Lecture 10
FIR filter design

- linear phase filter design
- magnitude filter design
- equalizer design

Finite impulse response (FIR) filter

$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau)$$

- $u : \mathbf{Z} \to \mathbf{R}$ is input signal; $y : \mathbf{Z} \to \mathbf{R}$ is output signal
- $h_i \in \mathbf{R}$ are filter coefficients; n is filter order or length

frequency response: a function $H : \mathbf{R} \to \mathbf{C}$ defined as

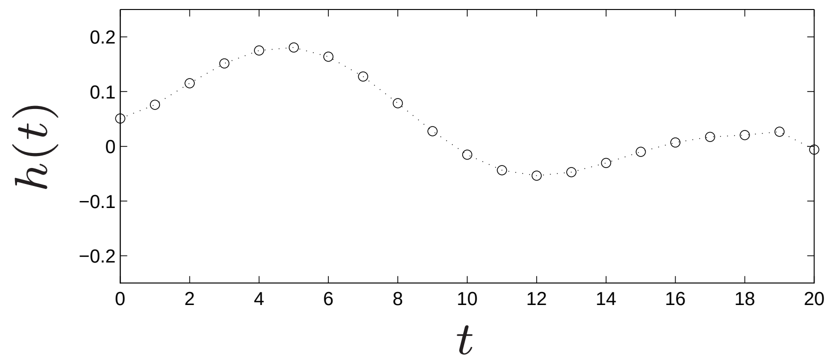
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-\mathrm{j}\omega} + \cdots + h_{n-1} e^{-\mathrm{j}(n-1)\omega} \quad \text{(with } \mathrm{j} = \sqrt{-1}\text{)} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{j} \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

periodic and conjugate symmetric; we only need to consider $\omega \in [0, \pi]$

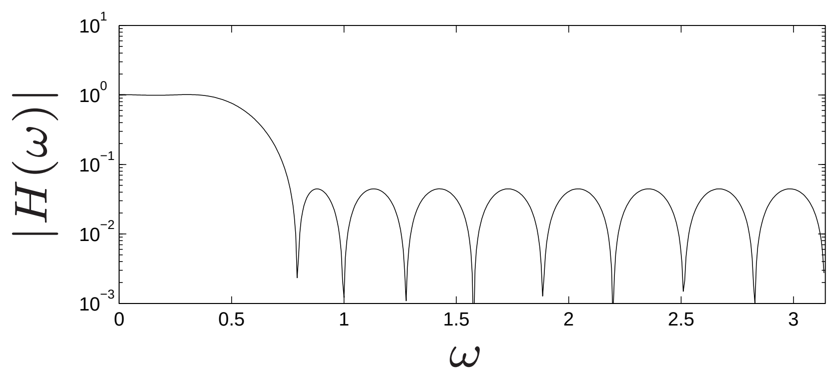
design problem: choose h_i so that H satisfies/optimizes specifications

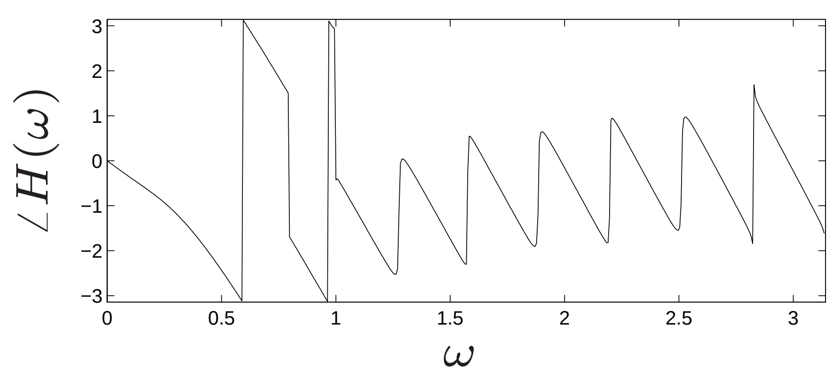
Example: lowpass FIR filter

impulse response (order $n = 21$)



frequency response: magnitude $|H(\omega)|$ and phase $\angle H(\omega)$





Linear-phase filters

suppose $n = 2N + 1$ is odd and impulse response is symmetric about h_N:

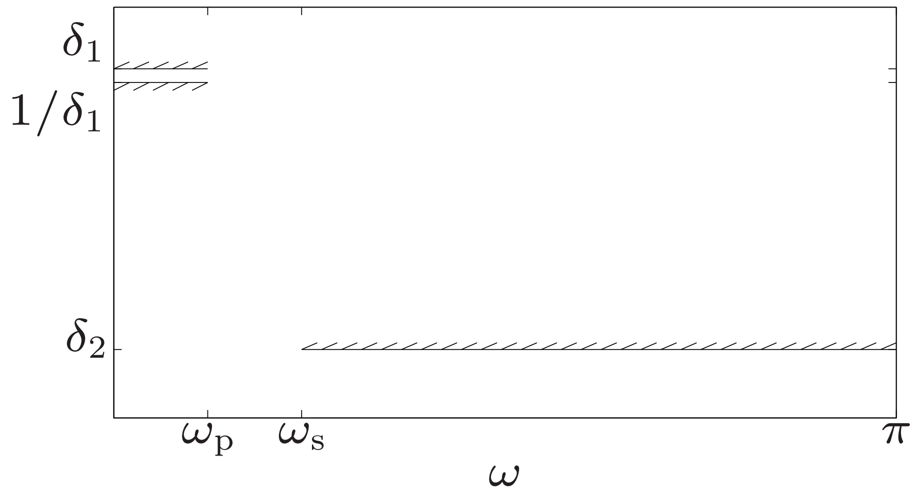
$$h_t = h_{n-1-t}, \qquad t = 0, \ldots, n-1$$

frequency response

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-\mathrm{j}\omega} + \cdots + h_{n-1} e^{-\mathrm{j}(n-1)\omega} \\ &= e^{-\mathrm{j}N\omega} \left(2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N\right) \\ &= e^{-\mathrm{j}N\omega} G(\omega) \end{aligned}$$

- term $e^{-\mathrm{j}N\omega}$ represents N-sample delay
- $G(\omega)$ is real-valued and $|H(\omega)| = |G(\omega)|$
- 'linear phase': $\angle H(\omega)$ is linear except for jumps of $\pm\pi$

Lowpass filter specifications



- maximum passband ripple ($\pm 20 \log_{10} \delta_1$ in dB):

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1 \quad \text{for } \omega \in [0, \omega_{\mathrm{p}}]$$

- minimum stopband attenuation ($-20 \log_{10} \delta_2$ in dB):

$$|H(\omega)| \leq \delta_2 \quad \text{for } \omega \in [\omega_{\mathrm{s}}, \pi]$$

Linear-phase lowpass filter design

- sample the frequency axis: $\omega_k = k\pi/K$, $k = 0, \ldots, K-1$
- assume without loss of generality that $G(0) > 0$, so ripple spec. is

$$1/\delta_1 \leq G(\omega_k) \leq \delta_1$$

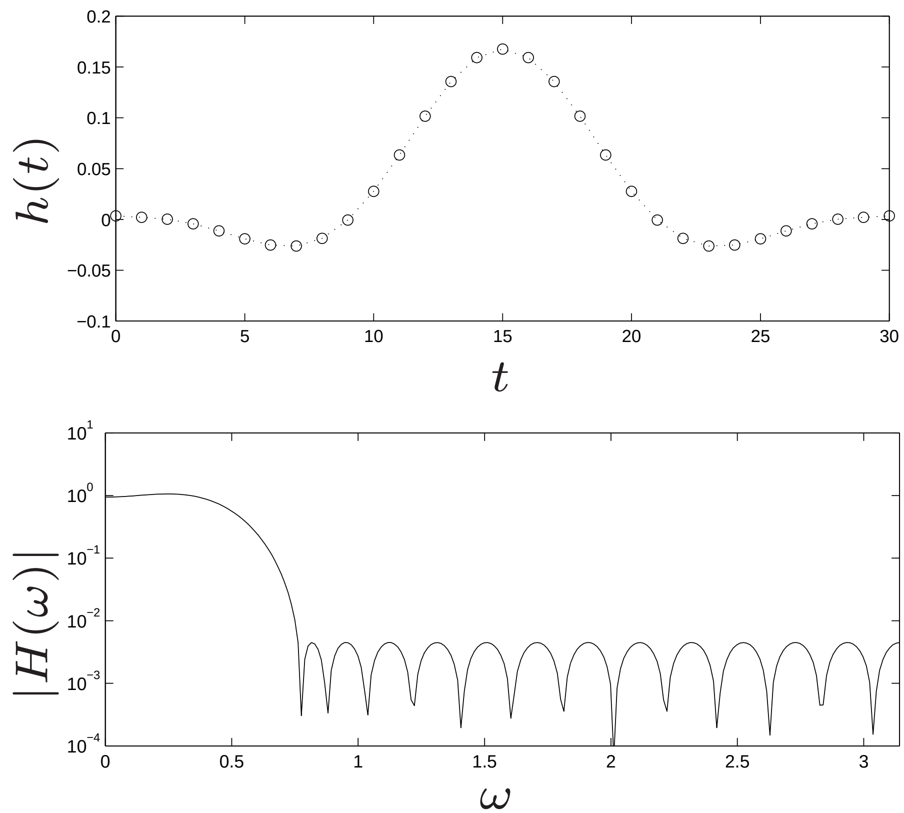
maximum stopband attenuation (for given passband ripple δ_1)

$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \leq G(\omega_k) \leq \delta_1 \quad \text{for } \omega_k \in [0, \omega_{\rm p}] \\ & -\delta_2 \leq G(\omega_k) \leq \delta_2 \quad \text{for } \omega_k \in [\omega_{\rm s}, \pi] \end{array}$$

- a linear program in variables h_i, δ_2
- known and used since 1960's
- can add other constraints, *e.g.*, $|h_i| \leq \alpha$

Example

- linear-phase filter of order $n = 31$
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- maximum ripple $\delta_1 = 1.059$ (± 0.5dB)



Variations

minimize passband ripple (variables δ_1, h)

$$\begin{array}{ll} \text{minimize} & \delta_1 \\ \text{subject to} & 1/\delta_1 \leq G(\omega_k) \leq \delta_1 \quad \text{for } \omega_k \in [0, \omega_\mathrm{p}] \\ & -\delta_2 \leq G(\omega_k) \leq \delta_2 \quad \text{for } \omega_k \in [\omega_\mathrm{s}, \pi] \end{array}$$

minimize transition bandwidth (variables ω_s, h)

$$\begin{array}{ll} \text{minimize} & \omega_\mathrm{s} \\ \text{subject to} & 1/\delta_1 \leq G(\omega_k) \leq \delta_1 \quad \text{for } \omega_k \in [0, \omega_\mathrm{p}] \\ & -\delta_2 \leq G(\omega_k) \leq \delta_2 \quad \text{for } \omega_k \in [\omega_\mathrm{s}, \pi] \end{array}$$

minimize filter order (variables N, h)

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & 1/\delta_1 \leq G(\omega_k) \leq \delta_1 \quad \text{for } \omega_k \in [0, \omega_\mathrm{p}] \\ & -\delta_2 \leq G(\omega_k) \leq \delta_2 \quad \text{for } \omega_k \in [\omega_\mathrm{s}, \pi] \end{array}$$

not LPs, but can be solved by bisection/LP feasibility problems

Outline

- linear phase filter design
- **magnitude filter design**
- equalizer design

Filter magnitude specifications

magnitude specification: a constraint

$$L(\omega) \le |H(\omega)| \le U(\omega) \qquad \forall\omega$$

$L, U : \mathbf{R} \to \mathbf{R}_+$ are given and

$$H(\omega) = \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{j} \sum_{t=0}^{n-1} h_t \sin t\omega$$

- arises in many applications, *e.g.*, audio, spectrum shaping
- not equivalent to linear inequalities in h_i (linear inequalities can not express the lower bound on absolute value)
- can change variables and convert to set of linear inequalities

Autocorrelation coefficients

definition: autocorrelation coefficients of $h = (h_0, \ldots, h_{n-1}) \in \mathbf{R}^n$

$$r_t = \sum_{\tau=0}^{n-1-t} h_\tau h_{\tau+t} \quad \text{(with } h_k = 0 \text{ for } k < 0 \text{ or } k \geq n\text{)}$$

$r_t = r_{-t}$ and $r_t = 0$ for $|t| \geq n$; hence suffices to specify $r = (r_0, \ldots, r_{n-1})$

Fourier transform of autocorrelation coefficients:

$$R(\omega) = \sum_\tau e^{-\mathrm{j}\omega\tau} r_\tau = r_0 + \sum_{t=1}^{n-1} 2r_t \cos \omega t = |H(\omega)|^2$$

magnitude specifications are *linear inequalities* in coefficients r_t:

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2 \quad \text{for } \omega \in [0, \pi]$$

Spectral factorization

when is $r \in \mathbf{R}^n$ the vector of autocorrelation coefficients of some $h \in \mathbf{R}^n$?

spectral factorization theorem: if and only if $R(\omega) \geq 0$ for all ω

- condition is an infinite set of linear inequalities in r
- many algorithms for spectral factorization (find h s.t. $R(\omega) = |H(\omega)|^2$)

consequence: to cast magnitude design problem as an LP,

- use $r = (r_0, \ldots, r_{n-1})$ as variable instead of $h = (h_0, \ldots, h_{n-1})$
- add spectral factorization condition as constraint: $R(\omega) \geq 0$ for all ω
- discretize the frequency axis
- optimize over r and use spectral factorization to recover h

Magnitude lowpass filter design

maximum stopband attenuation design (with variables r)

$$\begin{array}{ll} \text{minimize} & \gamma_2 \\ \text{subject to} & 1/\gamma_1 \leq R(\omega) \leq \gamma_1 \quad \text{for } \omega \in [0, \omega_\mathrm{p}] \\ & R(\omega) \leq \gamma_2 \quad \text{for } \omega \in [\omega_\mathrm{s}, \pi] \\ & R(\omega) \geq 0 \quad \text{for } \omega \in [0, \pi] \end{array}$$

(γ_i corresponds to δ_i^2 in original problem)

discretization: impose constraints at finite set of frequencies ω_k

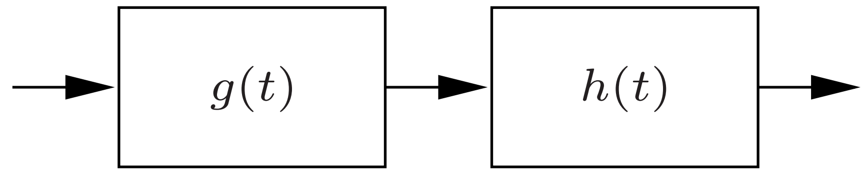
$$\begin{array}{ll} \text{minimize} & \gamma_2 \\ \text{subject to} & 1/\gamma_1 \leq R(\omega_k) \leq \gamma_1 \quad \text{for } \omega_k \in [0, \omega_\mathrm{p}] \\ & R(\omega_k) \leq \gamma_2 \quad \text{for } \omega_k \in [\omega_\mathrm{s}, \pi] \\ & R(\omega_k) \geq 0 \quad \text{for } \omega_k \in [0, \pi] \end{array}$$

this is a linear program in r, γ_2

Outline

- linear phase filter design
- magnitude filter design
- **equalizer design**

Equalizer design



(time-domain) equalization

- given g (unequalized impulse response), g_{des} (desired impulse response)
- design FIR equalizer h so that convolution $\tilde{g} = h * g$ approximates g_{des}

example

- g_{des} is pure delay D: $g_{\mathrm{des}}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$
- find equalizer h by solving

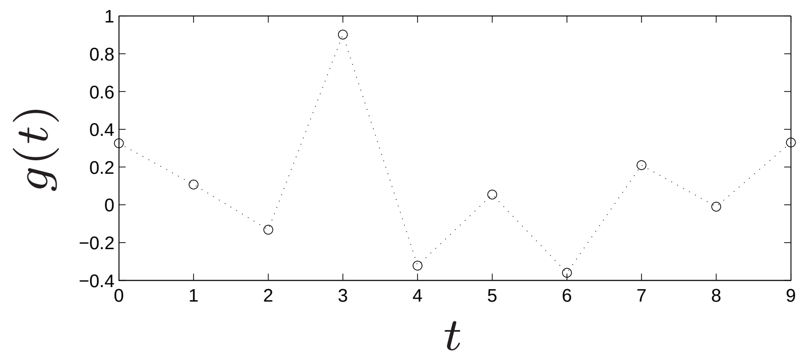
$$\begin{array}{ll} \text{minimize} & \max_{t \neq D} |\tilde{g}(t)| \\ \text{subject to} & \tilde{g}(D) = 1 \end{array}$$

this can be cast as an LP in the coefficients h_i

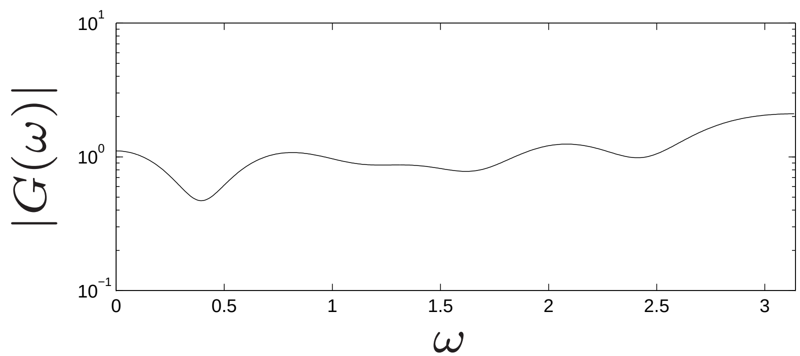
Example

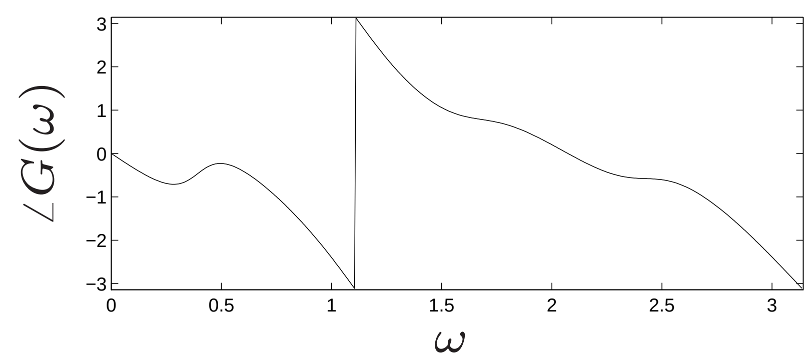
unequalized system (10th order FIR)

- impulse response



- frequency response magnitude

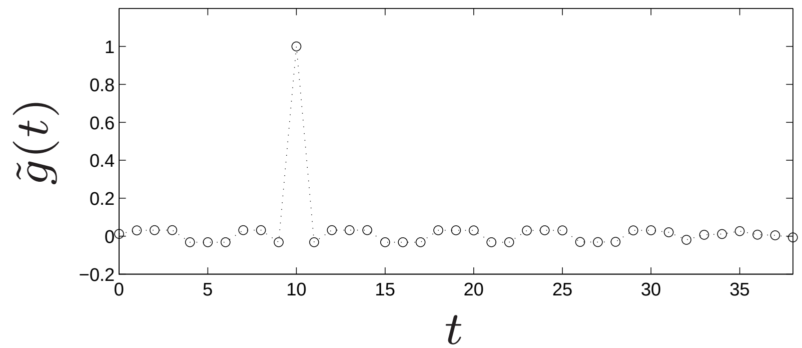




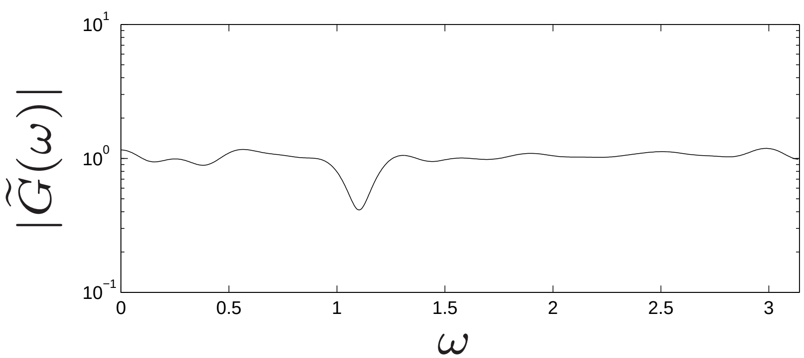
time-domain equalization (30th order FIR, $D = 10$)

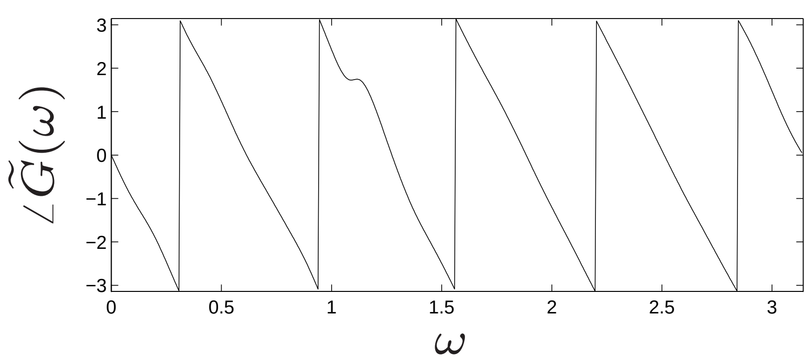
$$\text{minimize} \quad \max_{t \neq 10} |\tilde{g}(t)|$$

- equalized system impulse response

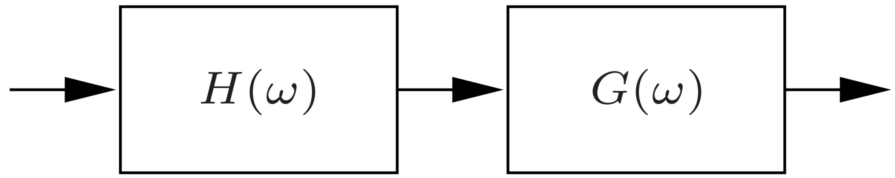


- equalized frequency response





Magnitude equalizer design



problem

- given system frequency response $G : [0, \pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G(\omega)H(\omega)| \approx 1$:

$$\text{minimize} \quad \max_{\omega \in [0,\pi]} \left| |G(\omega)H(\omega)|^2 - 1 \right|$$

LP formulation: use autocorrelation coefficients as variables

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| |G(\omega)|^2 R(\omega) - 1 \right| \leq \alpha \quad \text{for } \omega \in [0, \pi] \\ & R(\omega) \geq 0 \quad \text{for } \omega \in [0, \pi] \end{array}$$

after discretizing the frequency axis, we obtain an LP in r and α

Multi-system magnitude equalization

problem

- we are given M frequency responses $G_k : [0, \pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G_k(\omega)H(\omega)| \approx$ constant:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{k=1,\ldots,M} \max_{\omega\in[0,\pi]} \left| |G_k(\omega)H(\omega)|^2 - \gamma_k \right| \\ \text{subject to} & \gamma_k \geq 1, \quad k = 1, \ldots, M \end{array}$$

LP formulation: use autocorrelation coefficients as variables

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| |G_k(\omega)|^2 R(\omega) - \gamma_k \right| \leq \alpha \quad \text{for } \omega \in [0, \pi], \quad k = 1, \ldots, M \\ & R(\omega) \geq 0 \quad \text{for } \omega \in [0, \pi] \\ & \gamma_k \geq 1, \quad k = 1, \ldots, M \end{array}$$

after discretizing the frequency axis, we obtain an LP in γ_k, r, α

Example

- $M = 2$ systems, equalizer of order $n = 25$

- unequalized and equalized frequency responses