Special values of Riemann's zeta function

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March 6, 2013

Riemann's zeta function

If $s > 1$ is a real number, then the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

converges.

Proof: Compare the partial sum to an integral,

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq 1 + \int_1^N \frac{dx}{x^s} = 1 + \frac{1}{s-1}\left(1 - \frac{1}{N^{s-1}}\right) \leq 1 + \frac{1}{s-1}.$$

The resulting function $\zeta(s)$ is called *Riemann's zeta function*.

Was studied in depth by Euler and others before Riemann.

$\zeta(s)$ is named after Riemann for two reasons:

1. He was the first to consider allowing the s in $\zeta(s)$ to be a complex number $\neq 1$.
2. His deep 1859 paper "*Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*" ("*On the number of primes less than a given quantity*") made remarkable connections between $\zeta(s)$ and prime numbers.

In this talk we will discuss certain special values of $\zeta(s)$ for integer values of s.

In particular, we will discuss what happens at $s = 1$, 2 and -1.

Overview

1. The divergence of $\zeta(1)$
2. The identity $\zeta(2) = \pi^2/6$
3. The identity $\zeta(-1) = -1/12$

What happens as $s \to 1$?

The value $\zeta(s)$ diverges to ∞ as s approaches 1.

To see this, use an integral to bound the partial sums from below for $s > 1$:

$$\sum_{n=1}^{N} \frac{1}{n^s} \geq \int_1^{N+1} \frac{dx}{x^s} = \frac{1}{s-1}\left(1 - \frac{1}{(N+1)^{s-1}}\right).$$

It follows that $\zeta(s) \geq (s-1)^{-1}$ for $s > 1$.

In summary, so far we've seen that for $s > 1$,

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{1}{s-1} + 1.$$

Since the lower bound diverges as $s \to 1$, so does $\zeta(s)$.

This is related to the fact that the *Harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

Cute proof that the Harmonic series diverges

We consider the partial sum involving 2^k terms:

$$\begin{aligned}
\sum_{n=1}^{2^k} \frac{1}{n} &= 1 + \overbrace{\frac{1}{2}}^{2^0 \text{ terms}} + \overbrace{\frac{1}{3} + \frac{1}{4}}^{2^1 \text{ terms}} + \cdots + \overbrace{\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k}}^{2^{k-1} \text{ terms}} \\
&\geq 1 + \overbrace{\frac{1}{2}}^{2^0 \text{ terms}} + \overbrace{\frac{1}{4} + \frac{1}{4}}^{2^1 \text{ terms}} + \cdots + \overbrace{\frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k}}^{2^{k-1} \text{ terms}} \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{k-1}}{2^k} \\
&= 1 + \frac{k}{2}
\end{aligned}$$

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The identity $\zeta(2) = \pi^2/6$

If we apply the bounds

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{1}{s-1} + 1$$

from the previous part to $s = 2$ we deduce that

$$1 \leq \zeta(2) \leq 2.$$

But what number in this interval is

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots ?!$$

It turns out that

$$\zeta(2) = \frac{\pi^2}{6}.$$

In fact, more generally if $k \geq 1$ is any positive integer, then

$$\zeta(2k) = (-1)^{k+1}\frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

Here B_n is a rational number, the nth Bernoulli number, defined to be the coefficient of $X^n/n!$ in the series

$$\frac{X}{e^X - 1} = \sum_{n=0}^{\infty} B_n \frac{X^n}{n!}.$$

Thus, each value $\zeta(2k)$ is a rational multiple of π^{2k}.

If that isn't surprising to you, be aware of the following: the odd values $\zeta(2k+1)$ are not expected to be related to π in any significant algebraic way.

Why the even zeta values $\zeta(2k)$ are algebraically related to π and the odd values $\zeta(2k+1)$ are (probably) not is one unsolved problem in mathematics.

We'll now offer seven proofs that $\zeta(2) = \pi^2/6$, one for every day of the week.

First proof: An elementary trigonometric argument

First we note that for $\theta = \pi/(2N+1)$ one has

$$\cot^2(\theta) + \cot^2(2\theta) + \cdots + \cot^2(N\theta) = \frac{N(2N-1)}{3}.$$

For x in $(0, \pi/2)$ the inequality $\sin x < x < \tan x$ implies

$$\cot^2 x < \frac{1}{x^2} < \cot^2 x + 1.$$

Apply this to each of $x = \theta$, 2θ, 3θ, etc, and sum to deduce

$$\frac{N(2N-1)}{3} < \frac{1}{\theta^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right) < \frac{N(2N-1))}{3} + N.$$

Multiply by $\theta^2 = \pi^2/(2N+1)^2$ to deduce

$$\frac{\pi^2}{3}\frac{N(2N-1)}{(2N+1)^2} < \sum_{n=1}^{N}\frac{1}{n^2} < \frac{\pi^2}{3}\frac{N(2N-1)}{(2N+1)^2} + \pi^2\frac{N}{(2N+1)^2}$$

Since the upper and lower bounds both converge to the same limit as N grows, and the middle one converges to $\zeta(2)$, we deduce that

$$\zeta(2) = \frac{\pi^2}{3}\cdot\lim_{N\to\infty}\frac{N(2N-1)}{(2N+1)^2} = \frac{\pi^2}{3}\cdot\lim_{N\to\infty}\frac{1-\frac{1}{2N^2}}{2(1+\frac{1}{2N})^2} = \frac{\pi^2}{6}.$$

Second proof: Fourier series

The Fourier series expansion of x^2 is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2}.$$

Since $\cos(n\pi) = (-1)^n$ for integers n, evaluating at $x = \pi$ gives

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3} + 4\zeta(2).$$

Hence $\zeta(2) = \pi^2/6$.

Third proof: A double integral

We evaluate a certain double integral two ways. First,

$$\begin{aligned} I &= \int_0^1 \int_0^1 \frac{dxdy}{1-xy} \\ &= \sum_{n\geq 0} \int_0^1 \int_0^1 (xy)^n dxdy \\ &= \sum_{n\geq 1} \int_0^1 \frac{y^{n-1}}{n} dy \\ &= \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2). \end{aligned}$$

On the other hand, the substitutions $x = (\sqrt{2}/2)(u - v)$ and $y = (\sqrt{2}/2)(u + v)$ allow one to write

$$I = 4\int_0^{\sqrt{2}/2}\int_0^{u} \frac{dudv}{2 - u^2 + v^2} + 4\int_{\sqrt{2}/2}^{\sqrt{2}}\int_0^{\sqrt{2}-u} \frac{dudv}{2 - u^2 + v^2}.$$

Persistance and some trig substitutions allow one to evaluate both of the above integrals and show that

$$\zeta(2) = I = \underbrace{\frac{\pi^2}{18}}_{\text{first integral}} + \underbrace{\frac{\pi^2}{9}}_{\text{second integral}} = \frac{\pi^2}{6}.$$

Fourth proof: the residue theorem

The following can be proved using the residue theorem from complex analysis.

Theorem (Summation of rational functions)

Let P and Q be polynomials with $\deg Q \geq \deg P + 2$ and let $f(z) = P(z)/Q(z)$. Let $S \subseteq \mathbf{C}$ be the finite set of poles of f. Then

$$\lim_{N\to\infty} \sum_{\substack{k=-N \\ k\notin S}}^{N} f(k) = -\sum_{p\in S} \mathit{residue}_{z=p}(\pi f(z)\cot(\pi z)).$$

Let's take $f(z) = 1/z^2$. In this case $S = \{0\}$ and the theorem gives a formula for the sum

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2} = 2\zeta(2).$$

Since the polar set S consists only of 0, the preceding summation theorem shows us that this sum is nothing but

$$-\mathrm{residue}_{z=0}(\pi \cot(\pi z)/z^2).$$

That is, the theorem immediately gives us the formula

$$\zeta(2) = -\frac{\pi}{2} \cdot \mathrm{residue}_{z=0} \left(\frac{\cot(\pi z)}{z^2} \right).$$

We have

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{z^2}\left(\frac{a}{z} + b + cz + dz^2 + \cdots\right) = \frac{a}{z^3} + \frac{b}{z^2} + \overbrace{\frac{c}{z}}^{\text{residue}} + d + \cdots$$

and hence

$$\text{residue}_{z=0}\left(\frac{\cot(\pi z)}{z^2}\right) = \frac{1}{2}\cdot\frac{d^2}{dz^2}\left(z^3\cdot\frac{\cot(\pi z)}{z^2}\right)\Bigg|_{z=0} = -\frac{\pi}{3}.$$

Putting everything together shows that

$$\zeta(2) = -\frac{\pi}{2}\cdot\text{residue}_{z=0}\left(\frac{\cot(\pi z)}{z^2}\right) = \left(-\frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{3}\right) = \frac{\pi^2}{6}$$

Fifth proof: Weierstrass product

Let $P(X)$ be a polynomial of the form

$$P(X) = (1 + r_1 X)(1 + r_2 X) \cdots (1 + r_n X).$$

Then the coefficient of X in $P(X)$ is equal to

$$r_1 + r_2 + \cdots + r_n.$$

The sine function is *like* a polynomial: it has a Taylor series

$$\sin(X) = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!} + \cdots$$

and a Weierstrass product

$$\sin(X) = X \prod_{n=1}^{\infty} \left(1 - \frac{X^2}{(\pi n)^2}\right).$$

If we cancel X and let $Z = X^2$ then we deduce that

$$1 - \frac{Z}{3!} + \frac{Z^2}{5!} - \frac{Z^3}{7!} + \cdots = \prod_{n=1}^{\infty} \left(1 + \left(-\frac{1}{(\pi n)^2}\right) Z\right).$$

In analogy with polynomials, the identity

$$1 - \frac{Z}{3!} + \frac{Z^2}{5!} - \frac{Z^3}{7!} + \cdots = \prod_{n=1}^{\infty} \left(1 + \left(-\frac{1}{(\pi n)^2}\right) Z\right).$$

suggests that the coefficent of Z should be the sum of the reciprocal roots on the right. That is:

$$-\frac{1}{3!} = \sum_{n \geq 1} \frac{-1}{(\pi n)^2}$$

and hence $\zeta(2) = \frac{\pi^2}{6}$.

Sixth proof: moduli of elliptic curves

Let

$$\mathcal{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$$

and let $\mathrm{SL}_2(\mathbf{Z})$ act on $\mathcal{H}$ via fractional linear transformation. Then

$$\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}$$

is the coarse moduli space of elliptic curves, and one can show that

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}} \frac{dxdy}{y^2} = \frac{2\zeta(2)}{\pi}.$$

But this integral can be computed explicitely and is equal to $\pi/3$. Hence $\zeta(2) = \pi^2/6$.

Seventh proof: probabilistic (as in, this is probably a proof)

Euler used unique factorization to prove that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This *Euler product* is taken over all primes p.

The probability that an integer is divisible by p is $1/p$.

This is independent among numbers, so the probability that two integers are simultaneously divisible by p is $1/p^2$.

Recall: *coprime* integers share no common prime factors.

The probability $P(\text{coprime})$ that two random integers are coprime is the product over all primes p of the probability that they do not share the prime factor p.

Thus, the Euler product for $\zeta(s)$ shows that

$$P(\text{coprime}) = \prod_p \left(1 - \frac{1}{p^2}\right) = \zeta(2)^{-1}.$$

So to prove $\zeta(2) = \pi^2/6$, you just need to choose enough random pairs of integers and test whether they're coprime!

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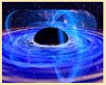
3. The identity $\zeta(-1) = -1/12$

In what sense does $1 + 2 + 3 + 4 + \cdots = -1/12$?

Talk:Zeta function regularization

From Wikipedia, the free encyclopedia



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Physics portal

Start — This article has been rated as **Start-Class** on the project's quality scale.

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On the face of this this article appears to be rubbish. Can anyone make sense of it? Billlion 15:53, 8 November 2005 (UTC)

I cleaned up the format and added some links, but it's not my field and I can't say anything about the content. It seems to closely follow Casimir_effect#Calculation, so maybe it could be merged or redirected there. Tom Harrison (talk) 17:01, 8 November 2005 (UTC)

I improve some explanations and make clear the relation to Casimir effect. --Enyokoyama (talk) 12:53, 6 January 2013 (UTC)

Figure: *Wikipedia* talk page for the article *Zeta function regularization*, February 25, 2013

Of course it's not literally true that the series

$$1 + 2 + 3 + 4 + 5 + \cdots$$

converges in the conventional sense of convergence.

There is a deeper truth hidden in the seemingly absurd claim that

$$1 + 2 + 3 + 4 + 5 + \cdots = -1/12.$$

Zeta as a function of a complex variable

As observed by Riemann, the sum defining the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

makes sense for all complex s with $\Re(s) > 1$.

Proof: If $s = x + iy$ with $x > 1$, note that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{e^{i\log(n)y}\, n^x}.$$

Since $\left|e^{i\log(n)y}\right| = 1$, this series converges absolutely if $\zeta(x)$ does. Since $x > 1$, we win.

Analyticity of $\zeta(s)$

The resulting complex zeta function is *analytic* (a.k.a. *complex differentiable*).

Proof: The partial sums are clearly analytic, being a finite sum of exponentials. It's not hard to prove that they converge uniformly on regions $\Re(s) \geq 1 + \varepsilon$ for $\varepsilon > 0$. A standard result in complex analysis then implies that $\zeta(s)$ is analytic in the region $\Re(s) > 1$.

Analytic continuation

Analytic functions are very rigid – they satisfy the property of *analytic continuation*. More precisely, one proves the following in a first course on complex analysis:

Theorem

Let $U \subseteq \mathbf{C}$ be an open subset and let f be analytic on U. Let $V \supset U$ denote a larger open subset, and assume further that V is connected. Then there exists at most one analytic function g on V such that $g|_U = f$.

Another fundamental contribution of Riemann to the study of $\zeta(s)$ is his proof that $\zeta(s)$ continues analytically to an analytic function on $\mathbf{C} - \{1\}$.

Since $\zeta(s)$ has a pole at $s = 1$, this is as good as it could be!

Note: outside the region $\Re(s) > 1$, the function $\zeta(s)$ is not defined by the usual summation. This distinction is crucial!

Functional equation

Still another fundamental contribution of Riemann to the study of $\zeta(s)$ is his proof of the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

where $\Gamma(s)$ denotes the gamma function defined via the integral

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$

Note that the functional equation relates $\zeta(-1)$ with $\zeta(2)$!

The value $\zeta(-1)$

So, if we plug in $s = -1$ to

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

we get

$$\begin{aligned}\zeta(-1) &= 2^{-1}\pi^{-2}\sin(-\pi/2)\Gamma(2)\zeta(2) \\ &= \left(\frac{-1}{2\pi^2}\right) \cdot 1! \cdot \left(\frac{\pi^2}{6}\right) \\ &= -\frac{1}{12}.\end{aligned}$$

Zeta function regularization

Physicists will often use this sort of technique to assign finite values to divergent series. Let

$$a_1 + a_2 + a_3 + \cdots$$

denote a possibly divergent series.

To assign it a finite value, define an associated zeta function:

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}.$$

If this converges and continues analytically to $s = -1$, then one can think of the value $\zeta_A(-1)$ as "acting like" the sum of the series $a_1 + a_2 + a_3 + \cdots$.

Stricly speaking, the sum

$$a_1 + a_2 + a_3 + \cdots$$

would not necessarily *converge* to $\zeta_A(-1)$ in any rigorous sense. Nevertheless, it turns out to be physically useful to assign such "zeta-regularized" values to certain divergent series.

The value $\infty!$

On this note, we'll end by discussing how to assign a "value" to $\infty!$. Here is a highly suspicious derivation:

$$\begin{aligned} \infty! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots \\ &= \exp\left(\sum_{n=1}^{\infty} \log(n)\right) \\ &= \exp\left(-\zeta'(0)\right) \end{aligned}$$

Since $\zeta(s)$ is analytic on $\mathbf{C} - \{1\}$, the value $\zeta'(0)$ is finite!

One can use the functional equation for $\zeta(s)$ to deduce that

$$-\zeta'(0) = \frac{1}{2}\log(2\pi).$$

Hence,

$$\infty! = \exp(-\zeta'(0)) = \exp((1/2)\log(2\pi)) = \sqrt{2\pi}$$

... right?

Thanks for listening!