The theory of manifolds   Lecture 1

In this lecture we will discuss two generalizations of the inverse function theorem. We'll begin by reviewing some linear algebra. Let

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

be a linear mapping and $[a_{i,j}]$ the $n \times m$ matrix associated with A. Then

$$A^t : \mathbb{R}^n \to \mathbb{R}^m$$

is the linear mapping associated with the transpose matrix $[a_{j,i}]$. For $k < n$ we define the *canonical submersions*

$$\pi : \mathbb{R}^n \to \mathbb{R}^k$$

to be the map $\pi(x_1, \dots, x_n) = (x_1, \dots, x_k)$ and the canonical immersion

$$\iota : \mathbb{R}^k \to \mathbb{R}^n$$

to be the map, $\iota(x_1, \dots, x_k) = (x_1, \dots x_k, 0, \dots 0)$. We leave for you to check that $\pi^t = \iota$.

Proposition 1. *If $A : \mathbb{R}^n \to \mathbb{R}^k$ is onto, there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $AB = \pi$.*

We'll leave the proof of this as an exercise.
Hint: Show that one can choose a basis, $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that

$$Av_i = e_i\,, \quad i = 1, \dots, k$$

is the standard basis of $\mathbb{R}^k$ and

$$Av_i = 0\,, \quad i > k\,.$$

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and set $Be_i = v_i$.

Proposition 2. *If $A : \mathbb{R}^k \to \mathbb{R}^n$ is one–one, there exists a bijective linear map $C : \mathbb{R}^n \to \mathbb{R}^n$ such that $CA = \iota$.*

Proof. The rank of $[a_{i,j}]$ is equal to the rank of $[a_{j,i}]$, so if if A is one–one, there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A^tB = \pi$.
Letting $C = B^t$ and taking transposes we get $\iota = \pi^t = CB$

□

Immersions and submersions

Definition 1. *Let E be an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map. f is a submersion at $p \in E$ if*

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^k$$

is onto.

f is an immersion at $p \in E$ if

$$Df(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

is injective.

Our first main result in this lecture is a non-linear version of Proposition 1.

Theorem 1 (Canonical submersion theorem)**.** *If f is a submersion at p and $f(p) = 0$, there exists a neighborhood, U of p in E, a neighborhood, V, of 0 in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ diffeomorphism, $g : V \to U$ such that $f \circ g = \pi$ and $g(0) = p$.*

Proof. Let $\tau_p : \mathbb{R}^n \to \mathbb{R}^n$ be the map, $x \to x + p$. Replacing f by $f \circ \tau_p$ we can assume $p = 0$. Let A be the linear map

$$Df(0) : \mathbb{R}^n \to \mathbb{R}^k \,.$$

By assumption this map is onto, so there exists a bijective linear map

$$B : \mathbb{R}^n \to \mathbb{R}^n$$

such that $AB = \pi$. And hence for $\widetilde{f} = f \circ B$ we have

$$D\widetilde{f}(0) = \pi \,.$$

Let $h : U \to \mathbb{R}^n$ be the map

$$h(x_1, \ldots, x_n) = (\widetilde{f}_1(x), \ldots, \widetilde{f}_k(x) \,,\, x_{k+1}, \ldots, x_n)$$

where the $\widetilde{f}_i$'s are the coordinate functions of $\widetilde{f}$. I'll leave for you to check that

$$Dh(0) \;=\; I \tag{1}$$

and

$$\pi \circ h \;=\; \widetilde{f} \,. \tag{2}$$

By (1) $Dh(0)$ is bijective, so by the inverse function theorem h maps a neighborhood, U of 0 in E diffeomorphically onto a neighborhood, V, of 0 in $\mathbb{R}^n$. Letting $g = B \circ h^{-1}$ we get from (2) $\pi = \widetilde{f} \circ h^{-1} = f \circ g$.

□

Our second main result is a non-linear version of Proposition 2. Let E be an open neighborhood of 0 in $\mathbb{R}^k$ and $f : E \to \mathbb{R}^n$ a $\mathcal{C}^\infty$-map.

Theorem 2 (Canonical immersion theorem)**.** *If f is an immersion at 0, there exists a neighborhood, V, of $f(0)$ in $\mathbb{R}^n$, a neighborhood, U, of 0 in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$-diffeomorphism $g : V \to U$ such that $\iota^{-1}(U) \subseteq E$ and $g \circ f = \iota$.*

Proof. Let $p = f(0)$. Replacing f by $\tau_{-p} \circ f$ we can assume that $f(0) = 0$. Since $Df(0) : \mathbb{R}^k \to \mathbb{R}^n$ is injective there exists a bijective linear map, $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $BDf(0) = \iota$, so if we define $\widetilde{f} = B \circ f$ we have $D\widetilde{f}(0) = \iota$. Let $\ell = n - k$ and let

$$h : U \times \mathbb{R}^\ell \to \mathbb{R}^n$$

be the map

$$h(x_1, \ldots, x_n) = \widetilde{f}(x_1, \ldots, x_k) + (0, \ldots, 0, x_{k+1}, \ldots, x_n) .$$

I'll leave for you to check that

$$Dh(0) = I \tag{3}$$

and

$$h \circ \iota = \widetilde{f} . \tag{4}$$

By (3) $Dh(0)$ is bijective, so by the inverse function theorem, h maps a neighborhood, U, of 0 in $E \times \mathbb{R}^\ell$ diffeomorphically onto a neighborhood, V, of 0 in $\mathbb{R}^n$. Let $g : V \to U$ be the inverse map composed with B. Then by (4), $\iota = h^{-1} \circ \widetilde{f} = g \circ f$. □

Problem set

1. Prove Proposition 1.
2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the map
$$(x_1, x_2, x_3) \to (x_1^2 - x_2^2 \,,\, x_2^2 - x_3^2) .$$
At what points $p \in \mathbb{R}^3$ is f a submersion?
3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map
$$(x_1, x_2) \to (x_1, x_2 \,,\, x_1^2, x_2^2) .$$
At what points, $p \in \mathbb{R}^2$, is f an immersion?

4. Let U and V be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $f : U \to V$ and $g : V \to \mathbb{R}^k$ be C^1-maps. Prove that if f is a submersion at $p \in U$ and g a submersion at $q = f(p)$ then $g \circ f$ is a submersion at p.

5. Let f and g be as in exercise 5. Suppose that g is a submersion at q. Show that $g \circ f$ is a submersion at p if and only if

$$\mathbb{R}^n = \mathrm{Span}\{\mathrm{Image}\, Df(p) + \mathrm{Kernel}\, Dg(q)\}$$

i.e., if and only if every vector, $\mathrm{v} \in \mathbb{R}^n$ can be written as a sum, $\mathrm{v} = \mathrm{v}_1 + \mathrm{v}_2$, where v_1 is in the image of $Df(p)$ and $Dg(q)(\mathrm{v}_2) = 0$.