The Lorentz force law must be modified to satisfy the Principle of Relativity

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Abstract Consideration of the relative motion of a bar magnet and a coil played a key role in Einstein's production of Special Relativity. In the frame where the bar magnet is moving and a coil is at rest, an EMF is generated in the coil as a consequence of the curl of E equation. In the frame where the bar magnet is at rest, an EMF is produced in the coil because its electrons are moving in a way such that the magnetic field is producing forces on them.

We consider the complementary situation where instead of a bar magnet generating a magnetic field we have a point charge generating an electric field. We will see that in order to satisfy the Principle of Relativity changes must be made to the Lorentz force law.

Keywords Principle of Relativity · Thought Experiment · Lorentz Force Law · Maxwell's Equations · Classical Physics · Classical Electromagnetism

1 Introduction

It is often currently thought that Einstein formulated Special Relativity as a consequence of him starting with the axioms that the Principle of Relativity is valid for electromagnetism and that Maxwell's Equations are correct. He did indeed reach the conclusions from such an axiomatization, but he apparently was led to such an axiomatization by considering a situation [1, 2] where a bar magnet and a conducting coil are in relative motion.[1]

In a frame where the bar magnet is moving and the conducting coil is at rest an electromotive force is generated in the coil because it is implied by Maxwell's $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ equation. In a frame where the coil is moving and the bar magnet is at rest, an electromotive force is generated in the coil because its electrons are moving through the magnetic field in such a way as to suffer a $q\,(\mathbf{v}/c) \times \mathbf{B}$ Lorentz

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[1] In his paper, Einstein's actual choice of terms were "magnet" and "conductor" rather than "bar magnet" and "coil". We use the latter choice of terms for purposes of improved clarity.

force. So it is seen that the Maxwell Equation and the Lorentz force law conspire to cause the system to physically obey a principle of velocity relativity.

Einstein, apparently satisfied with the important progress made by considering the situation where a magnetic field configuration is in relative motion to a coil, appears never to have considered the complementary situation where an electric field configuration is in relative motion to a coil. We will do so here, and discover that it leads to far ranging consequences.

2 A Crucial Insight Regarding Magnets and Coils in Relative Motion

Before considering the situation regarding a coil and a point electric charge in relative motion in different frames, let us carefully revisit the situation of a coil and bar magnet in relative motion in different frames. Crucial insights will be realized.

In the frame where the bar magnet is moving, it produces the EMF in the coil because of the $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ equation. It is often viewed that the differential $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ form of the Maxwell Equation is equivalent to the integral statement that the line integral of the electric field is equal to the time rate of change of flux of the magnetic field. This “flux” viewpoint is acceptable in this situation.

Now consider the frame where the bar magnet is at rest and the coil is moving. The electromotive force in the loop is due to the electrons in the coil cutting through magnetic field lines, producing a $q\,(\mathbf{v}/c) \times \mathbf{B}$ force. It is indeed the case that there is a change in magnetic field flux in this frame, but that such a change in flux is associated with an electromotive force in the coil is completely dependent on the $q\,(\mathbf{v}/c) \times \mathbf{B}$ effect. *Had there been no $q\,(\mathbf{v}/c) \times \mathbf{B}$ term in the Lorentz force law, there would be no EMF due to the “changing flux”.*

This last point is so crucial that we must repeat it. In the frame where the coil is moving, the EMF is generated because there is a $q\,(\mathbf{v}/c) \times \mathbf{B}$ term in the Lorentz force law. Without that term there would, under the actual laws, be no EMF associated with the changing flux of the magnetic field in that frame.

We can look at the situation in the following interesting way. If we are to accept the Principle of Relativity, the $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ Maxwell Equation *implies* that a $q\,(\mathbf{v}/c) \times \mathbf{B}$ term must exist in the Lorentz force law. *The $q\,(\mathbf{v}/c) \times \mathbf{B}$ term is actually implied by the $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ equation*[2] *according to the Principle of Relativity. We can actually derive the $q\,(\mathbf{v}/c) \times \mathbf{B}$ term.*

3 A Point Charge in Relative Motion to a Magnetizable Object

We can now turn our attention to the complementary situation where instead of having a bar magnet supplying a magnetic field, we have a point electric charge supplying an electric field.

[2] The $\nabla \cdot \mathbf{B} = 0$ equation is also needed. It ensures that the $\mathbf{B}$ field in the frame where the coil is moving is quantitatively exactly what is needed for $(\mathbf{v}/c) \times \mathbf{B}$ in that frame to correspond to the $\mathbf{E}$ field in the frame where the moving magnet induces the $\mathbf{E}$ field around the coil. Likewise, later in the paper when we discuss the role the $\nabla \times \mathbf{E}$ equation plays in the force law, the $\nabla \cdot \mathbf{E} = 0$ vacuum equation is also actually required.

Rather than having a coil of conducting material, we will have a half a loop of a magnetizable material with the topology of a horseshoe. In the frame where the charge is moving towards the half-loop, there is an increasing flux of electric field in the circular region bounded by the half-loop imaginarily extended to make a closed loop, and thus as a consequence of the $\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$ equation a magnetic field is generated tangentially around the loop. Being that the material is magnetizable, we create a horseshoe type magnet.

Now consider the situation in the frame where the point charge is at rest, and it is the half loop that is moving. From the Principle of Relativity the half loop must become magnetized. But how? What is the mechanism? The only electromagnetic field in this frame is the electric field, and it is not possible to construe it under the Lorentz force law to be able to apply the torque to the dipoles[3] in the magnetically permeable material in the way necessary to magnetize it.

The only electromagnetic field in this frame is the electric field, and it is not possible to construe it under the Lorentz force law to be able to apply the torque to the dipole in the magnetically permeable material in the way necessary to magnetize it.

We cannot say in this frame that because the flux of the electric field increased there was a tangential magnetic field induced. We saw that in the case of a coil moving towards a bar magnet that in that frame the increasing flux of the magnetic field through the coil caused the EMF through the *mechanism* of the $q\,(\mathbf{v}/c) \times \mathbf{B}$ force in the Lorentz force law. Had a $q\,(\mathbf{v}/c) \times \mathbf{B}$ term not existed in the Lorentz force law then there just would not have been an EMF induced in the moving coil as the flux of the magnetic field increased. *It cannot be stressed too strongly that the EMF in the situation where a coil moves towards a bar magnet was dependent in the most essential way on there being a $q\,(\mathbf{v}/c) \times \mathbf{B}$ force – without such a force there would not have been an EMF induced as the flux of $\mathbf{B}$ was caused to increase.* In that frame it is the $q\,(\mathbf{v}/c) \times \mathbf{B}$ that is responsible for the EMF. The situation regarding the point electric charge and the half loop of magnetizable material can only be resolved by including a $-q\,(\mathbf{v}/c) \times ((\mathbf{v}/c) \times \mathbf{E})$ term in the Lorentz force law, a term that plays a role analogous to the role played by the $q\,(\mathbf{v}/c) \times \mathbf{B}$ term in making the bar magnet and coil situation conform to the Principle of Relativity. This term provides the exact torque needed.[4]

4 A Point Charge in Relative Motion to a Device Firing Charges Perpendicular to that Motion

Let us consider yet another situation. We will have a point charge and an imaginary loop in relative motion to each other. The point charge is located at the point

[3] We are treating ferromagnetism, a quantum phenomenon, in a classical way. If the reader finds this troubling, then instead of doing the thought experiment with a horseshoe-shaped piece of ferromagnetically magnetizable material, he or she can do the thought experiment with a horseshoe-shaped container initially containing randomly oriented loops of current.

[4] Actually, throughout, for simplicity we will not be concerned about factors of $\frac{1}{\sqrt{1-(v/c)^2}}$. This might seem strange being that the effect uncovered here is quadratic in (v/c). However the effect introduced here is very qualitatively different from simple multiplication by a non-vector. For example, in Section 6 we will see that the effect changes whether a light bulb in a thought experiment lights up. No simple multiplication by $\frac{1}{\sqrt{1-(v/c)^2}}$ can do that.

($x = a$, $y = 0$, $z = 0$) and the imaginary loop is the locus of points ($x = b$, $y^2 + z^2 = c^2$, where c is a fixed constant that is the radius of the imaginary loop). The point ($x = b$, $y = 0$, $z = 0$) will be referred to as the center of the circle. In the frame where the point charge is at rest and the imaginary loop is moving, the quantity a is constant in time and the quantity b changes in time, while in the frame where the imaginary loop is at rest and the point charge is moving, the quantity b is constant in time and the quantity a changes in time. We will fire electrons from the center of the circle towards the imaginary loop – i.e. the electrons will be moving as if coming from the center of a tire, as if moving along imaginary spokes of the tire, moving towards the rim of the tire. We will also consider what happens if we fire electrons in the opposite way, moving towards the center rather than away from the center.

In the frame where the point charge is moving and the imaginary loop is at rest the increased flux of the electric field, via the $\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$ equation, leads to a tangential magnetic field around the loop region, and thus the electrons are either deflected towards the point charge or away from the point charge, depending on whether they are fired from the center of the circle towards the imaginary loop or they are fired from the imaginary loop towards the center of the circle.

Now consider this in the frame where the point charge is at rest and the imaginary loop is moving. According to the Principle of Relativity the fired electrons will be deflected towards or away from the point charge, depending on whether they are fired towards or away from the imaginary loop. As we saw in the case regarding the magnetizable half-loop, there is no way to explain the behavior under the standard Lorentz force law. And again, the addition of a $-q\,(\mathbf{v}/c)\times((\mathbf{v}/c)\times\mathbf{E})$ term perfectly resolves the situation.

When analyzing the situation with the bar magnet, we observed that the $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ Maxwell Equation, in conjunction with the Principle of Relativity, implies the existence of a $q\,(\mathbf{v}/c)\times\mathbf{B}$ term in the Lorentz force law. Here we see that the $\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$ Maxwell Equation, in conjunction with the Principle of Relativity, actually must imply that a $-q\,(\mathbf{v}/c)\times((\mathbf{v}/c)\times\mathbf{E})$ term must be added to the Lorentz force law.[5]

5 A Reciprocity Principle

Let us consider a situation where we have two particles on the x-axis, with Particle 1 being at rest and Particle 2 having a uniform velocity with non-vanishing components both in the x-direction and the y-direction. If one explicitly calculates the $-q\,(\mathbf{v}/c)\times((\mathbf{v}/c)\times\mathbf{E})$ force on Particle 2 one finds that it has a component in the y-direction. However, there does not appear to be any force on Particle 1 in the y-direction. This would imply a violation of Newton's Third Law.

The situation is remedied by stipulating that in addition to the electric field prescribed by Maxwell's Equations, Particle 2 generates an additional electric field

[5] The Linear Field Approximation of General Relativity put in the gravitomagnetism form has a similar form to the field equations of electromagnetism, and so the same sort of term we have uncovered for electromagnetism must occur there too. While General Relativity does have forces quadratic in the velocity, none match the required terms analogous to $-q\,(\mathbf{v}/c)\times((\mathbf{v}/c)\times\mathbf{E})$ term, and thus revision of General Relativity could not preserve the extraordinarily desirable Bianchi Identity structure. The situation is troubling and needs to be taken very seriously.

proportional to $q\frac{(\mathbf{v}/c)\times((\mathbf{v}/c)\times\mathbf{r}_{12})}{r^3}$, where $\mathbf{r}_{12}$ is the vector going from Particle 1 to Particle 2.[6]

Since we are forced to add forces generated by terms quadratic in the velocity, we must also construct rules whereby forces quadratic in the velocities of both of two moving the quantity s are administered in a way preserving Newton's Third Law. We will not go into the details here.

We should note that while it might seem that we have been proceeding in a speculative way, the opposite is true. Everything we have done throughout has been forced upon us to be in compliance with the Principle of Relativity and Newton's Third Law.

6 The Paradox of the Line Charge and Circuit

Consider a situation involving a line of charge and a circuit. The line of charge is on the y-axis and is at rest. The circuit is a square closed circuit of wire with a light bulb within the circuit, and lies in the xy-plane with two of the square's sides parallel to the y-axis, and the circuit has a velocity in the x-direction. It is undeniably clear that according to the Lorentz force law the light bulb does not light up.

Now consider that same situation except that both the line of charge and the circuit have y-velocities, y-velocities numerically equal for both the line charge and the circuit. (The circuit still has an x-velocity, just as it did in the first scenario.) If we apply the Lorentz force law, we see that because the line of charge moving in the y-direction produces a magnetic field in the z-direction of magnitude $\left(\frac{-2\lambda(v_y/c)}{x}\right)$, where v_y is the velocity in the y-direction of both the line of charge and the circuit, and λ is the charge density per unit length of the line of charge, the force in the y-direction on a charged particle in the circuit will be $q(v_x/c)\left(\frac{2\lambda(v_y/c)}{x}\right)$. The portion of the circuit closer to the line of charge, having a smaller value of x, will have a stronger force on its charges than the portion farther from the line of charge, and thus there will be a net EMF around the circuit, and the light bulb should light up.

This presents a serious paradox in that the second scenario was simply the first scenario under a Lorentz transformation! It is not acceptable that in one inertial frame the light bulb lights up while in another inertial frame it does not light up.

[6] Reciprocity necessary for the preservation of Newton's Third Law is something not to be taken for granted. The magnetic force on a Particle 2 generated by a Particle 1 has the form $q_2\,(\mathbf{v}_2/c)\times\mathbf{B_2}$ where q_2 is the charge of Particle 2, $\mathbf{v}_2$ is the velocity of Particle 2 and $\mathbf{B}_2$ is the magnetic field at the point where Particle 2 is that is generated by the motion of Particle 1. This goes as $q_2\frac{(\mathbf{v}_2/c)\times(q_1(\mathbf{v}_1/c)\times\mathbf{r}_{12})}{r^3}$. Likewise the force on Particle 1 goes as $q_1\frac{(\mathbf{v}_1/c)\times(q_2(\mathbf{v}_2/c)\times\mathbf{r}_{21})}{r^3}$. These quantities actually turn out to not always be equal to within a factor of minus one, thus presenting a serious problem.
This situation is not a physically meaningless mathematical oddity, but rather a situation genuinely needing modification – physical manifestations of the pathology can easily be created. For example, if we have Particle 1 at the origin moving in the x-direction, and Particle 2 on the y-axis (but not at the origin) moving in the y-direction, Particle 1 generates a magnetic field that can easily be seen under the Maxwell Lorentz regime to apply a force in the x -direction to Particle 2, while it can also easily be seen under the Maxwell Lorentz regime that Particle 2 exerts no force in the x-direction on Particle 1.

However, our $-q\,(\mathbf{v}/c) \times ((\mathbf{v}/c) \times \mathbf{E})$ force perfectly resolves the situation. In the frame where both the line of charge and the circuit have the non-vanishing velocity in the y-direction the line of charge produces an electric field of $(\frac{2\lambda}{x})$ and thus charges in the circuit experience a $-q\,(\mathbf{v}/c) \times ((\mathbf{v}/c) \times \mathbf{E})$ force from it of $-q(v_x/c)(\frac{(v_y/c)2\lambda}{x})$. This *precisely* cancels out the magnetic force, ensuring that in the frame where the line of charge and the circuit have the non-vanishing y-velocities the light bulb will not light up – just like it would not do in the other frame.

7 Possible Experimental Tests

The theoretical work here predicts very specific physical effects, and thus is well-suited for experimental testing. Indeed, several definitive experimental tests are possible.

Experiments verifying the Einstein discovery regarding the relative motion of a bar magnet and a coil are commonplace [3]. The complementary situation involving a point charge instead of a bar magnet is amenable to experiment – the magnetization of the horseshoe, as well as the electrons fired along imaginary spokes scenarios discussed earlier in the paper can be tested experimentally.

A more elemental version of the horseshoe scenario can be done where instead of a whole half-loop of iron dipoles being used, a single dipole composed of a current-carrying loop can have its torque measured as it travels along an electric field.

In a sense all the above experiments may seem more like tests of the Principle of Relativity than of the modification of the Lorentz force law – that is often going to be to some degree unavoidable being that the modifications we proposed actually are *inescapable* consequences of the Principle of Relativity. Nevertheless they are still tests for the existence of phenomena that the Lorentz force law implies could not possibly occur.

One can just directly measure the deflection of a beam of electrons in an electric field. Most interesting is when the beam is neither completely perpendicular nor completely parallel to the electric field. (This actually is no more than the "electrons fired along imaginary spokes" scenario in Section 4, without the imaginary tire-loop being present.) We predict a deflection caused by the $-q\,(\mathbf{v}/c) \times ((\mathbf{v}/c) \times \mathbf{E})$, a deflection with a clearly predicted magnitude and direction. This direct test can simply be done, and the presence of our predicted deflection not predicted by the Lorentz force law would be direct unequivocal experimental proof of our theory.

Another experimental test is the situation (discussed in Section 6) of a line of charge on the y-axis moving in the positive y direction with a loop that is in the xy-plane and contains a light bulb moving with the same y-velocity as the line charge but also having an x-velocity. The Lorentz force law predicts the light bulb will light up – the line charge will produce a magnetic field in the z-direction which will interact with the x-velocity of the loop, producing $q\,(\mathbf{v}/c) \times \mathbf{B}$ forces in the y direction that are not uniform with x being that $\mathbf{B}$ varies with x. However, our theory predicts that the $-q\,(\mathbf{v}/c) \times ((\mathbf{v}/c) \times \mathbf{E})$ force will exactly cancel out this effect, and the bulb will not light up. Because our prediction is so strikingly different from what would be expected, it could provide very dramatic confirmation of the theory.

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