CHAPTER NINE

Solutions for Section 9.1

Exercises

1. We draw a graph of $y = \cos t$ for $-\pi \leq t \leq 3\pi$ and trace along it on a calculator to find points at which $y = 0.4$. We read off the t-values at the points t_0, t_1, t_2, t_3 in Figure 9.1. If t is in radians, we find $t_0 = -1.159$, $t_1 = 1.159$, $t_2 = 5.124$, $t_3 = 7.442$. We can check these values by evaluating:

$$\cos(-1.159) = 0.40, \quad \cos(1.159) = 0.40, \quad \cos(5.124) = 0.40, \quad \cos(7.442) = 0.40.$$

Notice that because the cosine function is periodic, the equation $\cos t = 0.4$ has infinitely many solutions. The symmetry of the graph suggests that the solutions are related.

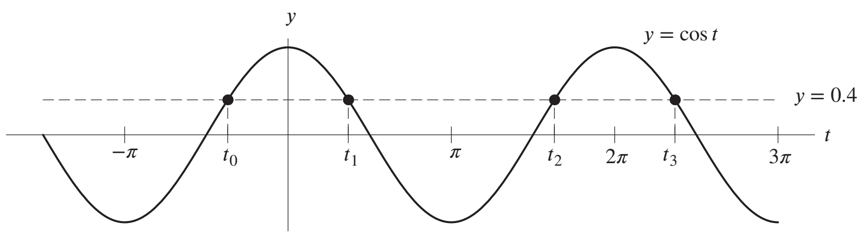


Figure 9.1: The points t_0, t_1, t_2, t_3 are solutions to the equation $\cos t = 0.4$

2. **(a)** Tracing along the graph in Figure 9.2, we see that the approximations for the two solutions are

$$t_1 \approx 1.88 \qquad \text{and} \qquad t_2 \approx 4.41.$$

Note that the first solution, $t_1 \approx 1.88$, is in the second quadrant and the second solution, $t_2 \approx 4.41$, is in the third quadrant. We know that the cosine function is negative in those two quadrants. You can check the two solutions by substituting them into the equation:

$$\cos 1.88 \approx -0.304 \qquad \text{and} \qquad \cos 4.41 \approx -0.298,$$

both of which are close to -0.3.

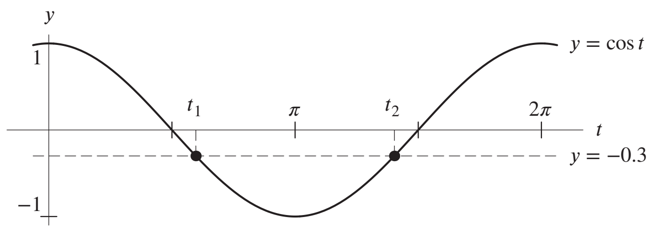


Figure 9.2: The angles t_1 and t_2 are the two solutions to $\cos t = -0.3$ for $0 \leq t \leq 2\pi$

(b) If your calculator is in radian mode, you should find

$$\cos^{-1}(-0.3) \approx 1.875,$$

which is one of the values we found in part (a) by using a graph. Using the $\boxed{\cos^{-1}}$ key gives only one of the solutions to a trigonometric equation. We find the other solutions by using the symmetry of the unit circle. Figure 9.3 shows that if $t_1 \approx 1.875$ is the first solution, then the second solution is

$$t_2 = 2\pi - t_1$$
$$\approx 2\pi - 1.875 \approx 4.408.$$

Thus, the two solutions are $t \approx 1.88$ and $t \approx 4.41$.

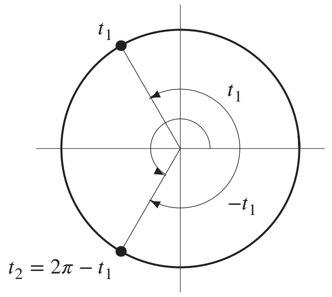


Figure 9.3: By the symmetry of the unit circle, $t_2 = 2\pi - t_1$

3. Graph $y = \sin\theta$ on $0 \le \theta \le 2\pi$ and locate the two points with y-coordinate 0.65. The θ-coordinates of these points are approximately $\theta = 0.708$ and $\theta = 2.434$. See Figure 9.4.

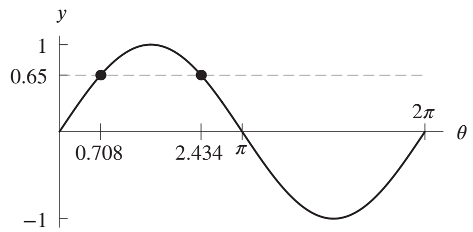


Figure 9.4

4. Graph $y = \tan x$ on $0 \le x \le 2\pi$ and locate the two points with y-coordinate 2.8. The x-coordinates of these points are approximately $x = 1.228$ and $x = 4.369$. See Figure 9.5.

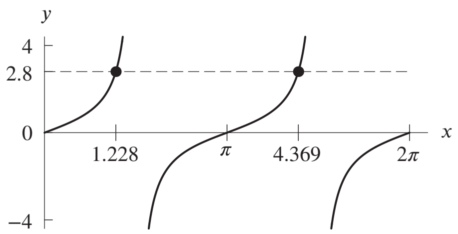


Figure 9.5

5. Graph $y = \cos t$ on $0 \le t \le 2\pi$ and locate the two points with y-coordinate -0.24. The t-coordinates of these points are approximately $t = 1.813$ and $t = 4.473$. See Figure 9.6.

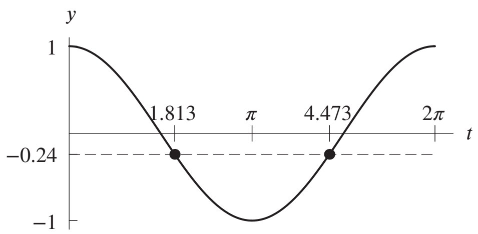


Figure 9.6

6. We have

$$\begin{aligned}5\sin(3\theta) &= 4\\ \sin(3\theta) &= \frac{4}{5}\\ (3\theta) &= \sin^{-1}\left(\frac{4}{5}\right)\\ 3\theta &= 53.130^\circ\\ \theta &= 0.3091 = 17.710^\circ.\end{aligned}$$

7. We have

$$\begin{aligned}9\tan(5\theta)+1 &= 10\\ 9\tan(5\theta) &= 9\\ \tan(5\theta) &= 1\\ 5\theta &= \tan^{-1}(1)\\ 5\theta &= 45^\circ\\ \theta &= 0.1571 = 9^\circ.\end{aligned}$$

8. We have

$$\begin{aligned}2\sqrt{3}\tan(2\theta)+1 &= 3\\ 2\sqrt{3}\tan(2\theta) &= 2\\ \tan(2\theta) &= \frac{2}{2\sqrt{3}}\\ \tan(2\theta) &= \frac{1}{\sqrt{3}}\\ 2\theta &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\\ 2\theta &= 30^\circ\\ \theta &= 0.2618 = 15^\circ.\end{aligned}$$

9. We have

$$\begin{aligned}3\sin\theta+3 &= 5\sin\theta+2\\ 1 &= 2\sin\theta\\ \frac{1}{2} &= \sin\theta\\ \sin^{-1}\left(\frac{1}{2}\right) &= \theta\\ 0.5236 = 30^\circ &= \theta.\end{aligned}$$

10. We have

$$\begin{aligned}6\cos(3\theta)+3 &= 4\cos(3\theta)+4\\ 2\cos(3\theta) &= 1\\ \cos(3\theta) &= \frac{1}{2}\\ 3\theta &= \cos^{-1}\left(\frac{1}{2}\right)\\ 3\theta &= 60^\circ\\ \theta &= 0.3491 = 20^\circ.\end{aligned}$$

11. We have

$$\begin{aligned}5\tan(4\theta)+4 &= 2(\tan(4\theta)+5)\\ 5\tan(4\theta)+4 &= 2\tan(4\theta)+10\\ 3\tan(4\theta) &= 6\\ \tan(4\theta) &= 2\\ 4\theta &= \tan^{-1}(2)\\ 4\theta &= 63.435^\circ\\ \theta &= 0.2768 = 15.859^\circ.\end{aligned}$$

12. Since the cosine function always has a value between -1 and 1, there are no solutions.

13. We use the inverse tangent function on a calculator to get $5\theta + 7 = -0.236$. Solving for θ, we get $\theta = -1.447$.

14. We divide both sides by 2, giving us $\sin(4\theta) = 0.3335$. We then use the inverse sine function on a calculator to get $4\theta = 0.340$, so $\theta = 0.085$.

Problems

15. **(a)** We know that $\cos(\pi/3) = 1/2$. From the graph of $y = \cos t$ in Figure 9.7, we see that $t = \pi/3$, $t = 5\pi/3$, $t = -\pi/3$, and $t = -5\pi/3$ are all solutions, as are any values of t obtained by adding or subtracting multiples of 2π to these values.

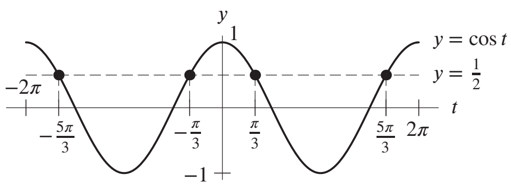


Figure 9.7: $y = \cos t$ has an infinite number of t-values for $y = 1/2$

(b) If we restrict our attention to the interval $0 \le t \le 2\pi$, we find two solutions, $t = \pi/3$ and $t = 5\pi/3$. To see why this is so, look at the unit circle in Figure 9.8. During one revolution around the circle, there are always two angles with the same cosine (or sine, or tangent), unless the cosine is 1 or -1. Therefore, we expect the equation $\cos t = 1/2$ to have two solutions in the interval $0 \le t \le 2\pi$.

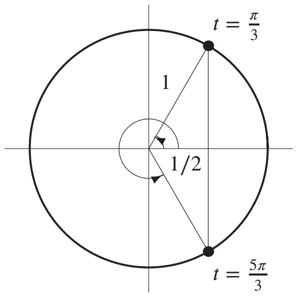


Figure 9.8: During one revolution around the unit circle, the two angles $\pi/3$ and $5\pi/3$ have the cosine value of $1/2$

16. We know that one solution to the equation $\cos t = 0.4$ is $t = \cos^{-1}(0.4) = 1.159$. From Figure 9.9, we see by the symmetry of the unit circle that another solution is $t = -1.159$. We know that additional solutions are given each time the angle t wraps around the circle in either direction. This means that

$$1.159 + 1\cdot 2\pi = 7.442 \qquad \text{wrap once around circle}$$

$$\begin{aligned} 1.159 + 2\cdot 2\pi &= 13.725 && \text{wrap twice around circle}\\ 1.159 + (-1)\cdot 2\pi &= -5.124 && \text{wrap once around circle the other way} \end{aligned}$$

and

$$\begin{aligned} -1.159 + 1\cdot 2\pi &= 5.124 && \text{wrap once around circle}\\ -1.159 + 2\cdot 2\pi &= 11.407 && \text{wrap once around circle}\\ -1.159 + (-1)\cdot 2\pi &= -7.442. && \text{wrap once around circle the other way} \end{aligned}$$

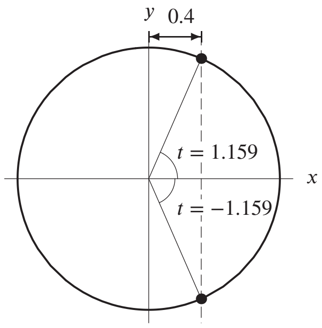


Figure 9.9: Two solutions to the equation $\cos t = 0.4$

17. We have $x = \cos^{-1}(0.6) = 0.927$. A graph of $\cos x$ shows that the second solution is $x = 2\pi - 0.927 = 5.356$.

18. Collecting the $\sin x$ terms on the left gives

$$\begin{aligned} 2\sin x &= 1 - \sin x\\ 3\sin x &= 1\\ \sin x &= 1/3\\ x &= \sin^{-1}(1/3) = 0.340. \end{aligned}$$

A graph of $\sin x$ shows the second solution is $x = \pi - 0.340 = 2.802$.

19. Multiplying both sides by $\cos x$ gives

$$\begin{aligned} 5\cos x &= 1/\cos x\\ 5\cos^2 x &= 1\\ \cos^2 x &= 1/5\\ \cos x &= \pm\sqrt{\frac{1}{5}}. \end{aligned}$$

Thus $x = \cos^{-1}\sqrt{1/5} = 1.107$ or $x = \cos^{-1}\left(-\sqrt{1/5}\right) = 2.034$. A graph of $\cos x$ shows that there are other solutions with $\cos x = \sqrt{1/5}$ given by $x = 2\pi - 1.107 = 5.176$ and with $\cos x = -\sqrt{1/5}$ given by $x = 2\pi - 2.034 = 4.249$. Thus the solutions are

$$1.107, 2.034, 4.249, 5.176.$$

20. Looking at the graph of $y = \sin(2x)$ in Figure 9.10, we see it crosses the line $y = 0.3$ four times between 0 and 2π, so there will be four solutions. Since $2x = \sin^{-1}(0.3) = 0.30469$, one solution is

$$x = \frac{0.30469}{2} = 0.1523.$$

The period of $y = \sin(2x)$ is π, so the other solutions in $0 \le x \le 2\pi$ are

$$x = 0.1523 + \pi = 3.294$$
$$x = \pi/2 - 0.1523 = 1.418$$
$$x = 3\pi/2 - 0.1523 = 4.560.$$

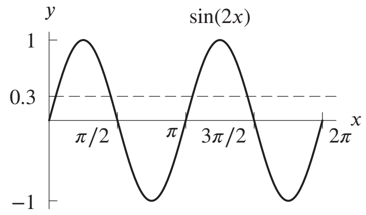


Figure 9.10

21. Since $\sin(x-1) = 0.25$, we know

$$x - 1 = \sin^{-1}(0.25) = 0.253.$$
$$x = 1.253.$$

Another solution for $x - 1$ is given by

$$x - 1 = \pi - 0.253 = 2.889$$
$$x = 3.889.$$

22. Since $5\cos(x+3) = 1$, we have

$$\cos(x+3) = \frac{1}{5} = 0.2$$
$$x + 3 = \cos^{-1}(0.2) = 1.3694$$
$$x = 1.3694 - 3 = -1.6306.$$

This value of x is not in the interval $0 \le x \le 2\pi$. To obtain values of x in this interval, we find values of $x+3$ in the interval between 3 and $3 + 2\pi$, that is between 3 and 9.283. These values of $x + 3$ are

$$x + 3 = 2\pi - 1.369 = 4.914$$
$$x + 3 = 2\pi + 1.369 = 7.653.$$

Thus

$$x = 1.914, 4.653.$$

23. See Figure 9.11. The solutions to the equation $\cos\theta = -0.4226$ are the angles corresponding to the points P and Q on the unit circle with $x = -0.4226$.

Since $\cos^{-1}(-0.4226) = 115°$, point P corresponds to 115°. Since P and Q both have reference angles of 65°, Q corresponds to $180° + 65° = 245°$. Thus the solutions of the equation are

$$\theta = 115° \text{ and } \theta = 180° + 65° = 245°,$$
$$\theta = 360° + 115° = 475° \text{ and } \theta = 360° + 245° = 605°.$$

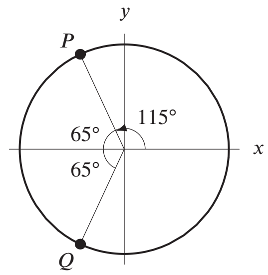


Figure 9.11: Points corresponding to angles with $\cos\theta = -0.4226$

24. **(a)** Since $\cos(65°) = 0.4226$, a calculator set in degrees gives $\cos^{-1}(0.4226) = 65°$. We see in Figure 9.12 that all the angles with a cosine of 0.4226 correspond either to the point P or to the point Q. We want solutions between 0° and 360°, so Q is represented by $360° - 65° = 295°$. Thus, the solutions are

$$\theta = 65° \qquad \text{and} \qquad \theta = 295°.$$

(b) A calculator gives $\tan^{-1}(2.145) = 65°$. Since $\tan\theta$ is positive in the first and third quadrants, the angles with a tangent of 2.145 correspond either to the point P or the point R in Figure 9.13. Since we are interested in solutions between 0° and 720°, the solutions are

$$\theta = 65°, \quad 245°, \quad 65° + 360°, \quad 245° + 360°.$$

That is

$$\theta = 65°, \quad 245°, \quad 425°, \quad 605°.$$

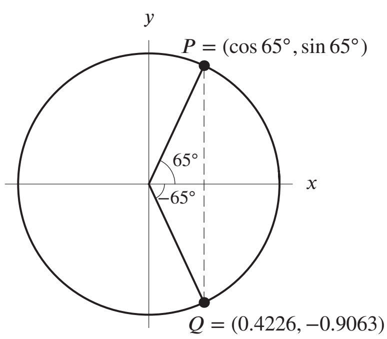


Figure 9.12: The angles 65° and −65°

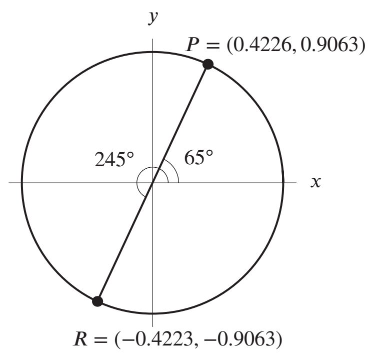


Figure 9.13: The angles 65° and 245°

25. By sketching a graph, we see that there are four solutions (see Figure 9.14). The first solution is given by $x = \cos^{-1}(0.6) = 0.927$, which is equivalent to the length labeled "b" in Figure 9.14. Next, note that, by the symmetry of the graph of the cosine function, we can obtain a second solution by subtracting the length b from 2π. Therefore, a second solution to the equation is given by $x = 2\pi - 0.927 = 5.356$. Similarly, our final two solutions are given by $x = 2\pi + 0.927 = 7.210$ and $x = 4\pi - 0.927 = 11.639$.

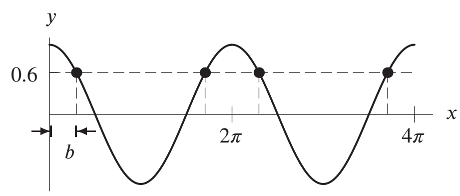


Figure 9.14

26. By sketching a graph, we see that there are two solutions (see Figure 9.15). The first solution is given by $x = \sin^{-1}(0.3) = 0.305$, which is equivalent to the length labeled "b" in Figure 9.15. Next, note that, by the symmetry of the graph of the sine function, we can obtain the second solution by subtracting the length b from π. Therefore, the other solution is given by $x = \pi - 0.305 = 2.837$.

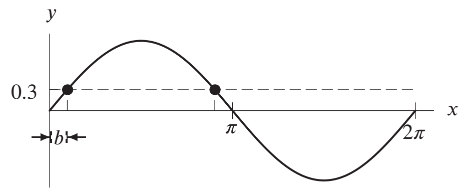


Figure 9.15

27. By sketching a graph, we see that there are two solutions (see Figure 9.16). The first solution is given by $x = \cos^{-1}(-0.7) = 2.346$, which corresponds to the leftmost point in Figure 9.16. To find the second solution, we first calculate the distance labeled "b" in Figure 9.16 to obtain $b = \pi - \cos^{-1}(-0.7) = 0.795$. Therefore, by the symmetry of the cosine function, the second solution is given by $x = \pi + b = 3.937$.

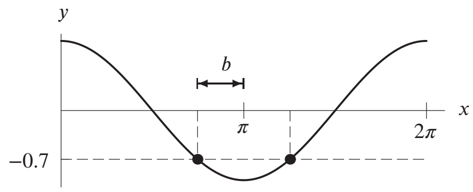


Figure 9.16

28. By sketching a graph, we see that there are four solutions (see Figure 9.17). To find the four solutions, we begin by calculating $\sin^{-1}(-0.8) = -0.927$. Therefore, the length labeled "b" in Figure 9.17 is given by $b = -\sin^{-1}(-0.8) = 0.927$. Now, using the symmetry of the graph of the sine function, we can see that the four solutions are given by $x = \pi + b = 4.069$, $x = 2\pi - b = 5.356$, $x = 3\pi + b = 10.352$, and $x = 4\pi - b = 11.639$.

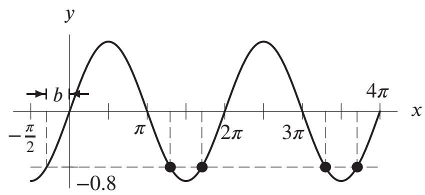


Figure 9.17

29. One solution is $\theta = \sin^{-1}(-\sqrt{2}/2) = -\pi/4$, and a second solution is $5\pi/4$, since $\sin(5\pi/4) = -\sqrt{2}/2$. All other solutions are found by adding integer multiples of 2π to these two solutions. See Figure 9.18.

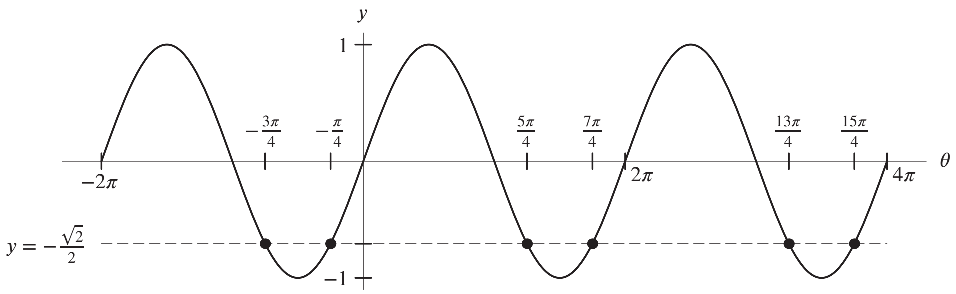


Figure 9.18

30. One solution is $\theta = \cos^{-1}(\sqrt{3}/2) = \pi/6$, and a second solution is $11\pi/6$, since $\cos(11\pi/6) = \sqrt{3}/2$. All other solutions are found by adding integer multiples of 2π to these two solutions. See Figure 9.19.

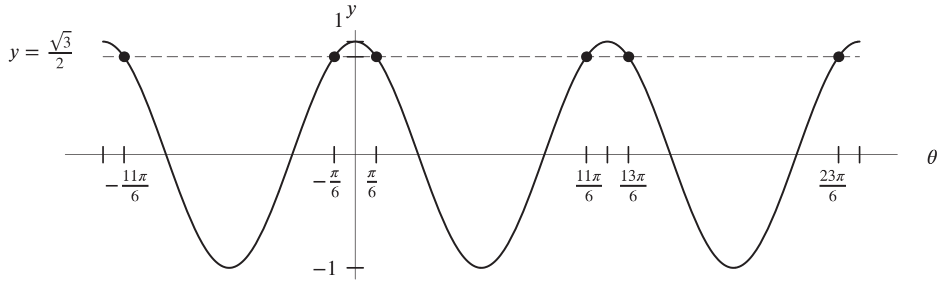


Figure 9.19

31. One solution is $\theta = \tan^{-1}(-\sqrt{3}/3) = -\pi/6$, All other solutions are found by adding integer multiples of π to $-\pi/6$. See Figure 9.20.

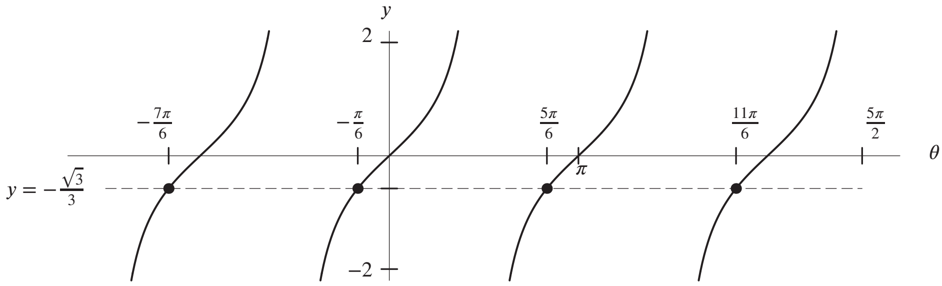


Figure 9.20

32.

$$\begin{aligned} \sec^2\alpha + 3\tan\alpha &= \tan\alpha \\ 1 + \tan^2\alpha + 3\tan\alpha &= \tan\alpha \\ \tan^2\alpha + 2\tan\alpha + 1 &= 0 \\ (\tan\alpha + 1)^2 &= 0 \\ \tan\alpha &= -1 \\ \alpha &= \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

33. From Figure 9.21 we can see that the solutions lie on the intervals $\frac{\pi}{8} < t < \frac{\pi}{4}$, $\frac{3\pi}{4} < t < \frac{7\pi}{8}$, $\frac{9\pi}{8} < t < \frac{5\pi}{4}$ and $\frac{7\pi}{4} < t < \frac{15\pi}{8}$. Using the trace mode on a calculator, we can find approximate solutions $t = 0.52$, $t = 2.62$, $t = 3.67$ and $t = 5.76$.

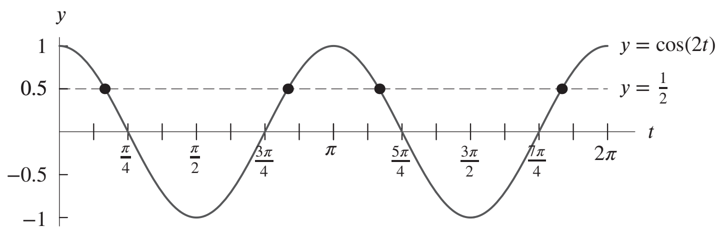


Figure 9.21

For a more precise answer we solve $\cos(2t) = \frac{1}{2}$ algebraically, giving $2t = \arccos(1/2)$. One solution is $2t = \pi/3$. But $2t = 5\pi/3$, $7\pi/3$, and $11\pi/3$ are also angles that have a cosine of $1/2$. Thus $t = \pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$ are the solutions between 0 and 2π.

34. To solve

$$\tan t = \frac{1}{\tan t}$$

we multiply both sides of the equation by $\tan t$. Multiplication gives us

$$\tan^2 t = 1 \qquad \text{or} \qquad \tan t = \pm 1.$$

From Figure 9.22, we see that there are two solutions for $\tan t = 1$, and two solutions for $\tan t = -1$; they are approximately $t = 0.79$, $t = 3.93$, and $t = 2.36$, $t = 5.50$.

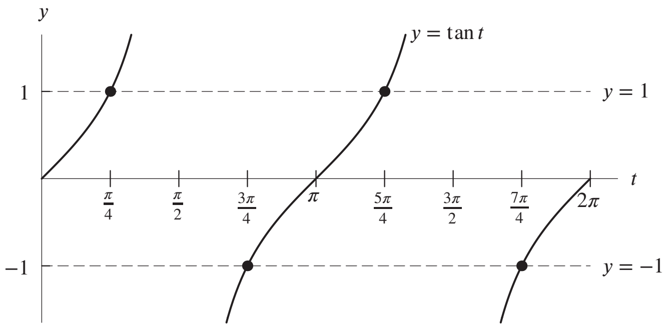


Figure 9.22

To find exact solutions, we have $t = \arctan(\pm 1) = \pm\pi/4$. There are other angles that have a tan of ± 1, namely $\pm 3\pi/4$. So $t = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$ are the solutions in the interval from 0 to 2π.

35. From Figure 9.23, we see that $2\sin t\cos t - \cos t = 0$ has four roots between 0 and 2π. They are approximately $t = 0.52$, $t = 1.57$, $t = 2.62$, and $t = 4.71$.

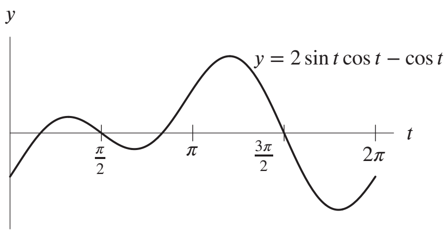


Figure 9.23

To solve the problem symbolically, we factor out $\cos t$:

$$2\sin t\cos t - \cos t = \cos t(2\sin t - 1) = 0.$$

So solutions occur either when $\cos t = 0$ or when $2\sin t - 1 = 0$. The equation $\cos t = 0$ has solutions $\pi/2$ and $3\pi/2$. The equation $2\sin t - 1 = 0$ has solution $t = \arcsin(1/2) = \pi/6$, and also $t = \pi - \pi/6 = 5\pi/6$. Thus the solutions to the original problem are

$$t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6} \text{ and } \frac{5\pi}{6}.$$

36. The solutions to the equation are at points where the graphs of $y = 3\cos^2 t$ and $y = \sin^2 t$ cross. From Figure 9.24, we see that $3\cos^2 t = \sin^2 t$ has four solutions between 0 and 2π; they are approximately $t = 1.05$, $t = 2.09$, $t = 4.19$, and $t = 5.24$.

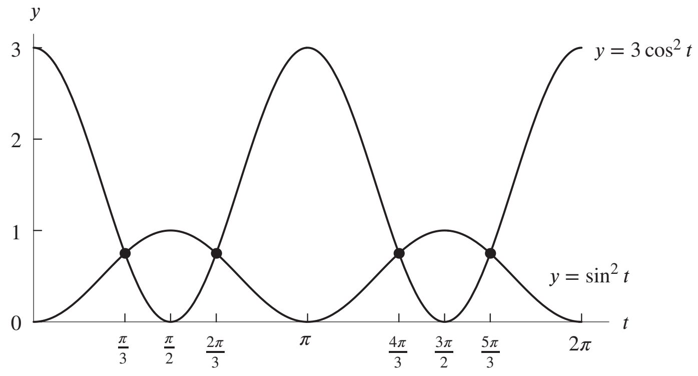


Figure 9.24

To solve $3\cos^2 t = \sin^2 t$ we divide both sides by $\cos^2 t$ and rewrite the equation as

$$3 = \tan^2 t \qquad \text{or} \qquad \tan t = \pm\sqrt{3}.$$

(Dividing by $\cos^2 t$ is valid only if $\cos t \neq 0$. Since $t = \pi/2$ and $t = 3\pi/2$ are not solutions, $\cos t \neq 0$.)

Using the inverse tangent and reference angles, we find that the solutions occur at the points

$$t = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3} \text{ and } \frac{5\pi}{3}.$$

37. Graph $y = 12 - 4\cos(3t)$ on $0 \leq t \leq 2\pi/3$ and locate the two points with y-coordinate 14. (See Figure 9.25.) These points have t-coordinates of approximately $t = 0.698$ and $t = 1.396$. There are six solutions in three cycles of the graph between 0 and 2π.

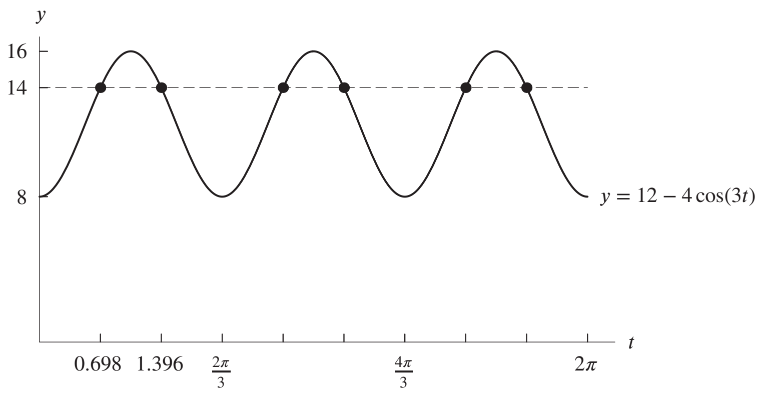


Figure 9.25

38. (a) Graph $y = 3 - 5\sin 4t$ on the interval $0 \leq t \leq \pi/2$, and locate values where the function crosses the t-axis. Alternatively, we can find the points where the graph $5\sin 4t$ and the line $y = 3$ intersect. By looking at the graphs of these two functions on the interval $0 \leq t \leq \pi/2$, we find that they intersect twice. By zooming in we can identify these points of intersection as roughly $t_1 \approx 0.16$ and $t_2 \approx 0.625$. See Figure 9.26.

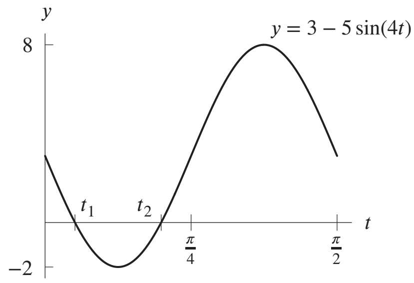


Figure 9.26

(b) Solve for $\sin(4t)$ and then use arcsine:

$$\begin{aligned} 5\sin(4t) &= 3 \\ \sin(4t) &= \frac{3}{5} \\ 4t &= \arcsin\left(\frac{3}{5}\right). \end{aligned}$$

So $t_1 = \dfrac{\arcsin(3/5)}{4} \approx 0.161$ is a solution. But the angle $\pi - \arcsin(3/5)$ has the same sine as $\arcsin(3/5)$. Solving $4t = \pi - \arcsin(3/5)$ gives $t_2 = \dfrac{\pi}{4} - \dfrac{\arcsin(3/5)}{4} \approx 0.625$ as a second solution.

39. (a) The maximum is \$100,000 and the minimum is \$20,000. Thus in the function

$$f(t) = A\cos(B(t-h)) + k,$$

the midline is

$$k = \frac{100,000 + 20,000}{2} = \$60,000$$

and the amplitude is

$$A = \frac{100,000 - 20,000}{2} = \$40,000.$$

The period of this function is 12 since the sales are seasonal. Since

$$\text{period } = 12 = \frac{2\pi}{B},$$

we have

$$B = \frac{\pi}{6}.$$

The company makes its peak sales in mid-December, which is month -1 or month 11. Since the regular cosine curve hits its peak at $t = 0$ while ours does this at $t = -1$, we find that our curve is shifted horizontally 1 unit to the left. So we have

$$h = -1.$$

So the sales function is

$$f(t) = 40,000\cos\left(\frac{\pi}{6}(t+1)\right) + 60,000 = 40,000\cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) + 60,000.$$

(b) Mid-April is month $t = 3$. Substituting this value into our function, we get

$$f(3) = \$40,000.$$

(c) To solve $f(t) = 60,000$ for t, we write

$$\begin{aligned} 60,000 &= 40,000\cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) + 60,000 \\ 0 &= 40,000\cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) \\ 0 &= \cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right). \end{aligned}$$

Since the cosine function takes on the value 0 at $\pi/2$ and $3\pi/2$, the angle $(\frac{\pi}{6}t + \frac{\pi}{6})$ equals either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Solving for t, we get $t = 2$ or $t = 8$. So in mid-March and mid-September the company has sales of $60,000 (which is the average or midline sales value).

40. **(a)** Let t be the time in hours since 12 noon. Let $d = f(t)$ be the depth in feet in Figure 9.27.

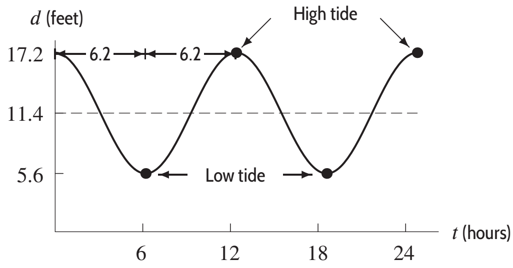


Figure 9.27

(b) The midline is $d = \dfrac{17.2 + 5.6}{2} = 11.4$ and the amplitude is $17.2 - 11.4 = 5.8$. The period is 12.4. Thus we get $d = f(t) = 11.4 + 5.8\cos\left(\frac{\pi}{6.2}t\right)$.

(c) We find the first t value when $d = f(t) = 8$:

$$8 = 11.4 + 5.8\cos\left(\frac{\pi}{6.2}t\right)$$

Using the $\cos^{-1}$ function, we have

$$\begin{aligned}\frac{-3.4}{5.8} &= \cos\left(\frac{\pi}{6.2}t\right)\\ \cos^{-1}\left(\frac{-3.4}{5.8}\right) &= \frac{\pi}{6.2}t\\ t &= \frac{6.2}{\pi}\cos^{-1}\left(\frac{-3.4}{5.8}\right) \approx 4.336 \text{ hours.}\end{aligned}$$

Since $0.336(60) \approx 20$ minutes, the latest time the boat can set sail is 4:20 pm.

41. The curve is a sine curve with an amplitude of 5, a period of 8 and a vertical shift of -3. Thus the equation for the curve is $y = 5\sin\left(\frac{\pi}{4}x\right) - 3$. Solving for $y = 0$, we have

$$\begin{aligned}5\sin\left(\frac{\pi}{4}x\right) &= 3\\ \sin\left(\frac{\pi}{4}x\right) &= \frac{3}{5}\\ \frac{\pi}{4}x &= \sin^{-1}\left(\frac{3}{5}\right)\\ x &= \frac{4}{\pi}\sin^{-1}\left(\frac{3}{5}\right) \approx 0.819.\end{aligned}$$

This is the x-coordinate of P. The x-coordinate of Q is to the left of 4 by the same distance P is to the right of O, by the symmetry of the sine curve. Therefore,

$$x \approx 4 - 0.819 = 3.181$$

is the x-coordinate of Q.

42. **(a)** The value of $\sin t$ will be between -1 and 1. This means that $k\sin t$ will be between $-k$ and k. Thus, $t^2 = k\sin t$ will be between 0 and k. So

$$-\sqrt{k} \le t \le \sqrt{k}.$$

(b) Plotting $2\sin t$ and t^2 on a calculator, we see that $t^2 = 2\sin t$ for $t = 0$ and $t \approx 1.40$.

(c) Compare the graphs of $k\sin t$, a sine wave, and t^2, a parabola. As k increases, the amplitude of the sine wave increases, and so the sine wave intersects the parabola in more points.

(d) Plotting $k\sin t$ and t^2 on a calculator for different values of k, we see that if $k \approx 20$, this equation will have a negative solution at $t \approx -4.3$, but that if k is any smaller, there will be no negative solution.

43. (a) In Figure 9.28, the earth's center is labeled O and two radii are extended, one through S, your ship's position, and one through H, the point on the horizon. Your line of sight to the horizon is tangent to the surface of the earth. A line tangent to a circle at a given point is perpendicular to the circle's radius at that point. Thus, since your line of sight is tangent to the earth's surface at H, it is also perpendicular to the earth's radius at H. This means that triangle OCH is a right triangle. Its hypotenuse is $r + x$ and its legs are r and d. From the Pythagorean theorem, we have

$$\begin{aligned} r^2 + d^2 &= (r + x)^2 \\ d^2 &= (r + x)^2 - r^2 \\ &= r^2 + 2rx + x^2 - r^2 = 2rx + x^2. \end{aligned}$$

Since d is positive, we have $d = \sqrt{2rx + x^2}$.

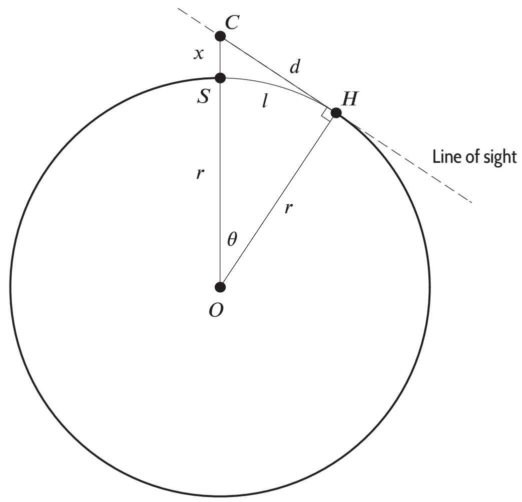


Figure 9.28

(b) We begin by using the formula obtained in part (a):

$$\begin{aligned} d &= \sqrt{2rx + x^2} \\ &= \sqrt{2(6{,}370{,}000)(50) + 50^2} \\ &\approx 25{,}238.908. \end{aligned}$$

Thus, you would be able to see a little over 25 kilometers from the crow's nest C.

Having found a formula for d, we will now try to find a formula for l, the distance along the earth's surface from the ship to the horizon H. In Figure 9.28, l is the arc length specified by the angle θ (in radians). The formula for arc length is

$$l = r\theta.$$

In this case, we must determine θ. From Figure 9.28 we see that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{r}{r + x}.$$

Thus,

$$\theta = \cos^{-1}\left(\frac{r}{r + x}\right)$$

since $0 \le \theta \le \pi/2$. This means that

$$\begin{aligned} l = r\theta &= r\cos^{-1}\left(\frac{r}{r + x}\right) \\ &= 6{,}370{,}000\cos^{-1}\left(\frac{6{,}370{,}000}{6{,}370{,}050}\right) \approx 25{,}238.776 \text{ meters.} \end{aligned}$$

There is very little difference—about 0.13 m or 13 cm—between the distance d that you can see and the distance l that the ship must travel to reach the horizon. If this is surprising, keep in mind that Figure 9.28 has not been drawn to scale. In reality, the mast height x is significantly smaller than the earth's radius r so that the point C in the crow's nest is very close to the ship's position at point S. Thus, the line segment d and the arc l are almost indistinguishable.

Solutions for Section 9.2

Exercises

1. No, these expressions are not equal everywhere. They have different amplitudes (1 and 3) and different periods ($2\pi/3$ and 2π).

The value of the two functions are different at $t = \pi/2$, since $\sin(3\pi/2) = -1$ and $3\sin(\pi/2) = 3$.

2. No, these expressions are not equal everywhere. (In fact, $\cos^2 t = -(\sin^2 t - 1)$.)

The value of the two functions are different at $t = 0$, since $\cos^2 0 = 1$ and $\sin^2 0 - 1 = -1$.

3. Yes, these expressions are equal everywhere. Expanding the first expression, we get

$$(\cos t - \sin t)(\cos t + \sin t) = \cos^2 t - \sin^2 t.$$

4. Yes, these expressions are equal everywhere they are defined. (They are both defined for all t except where $\cos t = 0$.) We use $\sec t = 1/\cos t$, giving

$$\sec t \sin t = \frac{1}{\cos t}\sin t = \frac{\sin t}{\cos t} = \tan t.$$

5. We use the relationship $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$. Substitute $\sin\theta = 1/4$:

$$\begin{aligned}\left(\frac{1}{4}\right)^2 + \cos^2\theta &= 1\\ \frac{1}{16} + \cos^2\theta &= 1\\ \cos^2\theta &= 1 - \frac{1}{16} = \frac{15}{16}\\ \cos\theta &= \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}.\end{aligned}$$

Because θ is in the first quadrant, $\cos\theta$ is positive, so $\cos\theta = \sqrt{15}/4$. To find $\tan\theta$, use the relationship

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}}.$$

6. We use the relationship $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$. Substitute $\sin\theta = 3/5$:

$$\begin{aligned}\left(\frac{3}{5}\right)^2 + \cos^2\theta &= 1\\ \frac{9}{25} + \cos^2\theta &= 1\\ \cos^2\theta &= 1 - \frac{9}{25} = \frac{16}{25}\\ \cos\theta &= \pm\frac{4}{5}.\end{aligned}$$

Because θ is in the second quadrant, $\cos\theta$ is negative, so $\cos\theta = -4/5$. To find $\tan\theta$, use the relationship

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -\frac{3}{4}.$$

7. We use the relationship $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$. Substitute $\cos\theta = -2/5$:

$$\begin{aligned}
\sin^2\theta + \left(-\frac{2}{5}\right)^2 &= 1\\
\sin^2\theta + \frac{4}{25} &= 1\\
\sin^2\theta &= 1 - \frac{4}{25} = \frac{21}{25}\\
\sin\theta &= \pm\frac{\sqrt{21}}{5}.
\end{aligned}$$

Because θ is in the third quadrant, $\sin\theta$ is negative, so $\sin\theta = -\sqrt{21}/5$. To find $\tan\theta$, use the relationship

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{21}/5}{-2/5} = \frac{\sqrt{21}}{2}.$$

8. We use the relationship $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$. Substitute $\cos\theta = 1/3$:

$$\begin{aligned}
\sin^2\theta + \left(\frac{1}{3}\right)^2 &= 1\\
\sin^2\theta + \frac{1}{9} &= 1\\
\sin^2\theta &= 1 - \frac{1}{9} = \frac{8}{9}\\
\sin\theta &= \pm\frac{\sqrt{8}}{3}.
\end{aligned}$$

Because θ is in the fourth quadrant, $\sin\theta$ is negative, so $\sin\theta = -\sqrt{8}/3$. To find $\tan\theta$, use the relationship

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{8}/3}{1/3} = -\sqrt{8}.$$

9. We have:

$$\begin{aligned}
\tan t\cos t - \frac{\sin t}{\tan t} &= \frac{\sin t}{\cos t}\cdot\cos t - \frac{\sin t}{\left(\frac{\sin t}{\cos t}\right)} \quad \text{because } \tan t = \tfrac{\sin t}{\cos t}\\
&= \sin t - \sin t\cdot\frac{\cos t}{\sin t}\\
&= \sin t - \cos t.
\end{aligned}$$

10. We have:

$$\begin{aligned}
2\cos t\,(3 - 7\tan t) &= 6\cos t - 14\cos t\cdot\tan t\\
&= 6\cos t - 14\cos t\cdot\frac{\sin t}{\cos t}\\
&= 6\cos t - 14\sin t.
\end{aligned}$$

11. We have:

$$\begin{aligned}
2\cos t - \cos t\,(1 - 3\tan t) &= 2\cos t - \cos t + 3\cos t\cdot\tan t\\
&= \cos t + 3\cos t\cdot\frac{\sin t}{\cos t}\\
&= \cos t + 3\sin t.
\end{aligned}$$

12. We have:

$$\begin{aligned}2\cos t\,(3\sin t - 4\tan t) &= 6\sin t\cos t - 8\tan t\cdot\cos t\\ &= 6\sin t\cos t - 8\frac{\sin t}{\cos t}\cdot\cos t\\ &= 6\sin t\cos t - 8\sin t.\end{aligned}$$

13. Writing $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we have

$$\frac{\sin 2\alpha}{\cos\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos\alpha} = 2\sin\alpha.$$

14. Writing $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\frac{\cos^2\theta - 1}{\sin\theta} = \frac{1 - \sin^2\theta - 1}{\sin\theta} = \frac{-\sin^2\theta}{\sin\theta} = -\sin\theta.$$

15. Writing $\cos 2t = \cos^2 t - \sin^2 t$ and factoring, we have

$$\frac{\cos 2t}{\cos t + \sin t} = \frac{\cos^2 t - \sin^2 t}{\cos t + \sin t} = \frac{(\cos t - \sin t)(\cos t + \sin t)}{\cos t + \sin t} = \cos t - \sin t.$$

16. Adding fractions gives

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta}.$$

17. Combining terms and using $\cos^2\phi + \sin^2\phi = 1$, we have

$$\frac{\cos\phi - 1}{\sin\phi} + \frac{\sin\phi}{\cos\phi + 1} = \frac{(\cos\phi - 1)(\cos\phi + 1) + \sin^2\phi}{\sin\phi(\cos\phi + 1)} = \frac{\cos^2\phi - 1 + \sin^2\phi}{\sin\phi(\cos\phi + 1)} = \frac{0}{\sin\phi(\cos\phi + 1)} = 0$$

18. Using $\tan t = \sin t/\cos t$, we have

$$\begin{aligned}\frac{1}{\sin t\cos t} - \frac{1}{\tan t} &= \frac{1}{\sin t\cos t} - \frac{1}{\sin t/\cos t}\\ &= \frac{1}{\sin t\cos t} - \frac{\cos t}{\sin t}\\ &= \frac{1-\cos^2 t}{\sin t\cos t}\\ &= \frac{\sin^2 t}{\sin t\cos t}\\ &= \frac{\sin t}{\cos t}\\ &= \tan t.\end{aligned}$$

19. We have: $\dfrac{\sin\sqrt{\theta}}{\cos\sqrt{\theta}} = \tan\sqrt{\theta}$.

20. We have:

$$\begin{aligned}\frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} &= 2\cdot\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\\ &= 2\tan\frac{\alpha}{2}.\end{aligned}$$

21. We have:

$$\frac{3\sin(\phi+1)}{4\cos(\phi+1)} = \frac{3}{4}\cdot\frac{\sin(\phi+1)}{\cos(\phi+1)}$$
$$= \frac{3}{4}\tan(\phi+1).$$

22. We have:

$$\frac{1}{\left(\dfrac{\cos\left(r^2-s^2\right)}{\sin\left(r^2-s^2\right)}\right)} = \frac{\sin\left(r^2-s^2\right)}{\cos\left(r^2-s^2\right)} = \tan\left(r^2-s^2\right).$$

23. We have

$$2\sin\left(\frac{2}{k+3}\right)\cdot\frac{5}{3\cos\left(\dfrac{2}{k+3}\right)} = \frac{10}{3}\cdot\frac{\sin\left(\dfrac{2}{k+3}\right)}{\cos\left(\dfrac{2}{k+3}\right)}$$
$$= \frac{10}{3}\tan\left(\frac{2}{k+3}\right).$$

24. We have:

$$\frac{2}{\cos\left(1-\dfrac{1}{z}\right)}\cdot\frac{\sin\left(1-\dfrac{1}{z}\right)}{3} = \frac{2}{3}\frac{\sin\left(1-\dfrac{1}{z}\right)}{\cos\left(1-\dfrac{1}{z}\right)}$$
$$= \frac{2}{3}\tan\left(1-\frac{1}{z}\right).$$

25. We solve the Pythagorean identity for $\sin\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1-\cos^2\theta$$
$$(\sin\theta)^2 = 1-\cos^2\theta.$$

If $\sin\theta \ge 0$,

$$\sin\theta = \sqrt{1-\cos^2\theta}.$$

If $\sin\theta < 0$,

$$\sin\theta = -\sqrt{1-\cos^2\theta}.$$

26. The relevant identities are $\cos^2\theta + \sin^2\theta = 1$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. See Table 9.1.

Table 9.1

θ in rad.	$\sin^2\theta$	$\cos^2\theta$	$\sin 2\theta$	$\cos 2\theta$
1	0.708	0.292	0.909	−0.416
$\pi/2$	1	0	0	−1
2	0.827	0.173	−0.757	−0.654
$5\pi/6$	1/4	3/4	$-\sqrt{3}/2$	1/2

Problems

27. Since $\cos 2t$ has period π and $\sin t$ has period 2π, if the result we want holds for $0 \le t \le 2\pi$, it holds for all t. So let's concentrate on the interval $0 \le t \le 2\pi$.

Solving $\cos 2t = 0$ gives $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

From the graph in Figure 9.29, we see $\cos 2t > 0$ for $0 \le t < \pi/4$, $3\pi/4 < t < 5\pi/4$, $7\pi/4 < t \le 2\pi$.

Solving $1 - 2\sin^2 t = 0$ gives

$$\sin^2 t = \frac{1}{2}$$
$$\sin t = \pm\frac{1}{\sqrt{2}},$$

so $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

From the graph of $y = \sin t$ and the lines $y = 1/\sqrt{2}$ and $y = -1/\sqrt{2}$ in Figure 9.30, we see that $-1/\sqrt{2} < \sin t < 1/\sqrt{2}$ on the same intervals that $\cos 2t > 0$.

Now if $-1/\sqrt{2} < \sin t < 1/\sqrt{2}$, then

$$\sin^2 t < \frac{1}{2}$$
$$1 - 2\sin^2 t > 0.$$

Thus, $\cos 2t$ and $1 - 2\sin^2 t$ have the same sign for all t.

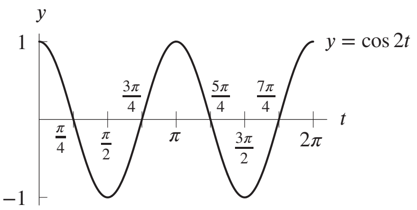


Figure 9.29

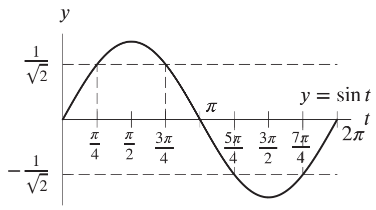


Figure 9.30

28. **(a)** $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2(1 - \cos^2\theta) = 1 - 2 + 2\cos^2\theta = 2\cos^2\theta - 1 = 2(\cos\theta)^2 - 1$.
(b) $\cos 2\theta = 1 - 2\sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = \cos^2\theta - \sin^2\theta = (\cos\theta)^2 - (\sin\theta)^2$.

29. We have

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}.$$

Dividing both top and bottom by $\cos^2\theta$ gives

$$\tan 2\theta = \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}} = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

30. We know that $\cos^2\theta = 1 - \sin^2\theta = 1 - (3/5)^2 = 16/25$, and since θ is in the second quadrant, $\cos\theta = -\sqrt{16/25} = -4/5$. Thus $\sin 2\theta = 2\sin\theta\cos\theta = 2(3/5)(-4/5) = -24/25$. Furthermore, $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2(3/5)^2 = 7/25$, and $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24}{25} \cdot \frac{25}{7} = \frac{-24}{7}$.

31. Multiply the denominator by $1 + \cos t$ to get $\sin^2 t$:

$$\begin{aligned}\frac{\sin t}{1 - \cos t} &= \frac{\sin t}{(1 - \cos t)}\frac{(1 + \cos t)}{(1 + \cos t)} \\ &= \frac{\sin t(1 + \cos t)}{1 - \cos^2 t} \\ &= \frac{\sin t(1 + \cos t)}{\sin^2 t} \\ &= \frac{1 + \cos t}{\sin t}.\end{aligned}$$

32. Get a common denominator:

$$\begin{aligned}\frac{\cos x}{1-\sin x}-\tan x &= \frac{\cos x}{1-\sin x}-\frac{\sin x}{\cos x}\\ &= \frac{\cos^2 x-\sin x(1-\sin x)}{(1-\sin x)(\cos x)}\\ &= \frac{\cos^2 x-\sin x+\sin^2 x}{(1-\sin x)(\cos x)}\\ &= \frac{1-\sin x}{(1-\sin x)\cos x}=\frac{1}{\cos x}.\end{aligned}$$

33. In order to get tan to appear, divide by $\cos x\cos y$:

$$\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}}=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

34. Using the trigonometric identity $\cos^2\theta=1-\sin^2\theta$, we have

$$\begin{aligned}\sin^2\theta-\cos^2\theta &= \sin\theta\\ \sin^2\theta-(1-\sin^2\theta) &= \sin\theta\\ 2\sin^2\theta-\sin\theta-1 &= 0\\ (2\sin\theta+1)(\sin\theta-1) &= 0\\ \sin\theta=-\frac{1}{2} \quad &\text{or} \quad \sin\theta=1.\end{aligned}$$

If $\sin\theta=1$, then $\theta=\pi/2$. On the other hand, if $\sin\theta=-1/2$, we first calculate the associated reference angle, which is $\sin^{-1}(1/2)=\pi/6$. Using a graph of the sine function on the interval $0\le\theta\le 2\pi$, we see that the two solutions to $\sin\theta=-1/2$ are given by $\theta=\pi+\pi/6=7\pi/6$ and $\theta=2\pi-\pi/6=11\pi/6$. Combining the above observations, we see that there are three solutions to the original equation: $\pi/2, 7\pi/6$, and $11\pi/6$.

35. Using the trigonometric identity $\sin(2\theta)=2\sin\theta\cos\theta$, we have

$$\begin{aligned}\sin(2\theta)-\cos\theta &= 0\\ 2\sin\theta\cos\theta-\cos\theta &= 0\\ \cos\theta(2\sin\theta-1) &= 0\\ \cos\theta=0 \quad &\text{or} \quad \sin\theta=\frac{1}{2}.\end{aligned}$$

If $\cos\theta=0$, then we have two solutions: $\theta=\pi/2$ and $\theta=3\pi/2$. On the other hand, if $\sin\theta=1/2$, we first calculate the associated reference angle, which is $\sin^{-1}(1/2)=\pi/6$. Using a graph of the sine function on the interval $0\le\theta\le 2\pi$, we see that the two solutions to $\sin\theta=1/2$ are given by $\theta=\pi/6$ and $\theta=\pi-\pi/6=5\pi/6$. Combining the above observations, we see that there are four solutions to the original equation: $\pi/2, 3\pi/2, \pi/6$ and $5\pi/6$.

36. Using the trigonometric identity $\sec^2\theta=\tan^2\theta+1$, we have

$$\begin{aligned}\sec^2\theta &= 1-\tan\theta\\ \tan^2\theta+1 &= 1-\tan\theta\\ \tan^2\theta+\tan\theta &= 0\\ \tan\theta(\tan\theta+1) &= 0\\ \tan\theta=0 \quad &\text{or} \quad \tan\theta=-1.\end{aligned}$$

If $\tan\theta=0$, then we have three solutions: $\theta=0$ and $\theta=\pi$ and $\theta=2\pi$. On the other hand, if $\tan\theta=-1$, we first calculate the associated reference angle, which is $\tan^{-1}(1)=\pi/4$. Using a graph of the tangent function on the interval $0\le\theta\le 2\pi$, we see that the two solutions to $\tan\theta=-1$ are given by $\theta=\pi-\pi/4=3\pi/4$ and $\theta=2\pi-\pi/4=7\pi/4$. Combining the above observations, we see that there are five solutions to the original equation: $0, \pi, 2\pi, 3\pi/4$, and $7\pi/4$.

37. Using the trigonometric identity $\tan(2\theta) = 2\tan\theta/(1-\tan^2\theta)$, we have

$$\begin{aligned}
\tan(2\theta) + \tan\theta &= 0\\
\frac{2\tan\theta}{1-\tan^2\theta} &= -\tan\theta\\
2\tan\theta &= -\tan\theta + \tan^3\theta\\
\tan^3\theta - 3\tan\theta &= 0\\
\tan\theta(\tan^2\theta - 3) &= 0\\
\tan\theta = 0 \quad &\text{or} \quad \tan\theta = \pm\sqrt{3}.
\end{aligned}$$

If $\tan\theta = 0$, then we have three solutions: $\theta = 0$ and $\theta = \pi$ and $\theta = 2\pi$. On the other hand, if $\tan\theta = \sqrt{3}$, we first calculate the associated reference angle, which is $\tan^{-1}(\sqrt{3}) = \pi/3$. Using a graph of the tangent function on the interval $0 \le \theta \le 2\pi$, we see that the two solutions to $\tan\theta = \sqrt{3}$ are given by $\theta = \pi/3$ and $\theta = \pi + \pi/3 = 4\pi/3$. Finally, if $\tan\theta = -\sqrt{3}$, we again have a reference angle of $\pi/3$, and the two solutions to $\tan\theta = -\sqrt{3}$ are given by $\theta = \pi - \pi/3 = 2\pi/3$ and $\theta = 2\pi - \pi/3 = 5\pi/3$. Combining the above observations, we see that there are seven solutions to the original equation: $0, \pi, \pi/3, 4\pi/3, 2\pi/3, 5\pi/3$, and 2π.

38. Not an identity. False for $x = 2$.

39. Not an identity. False for $x = 2$.

40. Not an identity. False for $x = 2$.

41. Not an identity. False for $x = \pi/2$.

42. Not an identity. False for $x = 2$.

43. If we let $x = 1$, then we have

$$\sin(x^2) = \sin(1^2) = 0.841 \neq 1.683 = 2\sin 1 = 2\sin x.$$

Therefore, since the equation is not true for $x = 1$, it is not an identity.

44. Identity. $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{1+2\cos^2 x - 1} = \dfrac{2\sin x\cos x}{2\cos^2 x} = \dfrac{\sin x}{\cos x} = \tan x.$

45. If we let $A = 1$, then we have

$$\frac{\sin(2A)}{\cos(2A)} = \frac{\sin 2}{\cos 2} = -2.185 \neq 3.115 = 2\tan 1 = 2\tan A.$$

Therefore, since the equation is not true for $A = 1$, it is not an identity.

46. We have

$$\begin{aligned}
\frac{\sin^2\theta - 1}{\cos\theta} &= \frac{-(1-\sin^2\theta)}{\cos\theta}\\
&= \frac{-\cos^2\theta}{\cos\theta}\\
&= -\cos\theta.
\end{aligned}$$

Therefore, the equation is an identity.

47. Not an identity. False for $x = 0$.

48. Identity. $\sin x\tan x = \sin x \cdot \dfrac{\sin x}{\cos x} = \dfrac{\sin^2 x}{\cos x} = \dfrac{1-\cos^2 x}{\cos x}.$

49. Working on the left side, we have

$$\begin{aligned}
\tan t + \frac{1}{\tan t} &= \frac{\sin t}{\cos t} + \frac{1}{\sin t/\cos t}\\
&= \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}\\
&= \frac{\sin^2 t + \cos^2 t}{\cos t\sin t}\\
&= \frac{1}{\sin t\cos t}.
\end{aligned}$$

Therefore, the left side equals the right side and the equation is an identity.

50. If we let $x = 1$, then we have

$$\sin\left(\frac{1}{x}\right) = \sin 1 = 0.841 \neq 0 = \sin 1 - \sin 1 = \sin 1 - \sin x.$$

Therefore, since the equation is not true for $x = 1$, it is not an identity.

51. Identity. $\dfrac{2\tan x}{1+\tan^2 x}\cdot\dfrac{\cos^2 x}{\cos^2 x} = \dfrac{2\sin x\cos x}{\cos^2 x+\sin^2 x} = \dfrac{\sin 2x}{1} = \sin 2x.$

52. If we let $\theta = 1$, then we have

$$\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} = \frac{\sin 1}{\cos 1} - \frac{\cos 1}{\sin 1} = 0.915 \neq -0.458 = \frac{\cos 2}{\sin 2} = \frac{\cos(2\theta)}{\sin(2\theta)}.$$

Therefore, since the equation is not true for $\theta = 1$, it is not an identity.

53. Identity. $\dfrac{1-\tan^2 x}{1+\tan^2 x}\cdot\dfrac{\cos^2 x}{\cos^2 x} = \dfrac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x} = \dfrac{\cos 2x}{1} = \cos 2x.$

54. **(a)** We can rewrite the equation as follows:

$$0 = \cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta.$$

Factoring, we get

$$(2\cos\theta - 1)(\cos\theta + 1) = 0.$$

Thus the solutions occur when $\cos\theta = -1$ or $\cos\theta = \frac{1}{2}$. These are special values of cosine. If $\cos\theta = -1$ then we have $\theta = 180^\circ$. If $\cos\theta = \frac{1}{2}$ we have $\theta = 60^\circ$ or 300°. Thus the solutions are

$$\theta = 60^\circ,\ 180^\circ,\ \text{and } 300^\circ.$$

(b) Using the Pythagorean identity, we can substitute $\cos^2\theta = 1 - \sin^2\theta$ and get

$$2(1 - \sin^2\theta) = 3\sin\theta + 3.$$

This gives

$$-2\sin^2\theta - 3\sin\theta - 1 = 0.$$

Factoring, we get

$$-2\sin^2\theta - 3\sin\theta - 1 = -(2\sin\theta + 1)(\sin\theta + 1) = 0.$$

Thus the solutions occur when $\sin\theta = -\frac{1}{2}$ or when $\sin\theta = -1$. If $\sin\theta = -\frac{1}{2}$, we have

$$\theta = \frac{7\pi}{6} \quad\text{and}\quad \frac{11\pi}{6}.$$

If $\sin\theta = -1$, we have

$$\theta = \frac{3\pi}{2}.$$

55. Note the hypotenuse of the triangle is $\sqrt{1+y^2}$.

(a) $y = \dfrac{y}{1} = \tan\theta.$

(b) $\cos\phi = \sin(\pi/2 - \phi) = \sin\theta.$

(c) Since $\cos\theta = \dfrac{1}{\sqrt{1+y^2}}$, we have $\sqrt{1+y^2} = \dfrac{1}{\cos\theta}$, or $1 + y^2 = \left(\dfrac{1}{\cos\theta}\right)^2$. (Alternatively, $1 + y^2 = 1 + \tan^2\theta$.)

(d) Triangle area $= \dfrac{1}{2}(\text{base})(\text{height}) = \dfrac{1}{2}(1)(y)$. But $y = \tan\theta$, so the area is $\frac{1}{2}\tan\theta$.

56. **(a)** By the Pythagorean theorem, the side adjacent to θ has length $\sqrt{1-y^2}$. So

$$\cos\theta = \sqrt{1-y^2}/1 = \sqrt{1-y^2}.$$

(b) Since $\sin\theta = y/1$, we have

$$\tan\theta = \frac{y}{\sqrt{1-y^2}}.$$

(c) Using the double-angle formula,

$$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2y^2.$$

(d) Supplementary angles have equal sines:

$$\sin(\pi - \theta) = \sin\theta = y.$$

(e) Since $\cos(\pi/2 - \theta) = y$, we have $\sin(\cos^{-1}(y)) = \sin(\pi/2 - \theta) = \sqrt{1 - y^2}$. So

$$\sin^2(\cos^{-1}(y)) = 1 - y^2.$$

57. We have $\cos\theta = x/3$, so $\sin\theta = \sqrt{1 - (x/3)^2} = \frac{\sqrt{9-x^2}}{3}$. Therefore,

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{9-x^2}}{3}\right)\left(\frac{x}{3}\right) = \frac{2x}{9}\sqrt{9-x^2}.$$

58. We have $\sin\theta = \frac{x+1}{5}$, so $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{x+1}{5}\right)^2 = 1 - \frac{2(x+1)^2}{25}$.

59. **(a)** Let $\theta = \cos^{-1} x$, so $\cos\theta = x$. Then, since $0 \le \theta \le \pi$, $\sin\theta = \sqrt{1-x^2}$ and $\tan\theta = \frac{\sqrt{1-x^2}}{x}$, and $\tan(2\cos^{-1} x) = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$. Now $1 - \tan^2\theta = 1 - \frac{1-x^2}{x^2} = \frac{2x^2-1}{x^2}$, so $\tan 2\theta = \frac{2\sqrt{1-x^2}}{x} \cdot \frac{x^2}{2x^2-1} = \frac{2x\sqrt{1-x^2}}{2x^2-1}$.

(b) Let $\theta = \tan^{-1} x$, so $\tan\theta = x$. Then $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, so $\sin(2\tan^{-1} x) = \sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{2x}{1+x^2}$.

60. We will use the identity $\cos(2x) = 2\cos^2 x - 1$, where x will be 2θ.

$$\begin{aligned}\cos 4\theta &= \cos(2x)\\ &= 2\cos^2 x - 1 \quad \text{(using the identity for } \cos(2x))\\ &= 2(2\cos^2\theta - 1)^2 - 1 \quad \text{(using the identity for } \cos(2\theta))\end{aligned}$$

61. First use $\sin(2x) = 2\sin x\cos x$, where $x = 2\theta$. Then

$$\sin(4\theta) = \sin(2x) = 2\sin(2\theta)\cos(2\theta).$$

Since $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = 2\cos^2\theta - 1$, we have

$$\sin 4\theta = 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1).$$

62. **(a)** Since $\pi/2 < t \le \pi$ we have $0 \le \pi - t < \pi/2$ so the double-angle formula for sine can be used for the angle $\theta = \pi - t$. Therefore $\sin 2\theta = 2\sin\theta\cos\theta$ tells us that

$$\sin 2(\pi - t) = 2\sin(\pi - t)\cos(\pi - t).$$

(b) By periodicity, $\sin 2(\pi - t) = \sin(2\pi - 2t) = \sin(-2t)$. By oddness, $\sin(-2t) = -\sin 2t$. Thus

$$\sin 2(\pi - t) = -\sin 2t.$$

(c) We have $\cos(\pi - t) = -\cos(-t) = -\cos t$, where the first equality is by what was given and the second by the evenness of cosine.

Similarly, we have $\sin(\pi - t) = -\sin(-t) = -(-\sin t) = \sin t$ where the first equality is by what was given and the second by the oddness of sine.

(d) Substitution of the results of parts (b) and (c) into part (a) shows that

$$-\sin 2t = 2\sin t(-\cos t).$$

Multiplication by -1 gives

$$\sin 2t = 2\sin t\cos t.$$

63. **(a)** Since $-\pi \le t < 0$ we have $0 < -t \le \pi$ so the double-angle formula for sine can be used for the angle $\theta = -t$. Therefore $\sin 2\theta = 2\sin\theta\cos\theta$ tells us that

$$\sin(-2t) = 2\sin(-t)\cos(-t).$$

(b) Since sine is odd, we have $\sin(-2t) = -\sin 2t$. Since sine is odd and cosine is even, we have

$$2\sin(-t)\cos(-t) = 2(-\sin t)\cos t = -2\sin t\cos t.$$

Substitution of these results into the results of part (a) shows that

$$-\sin(2t) = -2\sin t\cos t.$$

Multiplication by -1 gives

$$\sin 2t = 2\sin t\cos t.$$

64. **(a)** Since $\pi/2 < t \le \pi$ we have $0 \le \pi - t < \pi/2$ so the double-angle formula for cosine can be used for the angle $\theta = \pi - t$. Therefore $\cos 2\theta = 1 - 2\sin^2\theta$ tells us that

$$\cos 2(\pi - t) = 1 - 2\sin^2(\pi - t).$$

(b) By periodicity, $\cos 2(\pi - t) = \cos(2\pi - 2t) = \cos(-2t)$. By evenness, $\cos(-2t) = \cos 2t$. Thus

$$\cos 2(\pi - t) = \cos 2t.$$

(c) We have $1 - 2\sin^2(\pi - t) = 1 - 2(-\sin(-t))^2 = 1 - 2(\sin t)^2$ where the first equality is by what is given and the second by the oddness of sine.

(d) Substitution of the results of parts (b) and (c) into part (a) gives

$$\cos 2t = 1 - 2\sin^2 t.$$

65. **(a)** Since $-\pi \le t < 0$ we have $0 < -t \le \pi$ so the double-angle formula for cosine can be used for the angle $\theta = -t$. Therefore $\cos 2\theta = 1 - 2\sin^2\theta$ tells us that

$$\cos(-2t) = 1 - 2\sin^2(-t).$$

(b) Since cosine is even we have $\cos(-2t) = \cos 2t$. Since sine is odd we have

$$-2\sin^2(-t) = 1 - 2(-\sin t)^2 = 1 - 2\sin^2 t.$$

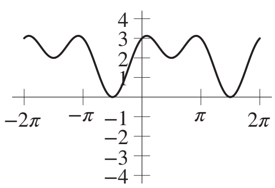
Substitution of these results into the results of part (a) gives

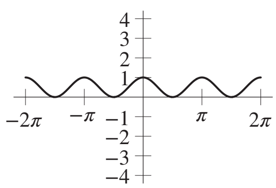
$$\cos 2t = 1 - 2\sin^2 t.$$

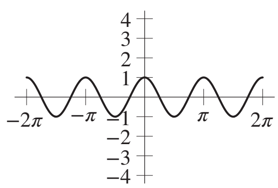
66. By graphing we can see which expressions appear to be identically equal. The graphs all show the same window, $-2\pi \le x \le 2\pi$, $-4 \le y \le 4$. The following pairs of expressions look identical:
a and *i*;
b and *l*;
c and *d* and *f*;
e and *g*;
h and *j*.

We can check the identities algebraically. For example, for *a* and *i*:

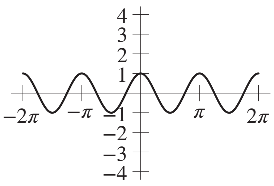
$$2\cos^2 t + \sin t + 1 = 2(1 - \sin^2 t) + \sin t + 1 = -2\sin^2 t + \sin t + 3.$$

(a)


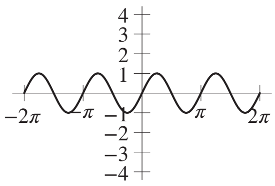
(b)


(c)


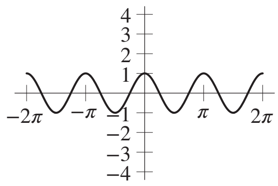
(d)



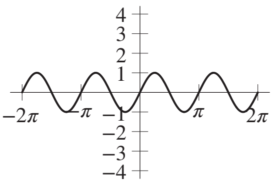
(e)



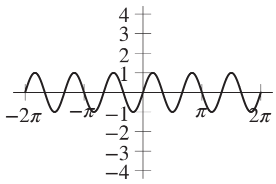
(f)



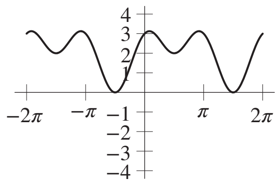
(g)



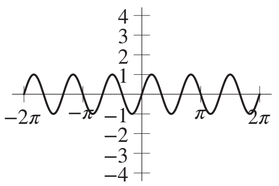
(h)



(i)



(j)



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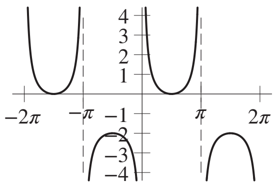
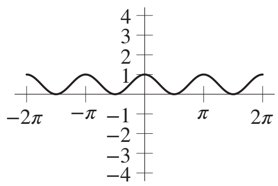
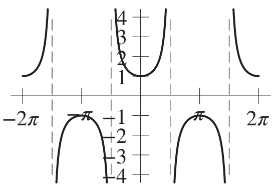


(l)



(m)



Solutions for Section 9.3

Exercises

1. Applying the sum-of-angles formula for sine, we have

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ &= (0.84)(0.39) + (0.54)(0.92) = 0.8244.\end{aligned}$$

2. Applying the difference-of-angles formula for sine, we have

$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B\\ &= (0.84)(0.39) - (0.54)(0.92) = -0.1692.\end{aligned}$$

3. Applying the sum-of-angles formula for cosine, we have

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= (0.54)(0.39) - (0.84)(0.92) = -0.5622.\end{aligned}$$

4. Applying the difference-of-angles formula for cosine, we have

$$\begin{aligned}\cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= (0.54)(0.39) + (0.84)(0.92) = 0.9834.\end{aligned}$$

5. Applying the sum-of-angles formula for sine, we have

$$\begin{aligned}\sin(S + T) &= \sin S \cos T + \cos S \sin T \\ &= \left(\frac{7}{25}\right)\left(\frac{8}{17}\right) + \left(\frac{24}{25}\right)\left(\frac{15}{17}\right) = \frac{416}{425}.\end{aligned}$$

6. Applying the difference-of-angles formula for sine, we have

$$\begin{aligned}\sin(S - T) &= \sin S \cos T - \cos S \sin T \\ &= \left(\frac{7}{25}\right)\left(\frac{8}{17}\right) - \left(\frac{24}{25}\right)\left(\frac{15}{17}\right) = -\frac{304}{425}.\end{aligned}$$

7. Applying the sum-of-angles formula for cosine, we have

$$\begin{aligned}\cos(S + T) &= \cos S \cos T - \sin S \sin T \\ &= \left(\frac{24}{25}\right)\left(\frac{8}{17}\right) - \left(\frac{7}{25}\right)\left(\frac{15}{17}\right) = \frac{87}{425}.\end{aligned}$$

8. Applying the difference-of-angles formula for cosine, we have

$$\begin{aligned}\cos(S - T) &= \cos S \cos T + \sin S \sin T \\ &= \left(\frac{24}{25}\right)\left(\frac{8}{17}\right) + \left(\frac{7}{25}\right)\left(\frac{15}{17}\right) = \frac{297}{425}.\end{aligned}$$

9. Write $\sin 15° = \sin(45° - 30°)$, and then apply the appropriate trigonometric identity.

$$\begin{aligned}\sin 15° &= \sin(45° - 30°) \\ &= \sin 45° \cos 30° - \sin 30° \cos 45° \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\end{aligned}$$

Similarly, $\sin 75° = \sin(45° + 30°)$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \sin 30° \cos 45° \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\end{aligned}$$

Also, note that $\cos 75° = \sin(90° - 75°) = \sin 15°$, and $\cos 15° = \sin(90° - 15°) = \sin 75°$.

10. Since 345° is 300° + 45°, we can use the sum-of-angle formula for sine and say that

$$\sin 345 = \sin 300 \cos 45 + \sin 45 \cos 300 = (-\sqrt{3}/2)(\sqrt{2}/2) + (\sqrt{2}/2)(1/2) = (-\sqrt{6} + \sqrt{2})/4.$$

11. Since 105° is 60° + 45°, we can use the sum-of-angle formula for sine and say that

$$\sin 105 = \sin 60 \cos 45 + \sin 45 \cos 60 = (\sqrt{3}/2)(\sqrt{2}/2) + (\sqrt{2}/2)(1/2) = (\sqrt{6} + \sqrt{2})/4.$$

12. Since 285° is 240° + 45°, we can use the sum-of-angle formula for cosine and say that

$$\cos 285 = \cos 240 \cos 45 - \sin 240 \sin 45 = (-1/2)(\sqrt{2}/2) - (-\sqrt{3}/2)(\sqrt{2}/2) = (-\sqrt{2} + \sqrt{6})/4.$$

13. **(a)** We have $\sin(15° + 42°) = \sin 15° \cos 42° + \sin 42° \cos 15° = 0.839$.
(b) See Figure 9.31.

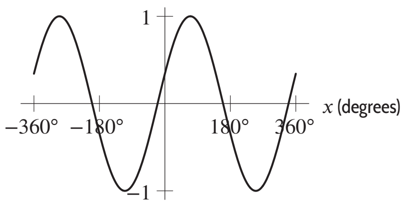


Figure 9.31

14. **(a)** We have $\sin(15° - 42°) = \sin 15° \cos 42° - \sin 42° \cos 15° = -0.454$.
(b) See Figure 9.32.

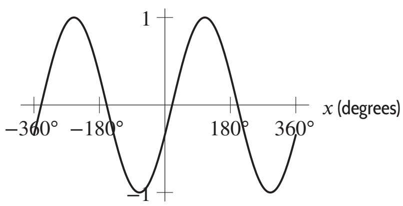


Figure 9.32

15. **(a)** We have $\cos(15° + 42°) = \cos 15° \cos 42° - \sin 15° \sin 42° = 0.545$.
(b) See Figure 9.33.

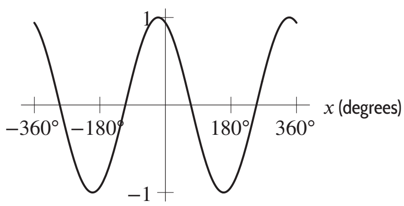


Figure 9.33

16. **(a)** We have $\cos(15° - 42°) = \cos 15° \cos 42° + \sin 15° \sin 42° = 0.891$.
(b) See Figure 9.34.

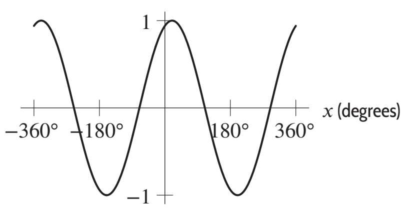


Figure 9.34

Problems

17. First, we note that the unlabeled side of the triangle has length $\sqrt{4-y^2}$.

(a) Using an angle difference formula, we have

$$\begin{aligned}\cos(\theta-\phi) &= \cos\theta\cos\phi+\sin\theta\sin\phi\\ &= \frac{\sqrt{4-y^2}}{2}\cdot\frac{y}{2}+\frac{y}{2}\cdot\frac{\sqrt{4-y^2}}{2}\\ &= \frac{y\sqrt{4-y^2}}{2}.\end{aligned}$$

(b) Using an angle difference formula, we have

$$\begin{aligned}\sin(\theta-\phi) &= \sin\theta\cos\phi-\cos\theta\sin\phi\\ &= \frac{y}{2}\cdot\frac{y}{2}-\frac{\sqrt{4-y^2}}{2}\cdot\frac{\sqrt{4-y^2}}{2}\\ &= \frac{y^2-(4-y^2)}{4}\\ &= \frac{y^2-2}{2}.\end{aligned}$$

(c) We have

$$\begin{aligned}\cos\theta-\cos\phi &= \frac{\sqrt{4-y^2}}{2}-\frac{y}{2}\\ &= \frac{\sqrt{4-y^2}-y}{2}.\end{aligned}$$

(d) Since $\theta+\phi=\pi/2$, we have

$$\sin(\theta+\phi)=\sin\left(\frac{\pi}{2}\right)=1.$$

18. **(a)** $\cos(t-\pi/2)=\cos t\cos\pi/2+\sin t\sin\pi/2=\cos t\cdot 0+\sin t\cdot 1=\sin t$.
(b) $\sin(t+\pi/2)=\sin t\cos\pi/2+\sin\pi/2\cos t=\sin t\cdot 0+1\cdot\cos t=\cos t$.

19. For the sine, we have $\sin 2t=\sin(t+t)=\sin t\cos t+\sin t\cos t=2\sin t\cos t$. This is the double-angle formula for sine.
For cosine, we have $\cos 2t=\cos(t+t)=\cos t\cos t-\sin t\sin t=\cos^2 t-\sin^2 t$. This is the double-angle formula for cosine.

20. **(a)** Applying the sum-of-angles formula for sine, we have

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ &= (0.67)(0.48)+(0.74)(0.87)=0.9654.\end{aligned}$$

(b) Applying the sum-of-angles formula for cosine, we have

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B\\ &= (0.74)(0.48)-(0.67)(0.87)=-0.2277.\end{aligned}$$

(c) Since $\tan\theta=\sin\theta/\cos\theta$, we have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{0.9654}{-0.2277}=-4.2398.$$

21. We have

$$\begin{aligned}\cos 3t &= \cos(2t+t)\\ &= \cos 2t\cos t - \sin 2t\sin t\\ &= (2\cos^2 t - 1)\cos t - (2\sin t\cos t)\sin t\\ &= \cos t((2\cos^2 t - 1) - 2\sin^2 t)\\ &= \cos t(2\cos^2 t - 1 - 2(1-\cos^2 t))\\ &= \cos t(2\cos^2 t - 1 - 2 + 2\cos^2 t)\\ &= \cos t(4\cos^2 t - 3)\\ &= 4\cos^3 t - 3\cos t,\end{aligned}$$

as required.

22. We are to prove that

$$\cos((n+1)\theta) = (2\cos\theta)(\cos(n\theta)) - \cos((n-1)\theta).$$

This holds if, and only if,

$$\cos((n+1)\theta) + \cos((n-1)\theta) = (2\cos\theta)(\cos(n\theta)).$$

Using the sum-of-cosines formula,

$$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2},$$

with $u = (n+1)\theta$ and $v = (n-1)\theta$ gives the result:

$$\begin{aligned}\cos((n+1)\theta) + 2\cos((n-1)\theta) &= 2\cos\frac{(n+1)\theta + (n-1)\theta}{2}\cos\frac{(n+1)\theta - (n-1)\theta}{2}\\ &= 2\cos(n\theta)\cos\theta.\end{aligned}$$

23. We manipulate the equation for the average rate of change as follows:

$$\begin{aligned}\frac{\cos(x+h)-\cos x}{h} &= \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= \frac{\cos x\cos h - \cos x}{h} - \frac{\sin x\sin h}{h}\\ &= \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).\end{aligned}$$

24. We manipulate the equation for the average rate of change as follows:

$$\begin{aligned}\frac{\sin(x+h)-\sin x}{h} &= \frac{\sin x\cos h + \sin h\cos x - \sin x}{h}\\ &= \frac{\sin x\cos h - \sin x}{h} + \frac{\sin h\cos x}{h}\\ &= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right).\end{aligned}$$

25. We manipulate the equation for the average rate of change as follows:

$$\begin{aligned}\frac{\tan(x+h)-\tan x}{h} &= \frac{\dfrac{\tan x + \tan h}{1 - \tan x\tan h} - \tan x}{h}\\ &= \frac{(\tan x + \tan h - \tan x + \tan^2 x\tan h)/(1 - \tan x\tan h)}{h}\\ &= \frac{\tan h + \tan^2 x\tan h}{(1 - \tan x\tan h)\cdot h}\end{aligned}$$

$$= \frac{\frac{\sin h}{\cos h} + \tan^2 x \cdot \frac{\sin h}{\cos h}}{\left(1 - \tan x \cdot \frac{\sin h}{\cos h}\right) \cdot h}$$

$$= \frac{(1+\tan^2 x)\frac{\sin h}{\cos h}}{\left(1 - \tan x \frac{\sin h}{\cos h}\right) \cdot h}$$

$$= \frac{\left(\frac{1}{\cos^2 x}\right) \cdot \frac{\sin h}{\cos h}}{\left(1 - \tan x \cdot \frac{\sin h}{\cos h}\right) \cdot h}$$

$$= \frac{\frac{1}{\cos^2 x} \cdot \sin h}{(\cos h - \tan x \sin h) \cdot h}$$

$$= \frac{\frac{1}{\cos^2 x} \cdot \sin h}{\cos h - \sin h \tan x} \cdot \left(\frac{1}{h}\right)$$

$$= \frac{1}{\cos^2 x} \frac{\sin h}{h} \cdot \frac{1}{\cos h - \sin h \tan x}.$$

26. **(a)** From $\cos 2u = 2\cos^2 u - 1$, we obtain $\cos u = \pm\sqrt{\frac{1+\cos 2u}{2}}$ and letting $u = \frac{v}{2}$, $\cos \frac{v}{2} = \pm\sqrt{\frac{1+\cos v}{2}}$.

(b) From $\tan \frac{1}{2}v = \frac{\sin \frac{1}{2}v}{\cos \frac{1}{2}v} = \frac{\pm\sqrt{\frac{1-\cos v}{2}}}{\pm\sqrt{\frac{1+\cos v}{2}}}$ we simplify to get $\tan \frac{1}{2}v = \pm\sqrt{\frac{1-\cos v}{1+\cos v}}$.

(c) The sign of $\sin \frac{1}{2}v$ is $+$, the sign of $\cos \frac{1}{2}v$ is $-$, and the sign of $\tan \frac{1}{2}v$ is $-$.

(d) The sign of $\sin \frac{1}{2}v$ is $-$, the sign of $\cos \frac{1}{2}v$ is $-$, and the sign of $\tan \frac{1}{2}v$ is $+$.

(e) The sign of $\sin \frac{1}{2}v$ is $-$, the sign of $\cos \frac{1}{2}v$ is $+$, and the sign of $\tan \frac{1}{2}v$ is $-$.

27. **(a)** Since ΔCAD and ΔCDB are both right triangles, it is easy to calculate the sine and cosine of their angles:

$$\sin\theta = \frac{c_1}{b}$$
$$\cos\theta = \frac{h}{b}$$
$$\sin\phi = \frac{c_2}{a}$$
$$\cos\phi = \frac{h}{a}.$$

(b) We can calculate the areas of the triangles using the formula Area $= \frac{1}{2}$Base $\cdot$ Height:

$$\text{Area } \Delta CAD = \frac{1}{2}c_1 \cdot h$$
$$= \frac{1}{2}(b\sin\theta)(a\cos\phi),$$
$$\text{Area } \Delta CDB = \frac{1}{2}c_2 \cdot h$$
$$= \frac{1}{2}(a\sin\phi)(b\cos\theta).$$

(c) We find the area of the whole triangle by summing the area of the two constituent triangles:

$$\text{Area } \Delta ABC = \text{Area } \Delta CAD + \text{Area } CDB$$

$$\begin{aligned}
&= \frac{1}{2}(b\sin\theta)(a\cos\phi) + \frac{1}{2}(a\sin\phi)(b\cos\theta)\\
&= \frac{1}{2}ab(\sin\theta\cos\phi + \sin\phi\cos\theta)\\
&= \frac{1}{2}ab\sin(\theta+\phi)\\
&= \frac{1}{2}ab\sin C.
\end{aligned}$$

28. (a) If B and C are acute angles, draw the altitude from A, dividing side a into two pieces, a_1 and a_2, as shown in Figure 9.35. Then $a_1 = b\cos C$ and $a_2 = c\cos B$, so $a = a_1 + a_2 = b\cos C + c\cos B$.

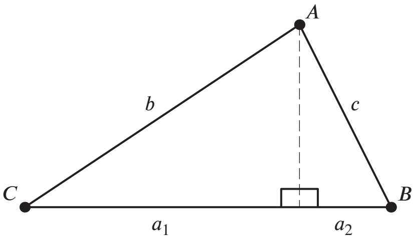


Figure 9.35

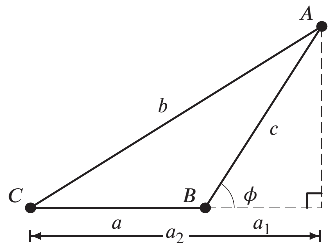


Figure 9.36

If one of the angles B or C (say B) is obtuse, draw the altitude from A as shown in Figure 9.36. Let a_1 be the extension of side a, so $a_2 = a_1 + a$, and let ϕ be the exterior angle at B, so that $\phi = 180° - B$. Therefore $\cos\phi = -\cos B$. Then $a_2 = b\cos C$ and $a_1 = c\cos\phi = -c\cos B$. Thus $a = a_2 - a_1 = b\cos C - (-c\cos B) = b\cos C + c\cos B$.

(b) From the Law of Sines, $\frac{a}{\sin A} = \frac{b}{\sin B}$, so $b = \frac{a}{\sin A}\cdot\sin B$, and similarly $c = \frac{a}{\sin A}\cdot\sin C$. Substituting these expressions into the result of part (a), we have

$$a = b\cos C + c\cos B = \frac{a}{\sin A}\sin B\cos C + \frac{a}{\sin A}\sin C\cos B,$$

or

$$a = \frac{a}{\sin A}(\sin B\cos C + \sin C\cos B).$$

Thus $\sin A = \sin B\cos C + \sin C\cos B$. But $A + B + C = 180°$, so $A = 180° - (B+C)$, and hence $\sin A = \sin(B+C)$. Therefore,

$$\sin(B+C) = \sin B\cos C + \sin C\cos B.$$

29. (a) The coordinates of P_1 are $(\cos\theta, \sin\theta)$; for P_2 they are $(\cos(-\phi), \sin(-\phi)) = (\cos\phi, -\sin\phi)$; for P_3 they are $(\cos(\theta+\phi), \sin(\theta+\phi))$; and for P_4 they are $(1, 0)$.

(b) The triangles P_1OP_2 and P_3OP_4 are congruent by the side-angle-side property because $\angle P_1OP_2 = \theta + \phi = \angle P_3OP_4$. Therefore their corresponding sides P_1P_2 and P_3P_4 are equal.

(c) We have

$$\begin{aligned}
(P_1P_2)^2 &= (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2\\
&= \cos^2\theta - 2\cos\theta\cos\phi + \cos^2\phi + \sin^2\theta + 2\sin\theta\sin\phi + \sin^2\phi\\
&= \cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi\\
&= 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)
\end{aligned}$$

We also have

$$\begin{aligned}
(P_3P_4)^2 &= (\cos(\theta+\phi) - 1)^2 + (\sin(\theta+\phi) - 0)^2\\
&= \cos^2(\theta+\phi) - 2\cos(\theta+\phi) + 1 + \sin^2(\theta+\phi)\\
&= 2 - 2\cos(\theta+\phi)
\end{aligned}$$

The distances P_1P_2 and P_3P_4 are the square roots of these expressions (but we will use the squares of the distances).

(d) $(P_3P_4)^2 = (P_1P_2)^2$ by part (b), so

$$\begin{aligned}
2 - 2\cos(\theta+\phi) &= 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)\\
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi.
\end{aligned}$$

Solutions for Section 9.4

Exercises

1. We have $A = \sqrt{8^2 + (-6)^2} = \sqrt{100} = 10$. Since $\cos\phi = 8/10 = 0.8$ and $\sin\phi = -6/10 = -0.6$, we know that ϕ is in the fourth quadrant. Thus,

$$\tan\phi = -\frac{6}{8} = -0.75 \qquad \text{and} \qquad \phi = \tan^{-1}(-0.75) = -0.644,$$

so $8\sin t - 6\cos t = 10\sin(t - 0.644)$.

2. We have $A = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$. Since $\cos\phi = 8/10 = 0.8$ and $\sin\phi = 6/10 = 0.6$ are both positive, ϕ is in the first quadrant. Thus,

$$\tan\phi = \frac{6}{8} = 0.75 \qquad \text{and} \qquad \phi = \tan^{-1}(0.75) = 0.644,$$

so $8\sin t + 6\cos t = 10\sin(t + 0.644)$.

3. Since $a_1 = -1$ and $a_2 = 1$, we have

$$A = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

and

$$\cos\phi = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\phi = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan\phi = -1.$$

From the signs of $\cos\phi$ and $\sin\phi$, we see that ϕ is in the second quadrant. Since

$$\tan^{-1}(-1) = -\frac{\pi}{4},$$

and the tangent function has period π, we take

$$\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Thus

$$-\sin t + \cos t = \sqrt{2}\sin\left(t + \frac{3\pi}{4}\right).$$

4. Since $a_1 = -2$ and $a_2 = 5$, we have

$$A = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$$

and

$$\cos\phi = \frac{-2}{\sqrt{29}} \quad \text{and} \quad \sin\phi = \frac{5}{\sqrt{29}} \quad \text{and} \quad \tan\phi = -\frac{5}{2}.$$

The signs of $\cos\phi$ and $\sin\phi$ show that ϕ must be in the second quadrant. Since

$$\tan^{-1}\left(-\frac{5}{2}\right) = -1.190$$

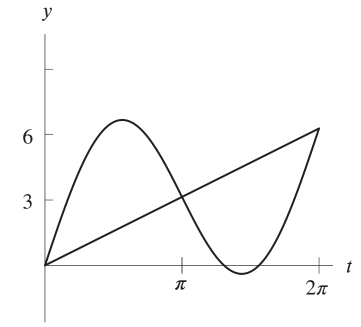
and the tangent function has period π, we take

$$\phi = \pi - 1.190 = 1.951.$$

Thus,

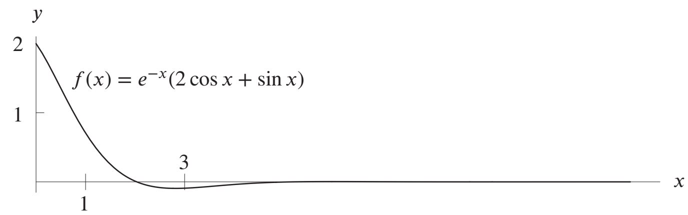
$$-2\sin 3t + 5\cos 3t = \sqrt{29}\sin(3t + 1.951).$$

5.



For the graphs to intersect, $t + 5\sin t = t$. So $\sin t = 0$, or t = any integer multiple of π.

6.



The maximum value of $f(x)$ is 2, which occurs when $x = 0$. The minimum appears to be $y \approx -0.094$, at $x \approx 2.820$.

7. The function $y = 10e^{-t/2}\cos 2\pi t$ matches graph (I). We see that the oscillation is damped less and less each period, so the amplitude (the function multiplying the cosine) must be decreasing and concave up. Only $e^{-t/2}$ has that characteristic.

8. The function $y = 10(1 - 0.01t^2)\cos 2\pi t$ matches graph (IV). We see that the oscillation is damped more and more each period, so the amplitude (the function multiplying the cosine) must be decreasing and concave down. Only $10(1 - 0.01t^2)$ has that characteristic.

9. The function $y = (10 - t)\cos 2\pi t$ matches graph (II). We see that the oscillation is damped equally each period, so the amplitude (the function multiplying the cosine) must be decreasing and linear. Only $10 - t$ has that characteristic.

10. The function $y = (10 - t)\cos t$ matches graph (III). It is hard to see how the oscillation is damped, but the period is very clearly not 1 as in the other graphs. If we draw $y = \pm(10 - t)$ on the graph very accurately, we can see that it just touches the peaks and troughs of the oscillating function.

11. **(a)** Substituting into f, we have

$$\begin{aligned} f(3) - f(2) &= \left(10{,}000 - 5000\cos\left(\frac{\pi}{6}\cdot 3\right)\right) - \left(10{,}000 - 5000\cos\left(\frac{\pi}{6}\cdot 2\right)\right) \\ &= -5000\cos\frac{\pi}{2} + 5000\cos\frac{\pi}{3} \\ &= 0 + 5000\left(\frac{1}{2}\right) = 2500. \end{aligned}$$

This means that the rabbit population increases by 2500 rabbits between March 1 (at $t = 2$) and April 1 (at $t = 3$).

(b) Using a graphing calculator to zoom in, or using the inverse cosine function, we see that $f(t) = 12{,}000$ for $t = 3.786$ and $t = 8.214$. This means that the rabbit population reaches 12,000 sometime during late April (at $t = 3.786$) and falls back to 12,000 sometime during early September (at $t = 8.214$).

Problems

12. Looking at the graph, which appears to be a damped oscillation, we first see that it starts at zero and goes up, so we begin with $\sin t$. Since there are 3 complete periods of oscillation from 0 to 6π, the period is 2π, and so we need only account for the changed amplitude of $\sin t$. The function $y = 3e^{-t/10}$ on the graph is the amplitude function, so we multiply it by $\sin t$ to get

$$y = 3e^{-t/10}\sin t.$$

13. **(a)** This is linear growth: $P = 5000 + 300t$.

(b) This is exponential growth: $P = 3200(1.04)^t$.

(c) The population is oscillating, so we can use a trigonometric model. Since the population starts at its lowest value, we use a reflected cosine function of the form

$$P(t) = -A\cos(Bt) + k.$$

Low = 1200; High = 3000.
Midline $k = \frac{1}{2}(3000 + 1200) = 2100$.
Amplitude $A = 3000 - 2100 = 900$.
The period is 5, so $5 = 2\pi/B$, and we have $B = 2\pi/5$. Thus,

$$P(t) = -900\cos\left(\frac{2\pi}{5}t\right) + 2100.$$

14. **(a)** We start the time count on Jan 1, so substituting $t = 0$ into $f(t)$ gives us the value of b, since both mt and $A\sin\frac{\pi t}{6}$ are equal to zero when $t = 0$. Thus, $b = f(0) = 20$. We see that in the 12-month period between Jan 1 and Jan 1 (a whole period of the periodic component), the value of the stock rose by \$30.00. Therefore, the linear component grows at the rate of \$30.00/year, or in terms of months, $30/12 = \$2.50$/month. So $m = 2.5$. Thus we have

$$P = f(t) = 2.5t + 20 + A\sin\frac{\pi t}{6}.$$

At an arbitrary data point, say (Apr 1, 37.50), we can solve for A. Since January 1 corresponds to $t = 0$, April 1 is $t = 3$. We have

$$37.50 = f(3) = 2.5(3) + 20 + A\sin\frac{3\pi}{6} = 7.5 + 20 + A\sin\frac{\pi}{2} = 27.5 + A.$$

Simplifying gives $A = 10$, and the function is

$$f(t) = 2.5t + 20 + 10\sin\frac{\pi t}{6}.$$

(b) The stock appreciates the most during the months when the sine function climbs the fastest. By looking at Figure 9.37 we see that this occurs roughly when $t = 0$ and $t = 11$, January and December.

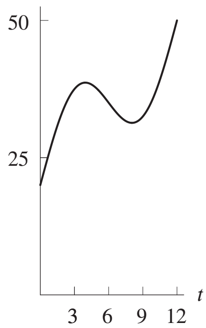


Figure 9.37

(c) Again, we look to Figure 9.37 to see when the graph actually decreases. It seems that the graph is decreasing roughly between the fourth and eighth months, that is, between May and September.

15. **(a)** We know that the minimum of $f(t)$ is 40 and that the maximum is 90. Thus the midline height is

$$\frac{90 + 40}{2} = 65$$

and the amplitude is

$$90 - 65 = 25.$$

We also know that $f(t)$ has a period of 24 hours and is at a minimum when $t = 0$. Thus a formula for $f(t)$ is

$$f(t) = 65 - 25\cos\left(\frac{\pi}{12}t\right).$$

(b) The amplitude is 30. The period is 24 hours, so the pattern repeats itself each day. The midline value is 80. So $g(t)$ goes up to a maximum of $80 + 30 = 110$ and down to a low of $80 - 30 = 50$ megawatts.

(c) From the graph in Figure 9.38, we can find that $t_1 \approx 4.160$ and $t_2 \approx 13.148$. Thus, the power required in both cities is the same at approximately 4 am and 1 pm.

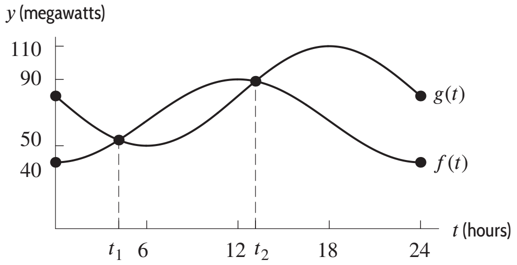


Figure 9.38

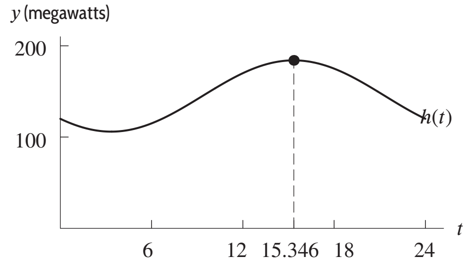


Figure 9.39: $h(t) = f(t) + g(t)$

(d) The function $h(t)$ in Figure 9.39 tells us the total amount of electricity required by both cities at a particular time of day. Using the trace key on a calculator, we find the maximum occurs at $t = 15.346$. So at 3:21 pm (since 0.346 hours is about 21 minutes) each day the most total power will be needed. The maximum power is 184.051 mw.

(e) Add the two functions to obtain

$$h(t) = 145 - \left(30\sin\left(\frac{\pi}{12}t\right) + 25\cos\left(\frac{\pi}{12}t\right)\right)$$

which can be written in the form

$$h(t) = 145 - A\sin\left(\frac{\pi}{12}t + \theta\right)$$

where $A = \sqrt{30^2 + 25^2} = \sqrt{1525} \approx 39.051$ and $\theta = \tan^{-1}(25/30) \approx 0.6947$. Thus

$$h(t) \approx 145 - 39.051\sin\left(\frac{\pi}{12}t + 0.6947\right).$$

The function has an exact maximum of $145 + \sqrt{1525}$ (about 184).

16. (a) As $t \to -\infty$, e^t gets very close to 0, which means that $\cos(e^t)$ gets very close to $\cos 0 = 1$. This means that the horizontal asymptote of f as $t \to -\infty$ is $y = 1$.

(b) As t increases, e^t increases at a faster and faster rate. We also know that as θ increases, $\cos\theta$ varies steadily between -1 and 1. But since e^t is increasing faster and faster as t gets large, $\cos(e^t)$ will vary between -1 and 1 at a faster and faster pace. So the graph of $f(t) = \cos(e^t)$ begins to wiggle back and forth between -1 and 1, faster and faster. Although f is oscillating between -1 and 1, f is not periodic, because the interval on which f completes a full cycle is not constant.

(c) The vertical axis is crossed when $t = 0$, so $f(0) = \cos(e^0) = \cos 1 \approx 0.540$ is the vertical intercept.

(d) Notice that the least positive zero of $\cos u$ is $u = \pi/2$. Thus the least zero t_1 of $f(t) = \cos(e^t)$ occurs where $e^{t_1} = \pi/2$ since e^t is always positive. So we have

$$e^{t_1} = \frac{\pi}{2}$$
$$t_1 = \ln\frac{\pi}{2}.$$

(e) We know that if $\cos u = 0$, then $\cos(u + \pi) = 0$. This means that if $\cos(e^{t_1}) = 0$, then $\cos\left(e^{t_1} + \pi\right) = 0$ will be the first zero of f coming after t_1. Therefore,

$$f(t_2) = \cos(e^{t_2}) = \cos\left(e^{t_1} + \pi\right) = 0.$$

This means that $e^{t_2} = e^{t_1} + \pi$. So

$$t_2 = \ln\left(e^{t_1} + \pi\right) = \ln(e^{\ln(\pi/2)} + \pi) = \ln\left(\frac{\pi}{2} + \pi\right) = \ln\left(\frac{3\pi}{2}\right).$$

Similar reasoning shows that the set of all zeros is $\{\ln(\frac{\pi}{2}), \ln(\frac{3\pi}{2}), \ln(\frac{5\pi}{2})...\}$.

17. **(a)** As $x \to \infty$, $\frac{1}{x} \to 0$ and we know $\sin 0 = 0$. Thus, $y = 0$ is the equation of the asymptote.

(b) As $x \to 0$ and $x > 0$, we have $\frac{1}{x} \to \infty$. This means that for small changes of x the change in $\frac{1}{x}$ is large. Since $\frac{1}{x}$ is a large number of radians, the function will oscillate more and more frequently as x becomes smaller.

(c) No, because the interval on which $f(x)$ completes a full cycle is not constant as x increases.

(d) $\sin\left(\frac{1}{x}\right) = 0$ means that $\frac{1}{x} = \sin^{-1}(0) + k\pi$ for k equal to some integer. Therefore, $x = \frac{1}{k\pi}$, and the greatest zero of $f(x) = \sin\frac{1}{x}$ corresponds to the smallest k, that is, $k = 1$. Thus, $z_1 = \frac{1}{\pi}$.

(e) There are an infinite number of zeros because $z = \frac{1}{k\pi}$ for all $k > 0$ are zeros.

(f) If $a = \frac{1}{k\pi}$ then the largest zero of $f(x)$ less then a would be $b = \frac{1}{(k+1)\pi}$.

18. Use the sum-of-angles identity for cosine on the second term to get

$$I\cos(\omega_c + \omega_d)t = I\cos\omega_c t\cos\omega_d t - I\sin\omega_c t\sin\omega_d t$$

and factor the term $\cos\omega_c t$ to show the equality.

19. **(a)** Types of video games are trendy for a length of time, during which they are extremely popular and sales are high, later followed by a cooling-down period as the users become tired of that particular game type. The game players then become interested in a different game type—and so on.

(b) The sales graph does not fit the shape of the sine or cosine curve, and we would have to say that neither of those functions would give us a reasonable model. However, from 1979–1989 the graph does have a basic negative cosine shape, but the amplitude varies.

(c) One way to modify the amplitude over time is to multiply the sine (or cosine) function by an exponential function, such as e^{kt}. So we choose a model of the form

$$s(t) = e^{kt}(-a\cos(Ct) + D),$$

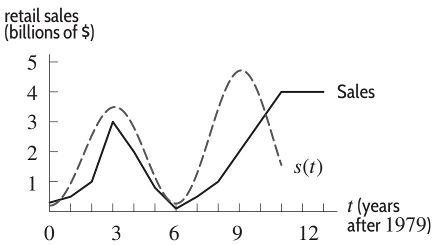
where t is the number of years since 1979. Note the $-a$, which is due to the graph looking like an inverted cosine at 1979. The average value starts at about 1.6, and the period appears to be about 6 years. The amplitude is initially about 1.4, which is the distance between the average value of 1.6 and the first peak value of 3.0. This means

$$s(t) = e^{kt}\left(-1.4\cos\left(\frac{2\pi}{6}t\right) + 1.6\right).$$

By trial and error on your graphing calculator, you can arrive at a value for the parameter k. A reasonable choice is $k = 0.05$, which gives

$$s(t) = e^{0.05t}\left(-1.4\cos\left(\frac{2\pi}{6}t\right) + 1.6\right).$$

(d)



Notice that even though multiplying by the exponential function does increase the amplitude over time, it does not increase the period. Therefore, our model $s(t)$ does not fit the actual curve all that well.

(e) The predicted 1993 sales volume is $f(14) = 4.632$ billion dollars.

20. **(a)** We have $\lambda = \frac{2\pi}{k} = \frac{2\pi}{2\pi} = 1$. Thus, the wavelength is 1 meter.

(b) The time for one wavelength to pass by is $\frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \frac{1}{2}$ of a second. Thus, two wavelengths pass by each second. The number of wavelengths which pass a point in a given unit of time is referred to as the frequency. It is sometimes written as 2 hertz (Hz), which equals 2 cycles per second.

(c) The shape of the rope at the instant when $t = 0$ is described by the sinusoidal function $y(x, 0) = 0.06\sin(2\pi x)$

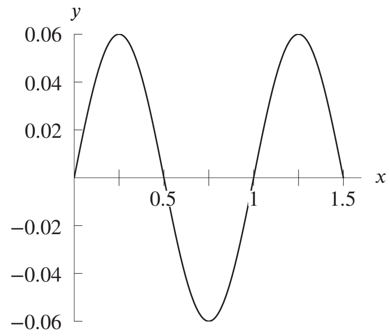


Figure 9.40

(d) Having the same graph means that $0.06\sin(2\pi x) = 0.06\sin(2\pi x - 4\pi t)$ for all x. In order for this to happen, $2\pi x$ and $2\pi x - 4\pi t$ must describe the same angle. This is the case any time $4\pi t$ is a multiple of 2π. Thus, $4\pi t = 2\pi k$ whenever $t = k/2$ for any integer k.

21. **(a)** We have:

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Dec'05–Jun'06}\end{array} = \frac{C(6)-C(0)}{6-0} = \frac{382-381}{6} = \frac{1}{6}\text{ ppm/month.}$$

Also,

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Dec'10–Jun'11}\end{array} = \frac{C(66)-C(60)}{66-60} = \frac{392-391}{6} = \frac{1}{6}\text{ ppm/month.}$$

And lastly,

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Mar–Sep'12}\end{array} = \frac{C(81)-C(75)}{81-75} = \frac{391-397}{6} = -1\text{ ppm/month.}$$

So, on average, the concentration of carbon dioxide increased by 0.167 parts per million per month during the first half of 2006 and 2011. It also decreased, on average, by 1 ppm per month during the middle months of 2012.

(b) $S(t)$ is a periodic function which measures the variation in the concentration of carbon dioxide due to the seasons. It does not take into account the long-term changes in the concentration of carbon dioxide.

(c) We have:

$$\begin{array}{c}\text{Average change in } S(t)\\ \text{over Dec'05–Jun'06}\end{array} = \frac{S(6)-S(0)}{6-0} = \frac{0-0}{6} = 0\text{ ppm/month.}$$

Also,

$$\begin{array}{c}\text{Average change in } S(t)\\ \text{over Dec'10–Jun'11}\end{array} = \frac{S(66)-S(60)}{66-60} = \frac{0-0}{6} = 0\text{ ppm/month.}$$

And lastly,

$$\begin{array}{c}\text{Average change in } S(t)\\ \text{over Mar–Sep'12}\end{array} = \frac{S(81)-S(75)}{81-75} = \frac{-3.5-3.5}{6} = \frac{-7}{6}\text{ ppm/month.}$$

So, on average, there was no change in the concentration of carbon dioxide due to seasonal variations during the first half of 2006 and 2011. This makes sense, because the carbon dioxide concentration grows and decays by 3.5 ppm during these periods. (Note that 3.5 ppm is the amplitude of $S(t)$.)

Also, seasonal changes caused an average decrease in carbon dioxide of about -1.167 ppm per month during the middle months of 2012.

(d) We have:

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Dec'05–Jun'06}\end{array} - \begin{array}{c}\text{Average change in } S(t)\\ \text{over Dec'05–Jun'06}\end{array} = \frac{1}{6} - 0 = \frac{1}{6}\text{ ppm/month.}$$

Also,

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Dec'10–Jun'11}\end{array} - \begin{array}{c}\text{Average change in } S(t)\\ \text{over Dec'10–Jun'11}\end{array} = \frac{1}{6} - 0 = \frac{1}{6}\text{ ppm/month.}$$

And lastly,

$$\begin{array}{c}\text{Average change in } C(t)\\ \text{over Mar–Sep'12}\end{array} - \begin{array}{c}\text{Average change in } S(t)\\ \text{over Mar–Sep'12}\end{array} = -1 + \frac{7}{6} = \frac{1}{6}\text{ ppm/month.}$$

So, the difference between the average concentration and average the seasonal variation is $1/6$ ppm/month over each of these common periods.

(e) The common difference between the two averages calculated in part (d), $1/6$ ppm/month, gives the average increase in the concentration of carbon dioxide per month. This value is the slope of the rising midline,

$$M(t) = 381 + t/6,$$

the portion of the concentration model $y = C(t) = S(t) + M(t)$ that captures the variation in the concentration of carbon dioxide not due to the seasons.

The slope of the rising midline tells us that, independently of seasonal changes, the concentration of carbon dioxide has been increasing by about $1/6$ ppm per month, on average, during the last few years.

Solutions for Section 9.5

Exercises

1. Though we could use the sum-of-angle and difference-of-angle formulas for cosine on each of the two parts, we can also use the formula for the sum of cosines:

$$\cos 165° - \cos 75° = -2 \sin \frac{165 + 75}{2} \sin \frac{165 - 75}{2} = -2 \sin 120 \sin 45 = -2 \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{2}.$$

2. Though we could use the sum-of-angle and difference-of-angle formulas for cosine on each of the two parts, we can also use the formula for the sum of cosines:

$$\cos 75° + \cos 15° = 2 \cos \frac{75 + 15}{2} \cos \frac{75 - 15}{2} = 2 \cos 45 \cos 30 = 2 \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}.$$

3. These cosine functions have the same amplitude and we use the relationship

$$\cos u + \cos v = 2 \cos \frac{u + v}{2} \cos \frac{u - v}{2}.$$

Letting $u = 5t$ and $v = 3t$, we have

$$\cos(5t) + \cos(3t) = 2 \cos \frac{5t + 3t}{2} \cos \frac{5t - 3t}{2} = 2 \cos(4t) \cos t.$$

4. These cosine functions have the same amplitude and we use the relationship

$$\cos u + \cos v = 2 \cos \frac{u + v}{2} \cos \frac{u - v}{2}.$$

Letting $u = 4t$ and $v = 6t$, we have

$$\cos(4t) + \cos(6t) = 2 \cos \frac{4t + 6t}{2} \cos \frac{4t - 6t}{2} = 2 \cos(5t) \cos t.$$

Notice that $\cos(-t) = \cos t$.

5. These sine functions have the same amplitude and we use the relationship

$$\sin u + \sin v = 2 \sin \frac{u + v}{2} \cos \frac{u - v}{2}.$$

Letting $u = 7t$ and $v = 3t$, we have

$$\sin(7t) + \sin(3t) = 2 \sin \frac{7t + 3t}{2} \cos \frac{7t - 3t}{2} = 2 \sin(5t) \cos(2t).$$

6. These sine functions have the same amplitude and we use the relationship

$$\sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2}.$$

Letting $u = 7t$ and $v = 3t$, we have

$$\sin(7t) - \sin(3t) = 2\cos\frac{7t+3t}{2}\sin\frac{7t-3t}{2} = 2\cos(5t)\sin(2t).$$

7. $\cos 35° + \cos 40° = 2\cos\left(\frac{35° + 40°}{2}\right)\cos\left(\frac{35° - 40°}{2}\right) = 1.585$. See Figure 9.41.

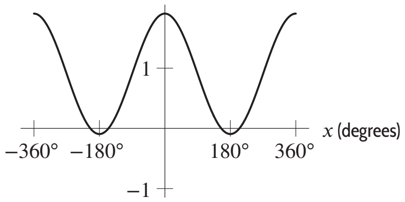


Figure 9.41

8. $\cos 35° - \cos 40° = -2\sin\left(\frac{35° + 40°}{2}\right)\sin\left(\frac{35° - 40°}{2}\right) = 0.053$. See Figure 9.42.

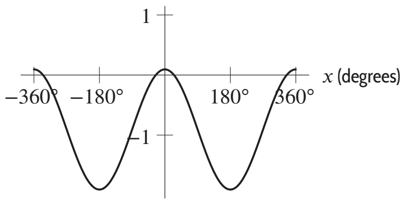


Figure 9.42

9. $\sin 35° + \sin 40° = 2\sin\left(\frac{35° + 40°}{2}\right)\cos\left(\frac{35° - 40°}{2}\right) = 1.216$. See Figure 9.43.

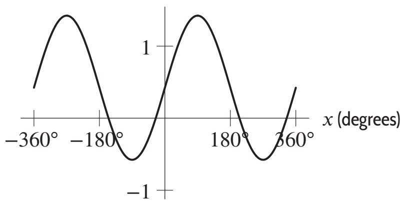


Figure 9.43

10. $\sin 35° - \sin 40° = 2\cos\left(\frac{35° + 40°}{2}\right)\sin\left(\frac{35° - 40°}{2}\right) = -0.069$. See Figure 9.44.

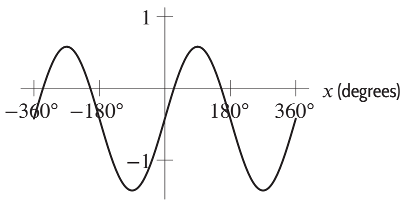


Figure 9.44

Problems

11. We can use the sum-of-angle identities

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

and subtract to obtain

$$\cos(\theta+\phi) - \cos(\theta-\phi) = -2\sin\theta\sin\phi.$$

We put $u = \theta+\phi$ and $v = \theta-\phi$ on the left side of the equation. Solving these simultaneous equations, we get $\theta = (u+v)/2$ and $\phi = (u-v)/2$. We put $\theta = (u+v)/2$ and $\phi = (u-v)/2$ on the right side to get

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

12. We can use the sum-of-angle identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$
$$\sin(\theta-\phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$$

and add to get

$$\sin(\theta+\phi) + \sin(\theta-\phi) = 2\sin\theta\cos\phi.$$

We put $u = \theta+\phi$ and $v = \theta-\phi$ on the left side of the equation. Solving these simultaneous equations, we get $\theta = (u+v)/2$ and $\phi = (u-v)/2$. We put $\theta = (u+v)/2$ and $\phi = (u-v)/2$ on the right side to get

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$

13. We start with

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$

Since $-\sin v = \sin(-v)$, we can write

$$\begin{aligned}\sin u - \sin v &= \sin u + \sin(-v)\\ &= 2\sin\left(\frac{u+(-v)}{2}\right)\cos\left(\frac{u-(-v)}{2}\right)\\ &= 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)\\ &= 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).\end{aligned}$$

14. We can use the identity $\sin u + \sin v = 2\sin((u+v)/2)\cos((u-v)/2)$. If we put $u = 4x$ and $v = x$ then our equation becomes

$$\begin{aligned}0 &= 2\sin\left(\frac{4x+x}{2}\right)\cos\left(\frac{4x-x}{2}\right)\\ &= 2\sin\left(\frac{5}{2}x\right)\cos\left(\frac{3}{2}x\right)\end{aligned}$$

The product on the right-hand side of this equation will be equal to zero precisely when $\sin(5x/2) = 0$ or when $\cos(3x/2) = 0$. We will have $\sin(5x/2) = 0$ when $5x/2 = n\pi$, for n an integer, or in other words for $x = (2\pi/5)n$. This will occur in the stated interval for $x = 2\pi/5$, $x = 4\pi/5$, $x = 6\pi/5$, and $x = 8\pi/5$. We will have $\cos(3x/2) = 0$ when $3x/2 = n\pi + \pi/2$, that is, when $x = (2n+1)\pi/3$. This will occur in the stated interval for $x = \pi/3$, $x = \pi$, and $x = 5\pi/3$. So the given expression is solved by

$$x = \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{\pi}{3}, \pi, \text{and} \frac{5\pi}{3}.$$

15. (a) First consider the height $h = f(t)$ of the hub of the wheel that you are on. This is similar to the basic Ferris wheel problem. Therefore $f_1(t) = 25 + 15\sin\left(\frac{\pi}{3}t\right)$, because the vertical shift is 25, the amplitude is 15, and the period is 6. Now the smaller wheel will also add or subtract height depending upon time. The difference in height between your position and the hub of the smaller wheel is given by $f_2(t) = 10\sin\left(\frac{\pi}{2}t\right)$ because the radius is 10 and the period is 4. Finally, adding the two together, we get:

$$f_1(t) + f_2(t) = f(t) = 25 + 15\sin(\frac{\pi}{3}t) + 10\sin(\frac{\pi}{2}t)$$

(b)

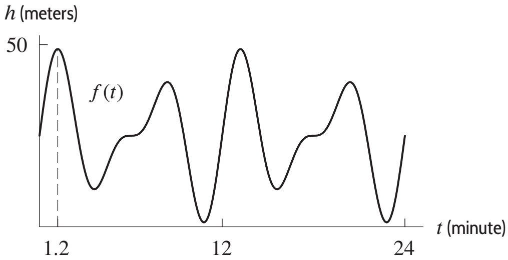


Figure 9.45

Looking at the graph shown in Figure 9.45, we see that $h = f(t)$ is periodic, with period 12. This can be verified by noting

$$\begin{aligned} f(t+12) &= 25 + 15\sin\left(\frac{\pi}{3}(t+12)\right) + 10\sin\left(\frac{\pi}{2}(t+12)\right) \\ &= 25 + 15\sin\left(\frac{\pi}{3}t + 4\pi\right) + 10\sin\left(\frac{\pi}{2}t + 6\pi\right) \\ &= 25 + 15\sin\left(\frac{\pi}{3}t\right) + 10\sin\left(\frac{\pi}{2}t\right) \\ &= f(t). \end{aligned}$$

(c) $h = f(1.2) = 48.776$ m.

Solutions for Section 9.6

Exercises

1. $5e^{i\pi}$

2. $e^{\frac{i3\pi}{2}}$

3. $0e^{i\theta}$, for any θ.

4. $2e^{\frac{i\pi}{2}}$

5. We have $(-3)^2 + (-4)^2 = 25$, and $\arctan(4/3) \approx 4.069$. So the number is $5e^{i4.069}$.

6. $\sqrt{10}e^{i\theta}$, where $\theta = \arctan(-3) \approx -1.249 + \pi = 1.893$ is an angle in the second quadrant.

7. $-5 + 12i$

8. $-11 + 29i$

9. $-3 - 4i$

10. $\frac{1}{4} - \frac{9i}{8}$

11. We have $(e^{i\pi/3})^2 = e^{i2\pi/3}$, thus $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

12. $3 - 6i$

13. We have $\sqrt{e^{i\pi/3}} = e^{(i\pi/3)/2} = e^{i\pi/6}$, thus $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$.

14. $\sqrt[4]{10}\cos\frac{\pi}{8} + i\sqrt[4]{10}\sin\frac{\pi}{8}$ is one solution.

Problems

15. One value of $\sqrt{4i}$ is $\sqrt{4e^{i\pi/2}} = (4e^{i\pi/2})^{1/2} = 2e^{i\pi/4} = 2\cos\frac{\pi}{4} + i2\sin\frac{\pi}{4} = \sqrt{2} + i\sqrt{2}$

16. One value of $\sqrt{-i}$ is $\sqrt{e^{i\frac{3\pi}{2}}} = (e^{i\frac{3\pi}{2}})^{\frac{1}{2}} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

17. One value of $\sqrt[3]{i}$ is $\sqrt[3]{e^{i\frac{\pi}{2}}} = (e^{i\frac{\pi}{2}})^{\frac{1}{3}} = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

18. One value of $\sqrt{7i}$ is $\sqrt{7e^{i\frac{\pi}{2}}} = (7e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \sqrt{7}e^{i\frac{\pi}{4}} = \sqrt{7}\cos\frac{\pi}{4} + i\sqrt{7}\sin\frac{\pi}{4} = \frac{\sqrt{14}}{2} + i\frac{\sqrt{14}}{2}$

19. One value of $\sqrt[4]{-1}$ is $\sqrt[4]{e^{i\pi}} = (e^{i\pi})^{1/4} = e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

20. One value of $(1+i)^{2/3}$ is $(\sqrt{2}e^{i\frac{\pi}{4}})^{2/3} = (2^{\frac{1}{2}}e^{i\frac{\pi}{4}})^{\frac{2}{3}} = \sqrt[3]{2}e^{i\frac{\pi}{6}} = \sqrt[3]{2}\cos\frac{\pi}{6} + i\sqrt[3]{2}\sin\frac{\pi}{6} = \sqrt[3]{2}\cdot\frac{\sqrt{3}}{2} + i\sqrt[3]{2}\cdot\frac{1}{2}$

21. One value of $(\sqrt{3}+i)^{1/2}$ is
$(2e^{i\frac{\pi}{6}})^{1/2} = \sqrt{2}e^{i\frac{\pi}{12}} = \sqrt{2}\cos\frac{\pi}{12} + i\sqrt{2}\sin\frac{\pi}{12} \approx 1.366 + 0.366i$

22. One value of $(\sqrt{3}+i)^{-1/2}$ is
$(2e^{i\frac{\pi}{6}})^{-1/2} = \frac{1}{\sqrt{2}}e^{i(-\frac{\pi}{12})} = \frac{1}{\sqrt{2}}\cos(-\frac{\pi}{12}) + i\frac{1}{\sqrt{2}}\sin(-\frac{\pi}{12}) \approx 0.683 - 0.183i$

23. Since $\sqrt{5} + 2i = 3e^{i\theta}$, where $\theta = \arctan\frac{2}{\sqrt{5}} \approx 0.730$, one value of $(\sqrt{5}+2i)^{\sqrt{2}}$ is $(3e^{i\theta})^{\sqrt{2}} = 3^{\sqrt{2}}e^{i\sqrt{2}\theta} = 3^{\sqrt{2}}\cos\sqrt{2}\theta + i3^{\sqrt{2}}\sin\sqrt{2}\theta \approx 3^{\sqrt{2}}(0.513) + i3^{\sqrt{2}}(0.859) \approx 2.426 + 4.062i$

24. Substituting $A_1 = 2 - A_2$ into the second equation gives

$$(1-i)(2-A_2) + (1+i)A_2 = 0$$

so

$$\begin{aligned} 2iA_2 &= -2(1-i) \\ A_2 &= \frac{-(1-i)}{i} = \frac{-i(1-i)}{i^2} = i(1-i) = 1+i \end{aligned}$$

Therefore $A_1 = 2 - (1+i) = 1 - i$.

25. Substituting $A_2 = i - A_1$ into the second equation gives

$$iA_1 - (i - A_1) = 3,$$

so

$$\begin{aligned} iA_1 + A_1 &= 3+i \\ A_1 &= \frac{3+i}{1+i} = \frac{3+i}{1+i}\cdot\frac{1-i}{1-i} = \frac{3-3i+i-i^2}{2} \\ &= 2-i \end{aligned}$$

Therefore $A_2 = i - (2-i) = -2 + 2i$.

26. If the roots are complex numbers, we must have $(2b)^2 - 4c < 0$ so $b^2 - c < 0$. Then the roots are

$$\begin{aligned} x = \frac{-2b \pm \sqrt{(2b)^2 - 4c}}{2} &= -b \pm \sqrt{b^2 - c} \\ &= -b \pm \sqrt{-1(c - b^2)} \\ &= -b \pm i\sqrt{c - b^2}. \end{aligned}$$

Thus, $p = -b$ and $q = \sqrt{c - b^2}$.

27. **(a)** The polar coordinates of i are $r = 1$ and $\theta = \pi/2$, as in Figure 9.46. Thus, the polar form of i is $i = re^{i\theta} = 1e^{i\pi/2} = e^{i\pi/2}$.
(b) Since $z = re^{i\theta}$ and $i = e^{i\pi/2}$, we have $iz = e^{i\pi/2} \cdot re^{i\theta} = re^{i(\theta+\pi/2)}$. The polar coordinates of z are (r, θ), while the polar coordinates of iz are $(r, \theta + \pi/2)$. The points z and iz are related as in Figure 9.47.

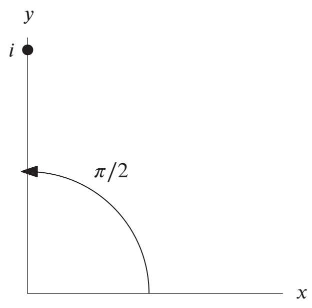


Figure 9.46

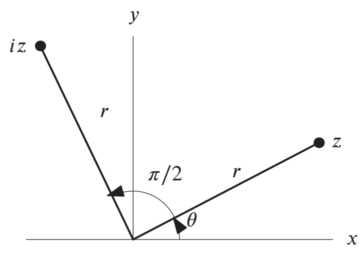


Figure 9.47

28. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i \sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating imaginary parts, we find

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

29. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i \sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating real parts, we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

30. Since $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$, we want to investigate $e^{-i\theta}$. By the definition of negative exponents,

$$e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{\cos\theta + i\sin\theta}.$$

Multiplying by the conjugate $\cos\theta - i\sin\theta$ gives

$$e^{-i\theta} = \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta.$$

Thus

$$\cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta,$$

so equating real parts, we see that cosine is even: $\cos(-\theta) = \cos\theta$.

31. Since $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$, we want to investigate $e^{-i\theta}$. By the definition of negative exponents,

$$e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{\cos\theta + i\sin\theta}.$$

Multiplying by the conjugate $\cos\theta - i\sin\theta$ gives

$$e^{-i\theta} = \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta.$$

Thus

$$\cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta,$$

so equating imaginary parts, we see that sine is odd: $\sin(-\theta) = -\sin\theta$.

32. The number $z = e^{2i}$ is given in polar form with $(r, \theta) = (1, 2)$. Two more sets of polar coordinates for z are $(1, 2 + 2\pi)$, and $(1, 2 + 4\pi)$. Three cube roots of z are given by

$$\begin{aligned}
\left(1e^{2i}\right)^{1/3} = 1^{1/3}e^{1/3\cdot 2i} &= e^{2i/3} = \cos(2/3) + i\sin(2/3)\\
&= 0.786 + 0.618i.\\
\left(1e^{(2+2\pi)i}\right)^{1/3} = 1^{1/3}e^{1/3\cdot(2+2\pi)i} &= e^{(2+2\pi)i/3} = \cos((2+2\pi)/3) + i\sin((2+2\pi)/3)\\
&= -0.928 + 0.371i.\\
\left(1e^{(2+4i)i}\right)^{1/3} = 1^{1/3}e^{1/3\cdot(2+4\pi)i} &= e^{(2+4\pi)i/3} = \cos((2+4\pi)/3) + i\sin((2+4\pi)/3)i\\
&= 0.143 - 0.909i.
\end{aligned}$$

33. One polar form for $z = -8$ is $z = 8e^{i\pi}$ with $(r, \theta) = (8, \pi)$. Two more sets of polar coordinates for z are $(8, 3\pi)$, and $(8, 5\pi)$. Three cube roots of z are given by

$$\begin{aligned}
\left(8e^{\pi i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot\pi i} &= 2e^{\pi i/3} = 2\cos(\pi/3) + i2\sin(\pi/3)\\
&= 1 + 1.732i.\\
\left(8e^{3\pi i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot 3\pi i} &= 2e^{\pi i} = 2\cos\pi + i2\sin\pi\\
&= -2.\\
\left(8e^{5\pi i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot 5\pi i} &= 2e^{5\pi i/3} = 2\cos(5\pi/3) + i2\sin(5\pi/3)\\
&= 1 - 1.732i.
\end{aligned}$$

34. Three polar forms for $z = 8$ have (r, θ) equal to $(8, 0)$, $(8, 2\pi)$, and $(8, 4\pi)$. Three cube roots of z are given by

$$\begin{aligned}
\left(8e^{0i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot 0i} &= 2e^{0i} = 2\cos 0 + i2\sin 0\\
&= 2.\\
\left(8e^{2\pi i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot 2\pi i} &= 2e^{2\pi i/3} = 2\cos(2\pi/3) + i2\sin(2\pi/3)\\
&= -1 + 1.732i.\\
\left(8e^{4\pi i}\right)^{1/3} = 8^{1/3}e^{1/3\cdot 4\pi i} &= 2e^{4\pi i/3} = 2\cos(4\pi/3) + i2\sin(4\pi/3)\\
&= -1 - 1.732i.
\end{aligned}$$

35. One polar form of the complex number $z = 1 + i$ has $(r, \theta) = (\sqrt{2}, \pi/4)$. Two more sets of polar coordinates for z are $(\sqrt{2}, \pi/4 + 2\pi) = (\sqrt{2}, 9\pi/4)$ and $(\sqrt{2}, \pi/4 + 4\pi) = (\sqrt{2}, 17\pi/4)$. Three cube roots of z are given by

$$\begin{aligned}
\left(\sqrt{2}e^{\pi i/4}\right)^{1/3} = \left(2^{1/2}\right)^{1/3}e^{1/3\cdot\pi i/4} &= 2^{1/6}e^{\pi i/12} = 2^{1/6}\cos(\pi/12) + i2^{1/6}\sin(\pi/12)\\
&= 1.084 + 0.291i.\\
\left(\sqrt{2}e^{9\pi i/4}\right)^{1/3} = \left(2^{1/2}\right)^{1/3}e^{1/3\cdot 9\pi i/4} &= 2^{1/6}e^{9\pi i/12} = 2^{1/6}\cos(9\pi/12) + i2^{1/6}\sin(9\pi/12)\\
&= -0.794 + 0.794i.\\
\left(\sqrt{2}e^{17\pi/i4}\right)^{1/3} = \left(2^{1/2}\right)^{1/3}e^{1/3\cdot 17\pi i/4} &= 2^{1/6}e^{17\pi i/12} = 2^{1/6}\cos(17\pi/12) + i2^{1/6}\sin(17\pi/12)\\
&= -0.291 - 1.084i.
\end{aligned}$$

36. By de Moivre's formula we have

$$(\cos\pi/4 + i\sin\pi/4)^4 = \cos(4\cdot\pi/4) + i\sin(4\cdot\pi/4) = -1 + i0 = -1.$$

37. By de Moivre's formula we have

$$(\cos 2\pi/3 + i\sin 2\pi/3)^3 = \cos(3\cdot 2\pi/3) + i\sin(3\cdot 2\pi/3) = 1 + i0 = 1.$$

38. By de Moivre's formula we have

$$(\cos 2 + i\sin 2)^{-1} = \cos(-2) + i\sin(-2) = \cos 2 - i\sin 2.$$

39. By de Moivre's formula we have

$$(\cos\pi/4 + i\sin\pi/4)^{-7} = \cos(-7\pi/4) + i\sin(-7\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

40. From the exponent rules, we know

$$e^{i\theta}\cdot e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1.$$

Using Euler's formula, we rewrite this as

$$\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}} = 1.$$

Multiplying out the left-hand side gives

$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta = 1.$$

Since $-i^2 = +1$, simplifying the left side gives the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1.$$

41. Using the exponent rules, we see from Euler's formula that

$$\begin{aligned} e^{i(\theta+\phi)} &= e^{i\theta}\cdot e^{i\phi} \\ &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi + \underbrace{i\cos\theta\sin\phi + i\sin\theta\cos\phi}_{i(\cos\theta\sin\phi+\sin\theta\cos\phi)} + \underbrace{i^2\sin\theta\sin\phi}_{-\sin\theta\sin\phi} \\ &= \underbrace{\cos\theta\cos\phi - \sin\theta\sin\phi}_{\text{Real part}} + i\underbrace{(\sin\theta\cos\phi + \cos\theta\sin\phi)}_{\text{Imaginary part}}. \end{aligned}$$

But Euler's formula also gives

$$e^{i(\theta+\phi)} = \underbrace{\cos(\theta+\phi)}_{\text{Real part}} + i\underbrace{\sin(\theta+\phi)}_{\text{Imaginary part}}.$$

Two complex numbers are equal only if their real and imaginary parts are equal. Setting real parts equal gives

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Setting imaginary parts equal gives

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta.$$

Solutions for Chapter 9 Review

Exercises

1. We have:

$$\begin{aligned} (1-\sin t)(1-\cos t) - \cos t\sin t &= \underbrace{1 - \cos t - \sin t + \sin t\cos t}_{(1-\sin t)(1-\cos t)} - \cos t\sin t \quad \text{multiply out} \\ &= 1 - \cos t - \sin t. \end{aligned}$$

2. We have:

$$\begin{aligned}2\cos t - 3\sin t - (2\sin t - 3\cos t) &= 2\cos t - 3\sin t - 2\sin t + 3\cos t\\ &= 5\cos t - 5\sin t.\end{aligned}$$

3. We have:

$$\begin{aligned}\sec t\sin t + 3\tan t &= \frac{1}{\cos t}\cdot\sin t + 3\tan t\\ &= \underbrace{\frac{\sin t}{\cos t}}_{\tan t} + 3\tan t\\ &= 4\tan t.\end{aligned}$$

4. We have:

$$\begin{aligned}\cot t\tan t\sin t &= \underbrace{\left(\frac{\cos t}{\sin t}\right)}_{\cot t}\cdot\underbrace{\left(\frac{\sin t}{\cos t}\right)}_{\tan t}\cdot\sin t\\ &= \sin t.\end{aligned}$$

5. We have:

$$\begin{aligned}\frac{\sec t}{\csc t} &= \frac{\left(\frac{1}{\cos t}\right)}{\left(\frac{1}{\sin t}\right)}\\ &= \frac{1}{\cos t}\cdot\sin t\\ &= \tan t.\end{aligned}$$

6. We have:

$$\begin{aligned}\frac{\cot t}{\csc t} &= \frac{\left(\frac{\cos t}{\sin t}\right)}{\left(\frac{1}{\sin t}\right)}\\ &= \frac{\cos t}{\sin t}\cdot\sin t\\ &= \cos t.\end{aligned}$$

7. We have:

$$\begin{aligned}(\sec t\cot t - \csc t\tan t)\sin t\cos t &= \sec t\cot t\sin t\cos t - \csc t\tan t\sin t\cos t \qquad \text{multiply out}\\ &= \underbrace{\left(\frac{1}{\cos t}\right)}_{\sec t}\underbrace{\left(\frac{\cos t}{\sin t}\right)}_{\cot t}\sin t\cos t - \underbrace{\left(\frac{1}{\sin t}\right)}_{\csc t}\underbrace{\left(\frac{\sin t}{\cos t}\right)}_{\tan t}\sin t\cos t\\ &= \cos t - \sin t.\end{aligned}$$

8. We have:

$$\begin{aligned}\frac{1}{2\csc t\cos t - 3\cot t} &= \frac{1}{2\underbrace{\left(\frac{1}{\sin t}\right)}_{\csc t}\cos t - 3\cot t}\\ &= \frac{1}{2\frac{\cos t}{\sin t} - 3\underbrace{\left(\frac{\cos t}{\sin t}\right)}_{\cot t}}\\ &= \frac{1}{-\left(\frac{\cos t}{\sin t}\right)} = -\frac{\sin t}{\cos t} = -\tan t.\end{aligned}$$

9. Using $1-\cos^2\theta = \sin^2\theta$, we have

$$\frac{1-\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta.$$

10. Writing $\cot x = \cos x/\sin x$ and $\csc x = 1/\sin x$, we have

$$\frac{\cot x}{\csc x} = \frac{\cos x}{\sin x}\cdot\frac{1}{1/\sin x} = \cos x.$$

11. Writing $\cos 2\phi = 2\cos^2\phi - 1$, we have

$$\frac{\cos 2\phi + 1}{\cos\phi} = \frac{2\cos^2\phi - 1 + 1}{\cos\phi} = \frac{2\cos^2\phi}{\cos\phi} = 2\cos\phi.$$

12. Multiplying out and using the fact that $1-\tan^2\theta = 1/\cos^2\theta$, we have

$$\cos^2\theta(1+\tan\theta)(1-\tan\theta) = \cos^2\theta(1-\tan^2\theta) = \cos^2\theta\cdot\frac{1}{\cos^2\theta} = 1.$$

13. $8-5i$

14. $(2+3i)(3+5i) = 6+9i+10i-15 = -9+19i$.

15. We have $(2)^2+(-4)^2 = 20$, and $\arctan(-2)\approx -1.107$. So the number is $\sqrt{20}e^{-1.107i}$.

16. We have $(3)^2+(5)^2 = 34$, and $\arctan(5/3)\approx 1.030$. So the number is $\sqrt{34}e^{1.030i}$.

Problems

17. We first solve for $\cos\alpha$,

$$\begin{aligned} 2\cos\alpha &= 1 \\ \cos\alpha &= \frac{1}{2} \\ \alpha &= \frac{\pi}{3},\ \frac{5\pi}{3}. \end{aligned}$$

18. We first solve for $\tan\alpha$,

$$\begin{aligned} \tan\alpha &= \sqrt{3} - 2\tan\alpha \\ 3\tan\alpha &= \sqrt{3} \\ \tan\alpha &= \frac{\sqrt{3}}{3} \\ \alpha &= \frac{\pi}{6},\ \frac{7\pi}{6} \end{aligned}$$

19. We first solve for $\tan\alpha$,

$$\begin{aligned} 4\tan\alpha + 3 &= 2 \\ 4\tan\alpha &= -1 \\ \tan\alpha &= -\frac{1}{4} \\ \alpha &= 2.897,\ 6.038. \end{aligned}$$

20. We first solve for $\sin\alpha$,

$$\begin{aligned} 3\sin^2\alpha + 4 &= 5 \\ 3\sin^2\alpha &= 1 \\ \sin^2\alpha &= \frac{1}{3} \\ \sin\alpha &= \pm\sqrt{\frac{1}{3}} \end{aligned}$$

$$\begin{array}{ll} \sin\alpha = \sqrt{\frac{1}{3}} & \sin\alpha = -\sqrt{\frac{1}{3}} \\ \alpha = 0.616,\ 2.526 & \alpha = 3.757,\ 5.668 \end{array}$$

21. **(a)** The graph resembles a cosine function with midline $k = 8$, amplitude $A = 10$, and period $p = 60$, so

$$y = 10\cos\left(\frac{2\pi}{60}x\right) + 8.$$

(b) We find the zeros at x_1 and x_2 by setting $y = 0$. Solving gives

$$\begin{aligned} 10\cos\left(\frac{2\pi}{60}x\right) + 8 &= 0 \\ \cos\left(\frac{2\pi}{60}x\right) &= -\frac{8}{10} \\ \frac{2\pi}{60}x &= \cos^{-1}(-0.8) \\ x &= \frac{60}{2\pi}\cos^{-1}(-0.8) \\ &= 23.8550. \end{aligned}$$

Judging from the graph, this is the value of x_1. By symmetry, $x_2 = 60 - x_1 = 36.1445$.

22. **(a)** The midline, $D = 10$, the amplitude $A = 4$, and the period 1, so

$$1 = \frac{2\pi}{B} \quad \text{and} \quad B = 2\pi.$$

Therefore the formula is

$$f(t) = 4\sin(2\pi t) + 10.$$

(b) Solving $f(t) = 12$, we have

$$\begin{aligned} 12 &= 4\sin(2\pi t) + 10 \\ 2 &= 4\sin(2\pi t) \\ \sin 2\pi t &= \frac{1}{2} \\ 2\pi t &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\ t &= \frac{1}{12}, \frac{5}{12}, \frac{13}{12}, \frac{17}{12} \end{aligned}$$

The spring is 12 centimeters from the ceiling at $1/12$ sec, $5/12$ sec, $13/12$ sec, $17/12$ sec.

23. **(a)** The average monthly temperature in Fairbanks is shown in Figure 9.48.

(b) These data are best modeled by a vertically reflected cosine curve, which is reasonable for something like temperature that oscillates with a 12-month period.

(c) The midline temperature is $(61.3 + (-11.5))/2 = 24.9$. The amplitude is $61.3 - 24.9 = 36.4$ Since the period is 12, we have $B = 2\pi/12 = \pi/6$. There is no horizontal shift, so $C = 0$. Hence our function is

$$f(t) = 24.9 - 36.4\cos\left(\frac{\pi}{6}t\right).$$

(d) To solve $f(t) = 32$, we must solve the equation

$$\begin{aligned}
32 &= 24.9 - 36.4\cos\left(\frac{\pi}{6}t\right) \\
7.1 &= -36.4\cos\left(\frac{\pi}{6}t\right) \\
\cos\left(\frac{\pi}{6}t\right) &= -0.1950.197 \\
\frac{\pi}{6}t &= \cos^{-1}(-0.195) = 1.767 \qquad \frac{\pi}{6}t = 2\pi - 1.767 = 4.516 \\
t &= 3.375 \qquad\qquad\qquad\qquad\quad t = 8.625
\end{aligned}$$

The temperature in Fairbanks reaches the freezing point in the middle of April and the middle of September.

(e) See Figure 9.49.

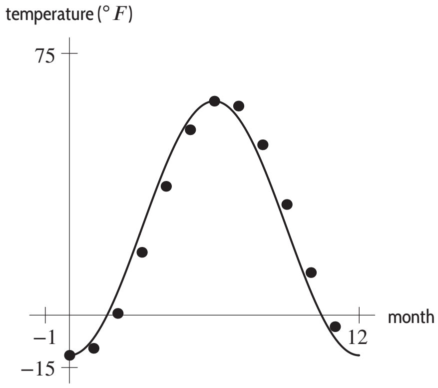


Figure 9.48

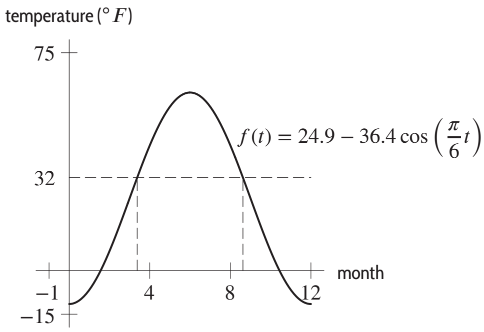


Figure 9.49

(f) The amplitude, period, and midline would be the same, but the function would be reflected vertically:

$$y = 24.9 + 36.4\cos\left(\frac{\pi}{6}t\right).$$

24. They are both right. The first student meant that $\sin 2\theta = 2\sin\theta$ is not an identity, meaning that it is not true for *all* θ. The second student had found one value for θ for which it was true.

25. Graphs of the four functions are in Figures 9.50–9.53. The graphs in Figures 9.51 and 9.52 suggest that $(\tan^2 x)(\sin^2 x)$ and $\tan^2 x - \sin^2 x$ may be identical.

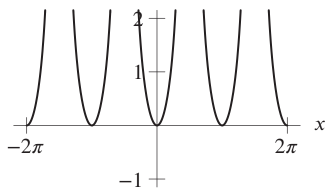


Figure 9.50: $\tan^2 x + \sin^2 x$

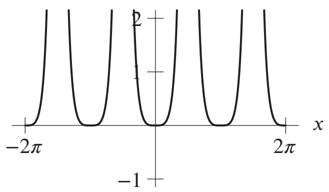


Figure 9.51: $(\tan^2 x)(\sin^2 x)$

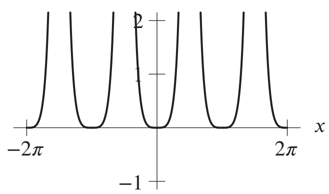


Figure 9.52: $\tan^2 x - \sin^2 x$

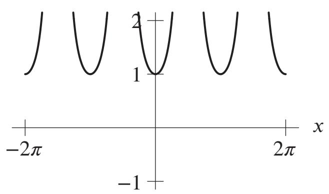


Figure 9.53: $\tan^2 x / \sin^2 x$

To prove the identity, use $\tan x = \sin x/\cos x$ to rewrite each side in terms of sine and cosine. We have

$$(\tan^2 x)(\sin^2 x) = \left(\frac{\sin x}{\cos x}\right)^2 (\sin^2 x) = \frac{\sin^4 x}{\cos^2 x}.$$

In addition,

$$\begin{aligned}\tan^2 x - \sin^2 x &= \left(\frac{\sin x}{\cos x}\right)^2 - \sin^2 x\\ &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x\\ &= \sin^2 x\left(\frac{1}{\cos^2 x} - 1\right)\\ &= \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}\\ &= \frac{\sin^2 x(\sin^2 x)}{\cos^2 x}\\ &= \frac{\sin^4 x}{\cos^2 x}.\end{aligned}$$

Since both expressions equal $\frac{\sin^4 x}{\cos^2 x}$, they are identical.

26. **(a)** $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{1} = y.$

(b) $\cos\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{\sqrt{1+y^2}}.$

(c) Since $\tan\theta = y$, we have $\theta = \tan^{-1} y$. Other answers are possible.

(d) $\cos(2\phi) = 2\cos\phi\sin\phi = 2\left(\frac{y}{\sqrt{1+y^2}}\right)\left(\frac{1}{\sqrt{1+y^2}}\right) = \frac{2y}{1+y^2}.$

27. Since $\csc\theta = 1/\sin\theta$, we have $\csc\theta = 1/(8/11) = 11/8$. Since $\cos^2\theta + \sin^2\theta = 1$,

$$\begin{aligned}\cos^2\theta + \left(\frac{8}{11}\right)^2 &= 1\\ \cos^2\theta &= 1 - \frac{64}{121}\\ \cos\theta &= \pm\sqrt{\frac{57}{121}}.\end{aligned}$$

Since $0 \le \theta \le \pi/2$, we know that $\cos\theta \ge 0$, so $\cos\theta = \sqrt{57/121}$. Since $\tan\theta = \sin\theta/\cos\theta$, we have $\tan\theta = (8/11)/\sqrt{57/121} = 8\sqrt{121}/11\sqrt{57} = 8/\sqrt{57}$.

28. Since $\sin\theta = 1/\csc\theta$, we have $\sin\theta = 1/94$. Using the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\cos^2\theta + \left(\frac{1}{94}\right)^2 &= 1\\ \cos^2\theta &= 1 - \frac{1}{94^2}\\ \cos\theta &= \pm\sqrt{\frac{8835}{8836}} = \pm\frac{\sqrt{8835}}{94}.\end{aligned}$$

Since $0 \le \theta \le \pi/2$, we know that $\cos\theta \ge 0$, so $\cos\theta = \sqrt{8835}/94$.

Using the identity $\tan\theta = \sin\theta/\cos\theta$, we see that $\tan\theta = (1/94)/(\sqrt{8835}/94) = 1/\sqrt{8835}$.

29. By the Pythagorean identity, we know that $\cos^2\theta + \sin^2\theta = 1$, so

$$\begin{aligned}0.27^2 + \sin^2\theta &= 1\\ \sin^2\theta &= 1 - 0.27^2\\ \sin\theta &= \pm\sqrt{0.927} \approx \pm 0.963.\end{aligned}$$

However, we only need one value, so we take $\sin\theta = 0.963$, Since $\tan\theta = \sin\theta/\cos\theta$, we have $\tan\theta = \sqrt{0.927}/0.27 \approx 3.566$.

30. No. Since $\sin\theta$ cannot be greater than 1, $\sin\theta = 3$ is impossible. The problem is that $\dfrac{\sin\theta}{\cos\theta}$ is a ratio, so $\sin\theta$ could be 0.3 and $\cos\theta$ could be 0.4. We only know the ratio is $3/4$.

31. We have $2/7 = \cos 2\theta = 2\cos^2\theta - 1$. Solving for $\cos\theta$ gives

$$\begin{aligned} 2\cos^2\theta &= \frac{9}{7} \\ \cos^2\theta &= \frac{9}{14} \end{aligned}$$

Since θ is in the first quadrant, $\cos\theta = +\sqrt{9/14} = 3/\sqrt{14}$.

32. We will use one of the three double-angle formulas for cosine to solve this equation algebraically. Since the equation involves $\sin\theta$, we will try the double-angle formula for cosine which involves the sine:

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

This gives

$$1 - 2\sin^2\theta = \sin\theta,$$

and we can rewrite this equation as follows:

$$2(\sin\theta)^2 + \sin\theta - 1 = 0.$$

Factoring, we have

$$(2\sin\theta - 1)(\sin\theta + 1) = 0.$$

The fact that the product $(2\sin\theta - 1)(\sin\theta + 1)$ equals zero implies that

$$2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta + 1 = 0.$$

Solving $\sin\theta + 1 = 0$, we have $\sin\theta = -1$. This means $\theta = 3\pi/2$. Solving the other equation, we have

$$\begin{aligned} 2\sin\theta - 1 &= 0 \\ \sin\theta &= \frac{1}{2}. \end{aligned}$$

This gives

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

In summary, there are three solutions for $0 \le \theta < 2\pi$: $\theta = \pi/6, 5\pi/6$, and $3\pi/2$. Figure 9.54 illustrates these solutions graphically as the points where the graphs of $\cos 2\theta$ and $\sin\theta$ intersect.

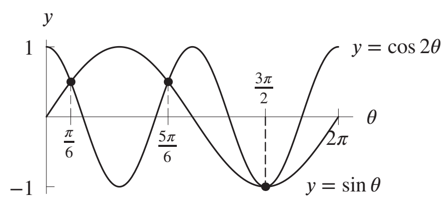


Figure 9.54: There are three solutions to the equation $\cos 2\theta = \sin\theta$, for $0 \le \theta < 2\pi$

33.

$$\begin{aligned} \cos(2\alpha) &= -\sin\alpha \\ 1 - 2\sin^2\alpha + \sin\alpha &= 0 \\ -2\sin^2\alpha + \sin\alpha + 1 &= 0 \\ 2\sin^2\alpha - \sin\alpha - 1 &= 0 \\ (2\sin\alpha + 1)(\sin\alpha - 1) &= 0 \end{aligned}$$

$$\begin{aligned} 2\sin\alpha + 1 &= 0 \\ 2\sin\alpha &= -1 \\ \sin\alpha &= -\frac{1}{2} \\ \alpha &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned} \qquad \begin{aligned} \sin\alpha - 1 &= 0 \\ \sin\alpha &= 1 \\ \alpha &= \frac{\pi}{2} \end{aligned}$$

34. Let $\theta = \cos^{-1}(\frac{5}{13})$. We can use the Pythagorean theorem and create a triangle with sides 5, 12, and 13 as in Figure 9.55. We see that $\sin\theta = \frac{12}{13}$. However we want $\sin(2\theta)$, so we use the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta = 2(\frac{12}{13})(\frac{5}{13}) = \frac{120}{169}$. You can check this on a calculator by finding that $\sin(2\cos^{-1}(5/13)) \approx 0.7100591716$, which is $120/169$ in decimal form.

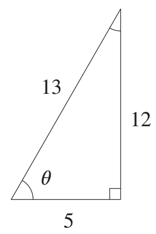


Figure 9.55

35. Start with the right side, which is more complex. Use a double-angle identity, so both sides are expressed as trigonometric functions of θ.

$$\begin{aligned} \frac{1-\cos 2\theta}{2\cos\theta\sin\theta} &= \frac{1-(2\cos^2\theta - 1)}{2\cos\theta\sin\theta} \\ &= \frac{2(1-\cos^2\theta)}{2\cos\theta\sin\theta} \\ &= \frac{2(\sin^2\theta)}{2\cos\theta\sin\theta} \\ &= \frac{\sin^2\theta}{\cos\theta\sin\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta. \end{aligned}$$

36. Start with the left side, which is more complex, and reduce it using the Pythagorean identity:

$$(\sin^2 2\pi t + \cos^2 2\pi t)^3 = (1)^3 = 1.$$

37. Start with the expression on the left and factor it as the difference of two squares, and then apply the Pythagorean identity to one factor.

$$\begin{aligned} \sin^4 x - \cos^4 x &= (\sin^2 x)^2 - (\cos^2 x)^2 \\ &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) \\ &= (\sin^2 x - \cos^2 x)(1) \\ &= (\sin^2 x - \cos^2 x). \end{aligned}$$

38. Because both sides are equally complicated, many different approaches are possible. We multiply the right side by $(1 + \sin\theta)/(1+\sin\theta)$ in order to introduce a factor of $1+\sin\theta$ to the numerator (the left side already has one) and to simplify the denominator. In the calculation which follows, we work only on the right side:

$$\begin{aligned}\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\\ &= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}\\ &= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\\ &= \frac{1+\sin\theta}{\cos\theta}.\end{aligned}$$

39. **(a)** $\cos\theta = x$.
(b) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$.
(c) $\tan^2\theta = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{\sqrt{1-x^2}}{x}\right)^2 = \frac{1-x^2}{x^2}$.
(d) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\sqrt{1-x^2}(x)$.
(e) $\cos(4\theta) = 1-2\sin^2(2\theta)$. Now we use part (d):
$\cos(4\theta) = 1-2(2x\sqrt{1-x^2})^2 = 1-8x^2(1-x^2)$.
(f) Using part (a), we see that $\cos^{-1}(x)=\theta$, so $\sin(\cos^{-1}x) = \sin(\theta) = \sqrt{1-x^2}$.

40. We first solve for $\cos\alpha$,

$$\begin{aligned}3\cos^2\alpha + 2 &= 3 - 2\cos\alpha\\ 3\cos^2\alpha + 2\cos\alpha - 1 &= 0\\ (3\cos\alpha - 1)(\cos\alpha + 1) &= 0\end{aligned}$$

$$\begin{aligned}3\cos\alpha - 1 &= 0 & \cos\alpha + 1 &= 0\\ \cos\alpha &= \tfrac{1}{3} & \cos\alpha &= -1\\ \alpha &= 1.231,\ 5.052 & \alpha &= \pi\end{aligned}$$

41. We first solve for $\sin\alpha$,

$$\begin{aligned}3\sin^2\alpha + 3\sin\alpha + 4 &= 3 - 2\sin\alpha\\ 3\sin^2\alpha + 5\sin\alpha + 1 &= 0\\ \sin\alpha &= \frac{-5\pm\sqrt{25-12}}{6}\\ \sin\alpha &= \frac{-5\pm\sqrt{13}}{6}\end{aligned}$$

$$\begin{aligned}\sin\alpha &= \tfrac{-5+\sqrt{13}}{6} & \sin\alpha &= \tfrac{-5-\sqrt{13}}{6}\\ \alpha &= 3.376,\ 6.049 & &\text{No solution } (-1\le\sin\alpha\le 1)\end{aligned}$$

42. Using the formula for $\cos(A+B)$ with $A=2\theta$ and $B=\theta$:

$$\begin{aligned}\cos 3\theta = \cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta(\sin^2\theta)\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

43. Start with the double-angle identity

$$\cos(2x) = 2\cos^2 x - 1.$$

Next, solve for $\cos x$:

$$\begin{aligned} 2\cos^2 x &= 1 + \cos(2x) \\ \cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\ \cos x &= \sqrt{\frac{1}{2}(1 + \cos(2x))}. \end{aligned}$$

Note that we chose the positive square root. We made this choice because we assumed that $0 \le \theta \le \pi/2$ which implies that $\cos x \ge 0$. Now we substitute $x = \theta/2$:

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}(1 + \cos\theta)}.$$

44. Since $0 < \ln x < \frac{\pi}{2}$ and $0 < \ln y < \frac{\pi}{2}$, the angles represented by $\ln x$ and $\ln y$ are in the first quadrant. This means that both their sine and cosine values will be positive. Since $\ln(xy) = \ln x + \ln y$, we can write

$$\sin(\ln(xy)) = \sin(\ln x + \ln y).$$

By the sum-of-angle formula we have

$$\sin(\ln x + \ln y) = \sin(\ln x)\cos(\ln y) + \cos(\ln x)\sin(\ln y).$$

Since cosine is positive, we have

$$\cos(\ln x) = \sqrt{1 - \sin^2(\ln x)} = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{\sqrt{8}}{3}$$

and

$$\cos(\ln y) = \sqrt{1 - \sin^2(\ln y)} = \sqrt{1 - \left(\frac{1}{5}\right)^2} = \frac{\sqrt{24}}{5}.$$

Thus,

$$\begin{aligned} \sin(\ln x + \ln y) &= \sin(\ln x)\cos(\ln y) + \cos(\ln x)\sin(\ln y) \\ &= \left(\frac{1}{3}\right)\left(\frac{\sqrt{24}}{5}\right) + \left(\frac{\sqrt{8}}{3}\right)\left(\frac{1}{5}\right) \\ &= \frac{\sqrt{24} + \sqrt{8}}{15} \\ &\approx 0.515. \end{aligned}$$

45. **(a)** The graph of $g(\theta) = \sin\theta - \cos\theta$ is shown in Figure 9.56.

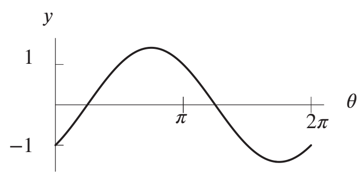


Figure 9.56

(b) We know $a_1 \sin t + a_2 \cos t = A\sin(t+\phi)$, where $A = \sqrt{a_1^2 + a_2^2}$ and $\tan\phi = a_2/a_1$. We let $a_1 = 1$ and $a_2 = -1$. This gives $A = \sqrt{2}$ and $\phi = \tan^{-1}(-1) = -\pi/4$, so

$$g(\theta) = \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right).$$

Note that we have chosen $B = 1$ and $k = 0$. We can check that this is correct by plotting the original function and $\sqrt{2}\sin(\theta - \pi/4)$ together.

We know that $\sin t = \cos(t - \pi/2)$, that is, the sine graph may be obtained by shifting the cosine graph $\pi/2$ units to the right. Thus,

$$g(\theta) = \sqrt{2}\cos\left(\theta - \frac{\pi}{4} - \frac{\pi}{2}\right) = \sqrt{2}\cos\left(\theta - \frac{3\pi}{4}\right).$$

46. **(a)** The population P_2 swings from a low of $3a - 2a = a$ to a high of $3a + 2a = 5a$, whereas P_1 swings from a low of $7a - a = 6a$ to a high of $7a + a = 8a$. Thus, P_2 has larger swings.

(b) The period of P_2 is $b + 2$, and the period of P_1 is b. Since P_2 has the longer period, it makes slower swings from low to high.

(c) At its smallest, $P_1 = 6a$, while at its largest, $P_2 = 5a$, so this statement describes P_1.

(d) The population P_2 reaches a low of a, where as P_1 reaches a low of $6a$, so P_2 appears more vulnerable to extinction.

47. See Figure 9.57. They appear to be the same graph. This suggests the truth of the identity $\cos t = \sin(t + \frac{\pi}{2})$.

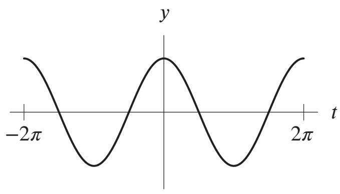


Figure 9.57: Graphs showing $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$

48. **(a)** $z_1 z_2 = (-3 - i\sqrt{3})(-1 + i\sqrt{3}) = 3 + (\sqrt{3})^2 + i(\sqrt{3} - 3\sqrt{3}) = 6 - i2\sqrt{3}$.

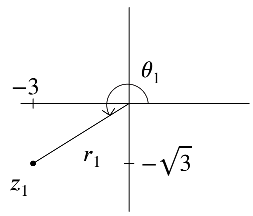
$$\frac{z_1}{z_2} = \frac{-3 - i\sqrt{3}}{-1 + i\sqrt{3}} \cdot \frac{-1 - i\sqrt{3}}{-1 - i\sqrt{3}} = \frac{3 - (\sqrt{3})^2 + i(\sqrt{3} + 3\sqrt{3})}{(-1)^2 + (\sqrt{3})^2} = \frac{i \cdot 4\sqrt{3}}{4} = i\sqrt{3}.$$

(b) We find (r_1, θ_1) corresponding to $z_1 = -3 - i\sqrt{3}$.

$r_1 = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$.

$\tan\theta_1 = \dfrac{-\sqrt{3}}{-3} = \dfrac{\sqrt{3}}{3}$, so $\theta_1 = \dfrac{7\pi}{6}$.

Thus $-3 - i\sqrt{3} = r_1 e^{i\theta_1} = 2\sqrt{3}\, e^{i\frac{7\pi}{6}}$.

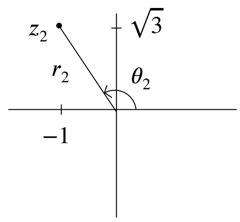


We find (r_2, θ_2) corresponding to $z_2 = -1 + i\sqrt{3}$.

$r_2 = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$;

$\tan\theta_2 = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$, so $\theta_2 = \dfrac{2\pi}{3}$.

Thus, $-1 + i\sqrt{3} = r_2 e^{i\theta_2} = 2e^{i\frac{2\pi}{3}}$.



We now calculate $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

$$\begin{aligned} z_1 z_2 &= \left(2\sqrt{3}e^{i\frac{7\pi}{6}}\right)\left(2e^{i\frac{2\pi}{3}}\right) = 4\sqrt{3}e^{i(\frac{7\pi}{6}+\frac{2\pi}{3})} = 4\sqrt{3}e^{i\frac{11\pi}{6}} \\ &= 4\sqrt{3}\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right] = 4\sqrt{3}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = 6 - i2\sqrt{3}. \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\sqrt{3}e^{i\frac{7\pi}{6}}}{2e^{i\frac{2\pi}{3}}} = \sqrt{3}e^{i(\frac{7\pi}{6}-\frac{2\pi}{3})} = \sqrt{3}e^{i\frac{\pi}{2}} \\ &= \sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i\sqrt{3}. \end{aligned}$$

These agree with the values found in (a).

49. **(a)** Since $z = 3 + 2i$, we have $iz = i(3 + 2i) = 3i + 2i^2 = -2 + 3i$. See Figure 9.58.

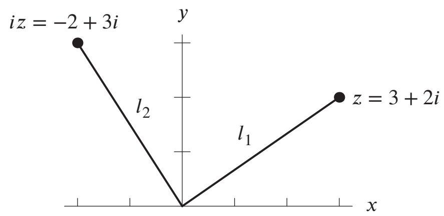


Figure 9.58

(b) The slope of the line l_1 in Figure 9.58 is $2/3$, and the slope of the line l_2 is $-3/2$. Since the product $(2/3)(-3/2) = -1$ of these two slopes is -1, the lines are perpendicular.

STRENGTHEN YOUR UNDERSTANDING

1. True, because the factor e^{-t} decreases the oscillations of $\cos t$ as t grows.

2. True. Substitute in $\tan\phi = \sin\phi/\cos\phi$ and simplify to see that this is an identity.

3. True. This is the definition of an identity.

4. False. An odd function would require $f(-\theta) = -f(\theta)$, but $f(-\frac{\pi}{4}) = f(\frac{\pi}{4}) = \dfrac{\sqrt{2}}{2}$.

5. True. This is an identity. Substitute using $\tan^2\theta = \sin^2\theta/\cos^2\theta$ and simplify to obtain $2 = 2\sin^2\theta + 2\cos^2\theta$. Divide by 2 to reach the Pythagorean identity.

6. False. This is not the Pythagorean identity, since the square is on the variable β. As a counterexample, let $\beta = 1$. Note that $\cos\beta^2 \neq (\cos\beta)^2$ and $\cos^2\beta = (\cos\beta)^2$.

7. True. First find $\sin\alpha$ and then use the double-angle formula. Since $\cos^2\alpha + \sin^2\alpha = 1$, we have $\sin^2\alpha = 1 - \frac{4}{9} = \frac{5}{9}$. Because α is in the third quadrant, its sine is negative, so $\sin\alpha = -\frac{\sqrt{5}}{3}$. We find $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2(-\frac{\sqrt{5}}{3})(-\frac{2}{3}) = \frac{4\sqrt{5}}{9}$.

8. False. Although true for the value $\theta = 45°$, it is not true for all values of θ. A counterexample is $\sin 0° \neq \cos 0°$.

9. True. There are many ways to prove this identity. We use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to substitute in the right side of the equation. This becomes $\frac{1}{2}(1 - (\cos^2\theta - \sin^2\theta))$. Now substitute using $1 - \cos^2\theta = \sin^2\theta$ (a form of the Pythagorean identity.) The right side then simplifies to $\sin^2\theta$, which is the left side.

10. False. A counterexample is $\cos(4\cdot 0°) = 1 \neq -\cos(-4\cdot 0°) = -1$.

11. False. For a counterexample, let $\theta = \phi = 90°$. Then $\sin(\theta + \phi) = \sin 180° = 0$, but $\sin\theta + \sin\phi = \sin 90° + \sin 90° = 1 + 1 = 2$.

12. False. For a counterexample let $\theta = 90°$. Then $\sin(2\theta) = \sin 180° = 0$, but $2\sin\theta = 2\sin 90° = 2$.

13. True. Start with the sine sum-of-angle identity:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

and let $\phi = \pi/2$, so

$$\sin(\theta + \pi/2) = \sin\theta\cos(\pi/2) + \sin(\pi/2)\cos\theta.$$

Simplify to

$$\sin(\theta + \pi/2) = \sin\theta \cdot 0 + 1 \cdot \cos\theta = \cos\theta.$$

14. False. We have $f(t) = \sin t \cos t = (1/2)\sin 2t$ which is periodic with period π.

15. True. We have $\sin(t - \pi) = \sin((t - \pi) + 2\pi) = \sin(t + \pi)$ where the first equality is by periodicity of the sine function.

16. False. For example, $t = \pi/2$ gives $\cos(t - \pi/2) = 1$ and $\cos(t + \pi/2) = -1$. The correct identity is $\cos(t - \pi/2) = -\cos(t + \pi/2)$.

17. True. Since $A\cos(Bt) = A\sin(Bt + \pi/2) = A\sin(B(t + \pi/(2B)))$, the graph of $A\cos(Bt)$ is a shift of $A\sin(Bt)$ to the left by $\pi/(2B)$.

18. True. We have

$$\begin{aligned} f(x + 2\pi) &= \sin 2(x + 2\pi) + \sin 3(x + 2\pi) \\ &= \sin(2x + 4\pi) + \sin(3x + 6\pi) \\ &= \sin 2x + \sin 3x = f(x) \end{aligned}$$

which shows that $f(x)$ is periodic. The period is 2π.

19. False. The sum of two nonzero sinusoidal functions with different periods is not a sinusoidal function. A graph of $f(x)$ on the interval $0 \le x \le 4\pi$ shows a more complicated shape that the graphs of a sinusoid.

20. False. If the periods are different the sum may not be sinusoidal. As a counterexample, graph $f(t) = \sin t + \sin 2t$, to see that $f(t)$ is not sinusoidal.

21. True. We use the assumption that a_1 and a_2 are nonzero. The amplitude of the single sine function is $A = \sqrt{a_1^2 + a_2^2}$. Thus, A is greater than either a_1 or a_2.

22. True. We have $f(t) = A\sin(2\pi t + \phi)$ where $A = \sqrt{3^2 + 4^2} = 5$, $\cos\phi = \frac{3}{5}$ and $\sin\phi = \frac{4}{5}$.

23. False. The amplitude of g increases exponentially.

24. True. The maximum of g occurs when $\sin(\pi t/12) = -1$, so the maximum value of the function is $55 + 10 = 65$.

25. True. Hertz is a measure of cycles per second and so a single cycle will take 1/60th of a second.

26. True. The zeros occur when $\cos(\pi t) + 1 = 0$, so $\cos(\pi t) = -1$. Hence, $\pi t = n\pi$, where n is an odd integer. Thus, $t = n$ is an odd integer.

27. False. The variable B affects the period; the variable A affects the amplitude.

28. True; divide by e^t, which is never zero, and obtain $\tan t = 2/3$.

29. True. Since $0 \le \cos^{-1}(x) \le \pi$, we have $0 \le \sin(\cos^{-1} x) \le 1$. Thus, $\cos^{-1}(\sin(\cos^{-1} x))$ is an angle θ whose cosine is between 0 and 1. In addition, we have $0 \le \theta \le \pi$, as this is part of the definition of $\cos^{-1} x$. Hence $0 \le \theta \le \pi/2$.

30. True; we need $a + b\sin t$ to be positive for the natural logarithm to be defined. Since $\sin t \ge -1$, we have $a + b\sin t \ge a - b > 0$.

31. True, since $\sqrt{a}$ is real for all $a \ge 0$.

32. True, since $(x - iy)(x + iy) = x^2 + y^2$ is real.

33. False, since $(1 + i)^2 = 2i$ is not real.

34. False. Let $f(x) = x$. Then $f(i) = i$ but $f(\bar{i}) = \bar{i} = -i$.

35. True. We can write any nonzero complex number z as $re^{i\beta}$, where r and β are real numbers with $r > 0$. Since $r > 0$, we can write $r = e^c$ for $c = \ln r$. Therefore, $z = re^{i\beta} = e^c e^{i\beta} = e^{c+i\beta} = e^w$ where $w = c + i\beta$ is a complex number.

36. False, since $(1 + 2i)^2 = -3 + 4i$.

37. True. This is Euler's formula, fundamental in higher mathematics.

38. True. $(1 + i)^2 = 1^2 + 2 \cdot 1 \cdot i + i^2 = 1 + 2i - 1 = 2i$.

39. True. Since $i^4 = 1$, we have $i^{101} = (i^4)^{25} i = 1 \cdot i = i$.