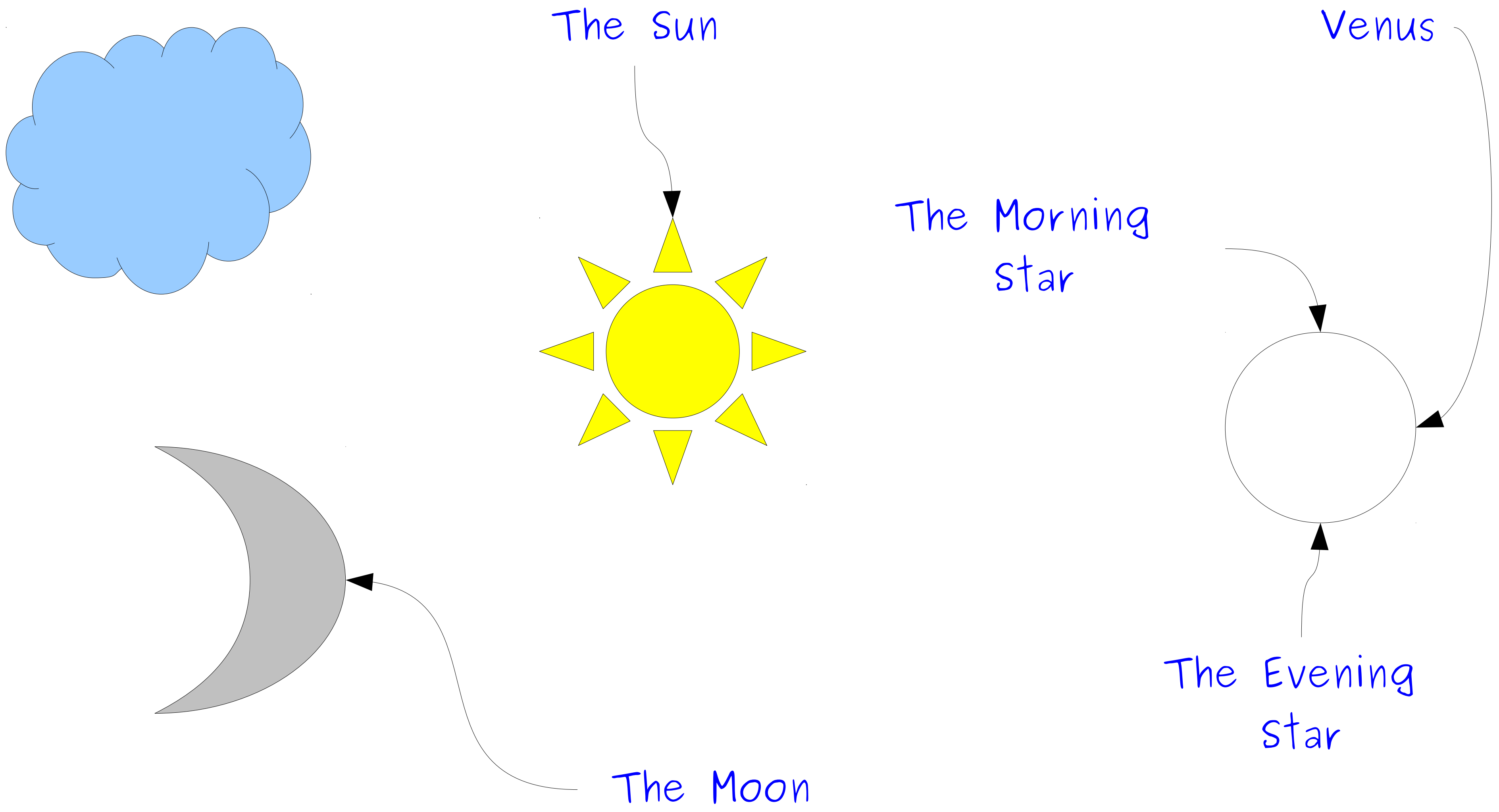
Mathematical Logic

Part Two

Problem set Three due in the box up front.

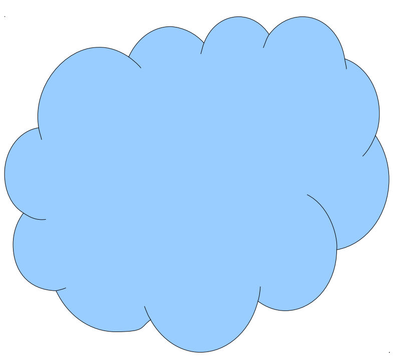
First-Order Logic

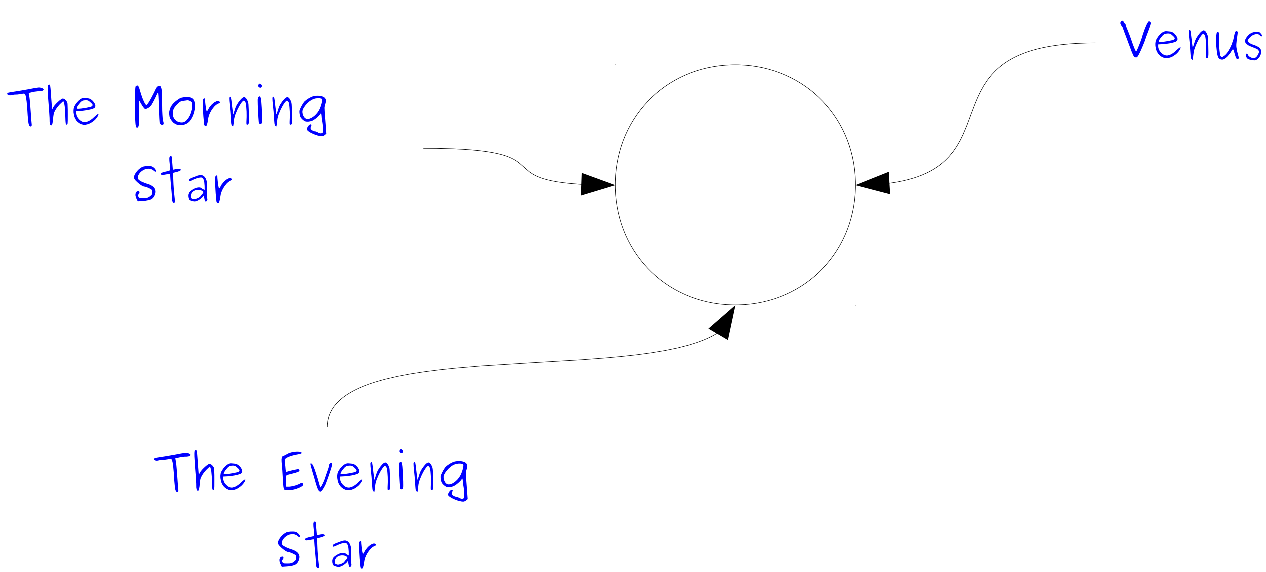
The Universe of First-Order Logic



First-Order Logic

- In first-order logic, each variable refers to some object in a set called the **domain of discourse**.
- Some objects may have multiple names.
- Some objects may have no name at all.





Propositional vs. First-Order Logic

- Because propositional variables are either true or false, we can directly apply connectives to them.

$$p \to q \qquad\qquad \neg p \leftrightarrow q \land r$$

- Because first-order variables refer to arbitrary objects, it does not make sense to apply connectives to them.

Venus → Sun ***137 ↔ ¬42***

- ***This is not C!***

Reasoning about Objects

- To reason about objects, first-order logic uses **predicates**.
- Examples:
  - $NowOpen(USGovernment)$
  - $FinallyTalking(House, Senate)$
- Predicates can take any number of arguments, but each predicate has a fixed number of arguments (called its **arity**)
- Applying a predicate to arguments produces a proposition, which is either true or false.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$LikesToEat(V, M) \land Near(V, M) \to WillEat(V, M)$$

$$Cute(t) \to Dikdik(t) \lor Kitty(t) \lor Puppy(t)$$

$$x < 8 \to x < 137$$

The notation $\mathbf{x < 8}$ is just a shorthand for something like **LessThan(*x*, 8)**. Binary predicates in math are often written like this, but symbols like $<$ are not a part of first-order logic.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\rightarrow$ and $\neg$ are.
- Examples:

$$MorningStar = EveningStar$$

$$Voldemort = TomMarvoloRiddle$$

- Equality can only be applied to **objects**; to see if **propositions** are equal, use $\leftrightarrow$.

For notational simplicity, define $\neq$ as

$$x \neq y \quad \equiv \quad \neg(x = y)$$

Expanding First-Order Logic

$$x < 8 \land y < 8 \rightarrow x + y < 16$$

Why is this allowed?

Functions

- First-order logic allows **functions** that return objects associated with other objects.
- Examples:

$$x + y$$

$$LengthOf(path)$$

$$MedianOf(x, y, z)$$

- As with predicates, functions can take in any number of arguments, but each function has a fixed arity.
- Functions evaluate to **objects**, not **propositions**.
- There is no syntactic way to distinguish functions and predicates; you'll have to look at how they're used.

How would we translate the statement

"For any natural number n,
n is even iff n^2 is even"

into first-order logic?

Quantifiers

- The biggest change from propositional logic to first-order logic is the use of **quantifiers**.
- A **quantifier** is a statement that expresses that some property is true for some or all choices that could be made.
- Useful for statements like "for every action, there is an equal and opposite reaction."

"For any natural number n,
n is even iff n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the **universal quantifier** and says "for any choice of n, the following is true."

The Universal Quantifier

- A statement of the form **$\forall x.\ \psi$** asserts that for **every** choice of x in our domain, ψ is true.
- Examples:

$$\forall v.\ (Puppy(v) \to Cute(v))$$

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow \neg Odd(n)))$$

$$Tallest(x) \to \forall y.\ (x \neq y \to IsShorterThan(y, x))$$

Some muggles are intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the **existential quantifier** and says "for some choice of m, the following is true."

The Existential Quantifier

- A statement of the form **$\exists x.\ \psi$** asserts that for **some** choice of x in our domain, ψ is true.
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

$$(\exists x.\ Appreciates(x, me)) \to Happy(me)$$

Operator Precedence (Again)

- When writing out a formula in first-order logic, the quantifiers $\forall$ and $\exists$ have precedence just below $\neg$.
- Thus

$$\forall x.\ P(x) \lor R(x) \to Q(x)$$

is interpreted as

$$((\forall x.\ P(x)) \lor R(x)) \to Q(x)$$

rather than

$$\forall x.\ ((P(x) \lor R(x)) \to Q(x))$$

Translating into First-Order Logic

A Bad Translation

All puppies are cute!

$$\forall x.\ (Puppy(x) \land Cute(x))$$

This should work for <u>*any*</u> choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!

$$\forall x.\ (Puppy(x) \to Cute(x))$$

This should work for <u>any</u> choice of x, including things that aren't puppies.

“Whenever $P(x)$, then $Q(x)$”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Another Bad Translation

Some blobfish is cute.

$$\exists x.\ (\mathit{Blobfish}(x) \to \mathit{Cute}(x))$$

Another Bad Translation

Some blobfish is cute.

$$\exists x.\ (Blobfish(x) \to Cute(x))$$

What happens if

1. The above statement is false, but
2. x refers to a cute puppy?

A Better Translation

Some blobfish is cute.

$$\exists x.\ (Blobfish(x) \land Cute(x))$$

What happens if

1. The above statement is false, but
2. x refers to a cute puppy?

“There is some $P(x)$ where $Q(x)$”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

The Takeaway Point

- Be careful when translating statements into first-order logic!
- $\forall$ is usually paired with $\to$.
  - Sometimes paired with $\leftrightarrow$.
- $\exists$ is usually paired with $\land$.

Time-Out For Announcements

Friday Four Square!

Today at 4:15PM at Gates

Problem Set Four

- Problem Set Four released today.
  - Checkpoint due on Monday.
  - Rest of the assignment due Friday.
  - Explore functions, cardinality, diagonalization, and logic!

Your Questions

What material is covered on the midterm?
Is it open-notes?

Hey Keith, how did you first get interested in math/computer science? Your enthusiasm is infectious but also somewhat curious.

Back to Logic!

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Everyone loves someone else."

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

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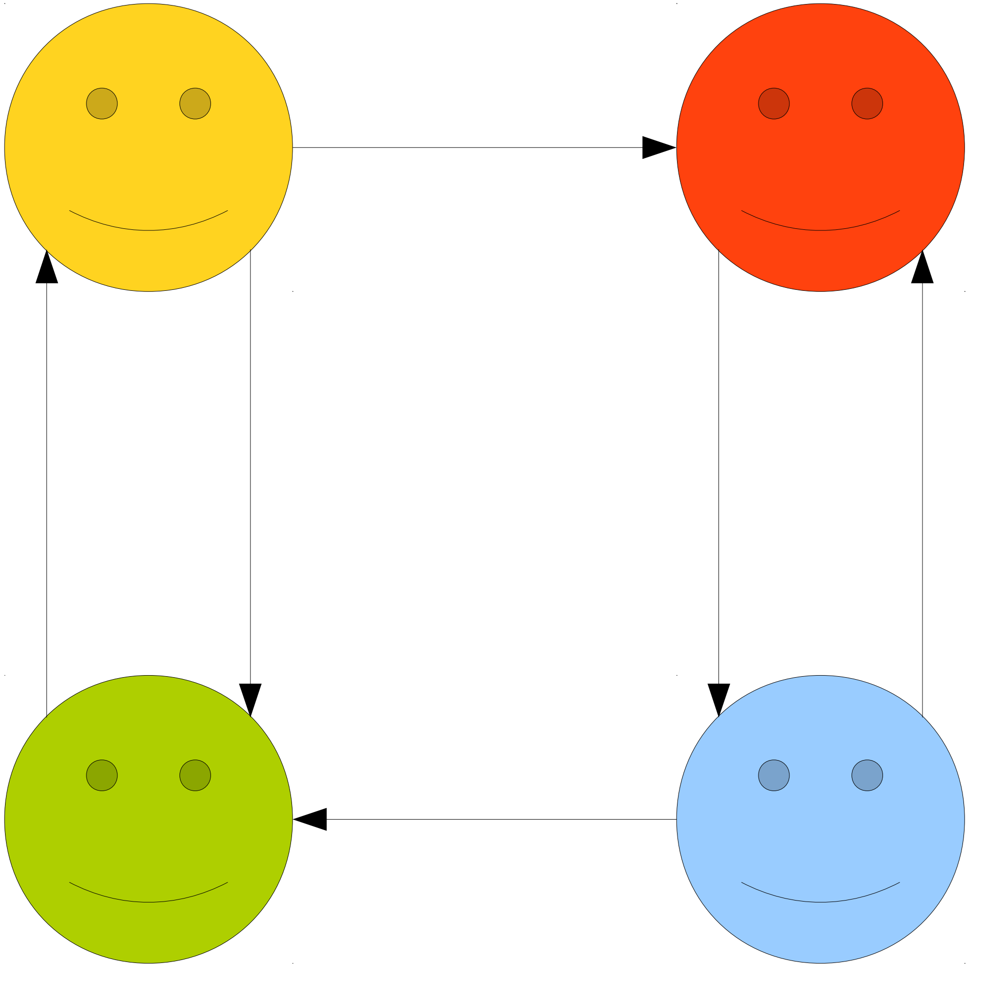
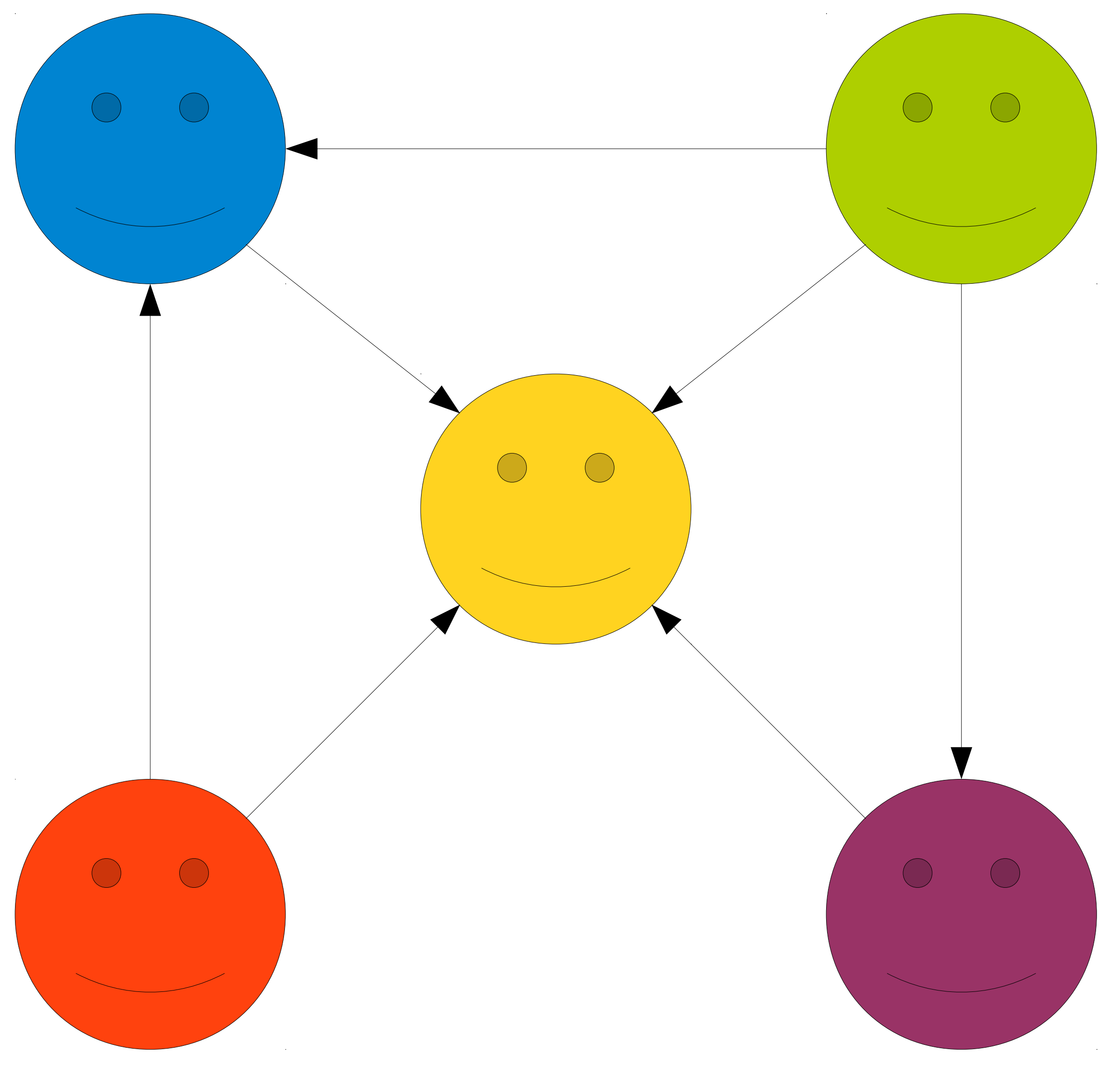
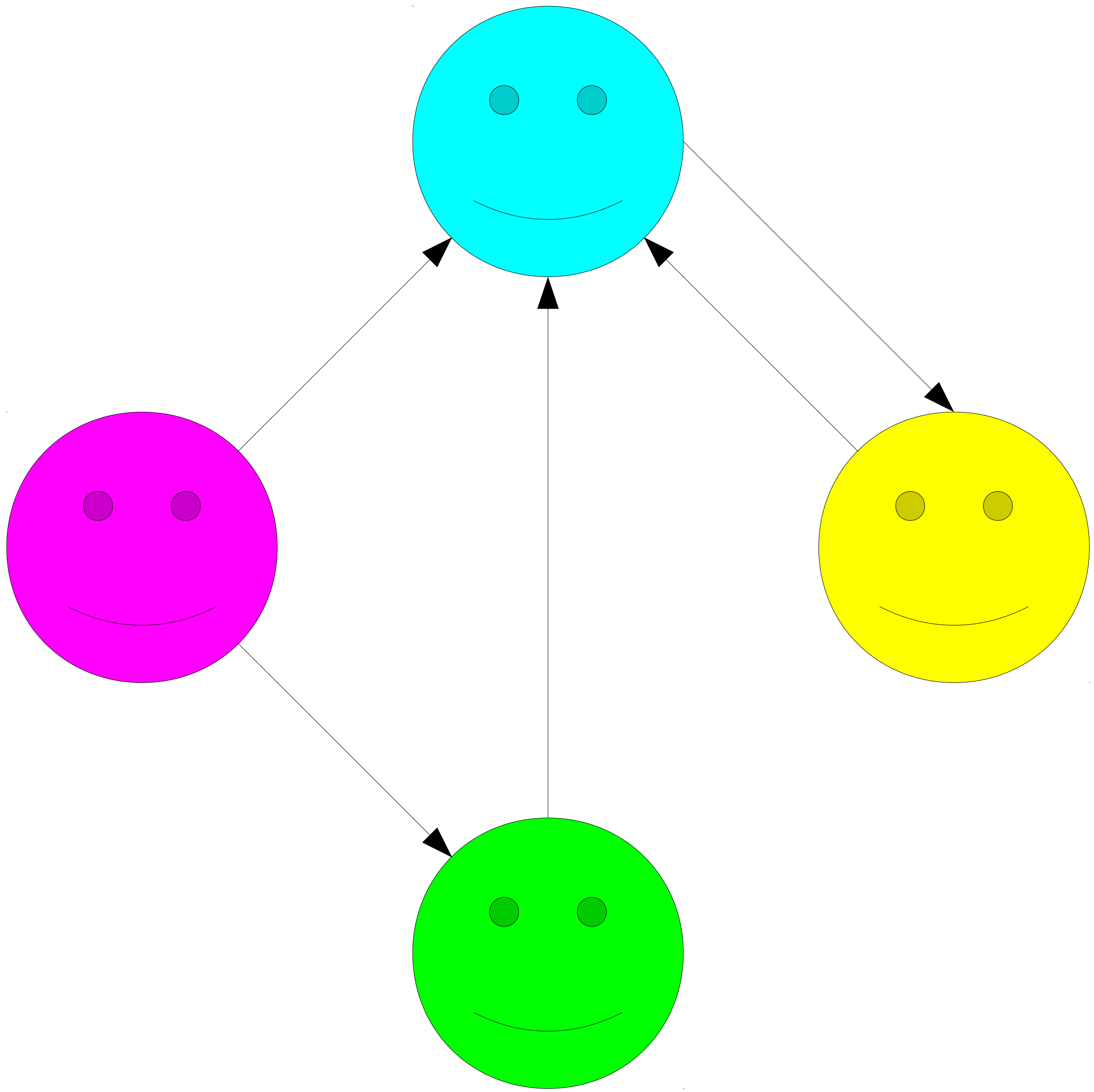
Everyone Loves Someone Else



There is Someone Everyone Else Loves



Everyone Loves Someone Else **and**
There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means "For any choice of x, there is **some** choice of y (possibly dependent on x) where $P(x, y)$ holds."

The statement

$$\exists y.\ \forall x.\ P(x, y)$$

means "There is some choice of y where for **any** choice of x, $P(x, y)$ holds."

Order matters when mixing existential and universal quantifiers!

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds."

- This is not technically a part of first-order logic; it is a shorthand for

$$\forall x.\ (x \in S \rightarrow P(x))$$

- How might we encode this concept?

$$\exists x \in S.\ P(x)$$

Answer: $\exists x.\ (x \in S \land P(x))$.

Note the use of ∧ instead of → here.

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.
- In CS103, please do not use variants of this syntax.
- Please don't do things like this:

$$\forall x \text{ with } P(x).\ Q(x)$$

$$\forall y \text{ such that } P(y) \land Q(y).\ R(y).$$

Translating into First-Order Logic

- First-order logic has great expressive power and is often used to formally encode mathematical definitions.
- Let's go provide rigorous definitions for the terms we've been using so far.

Set Theory

"Two sets are equal iff they contain the same elements."

$$\forall S.\ (Set(S) \to \forall T.\ (Set(T) \to (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T))$$

Many statements asserting a general claim is true are implicitly universally quantified.

Set Theory

"The **union** of two sets is the set containing all elements of both sets."

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\quad\quad \forall x.\ (x \in S \cup T \leftrightarrow x \in S \lor x \in T) \\
&\quad) \\
&)
\end{aligned}
$$

Relations

"R is a reflexive relation over A."

$$\forall a \in A.\ a\mathrm{R}a$$

Relations

"R is a symmetric relation over A."

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

Relations

"R is an antisymmetric relation over A."

$$\forall a \in A.\ \forall b \in A.\ (aRb \land bRa \to a = b)$$

Relations

"R is a transitive relation over A."

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

Negating Quantifiers

- We spent much of Wednesday's lecture discussing how to negate propositional constructs.
- How do we negate quantifiers?

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg \forall x.\ \varphi \equiv \exists x.\ \neg \varphi$$

$$\neg \exists x.\ \varphi \equiv \forall x.\ \neg \varphi$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from $\forall$ to $\exists$ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Analyzing Relations

"R is a binary relation over set A that is not reflexive"

$$\neg\forall a \in A.\ aRa$$
$$\exists a \in A.\ \neg aRa$$

"Some $a \in A$ is not related to itself by R."

Analyzing Relations

"R is a binary relation over A that is not antisymmetric"

$$\neg\forall x \in A.\ \forall y \in A.\ (xRy \land yRx \to x = y)$$
$$\exists x \in A.\ \neg\forall y \in A.\ (xRy \land yRx \to x = y)$$
$$\exists x \in A.\ \exists y \in A.\ \neg(xRy \land yRx \to x = y)$$
$$\exists x \in A.\ \exists y \in A.\ (xRy \land yRx \land \neg(x = y))$$
$$\exists x \in A.\ \exists y \in A.\ (xRy \land yRx \land x \neq y)$$

"Some $x \in A$ and $y \in A$ are related to one another by R, but are not equal"

Next Time

- **Formal Languages**
  - What is the mathematical definition of a problem?
- **Finite Automata**
  - What does a mathematical model of a computer look like?