A Framework for Robustness Analysis of Constrained Finite Receding Horizon Control

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Abstract

A framework for robustness analysis of input constrained finite receding horizon control is presented. Under the assumption of quadratic upper bounds on the finite horizon costs, we derive sufficient conditions for robust stability of the standard discrete-time linear-quadratic receding horizon control formulation. This is achieved by recasting conditions for nominal and robust stability as an implication between quadratic forms, lending itself to S-procedure tools which are used to convert robustness questions to tractable convex conditions. Robustness with respect to plant/model mismatch as well as for state measurement error is shown to reduce to the feasibility of linear matrix inequalities. Simple examples demonstrate the approach.

Keywords: predictive control, optimal control, linear systems, robustness, S-procedure, LMI.

1 Introduction

Receding horizon, moving horizon and model predictive control are names for a state feedback control technique where the control action is determined by solving an on-line optimization at each time step. The optimization involves the solution to a finite horizon open-loop control problem using a model of the true plant. The ability to easily incorporate constraints into the on-line optimization is the major advantage of receding horizon control. Unfortunately, a thorough theoretical analysis of its properties has proved to be a challenging task.

The difficulty with stability analysis of constrained receding horizon control can be attributed to the fact that it produces inherently nonlinear closed-loop systems, even when the plant is linear. Furthermore, this is complicated by the fact that, in general, a closed form expression for the controller and closed loop system does not exist. These difficulties are compounded even further when a mismatch exists between the true plant and the model of the plant used in the on-line optimization. Many authors have delved into the area of robustness of receding horizon control, generally approaching the problem from one of two viewpoints. One approach is to provide a robust formulation of receding horizon control by altering the on-line optimization to guarantee certain

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properties [10, 15, 4, 1, 28], while the other involves robustness analysis of more standard receding horizon implementations [25, 26, 8, 17, 18, 22, 21, 28, 16]. The approach presented in this paper tends to align better with the second point of view.

There are two distinguishing features of the problem we tackle. First, we consider control constrained receding horizon formulations using the simplest on-line optimizations. Since completing the optimization in a short amount of time is crucial, when constraints are linear, we strip the on-line optimization to the simplicity of a quadratic program, requiring no additional constraints such as terminal constraints or that the final predicted state reach a specified region. Hence, we consider schemes that are practically implementable. Secondly, we develop robustness *analysis* tools for these simplest schemes. That is, given a receding horizon controller we probe its stability and robustness properties.

While some other techniques require difficult on-line optimizations to obtain robustness results, our main difficulties are translated to off-line calculations. We make a key assumption to obtain our results, which is that quadratic upper bounds on the finite horizon cost can be calculated. This is nearly equivalent to finding a stabilizing control law that satisfies the constraints. When such a controller is already known to exist, what is the benefit of constrained receding horizon control? The justification tends to be that receding horizon control exhibits superior performance properties when compared to other constrained stabilizing controllers. This is why receding horizon control is often applied to open-loop stable plants, rather than just allowing the open-loop plant to evolve. Of course, it is not always true that receding horizon control performs well. In certain examples it actually destabilizes open-loop stable plants, hence a stability theory is necessary as well.

Our basic approach can be summarized as follows. We use the information in quadratic upper bounds on the finite horizon costs to write sufficient conditions for robust stability as implications between quadratic forms. The S-procedure [13, 14, 24] is then used to convert these to Linear Matrix Inequalities (LMIs), which are computationally tractable.

2 Problem Formulation

Let $\mathbb{R}^n$ denote the space of real n dimensional vectors and $\mathbb{R}^{n\times m}$ denote real matrices of size $n\times m$. The notation $(>,\geq,\leq,<)$ will be used to denote standard inequalities, with $(\succ,\succeq,\preceq,\prec)$ denoting matrix inequalities. That is, $A\succeq B$ if and only if $A-B$ is positive semidefinite.

2.1 Plant Models

We will refer to three different plant models: a nominal plant model and two uncertain representations. These are described below:

1. *Nominal:*
$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$
with $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$.

2. *Polytopic:*
$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k), \quad [\tilde{A},\tilde{B}] \in \,? \tag{2}$$
with $\tilde{A}\in\mathbb{R}^{n\times n}$, $\tilde{B}\in\mathbb{R}^{n\times m}$ and
$$? := Co\,\{[A_1,B_1],\ldots,[A_L,B_L]\} \tag{3}$$
where Co denotes the convex hull.

3. *Structured:*

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) &= C_1 x(k) + D_{11} w(k) + D_{12} u(k) \\ w(k) &= \Delta z(k) \end{aligned} \tag{4}$$

where $w(k) \in \mathbb{R}^{m_1}$ and the operator Δ is block diagonal: $\Delta = diag([\Delta_1, \ldots, \Delta_r])$, where each Δ_j is a memoryless time-varying matrix with $\|\Delta_j\|_2 := \overline{\sigma}(\Delta_j) \leq 1$, $j = 1 \ldots r$.

In all cases, we will assume the pair $[A, B]$ is controllable. In receding horizon control, an optimization is solved at every time step. This optimization is based on a plant model. In this paper, that model will always be the nominal model. We will then ask whether this controller based on the nominal plant model stabilizes all plants within the two uncertain models (polytopic and structured).

2.2 Receding Horizon Controller

A receding horizon controller is based on the following optimization which is solved at each time step:

$$\begin{aligned} J_i(x) := \min_{u(0),\ldots,u(i-1)} & \left[x^T(i) P_0 x(i) + \sum_{k=0}^{i-1} \left(x^T(k) Q x(k) + u^T(k) R u(k) \right) \right] \\ \text{subject to: } & x(k+1) = Ax(k) + Bu(k), \quad k = 0 \ldots i-1 \\ & x(0) = x \\ & c_u(u(k)) \leq 0, \quad k = 0 \ldots i-1 \end{aligned} \tag{5}$$

where $Q > 0$, $R > 0$, $P_0 > 0$ and i is the horizon length. The input constraint $c_u(\cdot) \leq 0$ is assumed to be convex and feasible in a neighborhood containing $u = 0$. (Note that we only consider control constraints. In Section 4 we discuss the difficulties associated with extending the results in this paper to state constraints.) To streamline notation, we will use the following definitions. Let

$$u_{[j,r]} := [u(j); \ldots; u(r)]$$

denote a sequence of control actions from j to r. The $*$ notation will be reserved to denote the optimal solution to (5) which is a function of the initial state x and horizon length i, i.e.

$$u^*_{[0,i-1]}(x, i) := [u^*(0; x, i); \ldots; u^*(i-1; x, i)]$$

corresponds to $J_i(x)$. Furthermore, we will use the notation $x^*(k; x, i)$, $k = 0, \ldots, i$ to denote the state trajectory obtained by applying the optimizing sequence $u^*_{[0,i-1]}(x, i)$ from the initial condition x through the nominal dynamics $x(k+1) = Ax(k) + Bu(k)$. The optimization in (5) can be written entirely in terms of the initial state x and the sequence of controls $u_{[0,i-1]}$ in which case it takes the form:

$$\begin{aligned} J_i(x) := & \inf_{\substack{u_{[0,i-1]} \\ c_u(u(k)) \leq 0, \quad k = 0 \ldots i-1}} \begin{bmatrix} x \\ u_{[0,i-1]} \end{bmatrix}^T H_i \begin{bmatrix} x \\ u_{[0,i-1]} \end{bmatrix} \\ = & \begin{bmatrix} x \\ u^*_{[0,i-1]}(x,i) \end{bmatrix}^T H_i \begin{bmatrix} x \\ u^*_{[0,i-1]}(x,i) \end{bmatrix} \end{aligned} \tag{6}$$

where H_i is the appropriately formulated matrix. The receding horizon controller is based on the optimization in (6). That is, at each time step k, the optimization (6) with a fixed horizon $i = N$ is solved for $u^*_{[0,N-1]}(x(k),N)$. The first control action $u^*(0;x(k),N)$ is implemented. At time $k+1$ the optimization is re-solved and the process repeats. From now on, we will fix the horizon at $i=N$ for our receding horizon controller, and analyze the stability and robustness of this scheme.

2.3 Notation and Assumptions

We will often find it convenient to stack the state $x(k)$ and control sequence $u_{[0,N]}$ together into a single vector. To transition between time k and time $k+1$ using the nominal plant model, we define the matrices Φ_j for $j = 1...N$ as follows.

$$\Phi_j := \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{jm} \end{bmatrix} \tag{7}$$

(where I_{jm} is the identity matrix of size $jm \times jm$ and m is the dimension of the control input) so that for the nominal plant model we have the following relationship

$$\begin{bmatrix} x(1) \\ u_{[1,j]} \end{bmatrix} = \begin{bmatrix} Ax(0) + Bu(0) \\ u_{[1,j]} \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{jm} \end{bmatrix} \begin{bmatrix} x(0) \\ u(0) \\ u_{[1,j]} \end{bmatrix} = \Phi_j \begin{bmatrix} x(0) \\ u_{[0,j]} \end{bmatrix}.$$

Additionally, define

$$\Phi_{[i,j]} := \prod_{k=j-i+1}^{j} \Phi_k$$

so that

$$\begin{bmatrix} x(i) \\ u_{[i,j]} \end{bmatrix} = \Phi_{[i,j]} \begin{bmatrix} x(0) \\ u_{[0,j]} \end{bmatrix}$$

(where $\prod_k^j(\cdot) = 1$ when $j > k$).

Throughout this paper, we will make use of the following key assumption:

Assumption 2.1 *Given a horizon length N, and a set $\mathcal{W}$, there exist matrices $U_i,\ i = 1 \dots N$ and a $\tilde{U}_1$ such that for all $x \in \mathcal{W}$,*

$$J_i(x^*((N-i);x,N)) \leq x^{*T}((N-i);x,N)U_i x^*((N-i);x,N), \quad i = 1 \dots N$$

and

$$J_1(x^*(N;x,N)) \leq x^{*T}(N;x,N)\tilde{U}_1 x^*(N;x,N).$$

This assumption states that we can construct quadratic upper bounds for the cost of the finite receding horizon objective along optimal trajectories beginning in some set $\mathcal{W}$. (Note that U_1 and $\tilde{U}_1$ essentially define the same bound, the only difference being that they might be valid in different regions: U_1 corresponding to $x^*(N-1;x,N)$ and $\tilde{U}_1$ corresponding to $x^*(N;x,N)$. In practice, they can often be chosen to be equal.) Furthermore, to be meaningful, these bounds must be tight in the sense that ideally they correspond to a stabilizing trajectory for the nominal plant. While this assumption is certainly restrictive, there are large classes of systems for which bounds can be found. Furthermore, these classes of systems are those that receding horizon control is typically applied to. Below we provide a brief outline:

Open-loop stable plants

When plants are open-loop stable, as is often the assumption in receding horizon control, the set $\mathcal{W}$ can be chosen as the entire state space, with the upper bound being the cost of the open-loop system. Specifically, U_i can be chosen as:

$$U_i = A^{iT} P_0 A^i + \sum_{k=0}^{i-1} A^{kT} Q A^k \tag{8}$$

and $\tilde{U}_1 = U_1$.

Marginally Stable/Unstable plants
In this case, no general procedure exists for determining upper bounds. Nevertheless, there are a number of standard approaches to this problem.

LQR. By choosing appropriate cost parameters in LQR, it is often possible to find a stabilizing controller that satisfies control and/or state constraints within some region of state space. This controller is used to generate upper bounds by evaluating its cost (the exact formula for U_i would be equation (8) with A replaced by $A+BK$ where K is the stabilizing LQR controller that satisfies the constraints). For so-called Asymptotically Null Controllable with Bounded Input (ANCBI) plants, an LQR controller can provide semi-global stability (i.e. stability for arbitrarily large but bounded sets of initial states). ANCBI plants have been considered in receding horizon literature [5, 27] and schemes for semi-global stabilization with linear control laws have been proposed in [11, 12, 23, 2].

LMIs. LMIs are a flexible tool for the computation of feedback control laws, even for constrained systems. In [10] LMIs are solved on-line to compute an infinite horizon state feedback that can satisfy magnitude and two-norm constraints on both the inputs and outputs. An alternate LMI formulation for some constrained systems has been proposed in [19] and numerous other problems may be found in [3].

Linear Programming. In [6] an algorithm is given for computing a constant upper bound through a linear program when state constraints define a bounded convex polytope. Their approach is easy to adapt to determine a quadratic upper bound. This can be done by either solving a linear program which is guaranteed to bound the cost at each vertex of the convex polytope (but requires further analysis to be guaranteed as an upper bound for the entire polytope) or solving an LMI requiring the upper bound to be valid on the entire boundary of the polytope, which establishes the required bound.

2.4 Application of Upper Bounds

We will use the bounds described in Assumption 2.1 in the following manner. Consider the optimal trajectory $x^*(i;x,N)$ $i = 0 \dots N$ generated by the solution to (6), $u^*_{[0,N-1]}(x,N)$ with $x \in \mathcal{W}$ where $\mathcal{W}$ is a set over which Assumption 2.1 holds. Along this trajectory, the following bounds are satisfied:

$$\begin{aligned} J_N(x^*(0;x,N)) &\leq (x^*(0;x,N))^T U_N x^*(0;x,N) \\ J_{N-1}(x^*(1;x,N)) &\leq (x^*(1;x,N))^T U_{N-1} x^*(1;x,N) \\ &\vdots \\ J_1(x^*(N-1;x,N)) &\leq (x^*(N-1;x,N))^T U_1 x^*(N-1;x,N). \end{aligned} \tag{9}$$

Therefore, if we define the set of all $u_{[0,N-1]}$ such that:

$$\begin{aligned}
\begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix}^T H_N \begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix} &\le (x^*(0;x,N))^T U_N x^*(0;x,N) \\
\begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix}^T \Phi_{[1,N-1]}^T H_{N-1} \Phi_{[1,N-1]} \begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix} &\le (x^*(1;x,N))^T U_{N-1} x^*(1;x,N) \\
&\vdots \\
\begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix}^T \Phi_{[N-1,N-1]}^T H_1 \Phi_{[N-1,N-1]} \begin{bmatrix} x \\ u_{[0,N-1]} \end{bmatrix} &\le (x^*(N-1;x,N))^T U_1 x^*(N-1;x,N)
\end{aligned} \tag{10}$$

then $u^*_{[0,N-1]}(x,N)$ is a member of this set. For notational convenience, define the matrices

$$\Pi_i := \Phi_{[i-1,N]}^T \left\{ \begin{bmatrix} U_i & 0 \\ 0 & 0_{(i+1)m} \end{bmatrix} - \begin{bmatrix} H_i & 0 \\ 0 & 0_m \end{bmatrix} \right\} \Phi_{[i-1,N]} \tag{11}$$

so that the inequality constraints in (10) can be written as:

$$\begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \Pi_i \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \ge 0, \qquad i = 1 \dots N. \tag{12}$$

Note that we have added another control move $u(N)$ which just multiplies zero. We will deal with this control move in a moment. For now, note that as stated above, $u^*_{[0,N-1]}(x,N)$ is a member of the set:

$$\left\{ u_{[0,N-1]} : \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \Pi_i \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \ge 0, \quad i = 1 \dots N \right\}. \tag{13}$$

A standard trick in stability analysis for receding horizon control is to compare the optimal cost at time k, with the cost corresponding to a feasible sequence at time $k+1$. If the cost at time $k+1$ is smaller then the optimal cost at time k, then $J_N(\cdot)$ is a Lyapunov function. We will use the same trick. At time k we will use the optimal sequence $u^*_{[0,N-1]}(x(k),N)$. At time $k+1$ we need a feasible sequence of N control moves. We will use $u^*_{[1,N-1]}(x(k),N)$ for the first $N-1$ moves. We are then left to choose a final feasible control action. This final control action is why we included the additional control action $u(N)$ in (13). We have complete freedom in choosing this control action so long as it is feasible. Possible choices include:

- $u(N) = 0$
- $u(N) = u^*(N-1;x,N)$
- $u(N) = u^*(0;x^*(N;x,N),1)$.

For this presentation, we will select the final choice $u(N) = u^*(0;x^*(N;x,N),1)$ which is the optimal control action corresponding to the cost $J_1(x^*(N;x,N))$. By Assumption 2.1,

$$J_1(x^*(N;x,N)) \le (x^*(N;x,N))^T \tilde{U}_1 x^*(N;x,N)$$

or if we consider the set of $u(N)$ such that

$$\begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \\ u(N) \end{bmatrix}^T \Phi_{[N,N]}^T H_1 \Phi_{[N,N]} \begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \\ u(N) \end{bmatrix} \le (x^*(N;x,N))^T \tilde{U}_1 x^*(N;x,N)$$

then $u^*(0; x^*(N; x, N), 1)$ is in this set. Let us define

$$\Pi_0 := \Phi_{[N,N]}^T \left\{ \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & 0_m \end{bmatrix} - H_1 \right\} \Phi_{[N,N]}. \tag{14}$$

Hence, we know that the sequence $[u^*_{[0,N-1]}(x, N), u^*(N; x^*(N; x, N), 1)]$ satisfies (13) and (14) for all $x \in \mathcal{W}$. Therefore, for $x \in \mathcal{W}$, if we define the set

$$\left\{ u_{[0,N]} : \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \Pi_i \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \dots N \right\} \tag{15}$$

then $[u^*_{[0,N-1]}(x, N), u^*(N; x^*(N; x, N), 1)]$ is a member of this set. A key step in our derivation of robust stability conditions will be to replace $[u^*_{[0,N-1]}(x, N), u^*(N; x^*(N; x, N), 1)]$ by the entire set of control actions in (15). Stability conditions are then verified over (15) which guarantees that they were true for $[u^*_{[0,N-1]}(x, N), u^*(N; x^*(N; x, N), 1)]$.

2.5 S-Procedure

A tool that we will call upon frequently often goes by the name *S-procedure* and can be summarized as follows:

> *A sufficient condition for the implication*
>
> $$z^T \Pi_0 z \geq 0, \dots, z^T \Pi_N z \geq 0 \Rightarrow z^T \Pi_s z \geq 0$$
>
> *to hold is for there to exist positive scalars* $\tau_i \geq 0, \quad i = 0 \dots N$ *such that*
>
> $$\sum_{i=0}^{N} \tau_i \Pi_i - \Pi_s \preceq 0.$$

This result is trivially proved by rewriting the above equation as $\sum_{i=0}^{N} \tau_i \Pi_i \preceq \Pi_s$ and multiplying on the left by z^T and the right by z.

2.6 Lyapunov Stability

Finally, we state the Lyapunov theorem that we will use to establish stability.

Theorem 2.1 *Consider the discrete time, free dynamic system $x(k+1) = f(x(k))$ where $f(0) = 0$. Suppose there exists a scalar function $V(x)$ such that $V(0) = 0$, V is positive definite, continuous, and $V(x) \to \infty$ when $\|x\| \to \infty$. Then, if there exists a positively invariant set $\mathcal{D}$ and a scalar function γ such that $\gamma(0) = 0$ and for all $x \in \mathcal{D}$, $x \neq 0$, we have $V(x(k)) - V(x(k+1)) \geq \gamma(x) > 0$, then the equilibrium $x = 0$ is asymptotically stable in $\mathcal{D}$.*

The proof of this result is easily adapted from [9]. We will apply this theorem as follows. The cost J_N will be our Lyapunov function V, and $f(x(k), k)$ will be the receding horizon controlled closed-loop system. It is obvious that J_N is a valid choice for V since it is positive definite, continuous (from convexity), and tends to infinity as the state tends to infinity. We are only left to show that it is decreasing along trajectories of the system. The following sections establish sufficient conditions to ensure this under polytopic, structured and measurement uncertainty.

3 Robust Stability

We are now prepared to derive sufficient conditions for robust stability in the form of linear matrix inequalities. Our approach is to write everything as quadratic forms and then apply the S-procedure. We begin by considering the polytopic model of an uncertain plant.

3.1 Polytopic Uncertainty

Recall the polytopic description of an uncertain plant model:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k), \quad [\tilde{A}, \tilde{B}] \in ?$$

with

$$? = Co\{[A_1, B_1], \ldots, [A_L, B_L]\}$$

where Co denotes the convex hull. We will derive sufficient conditions for robust stability of this system under the receding horizon controller. As stated in Section 2.6, our basic approach to stability is motivated by trying to determine when J_N can act as a Lyapunov function. Since it is positive definite, continuous, and tends to infinity as the state does, we are only left to show that it is decreasing along trajectories, or

$$J_N(x) - J_N(\tilde{A}x + \tilde{B}u^*(0; x, N)) \geq \epsilon \|x\|_2^2 \tag{16}$$

for all $[\tilde{A}, \tilde{B}] \in ?$ where ϵ is some small positive number. Unfortunately, it is not computationally tractable to test this condition exactly because it would involve solving the receding horizon optimization at every state. Instead, let us assume $x \in \mathcal{W}$ where $\mathcal{W}$ is a set in which Assumption 2.1 is satisfied. Now, we can follow a systematic approach which derives a sufficient LMI condition in place of (16).

1. *Replace the condition* $J_N(x) - J_N(\tilde{A}x + \tilde{B}u^*(0; x, N)) \geq \epsilon \|x\|_2^2$ *by a sufficient condition in terms of quadratic forms.*

   From (6) we can write

$$J_N(x) = \begin{bmatrix} x \\ u^*_{[0,N-1]}(x, N) \end{bmatrix}^T H_N \begin{bmatrix} x \\ u^*_{[0,N-1]}(x, N) \end{bmatrix}.$$

   Additionally, $J_N(\tilde{A}x + \tilde{B}u^*(0; x, N))$ can be bounded from above by the cost of applying the feasible control actions $[u^*_{[1,N-1]}(x, N), u^*(0; x^*(N; x, N), 1)]$ (see Section 2.4 for an explanation of this choice). Hence

$$\begin{aligned} & J_N(\tilde{A}x + \tilde{B}u^*(0; x, N)) \\ \leq & \begin{bmatrix} \tilde{A}x + \tilde{B}u^*(0; x, N) \\ u^*_{[1,N-1]}(x, N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix}^T H_N \begin{bmatrix} \tilde{A}x + \tilde{B}u^*(0; x, N) \\ u^*_{[1,N-1]}(x, N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix} \\ = & \begin{bmatrix} x \\ u^*_{[0,N-1]}(x, N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix}^T \begin{bmatrix} \tilde{A} & \tilde{B} & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix}^T H_N \begin{bmatrix} \tilde{A} & \tilde{B} & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix} \begin{bmatrix} x \\ u^*_{[0,N-1]}(x, N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix}. \end{aligned}$$

Therefore, if we define

$$\tilde{\Pi}_s := \begin{bmatrix} H_N & 0 \\ 0 & 0_m \end{bmatrix} - \begin{bmatrix} \tilde{A} & \tilde{B} & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix}^T H_N \begin{bmatrix} \tilde{A} & \tilde{B} & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix} - \begin{bmatrix} \epsilon I_n & 0 \\ 0 & 0_{(N+1)m} \end{bmatrix}, \tag{17}$$

it is possible to replace

$$J_N(x) - J_N(\tilde{A}x + \tilde{B}u^*(0;x,N)) \geq \epsilon \|x\|_2^2$$

by the sufficient condition

$$\begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix}^T \tilde{\Pi}_s \begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix} \geq 0. \tag{18}$$

2. *Write (18) as an implication:*

$$u_{[0,N]} = [u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)] \Rightarrow \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \tilde{\Pi}_s \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \geq 0 \tag{19}$$

for all $[\tilde{A}, \tilde{B}] \in ?$ *and* $x \in \mathcal{W}$.

3. *Replace the condition* $u_{[0,N]} = [u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)]$ *by the set (15).*

   Since $u_{[0,N]} = [u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)]$ is an element of the set (15), we can replace (19) by the sufficient condition

$$\begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \Pi_i \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \ldots N \Rightarrow \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \tilde{\Pi}_s \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \geq 0.$$

4. *Apply the S-procedure to convert this implication to an LMI:*

$$\sum_{i=0}^{N} \tau_i \Pi_i - \tilde{\Pi}_s \preceq 0, \qquad \tau_i \geq 0.$$

Finally, we argue by convexity that it is only necessary to check this linear matrix inequality on the vertices of ? to verify its satisfaction. Assume that

$$J_N(x) - J_N(A_i x + B_i u^*(0;x,N)) \geq \epsilon \|x\|_2^2, \quad i = 1 \ldots L. \tag{20}$$

But, J_N is a convex function and for fixed x,

$$\left\{ \tilde{A}x + \tilde{B}u^*(0;x,N) : \; [\tilde{A}, \tilde{B}] \in ? \right\}$$

is a convex polytope. Furthermore, convex functions achieve their maximum only on the vertices of polytopes. Hence, $J_N(\tilde{A}x + \tilde{B}u^*(0;x,N))$ achieves its maximum on one of the vertices of ?. Therefore it easily follows that if (20) is true on the vertices of ? then it is true for every plant in ? as well. This leads to the following theorem.

Theorem 3.1 *Let $\mathcal{W}$ be a set under which Assumption 2.1 holds with corresponding bounds U_i, $i = 1 \ldots N$ and $\tilde{U}_1$. Furthermore, let Π_i, $i = 1 \ldots N$ be given by (11), Π_0 by (14), and Π_s^l by (17) with $\tilde{A}$ and $\tilde{B}$ replaced by A_l and B_l, respectively. If there exist scalars $\tau_i^l \geq 0$, $i = 0 \ldots N$, $l = 1 \ldots L$ that satisfy the linear matrix inequalities*

$$\sum_{i=0}^{N} \tau_i^l \Pi_i - \Pi_s^l \preceq 0, \qquad l = 1 \ldots L \tag{21}$$

then the receding horizon controller of horizon N based on the nominal system (1) stabilizes every plant in the set ? $= Co\{[A_1, B_1], \ldots, [A_L, B_L]\}$ in any subset of $\mathcal{W}$ which is positively invariant under the uncertain closed loop dynamics.

This theorem states sufficient conditions for robust stability as the feasibility of LMIs. The general procedure outlined in this section for deriving sufficient LMI conditions for robust stability is the same for both structured and measurement uncertainty. Differences only occur in the details. Hence, we will proceed much more quickly through the following two sections which handle the structured and measurement uncertainty cases.

3.2 Structured Uncertainty

We now consider the structured uncertainty representation for uncertain systems:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) &= C_1 x(k) + D_{11} w(k) + D_{12} u(k) \\ w(k) &= \Delta z(k) \end{aligned}$$

where $w(k) \in \mathbb{R}^{m_1}$ and the operator Δ is block diagonal: $\Delta = diag([\Delta_1, \ldots, \Delta_r])$, where each Δ_j is a memoryless time-varying matrix with $\|\Delta_j\|_2 := \overline{\sigma}(\Delta_j) \leq 1$, $j = 1 \ldots r$. This can be rewritten as:

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \tag{22}$$

$$w_j^T(k) w_j(k) \leq (C_1 x(k) + D_{11} w(k) + D_{12} u(k))_j^T (C_1 x(k) + D_{11} w(k) + D_{12} u(k))_j \tag{23}$$

for $j = 1 \ldots r$. It is easy to see that the equations (23) are quadratic forms in $x(k)$, $w(k)$ and $u(k)$ which we will generically denote by:

$$\begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Upsilon_j \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad j = 1 \ldots r \tag{24}$$

with Υ_j the appropriate matrix representations of (23). (Note that we write Υ in terms of x, w and $u_{[0,N]}$, but equation (23) only involves x, w and $u(0)$. This is done for convenience of notation.)

Again, we wish to determine if

$$J_N(x) - J_N(Ax + B_1 w + B_2 u^*(0; x, N)) \geq \epsilon \|x\|_2^2 \tag{25}$$

for all w satisfying

$$\begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix}^T \Upsilon_j \begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0; x^*(N; x, N), 1) \end{bmatrix} \geq 0, \quad j = 1 \ldots r.$$

If we assume $x \in \mathcal{W}$ where $\mathcal{W}$ satisfies Assumption 2.1, then we can once again transform this into a sufficient linear matrix inequality in four steps:

1. *Replace the condition $J_N(x) - J_N(Ax + B_1 w + B_2 u^*(0;x,N)) \geq \epsilon \|x\|_2^2$ by a sufficient condition in terms of quadratic forms.*

   Similar to *step 2* in the polytopic case, define

$$\Pi_s^{\Delta} := \begin{bmatrix} H_N & 0 & 0 \\ 0 & 0_{m_1} & 0 \\ 0 & 0 & 0_m \end{bmatrix} - \begin{bmatrix} A & B_1 & B_2 & 0 \\ 0 & 0 & 0 & I_{Nm} \end{bmatrix}^T H_N \begin{bmatrix} A & B_1 & B_2 & 0 \\ 0 & 0 & 0 & I_{Nm} \end{bmatrix} - \begin{bmatrix} \epsilon I_n & 0 & 0 \\ 0 & 0_{m_1} & 0 \\ 0 & 0 & 0_{(N+1)m_1} \end{bmatrix} \tag{26}$$

   where the first term replaces $J_N(x)$, the second term bounds $J_N(Ax + B_1 w + B_2 u)$, and the final term replaces $\epsilon \|x\|_2^2$. Then a sufficient condition for (25) is

$$\begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix}^T \Pi_s^{\Delta} \begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix} \geq 0 \tag{27}$$

   for all w satisfying

$$\begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix}^T \Upsilon_j \begin{bmatrix} x \\ w \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix} \geq 0, \quad j = 1 \ldots r.$$

2. *Write (27) as an implication:*

$$\left. \begin{array}{c} u_{[0,N]} = [u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)] \\ \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Upsilon_j \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0, \; j = 1 \ldots r \end{array} \right\} \Rightarrow \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Pi_s^{\Delta} \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0. \tag{28}$$

3. *Replace the condition $u_{[0,N]} = [u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)]$ by the set of quadratic forms (15).*

   From (15) we know that $[u^*_{[0,N-1]}(x,N), u^*(0;x^*(N;x,N),1)]$ satisfies:

$$\left\{ u_{[0,N]} : \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix}^T \Pi_i \begin{bmatrix} x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \ldots N \right\}.$$

   Purely for notational reasons, we need to write these in terms of x, w and $u_{[0,N]}$, not just x and $u_{[0,N]}$ as above. Hence, define

$$\Pi_i^{\Delta} := \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix}^T \Pi_i \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix}, \qquad i = 0 \ldots N \tag{29}$$

so that we can write (15) equivalently as:

$$\left\{ u_{[0,N]} : \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Pi_i^\Delta \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \dots N \right\}.$$

This leads to the implication:

$$\left. \begin{array}{l} \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Pi_i^\Delta \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \dots N \\ \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Upsilon_j \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad j = 1 \dots r \end{array} \right\} \Rightarrow \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix}^T \Pi_s^\Delta \begin{bmatrix} x \\ w \\ u_{[0,N]} \end{bmatrix} \geq 0. \tag{30}$$

4. *Apply the S-procedure to convert this implication to an LMI:*

$$\sum_{i=1}^{r} \nu_i \Upsilon_i + \sum_{i=0}^{N} \tau_i \Pi_i^\Delta - \Pi_s^\Delta \preceq 0, \qquad \tau_i \geq 0, \quad \nu_j \geq 0.$$

Hence we have the following theorem:

Theorem 3.2 *Let $\mathcal{W}$ be a set under which Assumption 2.1 holds with corresponding bounds U_i, $i = 1 \dots N$ and $\tilde{U}_1$. Furthermore, let Π_i^Δ, $i = 0 \dots N$ be given by (29), Π_s^Δ by (26) and Υ_j, $j = 1 \dots r$ by (24). If there exist scalars $\tau_i \geq 0$, $i = 0 \dots N$ and scalars $\nu_j \geq 0$, $j = 1 \dots r$ that satisfy the linear matrix inequality*

$$\sum_{j=1}^{r} \nu_j \Upsilon_j + \sum_{i=0}^{N} \tau_i \Pi_i^\Delta - \Pi_s^\Delta \preceq 0 \tag{31}$$

then the receding horizon controller of horizon N based on the nominal system (1) robustly stabilizes the uncertain system (22-23) in any subset of $\mathcal{W}$ which is positively invariant under the uncertain closed loop system.

3.3 Measurement Uncertainty

In this section we deal with the issue of state measurement error. Let $x(k)$ denote the true state and $\hat{x}(k)$ denote our measurement of the state at time k. Furthermore, assume that the state measurement contains an error that can be characterized through a quadratic form:

$$\begin{bmatrix} \hat{x}(k) \\ x(k) \\ u_{[0,N]} \end{bmatrix}^T \Psi_j \begin{bmatrix} \hat{x}(k) \\ x(k) \\ u_{[0,N]} \end{bmatrix} > 0, \qquad j = 1 \dots r. \tag{32}$$

A simple example of such an error is: $\|\hat{x}(k) - x(k)\|_2 \leq 0.01 \|x(k)\|_2$.

This time we would like to show that

$$J_N(x) - J_N(Ax + Bu^*(0; \hat{x}, N) \geq \epsilon \|x\|_2^2$$

where $u^*_{[0,N-1]}(\hat{x},N)$ is the solution to

$$\inf_{\substack{u_{[0,N-1]} \\ c_u(u(k+i)) \le 0, \quad i=0\dots N-1}} \begin{bmatrix} \hat{x} \\ u_{[0,N-1]} \end{bmatrix}^T H_N \begin{bmatrix} \hat{x} \\ u_{[0,N-1]} \end{bmatrix}$$

and x is related to $\hat{x}$ through (32). That is, the on-line optimization is based upon the state measurement. This time we assume $x \in \mathcal{W}$ and $\hat{x} \in \mathcal{W}$ where $\mathcal{W}$ satisfies Assumption 2.1 and proceed in the following steps:

1. *Replace the condition $J_N(x) - J_N(Ax + Bu^*(0;\hat{x},N)) \ge \epsilon\|x\|_2^2$ by a sufficient condition in terms of quadratic forms.*

   Let M be a matrix that satisfies:

$$J_N(x) \ge J_N(\hat{x}) - \begin{bmatrix} \hat{x} \\ x \end{bmatrix}^T M \begin{bmatrix} \hat{x} \\ x \end{bmatrix} \tag{33}$$

   hence,

$$J_N(x) - J_N(Ax + Bu^*(0;\hat{x},N)) \ge J_N(\hat{x}) - \begin{bmatrix} \hat{x} \\ x \end{bmatrix}^T M \begin{bmatrix} \hat{x} \\ x \end{bmatrix} - J_N(Ax + Bu^*(0;\hat{x},N)).$$

   As in the polytopic and structured uncertainty case, we replace this with a sufficient condition in terms of quadratic forms. Define:

$$\begin{aligned}\hat{\Pi}_s := &\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \begin{bmatrix} H_N & 0 \\ 0 & 0_m \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & 0_{(N+1)m} \end{bmatrix} \\ &- \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix}^T H_N \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{Nm} \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix} \\ &- \begin{bmatrix} 0_n & 0 & 0 \\ 0 & \epsilon I_n & 0 \\ 0 & 0 & 0_{(N+1)m} \end{bmatrix}\end{aligned} \tag{34}$$

   where the first term replaces $J_N(\hat{x})$, the second term replaces the M term, the third term bounds $J_N(Ax + Bu^*(0;\hat{x},N))$, and the final term replaces $\epsilon\|x\|_2^2$, so that

$$\begin{bmatrix} \hat{x} \\ x \\ u^*_{[0,N-1]}(\hat{x},N) \\ u^*(0;x^*(N;\hat{x},N),1) \end{bmatrix}^T \hat{\Pi}_s \begin{bmatrix} \hat{x} \\ x \\ u^*_{[0,N-1]}(\hat{x},N) \\ u^*(0;x^*(N;\hat{x},N),1) \end{bmatrix} \ge 0 \tag{35}$$

   represents a sufficient condition for $J_N(x) - J_N(Ax + Bu^*(0;\hat{x},N)) \ge \epsilon\|x\|_2^2$. Furthermore, a sufficient condition for (33) is for M to satisfy

$$\begin{aligned}&\begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \end{bmatrix}^T H_N \begin{bmatrix} x \\ u^*_{[0,N-1]}(x,N) \end{bmatrix} \\ &\quad \ge \begin{bmatrix} \hat{x} \\ u^*_{[0,N-1]}(x,N) \end{bmatrix}^T H_N \begin{bmatrix} \hat{x} \\ u^*_{[0,N-1]}(x,N) \end{bmatrix} - \begin{bmatrix} \hat{x} \\ x \end{bmatrix}^T M \begin{bmatrix} \hat{x} \\ x \end{bmatrix}.\end{aligned}$$

We can write this as a quadratic form by defining:

$$\hat{\Pi}^M := \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \begin{bmatrix} H_N & 0 \\ 0 & 0_m \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix} - \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \begin{bmatrix} H_N & 0 \\ 0 & 0_m \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix} + \begin{bmatrix} M & 0 \\ 0 & 0_{(N+1)m} \end{bmatrix} \tag{36}$$

and requiring

$$\begin{bmatrix} \hat{x} \\ x \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix}^T \hat{\Pi}^M \begin{bmatrix} \hat{x} \\ x \\ u^*_{[0,N-1]}(x,N) \\ u^*(0;x^*(N;x,N),1) \end{bmatrix} \geq 0. \tag{37}$$

2. *Write (35) and (37) as implications:*

   We can write (35) as

$$\left.\begin{array}{l} u_{[0,N]} = [u^*_{[0,N]}(\hat{x},N), u^*(0;x^*(N;\hat{x},N),1)] \\ \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \Psi_j \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad j = 1 \dots r \end{array}\right\} \Rightarrow \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \hat{\Pi}_s \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0 \tag{38}$$

   and (37) as

$$\left.\begin{array}{l} u_{[0,N]} = [u^*_{[0,N]}(x,N), u^*(0;x^*(N;x,N),1)] \\ \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \Psi_j \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad j = 1 \dots r \end{array}\right\} \Rightarrow \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \hat{\Pi}^M \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0. \tag{39}$$

3. *Replace the conditions $u_{[0,N]} = [u^*_{[0,N]}(\hat{x},N), u^*(0;x^*(N;\hat{x},N),1)]$ and $u_{[0,N]} = [u^*_{[0,N]}(x,N), u^*(0;x^*(N;x,N),1)]$ by the set of quadratic forms (15).*

   If we define

$$\hat{\Pi}_i := \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \Pi_i \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}, \quad i = 0 \dots N \tag{40}$$

   and

$$\bar{\Pi}_i := \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}^T \Pi_i \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_{(N+1)m} \end{bmatrix}, \quad i = 0 \dots N \tag{41}$$

   where Π_i $i = 1 \dots N$ and Π_0 are from (11) and (14), respectively, then the implication (38) can be replaced by the sufficient condition:

$$\left.\begin{array}{l} \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \hat{\Pi}_i \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad i = 0 \dots N \\ \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \Psi_j \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0, \quad j = 1 \dots r \end{array}\right\} \Rightarrow \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix}^T \hat{\Pi}_s \begin{bmatrix} \hat{x} \\ x \\ u_{[0,N]} \end{bmatrix} \geq 0 \tag{42}$$

and (39) by:

$$\left.\begin{array}{l}\begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix}^T \bar{\Pi}_i \begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix} \geq 0, \quad i = 0\ldots N\\ \begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix}^T \Psi_j \begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix} \geq 0, \quad j = 1\ldots r\end{array}\right\} \Rightarrow \begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix}^T \hat{\Pi}^M \begin{bmatrix}\hat{x}\\ x\\ u_{[0,N]}\end{bmatrix} \geq 0. \quad (43)$$

4. *Apply the S-procedure to convert these implications to LMIs:*

$$\left.\begin{array}{l}\sum_{i=0}^{N} \tau_i \hat{\Pi}_i + \sum_{j=1}^{r} \mu_j \Psi_j - \hat{\Pi}_s \preceq 0, \quad \tau_i \geq 0, \quad \mu_j \geq 0\\ \sum_{i=0}^{N} \alpha_i \bar{\Pi}_i + \sum_{j=1}^{r} \beta_j \Psi_j - \hat{\Pi}^M \preceq 0, \quad \alpha_i \geq 0, \quad \beta_j \geq 0\end{array}\right\}.$$

We state this result as a theorem:

Theorem 3.3 *Let $\mathcal{W}$ be a set under which Assumption 2.1 holds with corresponding bounds U_i, $i = 1\ldots N$ and $\tilde{U}_1$. Furthermore, let $\hat{\Pi}_i$, $i = 0\ldots N$ be given by (40), $\bar{\Pi}_i$, $i = 0\ldots N$ by (41), $\hat{\Pi}_s$ by (34), $\hat{\Pi}^M$ by (36) and Ψ_j, $j = 1\ldots r$ describe the measurement error (32). Then, if there exist scalars $\tau_i \geq 0$, $i = 0\ldots N$, $\mu_j \geq 0$, $j = 1\ldots r$, $\alpha_i \geq 0$, $i = 1\ldots N$, $\beta_j \geq 0$, $j = 1\ldots r$, and a matrix $M^T = M$ that satisfy the linear matrix inequalities*

$$\left.\begin{array}{l}\sum_{i=0}^{N} \tau_i \hat{\Pi}_i + \sum_{j=1}^{r} \mu_j \Psi_j - \hat{\Pi}_s \preceq 0\\ \sum_{i=0}^{N} \alpha_i \bar{\Pi}_i + \sum_{j=1}^{r} \beta_j \Psi_j - \hat{\Pi}^M \preceq 0\end{array}\right\} \quad (44)$$

then the receding horizon controller of horizon N based on the state measurements $\hat{x}$ is stabilizing in any subset of $\mathcal{W}$ for which the measurements $\hat{x}$ are also contained within $\mathcal{W}$ and which is positively invariant for the closed loop system.

4 Discussion

In deriving our LMI results we proceed through four steps, each introducing some amount of conservativeness. How conservative, then, is the final result? It turns out that the final LMI conditions are actually less conservative than many existing approaches. Our sufficient LMI conditions for stability are based on checking whether the finite horizon cost J_N is decreasing. This approach is quite standard in receding horizon literature. As a result, some previous results can be reconciled with the LMI approach we have taken. Let us provide two examples to guide the reader.

The infinite horizon results of Rawlings and Muske [20] for open–loop stable plants are also guaranteed by the LMIs derived in this paper. This can be checked by using a terminal weight equal to the open-loop infinite horizon cost, and checking the appropriate LMIs. Additionally, the LMI approach can be used to go further and guarantee stability under other terminal weights, and under plant and measurement uncertainty.

As a second example, consider a scheme in which the on-line optimization requires a terminal constraint that the final state lie within a specified set. Furthermore, the terminal weight corresponds to the cost of applying a stabilizing linear controller ($u = Kx$) that satisfies the constraints within the terminal set [6]. In the nominal stability case, this information is equivalent to saying that $u(N) = Kx^*(N; x, N)$ is a valid choice for the control move $u(N)$. With this information we can substitute for $u(N)$ and write quadratic forms in terms of $x(k)$, $u_{[0,N-1]}$ and $x^*(N; x, N)$. Even without the upper bounds on any of the finite horizon costs that we have assumed in Assumption 2.1, this information alone will guarantee nominal stability from our LMIs since the matrix Π_s in terms of $x(k), u_{[0,N-1]}$ and $x^*(N; x, N)$ will be positive definite! One can then proceed further, with added difficulty, and attempt to analyze robustness properties, etc. We leave it to the reader to pursue other direct extensions.

We are also obliged to mention at least a plausible technique for dealing with the difficult question of finding a positively invariant subset of $\mathcal{W}$. One approach is as follows: First choose a set of initial conditions $\mathcal{I}$ and calculate a scalar upper bound $\mathcal{S}$ for J_N over the set $\mathcal{I}$. If J_N is indeed a Lyapunov function then the set $\mathcal{W} = \{x : J_N(x) \leq \mathcal{S}\}$ is positively invariant. But, a priori we don't know whether J_N is a Lyapunov function. So, instead calculate a lower bound $x^T L x$ for J_N by solving the unconstrained problem (a Riccati equation). Then if we verify the LMI conditions in the stability theorems over the set $\{x : x^T L x \leq \mathcal{S}\}$, (which contains $\mathcal{W}$), then J_N is a Lyapunov function in the set $\mathcal{W}$ which is positively invariant and contains our initial conditions $\mathcal{I}$. Hence, we are stable from any initial condition in $\mathcal{I}$. We have used this technique for the second example in Section 5. More details can be found in [19].

We have also utilized the following freedom in our numerical examples. Note that we include the initial term $x^T Q x$ in the cost $J_N(x)$ (5) even though it has no effect on the optimizing solution $u^*_{[0,N-1]}(x, N)$ and hence no effect on the receding horizon control law. In fact, the Q corresponding to this first term can be used as a free variable in the Π_s term in the LMIs we derived. This corresponds to testing Lyapunov functions J_N, but parameterized by this Q, and can further reduce the conservativeness of the LMI conditions.

Finally, we mention state constraints. When there is no uncertainty, our LMI approach will work for state constraints. Again there is the difficult issue of computing quadratic upper bounds as in Assumption 2.1. On the other hand, when uncertainty is present, the results do not hold. This is because the receding horizon controller is based upon an optimization using the nominal plant model. Hence, state constraints will be satisfied if the nominal plant model is used, but not necessarily if the uncertain model is used. So-called soft state constraints [7] can be used to get around this problem.

5 Numerical Examples

In this section we demonstrate the LMI computations on both an open-loop stable and unstable plant. These examples provide a simple illustration of the presented approach.

5.1 Example 1: An open-loop stable system

Consider the following stable dynamics:

$$x(k+1) = \begin{pmatrix} 4/3 + d & -2/3 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \tag{45}$$

subject to the saturation constraint, $|u| \leq \psi$ where d is an unknown parameter that lies in $d \in [-\theta,\ \theta]$ (and θ is yet to be specified). We consider the following cost parameters:

$$Q = \begin{pmatrix} 1 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{3}{2} \end{pmatrix}, \qquad R = 1.$$

Since this example is open-loop stable, we will compute the upper bounds U_j required for (10) by simply calculating the cost accumulated from the open-loop system (see (8)). Using upper bounds calculated in this manner allows for *global* stability (i.e., $\mathcal{W} = \mathbb{R}^n$) to be determined *independent of the level of saturation* (since the upper bounds are independent of the constraint). We will analyze receding horizon formulations with the following terminal weights:

- $P_0 = Q$. (Note that this terminal weight cannot correspond to the cost associated with an extension by any stabilizing controller.)
- $P_0 = U_\infty = \begin{bmatrix} 7.1667 & -4.2222 \\ -4.2222 & 4.6852 \end{bmatrix}$ where U_∞ denotes the infinite horizon cost of the open loop stable system:
$$U_\infty = Q + A^T U_\infty A.$$

• **Robustness Results**

For a fixed horizon length, we will determine (by checking the feasibility of the LMI given in Theorem 3.1 for various values of θ) the largest value of θ (which defines the range for the unknown parameter d) that will be guaranteed stable under the receding horizon policy based on the nominal system ($d = 0$). The results are given in Table 1.

	θ	
Horizon	$P_0 = Q$	$P_0 = U_\infty$
$N = 3$	0.12	0.15
$N = 6$	0.16	0.15
$N = 10$	0.15	0.15

Table 1: Largest value of θ for which robust stability is guaranteed

For each horizon tested and both terminal weights we find that the receding horizon controller can tolerate variations from the nominal plant in d of more than 0.1 without jeopardizing stability. For the terminal weight U_∞ which corresponds to an infinite horizon approach, the horizon length has little effect, if any, on the robustness results obtained for this controller. This is due to the fact that both the terminal weight and the upper bounds are computed from the cost of the open-loop system and provide much of the same information. Hence, extending the horizon in this case does not introduce new information which could improve (or worsen) the analysis of robustness properties.

• **Measurement Error**

Finally, we will use our results to determine the amount of state measurement error that can be tolerated while stability is still maintained. Measurement error will be modeled as:

$$\|\hat{x} - x\|_2 \leq e\|x\|_2.$$

	e	
Horizon	$P_0 = Q$	$P_0 = U_\infty$
$N = 3$	0.07	0.08
$N = 6$	0.08	0.08
$N = 10$	0.08	0.08

Table 2: Largest value of e for which robust stability is guaranteed

Thm. 3.3 was used to determine the largest possible values for e for a given terminal weight P_0 and horizon length N. The results are supplied in Table 2.

For the infinite horizon terminal weight U_∞, using the same reasoning as in the robustness analysis case, it is not surprising that the results were the same regardless of horizon length. Clearly this is more an artifact of the upper bounds being computed from the open loop cost than any intrinsic property of infinite horizon controllers. For the terminal weight $P_0 = Q$, various horizons produce distinct results. For example, a horizon of $N = 3$ is guaranteed stable up to errors of 7%, while for the other horizons in the table, Thm. 3.3 allows for 8% errors.

It is important to note that the results obtained in Example 1 used the simplest method for computing the upper bounds U_i. Had more sophisticated and tighter bounds been used, the stability and robustness results would have been more informative.

5.2 Example 2: An open-loop unstable system

Consider the unstable system:

$$x(k+1) = \begin{pmatrix} 1 & 1.1+d \\ -1.1 & 1+d \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)$$

with $d \in [-\theta, \theta]$ and subject to the following constraint: $|u(k)| \leq 2$. For cost parameters we will use:

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad R = 1.$$

With this form of constraint, it is not possible to stabilize the system globally, hence we will consider initial conditions in the set: $\mathcal{I} = \{x | \; x^T x \leq 1\}$. Furthermore, since the system is open-loop unstable, we have used LMIs to compute upper bounds for the finite horizon costs and determine the positively invariant set $\mathcal{W}$ that contains $\mathcal{I}$ (see [19] for details). Finally, we will test three terminal weights:

- $P_0 = Q$
- $P_0 = L_\infty$ where L_∞ solves the algebraic Riccati equation and corresponds to the optimal cost of the *unconstrained system*

$$L_\infty = \begin{bmatrix} 3.9313 & 3.7919 \\ 3.7919 & 9.8283 \end{bmatrix}.$$

- $P_0 = 2L_\infty$

• Robustness Results

For this example, we determined the maximum range in which the parameter d could vary while stability was still guaranteed by Thm. 3.1. The results are supplied in Table 3.

	θ	
Horizon	$P_0 = L_\infty$	$P_0 = 2L_\infty$
$N = 2$	0.06	0.13
$N = 3$	0.04	0.05
$N = 6$	0.02	0.02

Table 3: Largest value of θ for which stability can be guaranteed by Thm. 3.1

The results show that for these cost parameters, horizon lengths, and form of plant uncertainty, we cannot guarantee a large degree of robustness, except for the largest terminal weight $2L_\infty$, and the shortest horizon $N = 2$.

6 Conclusion

We presented a new approach to the analysis of constrained finite receding horizon control for the class of systems for which quadratic upper bounds on the finite horizon costs can be calculated. This approach, based on viewing stability as an implication between quadratic forms, allows stability and robustness properties of the standard quadratic program based receding horizon scheme to be determined by LMIs. As a robustness analysis tool it was shown to effectively handle plant uncertainty in both polytopic or structured uncertainty representations, as well as state measurement errors. This framework is potentially applicable to a wide assortment of receding horizon schemes.

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