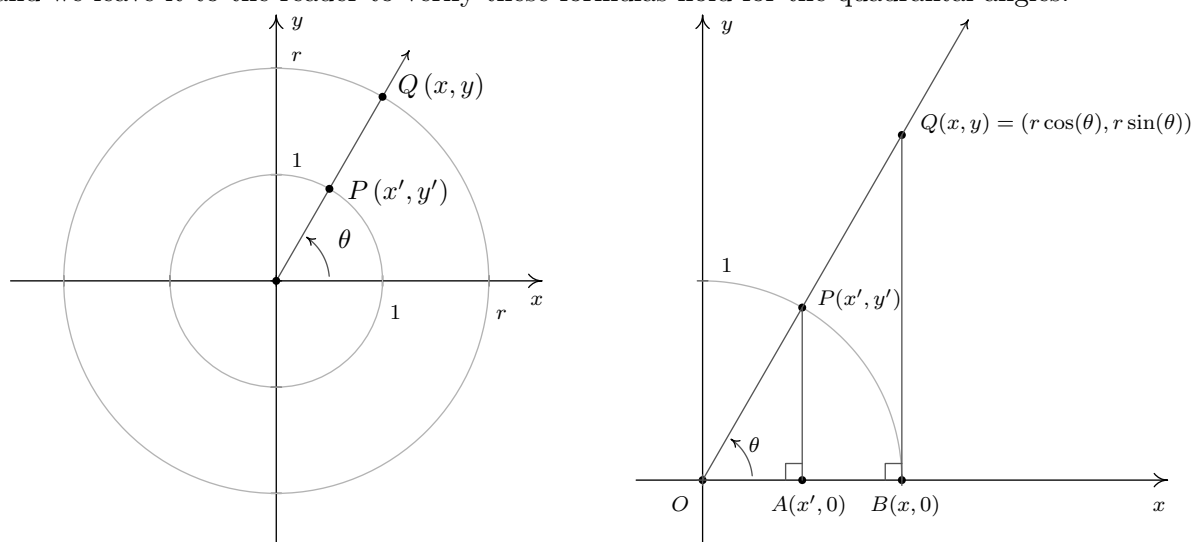
While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers, k . While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers k . (Can you see why this works from the diagram?) \square

One of the key items to take from Example 10.2.5 is that, in general, solutions to trigonometric equations consist of infinitely many answers. To get a feel for these answers, the reader is encouraged to follow our mantra from Chapter 9 - that is, 'When in doubt, write it out!' This is especially important when checking answers to the exercises. For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k . While on the surface, this family may look

different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k , we leave it to the reader to show they represent the same list of angles.

10.2.1 BEYOND THE UNIT CIRCLE

We began the section with a quest to describe the position of a particle experiencing circular motion. In defining the cosine and sine functions, we assigned to each angle a position on the *Unit Circle*. In this subsection, we broaden our scope to include circles of radius r centered at the origin. Consider for the moment the *acute* angle θ drawn below in standard position. Let $Q(x, y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two right triangles, $\triangle OPA$ and $\triangle OQB$. These triangles are similar,¹⁰ thus it follows that $\frac{x}{x'} = \frac{r}{1} = r$, so $x = rx'$ and, similarly, we find $y = ry'$. Since, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of Q to be $x = r \cos(\theta)$ and $y = r \sin(\theta)$. By reflecting these points through the x -axis, y -axis and origin, we obtain the result for all non-quadrantal angles θ , and we leave it to the reader to verify these formulas hold for the quadrantal angles.



Not only can we describe the coordinates of Q in terms of $\cos(\theta)$ and $\sin(\theta)$ but since the radius of the circle is $r = \sqrt{x^2 + y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of Q . These results are summarized in the following theorem.

THEOREM 10.3. Suppose $Q(x, y)$ is the point on the terminal side of an angle θ , plotted in standard position, which lies on the circle of radius r , $x^2 + y^2 = r^2$. Then $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

¹⁰Do you remember why?

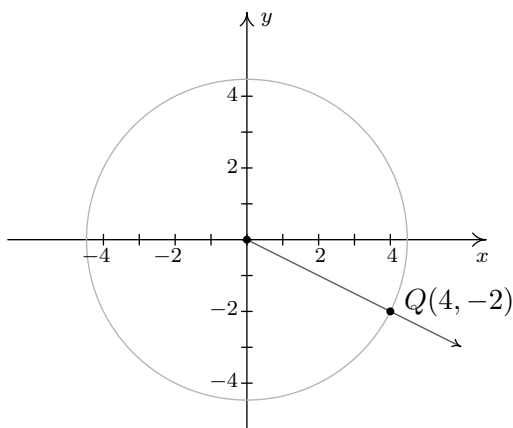
Note that in the case of the Unit Circle we have $r = \sqrt{x^2 + y^2} = 1$, so Theorem 10.3 reduces to our definitions of $\cos(\theta)$ and $\sin(\theta)$.

EXAMPLE 10.2.6.

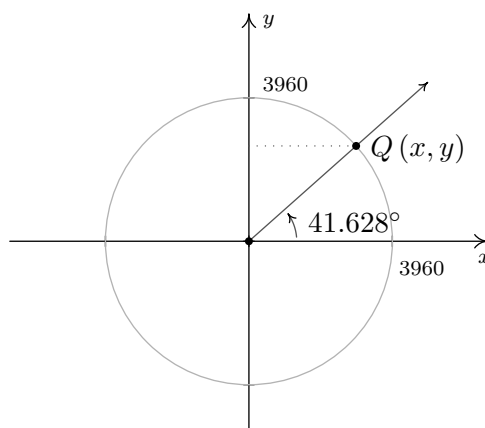
1. Suppose that the terminal side of an angle θ , when plotted in standard position, contains the point $Q(4, -2)$. Find $\sin(\theta)$ and $\cos(\theta)$.
2. In Example 10.1.5 in Section 10.1, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the radius of the Earth at the Equator is approximately 3960 miles.

SOLUTION.

1. Using Theorem 10.3 with $x = 4$ and $y = -2$, we find $r = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$ so that $\cos(\theta) = \frac{x}{r} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}$ and $y = \frac{y}{r} = \frac{-2}{2\sqrt{5}} = -\frac{\sqrt{5}}{5}$.
2. Assuming the Earth is a sphere, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the x -axis, the value we seek is the x -coordinate of the point $Q(x, y)$ indicated in the figure below.



The terminal side of θ contains $Q(4, -2)$



A point on the Earth at 41.628° N

Using Theorem 10.3, we get $x = 3960 \cos(41.628^\circ)$. Using a calculator in 'degree' mode, we find $3960 \cos(41.628^\circ) \approx 2960$. Hence, the radius of the Earth at North Latitude 41.628° is approximately 2960 miles.

```

NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi r<^&theta
FULL HORIZ G-T
SETCLOCK 10/07/01 21:55

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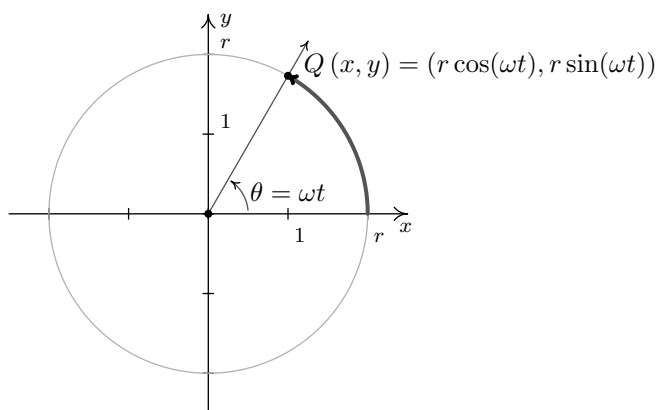
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3960cos(41.628)
2959.995241

```

□

Theorem 10.3 gives us what we need to describe the position of an object traveling in a circular path of radius r with constant angular velocity ω . Suppose that at time t , the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r, 0)$ when $t = 0$, the angle θ is in standard position. By definition, $\omega = \frac{\theta}{t}$ which we rewrite as $\theta = \omega t$. According to Theorem 10.3, the location of the object $Q(x, y)$ on the circle is found using the equations $x = r \cos(\theta) = r \cos(\omega t)$ and $y = r \sin(\theta) = r \sin(\omega t)$. Hence, at time t , the object is at the point $(r \cos(\omega t), r \sin(\omega t))$.¹¹



Equations for Circular Motion

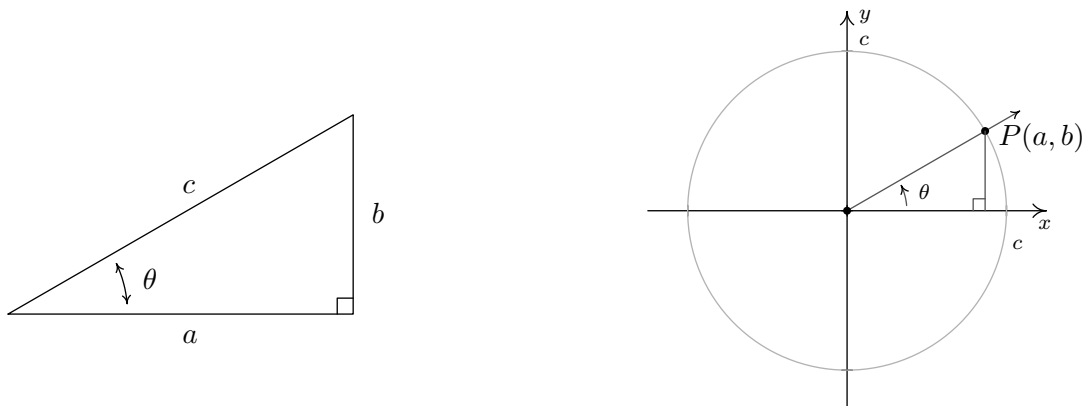
EXAMPLE 10.2.7. Suppose we are in the situation of Example 10.1.5. Find the equations of motion of Lakeland Community College as the earth rotates.

SOLUTION. From Example 10.1.5, we take $r = 2960$ miles and $\omega = \frac{\pi}{12 \text{ hours}}$. Hence, the equations of motion are $x = r \cos(\omega t) = 2960 \cos\left(\frac{\pi}{12}t\right)$ and $y = r \sin(\omega t) = 2960 \sin\left(\frac{\pi}{12}t\right)$, where x and y are measured in miles and t is measured in hours. \square

In addition to circular motion, Theorem 10.3 is also the key to developing what is usually called ‘right triangle’ trigonometry.¹² As we shall see in the sections to come, many applications in trigonometry involve finding the measures of the angles in, and lengths of the sides of, right triangles. Indeed, we made good use of some properties of right triangles to find the exact values of the cosine and sine of many of the angles in Example 10.2.1, so the following development shouldn’t be that much of a surprise. Consider the generic right triangle below with corresponding acute angle θ . The side with length a is called the side of the triangle *adjacent* to θ ; the side with length b is called the side of the triangle *opposite* θ ; and the remaining side of length c (the side opposite the right angle) is called the hypotenuse. We now imagine drawing this triangle in Quadrant I so that the angle θ is in standard position with the adjacent side to θ lying along the positive x -axis.

¹¹If the object does not start at $(r, 0)$ when $t = 0$, the equations of motion need to be adjusted accordingly. If $t_0 > 0$ is the first time the object passes through the point $(r, 0)$, it can be shown the position of the object is given by $x = r \cos(\omega(t - t_0))$ and $y = r \sin(\omega(t - t_0))$.

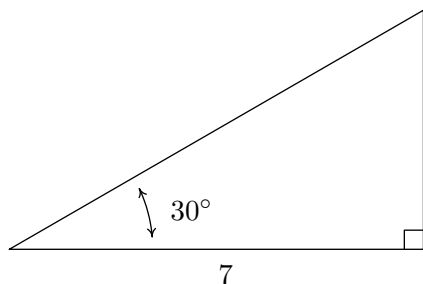
¹²You may have been exposed to this in High School.



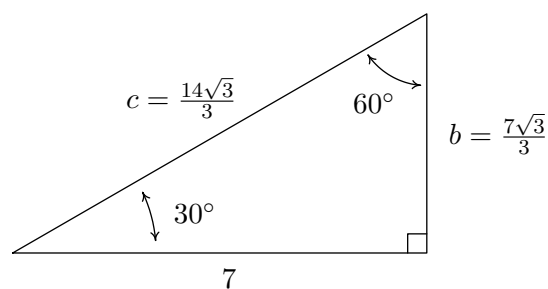
According to the Pythagorean Theorem, $a^2 + b^2 = c^2$, so that the point $P(a, b)$ lies on a circle of radius c . Theorem 10.3 tells us that $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, so we have determined the cosine and sine of θ in terms of the lengths of the sides of the right triangle. Thus we have the following theorem.

THEOREM 10.4. Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a , the length of the side opposite θ is b , and the length of the hypotenuse is c , then $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$.

EXAMPLE 10.2.8. Find the measure of the missing angle and the lengths of the missing sides of:

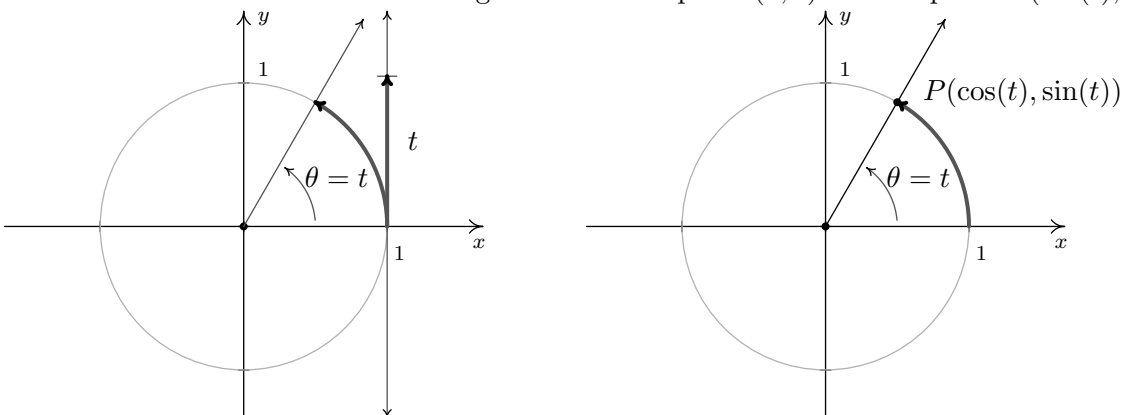


SOLUTION. The first and easiest task is to find the measure of the missing angle. Since the sum of angles of a triangle is 180° , we know that the missing angle has measure $180^\circ - 30^\circ - 90^\circ = 60^\circ$. We now proceed to find the lengths of the remaining two sides of the triangle. Let c denote the length of the hypotenuse of the triangle. By Theorem 10.4, we have $\cos(30^\circ) = \frac{7}{c}$, or $c = \frac{7}{\cos(30^\circ)}$. Since $\cos(30^\circ) = \frac{\sqrt{3}}{2}$, we have, after the usual fraction gymnastics, $c = \frac{14\sqrt{3}}{3}$. At this point, we have two ways to proceed to find the length of the side opposite the 30° angle, which we'll denote b . We know the length of the adjacent side is 7 and the length of the hypotenuse is $\frac{14\sqrt{3}}{3}$, so we could use the Pythagorean Theorem to find the missing side and solve $(7)^2 + b^2 = \left(\frac{14\sqrt{3}}{3}\right)^2$ for b . Alternatively, we could use Theorem 10.4, namely that $\sin(30^\circ) = \frac{b}{c}$. Choosing the latter, we find $b = c \sin(30^\circ) = \frac{14\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{7\sqrt{3}}{3}$. The triangle with all of its data is recorded below.



□

We close this section by noting that we can easily extend the functions cosine and sine to real numbers by identifying a real number t with the angle $\theta = t$ radians. Using this identification, we define $\cos(t) = \cos(\theta)$ and $\sin(t) = \sin(\theta)$. In practice this means expressions like $\cos(\pi)$ and $\sin(2)$ can be found by regarding the inputs as angles in radian measure or real numbers; the choice is the reader's. If we trace the identification of real numbers t with angles θ in radian measure to its roots on page 604, we can spell out this correspondence more precisely. For each real number t , we associate an oriented arc t units in length with initial point $(1, 0)$ and endpoint $P(\cos(t), \sin(t))$.



In the same way we studied polynomial, rational, exponential, and logarithmic functions, we will study the trigonometric functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$. The first order of business is to find the domains and ranges of these functions. Whether we think of identifying the real number t with the angle $\theta = t$ radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both the cosine and sine functions are defined for all real numbers t . In other words, the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is $(-\infty, \infty)$. Since $\cos(t)$ and $\sin(t)$ represent x - and y -coordinates, respectively, of points on the Unit Circle, they both take on all of the values between -1 and 1 , inclusive. In other words, the range of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is the interval $[-1, 1]$. To summarize:

THEOREM 10.5. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \cos(t)$
 - has domain $(-\infty, \infty)$
 - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
 - has domain $(-\infty, \infty)$
 - has range $[-1, 1]$

Suppose, as in the Exercises, we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. As we have already mentioned, the distinction between t as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the exact same manner¹³ as we did in Example 10.2.5 number 2. Our solution is only cosmetically different in that the variable used is t rather than θ : $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, k . We will study the cosine and sine functions in greater detail in Section 10.5. Until then, keep in mind that any properties of cosine and sine developed in the following sections which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

¹³Well, to be pedantic, we would be technically using ‘reference numbers’ or ‘reference arcs’ instead of ‘reference angles’ – but the idea is the same.