CHAPTER 5

Continuity & Momentum Equations

In this chapter we derive the continuity equation and the momentum equations for a fluid using the macroscopic continuum approach.

Continuity Equation: Consider an Eulerian volume V. The mass inside V is given by $M = \int \rho \mathrm{d}V$. The rate at which this mass increases has to be balanced by the rate at which mass flows in or out of V. This implies that

$$\frac{\partial}{\partial t}\int_V \rho \, \mathrm{d}V = -\int_S \rho\vec{u}\cdot \mathrm{d}\vec{S} = -\int_V \nabla\cdot(\rho\vec{u})\,\mathrm{d}V$$

where the last equality follows from Gauss' divergence theorem, and the minus sign is required becuase $\mathrm{d}\vec{S}$ is directed outwards. Since this has to hold for any volume V, we obtain the continuity equation:

	Vector Notation	Index Notation
Lagrangian:	$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\nabla\cdot\vec{u} = 0$	$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\frac{\partial u_i}{\partial x_i} = 0$
Eulerian:	$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$	$\frac{\partial\rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$

NOTE: A liquid is (to good approximation) an incompressible fluid which means that $\nabla\rho = 0$ and that $\partial\rho/\partial t = 0$. The continuity equation then shows that $\nabla\cdot\vec{u} = 0$ (i.e., the flow is divergence free, and thus solenoidal) and that $\mathrm{d}\rho/\mathrm{d}t = 0$ (i.e, the density of each fluid element is conserved over time. It is important to distinguish an incompressible fluid from an incompressible flow, which is defined by $\mathrm{d}\rho/\mathrm{d}t = 0$. The continuity equation shows that an incompressible flow is also divergence free (i.e,, has $\nabla\cdot\vec{u} = 0$). However, it is not necessarily true that $\nabla\rho = 0$ or that $\partial\rho/\partial t = 0$. Hence, a compressible fluid can undergo incompressible flow, but not vice versa.

Momentum Equations: In order to derive the momentum equations, we start with Newton's second law of motion

$$\vec{F} = m\,\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

where $\vec{p} = m\,\vec{v}$. Consider a fluid element $\delta V = \int \mathrm{d}V$ in an external force field $\vec{F}_{\mathrm{ext}}$. In most astrophysical cases of interest to us, this external force will be gravity, which is conservative, so that we can write $\vec{F}_{\mathrm{ext}} = -m\,\nabla\Phi$, with Φ the Newtonian gravitational potential. The momentum of our fluid element is given by $\vec{p} = \int \rho\,\vec{u}\,\mathrm{d}V$. We thus can write Newton's second law of motion as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\int \rho\vec{u}\mathrm{d}V\right] = \int \vec{F}'\,\mathrm{d}V$$

where $\vec{F}'$ is the total force (including the external one) per unit volume acting on our fluid element.

Let us first consider the term on the left. Note that you may not take the derivative inside of the integral! After all, V is the Lagrangian volume of our fluid element, and is thus a function of time. Instead, we make the assumption that the fluid element is sufficiently small that we may neglect changes in $\rho\vec{u}$ across its volume, so that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\int \rho\vec{u}\mathrm{d}V\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\,\vec{u}\,\delta V\right) = \rho\,\delta V\,\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}$$

where the second equality follows from the fact that $\mathrm{d}(\rho\delta V)/\mathrm{d}t = 0$ (i.e., the mass of our fluid element is conserved).

Next we work out the $\int \vec{F}'\mathrm{d}V$ term. The total force acting on our fluid element consists of two components. The external force $\vec{F}_{\mathrm{ext}}$ and a surface force due to the fluid's pressure. In the case of gravity, we have that $\vec{F}'_{\mathrm{ext}} = -\rho\,\nabla\Phi$. For the contribution due to the fluid's pressure, we assume for now that the pressure is isotropic (in the next chapter we will relax this assumption).

The pressure force acting on an infinitessimal surface element of our fluid element is $-P\mathrm{d}\vec{S}$, where the minus sign arises because $\mathrm{d}\vec{S}$ is directed outwards and $\vec{F}$ is the force acting on our fluid element. Hence, the total pressure force acting on our fluid element in (cartesian) direction $\hat{n}$ is

$$\vec{F} \cdot \hat{n} = -\int_S P\,\hat{n} \cdot \vec{S} == \int_V \nabla \cdot (P\,\hat{n})\,\mathrm{d}V$$

where we have used Gauss' divergence theorem. Hence, per unit volume we have that

$$\vec{F}' \cdot \hat{n} = -\nabla \cdot (P\,\hat{n}) = -P\,\nabla \cdot \hat{n} - \nabla P \cdot \hat{n} = -\nabla P \cdot \hat{n}$$

where the last equality follows from the fact that $\hat{n}$ is a unit vector in a constant direction. Combining all the above, and using that $\int \vec{F}'\mathrm{d}V \approx \vec{F}'\delta V$ we obtain that

$$\rho \delta V \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} \cdot \hat{n} = -\rho \nabla \Phi \cdot \hat{n} \delta V - \nabla P \cdot \hat{n} \delta V$$

Since this must be true for any volume element δV, and along any direction $\hat{n}$, we have that

	Vector Notation	**Index Notation**
Lagrangian:	$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} - \nabla \Phi$	$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$
Eulerian:	$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi$	$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$

The continuity and momentum equations derived above (sometimes in combination with the energy equation to be derived in Chapter 14) are called the **Euler Equations**. As we will see, they describe a fluid in which viscosity can be ignored (called an **inviscid fluid**)..