1 Taylor Series: functions of a single variable

Recall that smooth functions $f(x)$ of one variable have convergent Taylor series. The Taylor series of such a function about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_n \qquad (1)$$

where the remainder term $R_n \to 0$ as $n \to \infty$ for all x in the interval of convergence. Such expansions can be used to tell how a function behaves for x near a. When $a = 0$ the Taylor series is also called the MacLaurin series of $f(x)$. Some common series are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{all } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad |x| < 1\ ,\ x = 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x| \leq 1$$

It is important to emphasize that the Taylor series is "about" a point. For example, the Taylor series of $f(x) = \ln(1+x)$ about $x = 0$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

If you truncate the series it is a good approximation of $\ln(1+x)$ near $x = 0$. Using (1), the Taylor series of $f(x) = \ln(1+x)$ about $x = 1$ is

$$\ln(1+x) = \ln(2) + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{24}(x-1)^3 + \cdots$$

If you truncate this series it is a good approximation of $\ln(1+x)$ near $x = 1$.

2 Taylor series: functions of two variables

If a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is sufficiently smooth near some point $(\bar{x}, \bar{y})$ then it has an m-th order Taylor series expansion which converges to the function as $m \rightarrow \infty$. Expressions for m-th order expansions are complicated to write down. For our purposes we will only need second order expansions so we state a related Theorem here:

Theorem 2.1 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and assume that f and all its derivatives up to third-order are continous on some neighbourhood $N_r(a)$, $a = (a_1, a_2)$. For each $x = (x_1, x_2) \in N_r(a)$*

$$\begin{aligned} f(x) &= f(a) + f_{x_1}(a)(x_1 - a_1) + f_{x_2}(a)(x_2 - a_2) \\ &+ \frac{1}{2!}\left(f_{x_1x_1}(a)(x_1 - a_1)^2 + 2f_{x_1x_2}(a)(x_1 - a_1)(x_2 - a_2) + f_{x_2x_2}(a)(x_2 - a_2)^2\right) + R_3 \end{aligned}$$

where the remainder term $R_3 \rightarrow 0$ as $x \rightarrow a$.

Notationally there are many ways to write out Taylor series. For $f = f(x_1, x_2)$ one can define the gradient of f as

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$$

The Hessian $H_f(x)$ of f is defined as

$$H_f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Note that H_f is a symmetric matrix. With these definitions, the expansion in the Theorem above can be written

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2!}(\zeta - a)^T H_f(a)(\zeta - a) + R_3$$

where $\cdot$ is the dot product. Here, the linear approximation [1]of f is

$$f(x) \simeq f(a) + \nabla f(a) \cdot (x - a)$$

and the remainder term is $R_2(\zeta) = \frac{1}{2!}(\zeta - a)^T H_f(a)(\zeta - a)$. The second-order Taylor series of f about a is

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2!}(x - a)^T H_f(a)(x - a) + R_3$$

where the exact form of the remainder term R_3 is a complicated expression. Here we will simply write:

$$R_3 = O(\| x - a \|^3) = O(3).$$

Example 1 *Let $f(x) = x_1^2 + e^{x_1} - x_1x_2 + 3x_2$ and $a = (0, 0)$. Then,*

$$\nabla f(x) = (2x_1 + e^{x_1} - x_2, -x_1 + 3)$$

so that

$$\nabla f(x) = (1, 3)$$

The Hessian is

$$H_f(x) = \begin{bmatrix} 2 + e^{x_1} & -1 \\ -1 & 0 \end{bmatrix}$$

[1] or first-order Taylor series approximation

so that

$$H_f(a) = \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

Since $f(a) = 1$,

$$f(x) = 1 + (1,3) \cdot x + \frac{1}{2!} x^T \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} x + O(\| x \|^3)$$

Longhand,

$$f(x) = 1 + x_1 + 3x_2 + \frac{3}{2} x_1^2 - x_1 x_2 + O(\| x \|^3)$$

Example 2 *Do several in class*

3 Implicit Function Theorem (IFT): Single variable

Theorem 3.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, \mu)$ and that f, f_x and f_μ are continuous on a neighbourhood $(x, \mu) = (x^*, \mu^*)$. If*

i) $f(x^*, \mu^*) = 0$

ii) $f_\mu(x^*, \mu^*) \neq 0$

then there is a unique function $\bar{\mu}(x)$ such that

$$f(x, \bar{\mu}(x)) = 0 \tag{2}$$

for all x in a neighbourhood of $x = x^*$.

Basically the theorem guarantees $f(x, \mu) = 0$ has a unique solution $\bar{\mu}(x)$ for all x near x^*. For example, consider

$$f(x, \mu) = e^{\mu x} - x + \mu - 1$$

Then

$$f_\mu(x, \mu) = x e^{\mu x} + 1$$

from which

$$f(0, 0) = 0 \qquad f_\mu(0, 0) = 1 \neq 0$$

By the IFT (with $(x^* = 0, \mu^* = 0)$

$$\dot{x} = f(x, \mu)$$

has a unique fixed point for all x near $x^* = 0$. Note that this precludes the possibility of any transcritical bifurcations for such x values (Why?).

Note that if f is smooth enough one can differentiate (2) in x to get

$$f_x(x, \bar{\mu}) + f_\mu(x, \bar{\mu}) \frac{d\bar{\mu}}{dx} = 0$$

Were $f_\mu = 0$ then (x, μ) would have to solve $f = 0$ and $f_x = 0$. That would mean (x, μ) would a "point" and μ could not vary in x!! This explains the importance of the (IFT) hyptothesis.

Lastly, when (IFT) hypotheses are satisfied we know

$$\bar{\mu}'(x) = \frac{d\bar{\mu}}{dx} = -\frac{f_x(x, \bar{\mu})}{f_\mu(x, \bar{\mu})}$$

For saddle-nodes with quadratic tangency we need $\bar{\mu}'(x^*) = 0$ hence $f_x = 0$ (nonhyperbolicity) is required at saddle node bifurcations.

4 Hyperbolicity and hyperbolic fixed points

Definition 4.1 *A fixed point $\bar{x}$ of $\dot{x} = f(x, \mu)$ is hyperbolic if $\frac{\partial f}{\partial x}(\bar{x}, \mu) \neq 0$.*

Hyperbolicity of fixed points is at the very heart of bifurcation theory. All the basic bifurcation saddle node, transcritical, pitchfork bifurcations occur (respectively) at nonhyperbolic fixed points $(x^*, \mu^*) = (0, 0)$

$$\begin{aligned} \dot{x} &= \mu - x^2 \\ \dot{x} &= \mu x - x^2 \\ \dot{x} &= x(\mu - x^2) \end{aligned}$$

5 Local Bifurcation Theory - some simple results

Here we summarize some theorems for "generic" saddle-node, transcritical and pitchfork bifurcations of

$$\dot{x} = f(x, \mu) \quad , \quad x, \mu \in \mathbb{R}.$$

Proofs use the implicit function theorem, near identity transformations and various other transformations. In all of the the theorems below we assume that f has continuous (mixed) derivatives up to third order (fourth for pitchforks) for all (μ, x) near the bifurcation point (μ^*, x^*).

Theorem 5.1 *If there is a pair (μ^*, x^*) for which*

$$\begin{aligned} f(x^*, \mu^*) &= 0 && (3) \\ f_x(x^*, \mu^*) &= 0 && (4) \\ f_\mu(x^*, \mu^*) &\neq 0 && (5) \\ f_{xx}(x^*, \mu^*) &\neq 0 && (6) \end{aligned}$$

then $\dot{x} = f(x, \mu)$ has a saddle-node bifurcation with quadratic tangency at (μ^*, x^*).

Transcritical (2-branch)

Theorem 5.2 *If there is a pair (μ^*, x^*) for which*

$$\begin{aligned} f(x^*, \mu^*) &= 0 && (7) \\ f_x(x^*, \mu^*) &= 0 && (8) \\ f_\mu(x^*, \mu^*) &= 0 && (9) \\ f_{x\mu}(x^*, \mu^*) &\neq 0 && (10) \\ f_{xx}(x^*, \mu^*) &\neq 0 && (11) \end{aligned}$$

then $\dot{x} = f(x, \mu)$ has a (2 branch) transcritical bifurcation at (μ^*, x^*).

Pitchfork (Quadratic Tangency)

Theorem 5.3 *If there is a pair (μ^*, x^*) for which*

$$\begin{aligned} f(x^*, \mu^*) &= 0 && (12) \\ f_x(x^*, \mu^*) &= 0 && (13) \\ f_\mu(x^*, \mu^*) &= 0 && (14) \\ f_{xx}(x^*, \mu^*) &= 0 && (15) \\ f_{x\mu}(x^*, \mu^*) &\neq 0 && (16) \\ f_{xxx}(x^*, \mu^*) &\neq 0 && (17) \end{aligned}$$

then $\dot{x} = f(x, \mu)$ has a pitchfork bifurcation with quadratic tangency at (μ^*, x^*).

5.1 Saddle-Node Local Theory

Suppose that on the locus of equilibria $\mu = \mu(x)$, i.e. μ is a function of x in a neighbourhood of the suspect bifurcation point. If the bifurcation occurs at (μ^*, x^*) then by letting

$$X = x - x^* \quad , \quad \eta = \mu - \mu^*$$

the differential equation for X is

$$\dot{X} = F(X, \eta) = f(X + x^*, \eta + \mu^*)$$

which has a bifurcation at $(X, \eta) = (0, 0)$. Thus, without any loss of generality we will assume that the bifurcation occurs at $(\mu, x) = (0, 0)$.

In this setting $f(0, 0) = 0$, $\mu(0) = 0$ and

$$f(x, \mu(x)) = 0$$

for all x near the bifurcation point. Differentiating this expression in x yields

$$f_x(x, \mu(x)) + f_\mu(x, \mu(x))\frac{d\mu}{dx} = 0 \tag{18}$$

which when solved for $\mu'(x) = \frac{d\mu}{dx}$ gives

$$\mu'(x) = -\frac{f_x(x, \mu(x))}{f_\mu(x, \mu(x))}.$$

One condition which must be satisfied at a quadratic tangency is that $\mu'(0) = 0$. Given the formula above, this can only happen if

$$f_x(0, 0) = 0 \tag{19}$$

$$f_\mu(0, 0) \neq 0. \tag{20}$$

Together, the conditions (19)-(20) imply that $x = 0$ is a nonhyperbolic fixed point. For there to be a quadratic tangency, however, we also need to have $\mu''(0) \neq 0$. By differentiating (18) in x, evaluating the resulting expression at $x = 0$ and using (19) it can be shown that

$$f_{xx}(0, 0) + f_\mu(0, 0)\mu''(0) = 0.$$

Because $f_\mu(0, 0) \neq 0$, $\mu''(0) \neq 0$ only if

$$f_{xx}(0, 0) \neq 0. \tag{21}$$

In conclusion, a saddle-node bifurcation with quadratic tangency will exist if the three conditions (19)-(21) are satisfied. We can also deduce that the Taylor series expansion of f about such a bifurcation point will have the form

$$f(x, \mu) = a_0\mu + a_1x^2 + a_2x\mu + a_3\mu^2 + O(3)$$

for some constants $a_0 \neq 0$,$a_1 \neq 0$, a_2 and a_3. Here, $O(3)$ is notation to indicate higher order terms in the Taylor series, i.e., $x^3, x^2\mu, \ldots$. In the next section we will discuss how one can simplify this expression to create what is called the "normal form" for the bifurcation.

Theorem 5.4 *Let $\dot{x} = f(x, \mu)$ and assume that for all (μ, x) near some point (μ^*, x^*) f has continuous (mixed) derivatives up to and including third order, i.e., $f_x, f_\mu, f_{xx}, \ldots, f_{x\mu\mu}, f_{\mu\mu\mu}$. If*

$$f(x^*, \mu^*) = 0 \tag{22}$$

$$f_x(x^*, \mu^*) = 0 \tag{23}$$

$$f_\mu(x^*, \mu^*) \neq 0 \tag{24}$$

$$f_{xx}(x^*, \mu^*) \neq 0 \tag{25}$$

then $\dot{x} = f(x, \mu)$ has a saddle-node bifurcation with quadratic tangency at (μ^*, x^*).

Example: Consider

$$\dot{x} = f(x,\mu) = \mu - x - e^{-x}.$$

A nonhyperbolic equilibria exists at any pair of (μ, x) *such that*

$$\begin{aligned} f = 0 \quad &\Leftrightarrow \quad \mu - x - e^{-x} = 0 \\ f_x = 0 \quad &\Leftrightarrow \quad -1 + e^{-x} = 0 \end{aligned}$$

Thus $(\mu^*, x^*) = (1,0)$ *is a bifurcation point. Since* $f_{xx}(x,\mu) = -e^{-x}$, $f_{xx}(0,1) = -1 \neq 0$. *Lastly,* $f_\mu = 1 \neq 0$ *so that a saddle-node bifurcation of quadratic tangency occurs at* $(\mu^*, x^*) = (1,0)$, *i.e.,*

$$f = 0 \quad , \quad f_x = 0 \quad , \quad f_\mu \neq 0 \quad , \quad f_{xx} \neq 0.$$

The (2-variable) Taylor series of $f(x,\mu)$ *about* $(\mu^*, x^*) = (1,0)$ *is*

$$f(x,\mu) = (\mu - 1) - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$$

so that

$$\dot{x} = \eta - \frac{1}{2}x^2 + O(x^3)$$

where $\eta = \mu - 1$ *is a new parameter. For this example, the equation above is in "normal form".*

Note that some saddle-node bifurcations do not result from "quadratic" tangencies. For example,

$$\dot{x} = \mu + x^4$$

has a saddle-node bifurcation at $(\mu, x) = (0,0)$ even though $f_{xx}(0,0) = 0$. This is an example of a "quartic" tangency. Clearly other more complicated variants can occur as well. For example, just consider what happens at $(\mu, x) = (0,0)$ if

$$\dot{x} = (x^2 - \mu)(x^2 - 4\mu).$$

Also, some bifurcations that are quadratic are not a result of a saddle-node bifurcation. For example, consider

$$\dot{x} = f(x,\mu) = \sqrt{\mu} - x \quad , \quad \mu \geq 0,$$

which has a sole branch of fixed points $x = \sqrt{\mu}$ and a quadratic tangency at $x = 0$. At $(\mu^*, x^*) = (0,0)$ two branches of fixed points do not coalescece. This example does not violate Theorem 5.4 since $f_\mu(0,0)$ is not defined, let alone continuous for all (μ, x) near $(\mu^*, x^*) = (0,0)$.

5.2 Transcritical Local Theory

Suppose that two disjoint branches of fixed points intersect at a transcritical bifurcation point. From Figure 1 it is easy to see that near (TC), $f(x,\mu) = 0$ does not imply that there is a single function which describes both branches of fixed points.

Considering the implicit function theorem (IFT) f_μ must vanish at a (TC) bifurcation point. Alternately by swapping x and μ in the theorem, f_x must also vanish. The latter implies that fixed points must (like at saddle node bifurcations) be nonhyperbolic at transcritical bifurcations. Like saddle node bifurcations, there are theorems which place sufficiency conditions on f assuring that transcritical bifurcations exist.

Theorem 5.5 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,\mu)$ *have continuous derivatives of all orders up to and including degree three, i.e.,* $f, f_\mu, \ldots f_{xx\mu}$ *near* (μ^*, x^*). *If*

$$\begin{aligned} f(x^*, \mu^*) &= 0 && (26) \\ f_x(x^*, \mu^*) &= 0 && (27) \\ f_\mu(x^*, \mu^*) &= 0 && (28) \\ f_{x\mu}(x^*, \mu^*) &\neq 0 && (29) \\ f_{xx}(x^*, \mu^*) &\neq 0 && (30) \end{aligned}$$

then $\dot{x} = f(x,\mu)$ *has a (2 branch) transcritical bifurcation at* (μ^*, x^*).

Figure 1: Generic transcritical bifurcation showing that $f(x, \mu) = 0$ does not imply $\mu = \mu(x)$ or $x = x(\mu)$.

Example: Let

$$\dot{x} = f(x, \mu) = \mu ln(x) + x - 1$$

The first three conditions of the Theorem are:

$$f = \mu ln(x) + x - 1 = 0 \tag{31}$$

$$f_x = \frac{\mu}{x} + 1 = 0 \tag{32}$$

$$f_\mu = ln(x) = 0 \tag{33}$$

Solving (32)-(33) one finds $(\mu^*, x^*) = (-1, 1)$ *as a candidate for a (TC). It is easily verified that (31) is also satisfied and that*

$$\begin{aligned} f_{x\mu} &= \frac{1}{x} \neq 0 \\ f_{xx} &= -\frac{\mu}{x^2} \neq 0 \end{aligned}$$

at $(\mu^*, x^*) = (-1, 1)$. *By the Theorem one concludes there is a (2-branch) transcritical bifurcation point at* $(\mu^*, x^*) = (-1, 1)$.

Much like saddle node bifurcations, such Theorems lead to normal forms for (TC) bifurcations. For example, if (μ^*, x^*) is a pair which satisfies the hypotheses of the Theorem then

$$\dot{x} = \frac{1}{2} f_{\mu\mu}(\mu^*, x^*)(\mu - \mu^*)^2 + f_{x\mu}(\mu^*, x^*)(x - x^*)(\mu - \mu^*) + \frac{1}{2} f_{xx}(\mu^*, x^*)(x - x^*)^2 + O(3)$$

Making the definitions

$$y = x - x^* \quad , \quad \eta = \mu - \mu^*$$

one has the normal form

$$\dot{y} = a\eta^2 + b\eta y + cy^2 + O(3)$$

where a, b, c are constants and $bc \neq 0$.

Example: For the previous example

$$\dot{x} = f(x, \mu) = \mu ln(x) + x - 1$$

where $(\mu^*, x^*) = (-1, 1)$ *is a (TC) bifurcation point,*

$$\dot{x} = (x - 1)(\mu + 1) + \frac{1}{2}(x - 1)^2 + O(3)$$

leading to the normal form

$$\dot{y} = y\left(\eta + \frac{1}{2}y\right) + O(3)$$