ME 352 Supplemental Notes:

Infinite and Truncated Series

1 Learning objectives

After studying these notes you should...

- Be able to define an infinite series
- Be able to distinguish geometric series from a power series
- Be able to write the generic formula for a Taylor series
- Be able to write the first three terms of the series representations of e^x, $\sin(x)$, and $\cos(x)$.

2 Infinite Series: A Review

2.1 Definitions

Sequence: a function whose domain is a set of positive integers

$$(n, f(n)) : n = 1, 2, 3, ...$$

Example: $f(n) = 1/n$

Example: (Fibonacci)

$$\begin{array}{ll} f(1) = 1 & (n = 1) \\ f(2) = 1 & (n = 2) \\ f(n) = f(n-2) + f(n-1) & n = 3, 4, \ldots \end{array}$$

Exercise: Write out the first ten terms of the Fibonacci series

2.2 Limit of a Sequence

- $\lim_{n\to\infty} f(n) = L$
- Limit of $f(n)$ exists only if its graph has an asymptote
- Limit may or may not exist

Example:

$f(n) = 1/n$ has the limit 0

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \ldots$$

Example:

$f(n) = n/(n+1)$ has the limit 1

$$\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \ldots$$

2.3 Sequence of Partial Sums

$$\begin{aligned}
s_1 &= u_1 \\
s_2 &= u_1 + u_2 \\
s_3 &= u_1 + u_2 + u_3 \\
\ldots & \\
s_n &= u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k
\end{aligned}$$

Each member of the sequence is a sum of n terms. The sequence can be defined recursively

$$\begin{aligned}
s_1 &= u_1 \\
s_n &= s_{n-1} + u_n \qquad n > 1
\end{aligned}$$

where u_n is the nth term.

2.4 Infinite Series

A series (usually partial sums) with an infinite number of terms

Example:

$$1 + 2 + 3 + 4 + \ldots$$

Example:

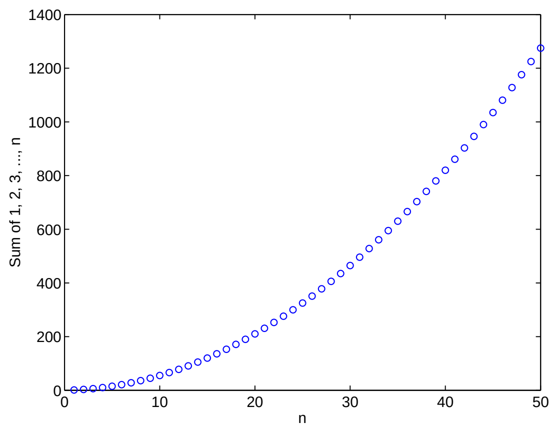
$$1 + 1/2 + 1/3 + 1/4 + \ldots$$

2.5 Convergence of Infinite Series

If a series converges, it has a limit. However, the existence of a limit is a *necessary* condition, not a *sufficient* condition

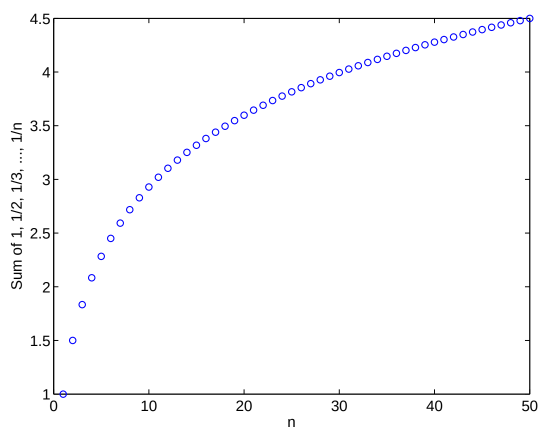
Example

$f(n) = 1 + 2 + 3 + 4 + \ldots n$ *does not* converge.



Example

$f(n) = 1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n$ *does not* converge.



A series

$$u_1 + u_2 + \ldots + u_n + \ldots$$

will not converge unless $\lim_{n\to\infty} u_n = 0$. This is a *necessary*, not sufficient condition

2.6 Geometric Series

An infinite series of the form

$$a + ax + ax^2 + ax^3 + \ldots + ax^n + \ldots$$

is called a *Geometric Series*. If $a \neq 0$, then the ratio of successive terms is x, i.e.

$$\frac{ax^n}{ax^{n-1}} = x$$

Consider the sum of the first n terms of the geometric series

$$s_n = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} \tag{1}$$

Multiply both sides by x

$$xs_n = ax + ax^2 + ax^3 + ax^4 + \ldots + ax^n \tag{2}$$

Subtract Equation (2) from Equation (1)

$$\begin{aligned}(1-x)s_n &= a - ax^n \\ &= a(1-x^n)\end{aligned}$$

If $x \neq 1$ divide both sides by $1 - x$ to get

$$s_n = \frac{a(1-x^n)}{1-x} \qquad (x \neq 1) \tag{3}$$

Now, assume that $|x| < 1$ and take the limit as $n \to \infty$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-x^n)}{1-x} = \frac{a}{1-x} \qquad |x| < 1$$

Summary If $|x| < 1$, then

$$a + ax + ax^2 + ax^3 + \ldots + ax^n + \ldots = \frac{a}{1-x}$$

2.7 Power series expansions

An power series is an expression of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots$$

The geometric series is a power series with all $a_k = a$, where a is a constant. Note that a truncated power series is just a polynomial.

2.8 Taylor series expansions

Taylor series expansions are special power series designed to approximate a function.

Given $y = f(x)$, we seek polynomials of the form

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

such that $f_n(x)$ is a good approximation to $f(x)$.

Consider the constant approximation to $f(x)$

$$f_0(x) = a_0$$

The best choice of a_0 is the value of the function at some point. Designate $\tilde{x}$ as the point where $f_0(x)$ and $f(x)$ are supposed to agree. Hence, $a_0 = f(\tilde{x})$

Next consider the linear approximation to $f(x)$

$$f_1(x) = a_0 + a_1 x$$

We want the good agreement at $\tilde{x}$ so rewrite this as

$$f_1(x - \tilde{x}) = a_0 + a_1(x - \tilde{x})$$

The best choice of a_0 is once again $f(\tilde{x})$. Geometric reasoning shows that the best choice of a_1 is the slope of the function $f(x)$ at $x = \tilde{x}$. Therefore, the linear approximation to $f(x)$ near $\tilde{x}$ is

$$f_1(x = \tilde{x}) = f(\tilde{x}) + (x - \tilde{x}) \left.\frac{df}{dx}\right|_{x=\tilde{x}}$$

Repeating this argument gives the Taylor series with remainder

$$f(x) = f(\tilde{x}) + (x - \tilde{x}) \left.\frac{df}{dx}\right|_{x=\tilde{x}} + \frac{(x - \tilde{x})^2}{2} \left.\frac{d^2 f}{dx^2}\right|_{x=\tilde{x}} + \frac{(x - \tilde{x})^3}{3!} \left.\frac{d^3 f}{dx^3}\right|_{x=\tilde{x}} + \ldots + R_n(x, \tilde{x})$$

2.9 Series Expansions for e^x, $\sin(x)$ and $\cos(x)$

The following series converge for all $-\infty < x < \infty$. However, it is not practical to evaluate these series for large x

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^k x^{2k}}{(2k)!} + \ldots.$$