14.7 Using Double- and Half-Angle Formulas

Goals
- Evaluate expressions using double- and half-angle formulas.
- Use double- and half-angle formulas to solve real-life problems.

Your Notes

DOUBLE-ANGLE AND HALF-ANGLE FORMULAS

Double-Angle Formulas

$\cos 2u = \cos^2 u - \sin^2 u$ $\quad\quad$ $\sin 2u = 2 \sin u \cos u$

$\cos 2u = 2\cos^2 u - 1$ $\quad\quad$ $\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u}$

$\cos 2u = 1 - 2\sin^2 u$

Half-Angle Formulas

$\sin \dfrac{u}{2} = \pm\sqrt{\dfrac{1 - \cos u}{2}}$ $\quad\quad$ $\tan \dfrac{u}{2} = \dfrac{1 - \cos u}{\sin u}$

$\cos \dfrac{u}{2} = \pm\sqrt{\dfrac{1 - \cos u}{2}}$ $\quad\quad$ $\tan \dfrac{u}{2} = \dfrac{\sin u}{1 + \cos u}$

Example 1 ***Evaluating Trigonometric Expressions***

Find the exact value of $\tan \dfrac{\pi}{12}$.

Solution

Use the fact that $\dfrac{\pi}{12}$ is half of $\dfrac{\pi}{6}$.

$$\tan \frac{\pi}{12} = \tan \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{1 - \cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= 2 - \sqrt{3}$$

Your Notes

Example 2 *Evaluating Trigonometric Expressions*

For $\cos u = \frac{5}{13}$ with $0 < u < \frac{\pi}{2}$, find (a) $\sin \frac{u}{2}$ and (b) $\sin 2u$.

a. Because $\frac{u}{2}$ is in Quadrant I, $\sin \frac{u}{2}$ is positive.

$$\sin \frac{u}{2} = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 - \frac{5}{13}}{2}} = \sqrt{\frac{4}{13}}$$

$$= \frac{2\sqrt{13}}{13}$$

b. Use a Pythagorean identity to conclude that $\sin u = \frac{12}{13}$.

$$\sin 2u = 2 \sin u \cos u = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$$

Checkpoint **Complete the following exercises.**

1. Find the exact value of $\tan \frac{7\pi}{8}$.	**2.** Given $\cos u = -\frac{20}{29}$ with $\frac{\pi}{2} < u < \pi$, find $\sin 2u$.
$-\sqrt{2} + 1$	$-\frac{840}{841}$

Example 3 *Verifying a Trigonometric Identity*

Verify the identity $\cos 3x = \cos^3 x - 3 \sin^2 x \cos x$.

$\cos 3x = \cos (2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (\cos^2 x - \sin^2 x)\cos x - (2 \sin x \cos x)\sin x$

$= \cos^3 x - \sin^2 x \cos x - 2 \sin^2 x \cos x$

$= \cos^3 x - 3 \sin^2 x \cos x$

Your Notes

Example 4 *Solving a Trigonometric Equation*

Solve $\tan \frac{x}{2} = \sin x$ for $0 \le x < 2\pi$.

$\tan \frac{x}{2} = \sin x$	Write original equation.
$\frac{1 - \cos x}{\sin x} = \sin x$	Use a half-angle formula.
$1 - \cos x = \sin^2 x$	Multiply each side by $\sin x$.
$1 - \cos x = 1 - \cos^2 x$	Use a Pythagorean identity.
$\cos^2 x - \cos x = 0$	Subtract $(1 - \cos^2 x)$ from each side.
$\cos x(\cos x - 1) = 0$	Factor.
$\cos x = 0$ or $\cos x - 1 = 0$	Zero product property
$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos x = 1$, $x = 0$	

✔ ***Checkpoint*** **Complete the following exercises.**

3. Verify the identity $\sin 4x = 4 \sin x \cos x(1 - 2 \sin^2 x)$.

Check students' work.

4. Solve $\cos 2x + \sin x = 0$ for $0 \le x < 2\pi$.

$\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Homework