

Math 111: Derivation of Trigonometric Identities

Many of the trigonometric identities can be derived in succession from the identities:

$$\sin(-\theta) = -\sin \theta, \quad (1)$$

$$\cos(-\theta) = \cos \theta, \quad (2)$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha, \text{ and} \quad (3)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (4)$$

The first and second identities indicate that \sin and \cos are odd and even functions, respectively. Suppose that $\beta = -w$, then (3) simplifies to

$$\begin{aligned} \sin(\alpha + (-w)) &= \sin \alpha \cos(-w) + \sin(-w) \cos \alpha && \text{by (3)} \\ &= \sin \alpha \cos w - \sin w \cos \alpha && \text{by (1) and (2)} \end{aligned}$$

Since w is an arbitrary label, then β will do as well. Hence,

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha \quad (5)$$

Similarly, equation (4) simplifies as

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (6)$$

To find the identity for $\tan(\alpha + \beta)$, divide (3) by (4) as follows:

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \sin \beta \cos \alpha}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \quad (7)$$

Divide both the top and bottom of (7) by $\cos \alpha \cos \beta$ results with

$$\tan(\alpha + \beta) = \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \beta \cos \alpha}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (8)$$

Because $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$, then it follows that

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}. \quad (9)$$

The Double Angle identities can be derived from equations (3) and (4). Suppose $\alpha = \beta = \theta$, then (3) simplifies as

$$\sin(\theta + \theta) = \sin \theta \cos \theta + \sin \theta \cos \theta.$$

Hence,

$$\sin(2\theta) = 2 \sin \theta \cos \theta. \quad (10)$$

Similarly,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \text{ and} \quad (11)$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \quad (12)$$

The first of the Pythagorean identities can be found by setting $\alpha = \beta = \theta$ in (6). Hence,

$$\cos(\theta - \theta) = \sin \theta \sin \theta + \cos \theta \cos \theta.$$

So,

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (13)$$

Dividing both sides of (13) by $\cos^2 \theta$ yields

$$\tan^2 \theta + 1 = \sec^2 \theta. \quad (14)$$

Dividing both sides of (13) by $\sin^2 \theta$ yields

$$1 + \cot^2 \theta = \csc^2 \theta. \quad (15)$$

Equations (11) and (13) can generate the Power Reducing identities. Using $\cos^2 \theta = 1 - \sin^2 \theta$, (11) can be written as

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta.$$

Solving the above equation for $\sin^2 \theta$ yields

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}. \quad (16)$$

Similarly,

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}, \text{ and} \quad (17)$$

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}. \quad (18)$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\begin{aligned} \sin(\alpha - \beta) + \sin(\alpha + \beta) &= \sin \alpha \cos \beta - \sin \beta \cos \alpha + \sin \alpha \cos \beta + \sin \beta \cos \alpha \\ \sin(\alpha - \beta) + \sin(\alpha + \beta) &= 2 \sin \alpha \cos \beta. \end{aligned}$$

Hence,

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]. \quad (19)$$

Similarly,

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha - \beta) - \sin(\alpha + \beta)], \quad (20)$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)], \text{ and} \quad (21)$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]. \quad (22)$$

Substituting $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$ into (19) yields:

$$\begin{aligned} &\frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] = \sin \alpha \cos \beta \\ \implies &\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin \alpha \cos \beta \\ \implies &\sin \left(\frac{u+v}{2} - \frac{u-v}{2} \right) + \sin \left(\frac{u+v}{2} + \frac{u-v}{2} \right) = 2 \sin \left(\frac{u+v}{2} \right) \cos \left(\frac{u-v}{2} \right) \\ \implies &\sin u + \sin v = 2 \sin \left(\frac{u+v}{2} \right) \cos \left(\frac{u-v}{2} \right) \end{aligned}$$

Since u and v are arbitrary labels, then α and β will do just as well. Hence,

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \quad (23)$$

Similarly, replacing α by $\frac{\alpha+\beta}{2}$ and β by $\frac{\alpha-\beta}{2}$ into (20), (21), and (22) yields

$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right) \quad (24)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \quad (25)$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right) \quad (26)$$