Math 417 – Sections 53 & 54 Solutions

1. To find the Maclaurin Series for $f(z) = z\cosh(z^2)$, we start with the Maclaurin Series for $\cosh z$ on p. 187:

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

Substituting z with z^2, we have:

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z| < \infty)$$

Multiplying by z, we have the Maclaurin Series for $f(z)$:

$$z\cosh(z^2) = z\sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty)$$

3. For the function:

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}$$

we know that the Maclaurin Series converges for

$$\left|\frac{z^4}{9}\right| < 1 \quad \Rightarrow \quad |z| < 9^{1/4} = \sqrt{3}$$

Using the geometric series expansion, we have:

$$\begin{aligned}
f(z) &= \frac{z}{9} \cdot \frac{1}{1-(-z^4/9)} \\
&= \frac{z}{9}\left[1 - \frac{z^4}{9} + \left(\frac{z^4}{9}\right)^2 - \dots\right] \\
&= \frac{z}{9} - \frac{z^5}{9^2} + \frac{z^9}{9^3} - \frac{z^{13}}{9^4} + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+1}}{9^{n+1}} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+1}}{3^{2n+2}} \qquad (|z| < \sqrt{3})
\end{aligned}$$

6. To find the Maclaurin Series for $f(z) = \sin(z^2)$, we start with the Maclaurin Series of $\sin z$:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

and replace z with z^2:

$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \dots$$

Note that the Maclaurin Series does not contain terms with odd powers, even powers that are multiples of 4, and the constant term. Therefore,

$$f^{(4n)}(0) = 0 \qquad \text{and} \qquad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \dots)$$