Modeling Potential Flow Using Laurent Series Expansions and Boundary Elements

T. R. Dean,[1] **T. V. Hromadka II,**[2] **Thomas Kastner,**[2] **Michael Phillips**[2]
[1] *Department of Electrical Engineering and Computer Science, United States Military Academy, West Point, New York 10996*

[2] *Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996*

Received 11 January 2010; accepted 10 July 2010
Published online 27 October 2010 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.20636

The fundamental underpinnings of the well-known Bernoulli's Equation, as used to describe steady state two-dimensional flow of an ideal incompressible irrotational flow, are typically described in terms of partial differential equations. However, it has been shown that the Cauchy Integral Theorem of standard complex variables also explain the Bernoulli's Equation and, hence, can be directly used to model problems of ideal fluid flow (or other potential problems such as electrostatics among other topics) using approximation function techniques such as the complex variable boundary element method (CVBEM). In this article, the CVBEM is extended to include Laurent Series expansions about singular points located outside of the problem domain union boundary. It is shown that by including such negatively powered complex monomials in the CVBEM formulation, considerable power is introduced to model potential flow problems. Published 2010 Wiley Periodicals, Inc.* Numer Methods Partial Differential Eq 28: 573–586, 2012

Keywords: complex variables; CVBEM, complex variable boundary elements; Laurent Series; potential flow

I. INTRODUCTION

The linkage between complex variables and potential flow theory is described in Simmonds [1] among other references and provides another basis for using the Cauchy Integral of complex variables theory to describe potential flow such as the analysis of two-dimensional ideal flow of incompressible irrotational fluid. The complex variable boundary element method or CVBEM is an approximation of the Cauchy Integral where spline basis functions are used to describe a global trial function which, in turn, is used to approximate the boundary values of the potential problem. The global trial function is then used as the approximation of the potential function evaluated on the problem boundary, which becomes the integrand in the Cauchy Integral. Because the global trial function is continuous on the problem boundary, the resulting Cauchy Integral is analytic

Correspondence to: T. V. Hromadka II, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996 (e-mail: tedhromadka@yahoo.com)
Additional Supporting Information may be found in the online version of this article.

throughout the problem domain, guaranteeing an approximation function that exactly satisfies the Laplace equation throughout the problem domain, which is a property not achieved by the widely used numerical techniques such as finite difference and finite element methods. Also, from the Maximum Modulus theorem of complex variables, the maximum value of the magnitude of the analytic function occurs on the problem boundary, which means the maximum error between the CVBEM approximation and the exact solution to the boundary value problem must occur on the problem boundary, and therefore, the magnitude of the approximation error can be determined, which is another property not afforded by the usual domain numerical schemes of finite difference and finite element methods. With the availability of mathematical computer programs such as Mathematica, among others, the CVBEM can be readily implemented and computer graphics utilized that provide remarkable insight into approximation of potential problems such as ideal fluid flow.

Other numerical approximation techniques exist that are similar to the CVBEM. For example, the method of fundamental solutions utilizes a linear combination of a set of particular solutions (i.e., "fundamental solutions") to the subject linear partial differential equation and then evaluates the multiplicative coefficients by matching problem boundary conditions. The work of Obrist et al. [2] is an example of this modeling technique and considers applications to the LaPlace equation. The choice of fundamental solutions is open to the modeler. However, in the CVBEM and the current Laurent Series extension, the basis functions used are of a particular type and are developed from numerical integration of the Cauchy integral equation. Another similar numerical technique is the method of complex panels (or the Vortex Panel technique, such as presented in Ref. [3] and has found use in the modeling aerodynamics and flow field phenomena). An applet for using the Vortex Panel method is found at http://www.engapplets.vt.edu/fluids/vpm/vpminfo.html. Again, the CVBEM develops the numerical approximation by use of a numerical solution of the Cauchy integral equation and as presented in this work, by inclusion of additional complex variable basis functions such as developed in a Laurent series expansion.

Background into the CVBEM and related topics is provided in Hromadka and Whitley [4] and in three dimensions in Hromadka [5], among other references. A recent review of the CVBEM is found in the special issue of Engineering Analysis with Boundary Elements (see volume 30 (2006), issue 12). Recently, program Mathematica has been used to model potential flow problems using the CVBEM [6] or its variant, the complex polynomial method [7]. In the development of the CVBEM to date, little if any attention has been paid toward use of basis functions of the type found in the expansion of complex variable Laurent Series, involving negative powers of complex monomials. In this work, the CVBEM is used as the foundation for including negative powers of complex monomials as additional basis functions, with points of expansion located such as to optimize the reduction in modeling error in fitting problem boundary conditions. This is accomplished by examining a set of possible Laurent Series poles and assessing the goodness of fit in using each expansion point, one by one, in fitting the problem boundary conditions. It is found that this new approach to modeling potential problems provides considerable strength in modeling potential problems such as those occurring in ideal fluid flow.

II. CVBEM FORMULATION AND CPM FORMULATION

The complex variable boundary element method (or CVBEM) results in a series expansion of the form:

$$\hat{\omega}(z) = \sum_{j=1}^{n} (z - z_j)\mathrm{Ln}(z - z_j) \tag{1}$$

where points z_j are nodal points where complex logarithms are centered with branch-cuts oriented to lie exterior to the study region.

The complex polynomial method (e.g., Bohannon and Hromadka [7]) is a series of complex monomials of the form:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j z^j \tag{2}$$

In both of these complex variable approximation methods, collocation may be used to determine the complex coefficients used in the respective coefficients, or a least-squares best fit to the boundary condition values can be used in a Hilbert Space setting. Details of these methods and computer programs to implement either of these modeling approaches can be found in the cited literature.

It is noted that in the CPM, the resulting approximation function is entire (i.e., is analytic throughout the entire complex plane), whereas the CVBEM is not analytic along branch points and corresponding branch-cuts of the complex logarithm function (therefore, branch-cuts are oriented to lie outside of the study region). It is also noted that the CVBEM involves use of a set of nodal points where complex logarithms are evaluated and that these point locations are subject to optimization such as discussed in Dean and Hromadka [6].

III. LAURENT SERIES FORMULATION

The complex variable boundary element method uses basis functions of the form of complex polynomials (to the degree of the trial function used to interpolate boundary values in the global trial function) and also products of complex polynomials with complex logarithms (e.g., Hromadka and Whitley [4]). The complex logarithms are manipulated so that their respective branch-cuts lie outside of the problem domain. The complex polynomial method (e.g., Bohannon and Hromadka [7]) uses complex monomials as the basis functions. In this work, basis functions as obtained from a partial sum of the Laurent series expansion are included with the basis functions used in either of the above two complex variable approximation methods.

In a Laurent series expansion, complex variable monomials are summed including both positive and negative powers, expanded about a particular expansion point. The typical expansion about point z_0 is

$$\omega(z) = \sum_{j=-m}^{n} c_j (z - z_0)^j \tag{3}$$

In the above Eq. (3), a partial sum of the Laurent series is shown where n is the number of positive powered complex monomials and m is the number of negative powered complex monomials. It is noted that although the above equation is expanded about point z_0, expansion about other points follows the same type of expansion relationship as seen for the expansion about point z_0.

The theory of Laurent series, including regions of convergence and analyticity, can be found in numerous texts including Mathews and Howell [8], among others. The series expansion involves both positive and negative powers of complex monomials all expanded about the point z_0. From the series expansion, it is noted that point z_0 is a singularity. However, all of these terms from the series expansion are analytic in the study region if point z_0 lies exterior of the study region. Furthermore, several expansion points can be used in an approximation effort, resulting in several partial sums of different Laurent series expansions. For the case of several expansion points, the ensemble of negatively powered complex monomials are all linearly independent functions, and, therefore, can be included in a basis set of functions in the approximation effort.

From Eq. (3), the Laurent series expansion (about arbitrary expansion point z_0) involves complex monomials with negative and positive powers. If only the positive powers are used, then the resulting approximation function is a complex polynomial (as expansions about several points z_j are not linearly independent) and the complex polynomial method (e.g., Bohannon [7]) results. The opportunity to use several expansion points occurs when the negatively powered complex monomials are used in the approximation function. Consequently, for several expansion points, the ensemble of negatively powered complex monomials are all linearly independent and form a larger dimension basis where linear combinations of these basis functions are analytic throughout the entire complex plane less the set of expansion points (which is why locations of such expansion points should generally be exterior of the study region). In comparison, the CVBEM expansion of Eq. (1) involves a different family of basis functions that are analytic throughout the entire complex plane less the set of branch points and corresponding branch cuts (which are all oriented to not intersect with each other and all lie outside of the study region). The CVBEM can be implemented using not only the basis functions shown in Eq. (1) but also including the positively and negatively powered complex monomials of the Laurent series. Of course, only the negatively powered monomials form a linearly independent set of functions. To isolate out the approximation function strength of the negatively powered complex monomials in the Laurent series expansion, the remainder of this articles deals with development of approximation functions using only the negatively powered complex monomials. It is shown that this specialized basis function set provides considerable computational power in approximating difficult potential problems. Therefore, a CVBEM based upon using the basis functions from both families shown in Eqs. (2) and (3) provides even higher approximation function accuracy and capability.

In this work, three new concepts are introduced in the modeling effort:

a. Consideration of the number of terms used in the partial sum of a Laurent series expansions (for example, assessment of approximation improvement by including the Laurent series terms for a larger partial sum);
b. the use of several Laurent series expansion by using several expansion points (or poles) simultaneously;
c. optimization of the location of Laurent series expansion points.

The first new concept (a) is handled by evaluating the modeling error in matching boundary condition values with respect to using more terms in the Laurent series expansion. For example, a series of modeling approximations is made by using successively more terms of the Laurent series until little improvement is observed (within a tolerance) in matching boundary condition values by the resulting approximation function. That is, a "trade-off" between using more terms from the Laurent series versus improvement in modeling accuracy is assessed. Once a relative optimum number of terms is determined, another pole is then introduced into the approximation function to be similarly dealt with. In this way, more Laurent series poles are introduced until the target error tolerance in matching boundary condition values is achieved. This concept is explored in example problem 1.

The use of multiple Laurent series expansions is advantages for multiple reasons. By including additional Laurent series, the complexity of the model is reduced due to the fact that more terms can be included at a lower order. Additionally, the accuracy of a model can be improved by locating a pole near where a singularity exists in a problem. This concept is demonstrated in example problem 2.

The third new concept (c) is discussed below.

IV. OPTIMIZATION OF LAURENT SERIES EXPANSION POINT (POLE) LOCATIONS

Because Laurent series poles can be established at any location exterior of the study region (to avoid introducing a singularity inside the study region), the choice of where to locate such a pole is an issue that can be handled by assessing modeling error in matching boundary condition values versus test location. Many schemes for assessing such points are possible. The scheme used in this work is to set a grid of test points located outside of the study region, and then to work with each test point, one by one, and determine the respective modeling error in fitting the problem boundary condition values. The final location for the next pole location is that test point that corresponds to the minimum modeling error for the measure chosen (e.g., a maximum error magnitude is used in this work). Because the computational effort involved in optimizing expansion point locations is large, the set up of the grid of test point locations needs to be carefully laid out so as to not arbitrarily use a dense grid of test points.

For example, the procedure used in this article to assess locations for the Laurent series pole locations for the Example Problem 1 (ideal fluid flow in a 90-degree bend around a unit radius cylinder) is to rotate a fixed test point grid template, shown in Figure 5, about the problem domain. Each test point is considered, one at a time, as the test pole location, and the corresponding error in matching boundary conditions is determined. After all the test points are evaluated, the corresponding modeling error in matching boundary conditions can be plotted against distance of the test pole location from the problem domain, as shown in Figure 6. From Figure 6, the best estimate of the pole location can be determined for the given positioning of the template. Then, other template positions can be similarly evaluated. Once the best pole location is determined, that location is held fixed, and then the next pole location is considered by introducing another pole for another Laurent expansion.

V. MATHEMATICA CODE

The above concepts are developed on computer program Mathematica which, like other similar programs, such as Matlab, provides considerable advantages to other computer programming languages (e.g., FORTRAN, C) in handling complex variable and other detailed mathematical manipulations, as well as simplified coding requirements to solve complex mathematical equations as well as provide highly accurate computations and extensive graphical capabilities.

The use of program Mathematica does have disadvantages. A program run in Mathematica will run slower than a similar compiled program written in C or FORTRAN. Additionally, while Mathematica provides many built-in algorithms, the amount of control of the algorithms used and the algorithms themselves is somewhat restricted. The use of Mathematica also reduces the portability of the program, as the program can only be run with a Mathematica license.

Although this article considers Laurent series expansions in general, which can be used to accommodate a variety of boundary conditions such as specified values of the potential or the stream function, or a combinations of both, only the Dirichlet problem (i.e., specified values of the potential function) is considered in this article. From the expansion of the Laurent series shown in Eq. (3), the complex coefficients are each composed of two real numbers by $c_j = a_j + ib_j$ for coefficient j. Therefore, each complex coefficient requires two points of collocation to uniquely determine the corresponding component values of the coefficients. However, as the problem considered is a Dirichlet problem, stream function values are not needed as boundary conditions and so the constant b_0 of $c_0 = a_0 + ib_0$ can be arbitrarily set to zero reducing the collocation point count by one. (It is noted that other means of determining the complex coefficients can be used instead of collocation, such as a least squares approach to minimizing the difference between approximation and specified values along the problem boundary.) This step of setting $b_0 = 0$ is used in the Mathematica code found in Supporting Information Appendix A.

Basis functions are expanded using the ComplexExpand function on Mathematica and separated into their real and imaginary components. From here, the modeling procedure now moves toward solving a matrix system to determine the $2N + 1$ real-valued coefficient components by collocating the approximation function to equal the boundary condition values at each of the $2N + 1$ evaluation points. After solving the square $(2N + 1) \times (2N + 1)$ matrix system, the real-valued coefficients are substituted back into the underlying approximation function, resulting in an analytic function defined over the problem domain union problem boundary. Furthermore, these same $2N + 1$ coefficient real values can be directly used to develop the conjugate stream function, for use in developing flow nets and other graphical plots.

The complete Mathematica code is contained in Supporting Information Appendix A.

VI. CONSIDERATIONS OF CONVERGENCE

In other work (for example, see Whitley and Hromadka (2009)), it is shown that there exists a complex polynomial that can approximate pointwise arbitrarily close to the exact solution of a potential problem. Because the sum of the partial sums of Laurent series expansions can be rewritten as a product of a complex polynomial with a complex rational function (by rewriting the sum of partial sums in terms of having a common denominator), then the approximation effort can be viewed as approximating the product of the reciprocal of the above rational function with the original potential function to be approximated. Therefore, the use of additional Laurent series expansions (by using several expansion points) is analogous to using a larger powered complex polynomial. Of course, an advantage to using more Laurent series expansion points instead of higher powered complex monomials is the bypassing of the limitations of the underlying computer program capability in dealing with high powers of monomials.

VII. EXAMPLE PROBLEMS

A. Example Problem 1: Ideal Fluid Flow

The two-dimensional flow of an ideal irrotational and incompressible fluid in a 90° bend around a unit radius cylinder is a modeling problem that involves considerable complexity. To demonstrate the approximation efficiency of the Laurent series approach, only the negative power terms of complex monomials is used in the example problem. Figure 1 shows the problem domain considered, where Ω is the problem domain and Γ is the problem boundary. In this figure, the unit radius cylinder is seen in the lower left, while the extent of the problem domain is seen as the other straight line segments in the first quadrant. The exact solution to this particular problem is well-known as $\omega(z) = z^2 + z^{-2}$, as described in [1], and, therefore, can be used to provide not only the problem boundary conditions to be used in the modeling application but also for comparing with the modeling results to assess the modeling accuracy.

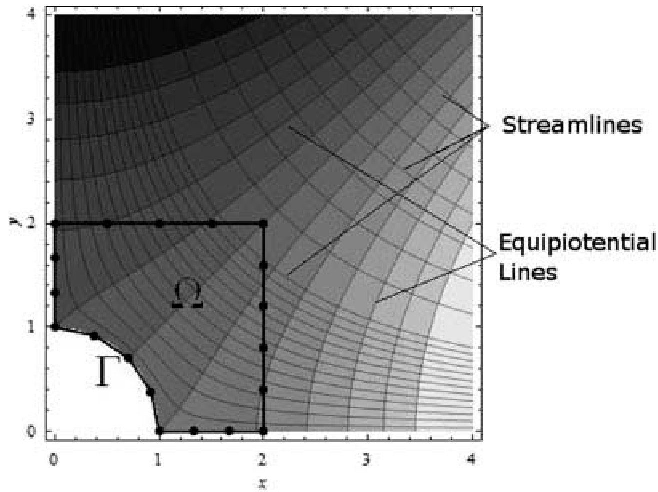

FIG. 1. The exact solution for example problem 1. The flow is modeled by $\omega(z) = z^2 + z^{-2}$. Ω is the problem domain and Γ is the problem boundary. Dots represent the evaluation points.

Figure 1 also shows the combined plots of the exact solution streamlines and equipotentials, forming the flow-net for the example problem over the problem domain considered. Using Fig. 1, values of the potential function (from the exact solution) can be readily determined and then used as the Dirichlet problem boundary conditions at the collocation point locations on the problem boundary. For the example problem, the collocation points are specified along the problem boundary.

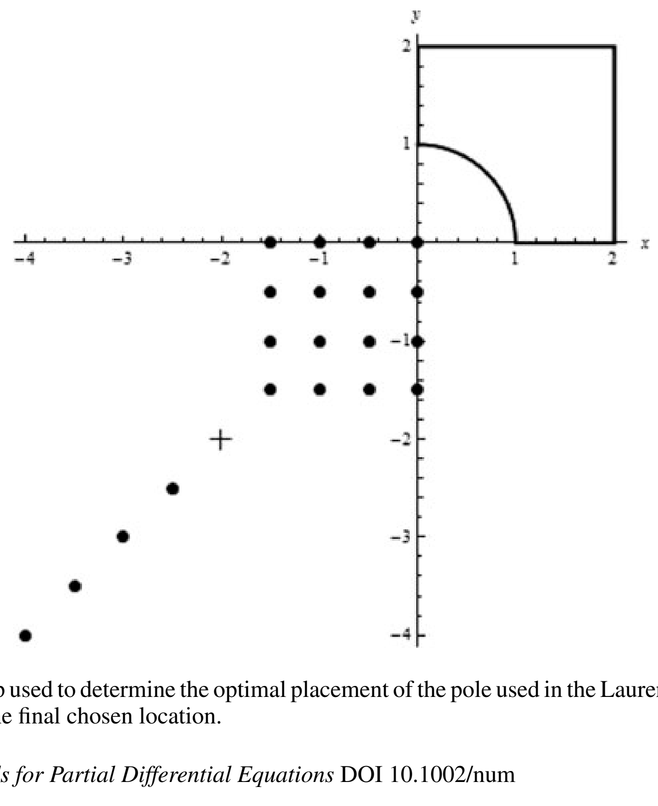

FIG. 2. Test setup used to determine the optimal placement of the pole used in the Laurent series expansion. The cross marks the final chosen location.

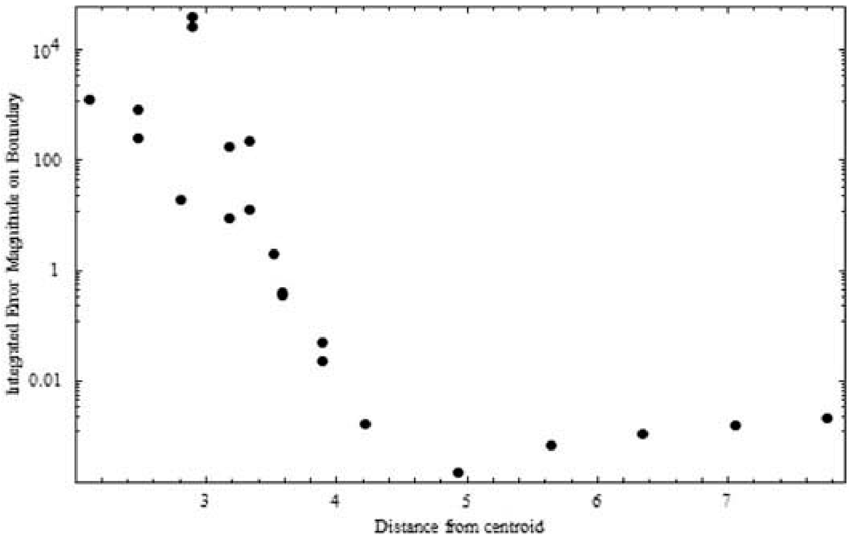

FIG. 3. Resulting squared error on the problem boundary versus distance from the center of the problem domain.

Using the test pole location template of Fig. 2, a partial sum of the first 10 terms of the Laurent series was determined using each test pole location of Fig. 2, and the corresponding error in matching boundary conditions at the collocation points (Fig. 3) determined for assessing the optimum location for a pole. By rotating the test pole template about the problem domain, other assessments can be made.

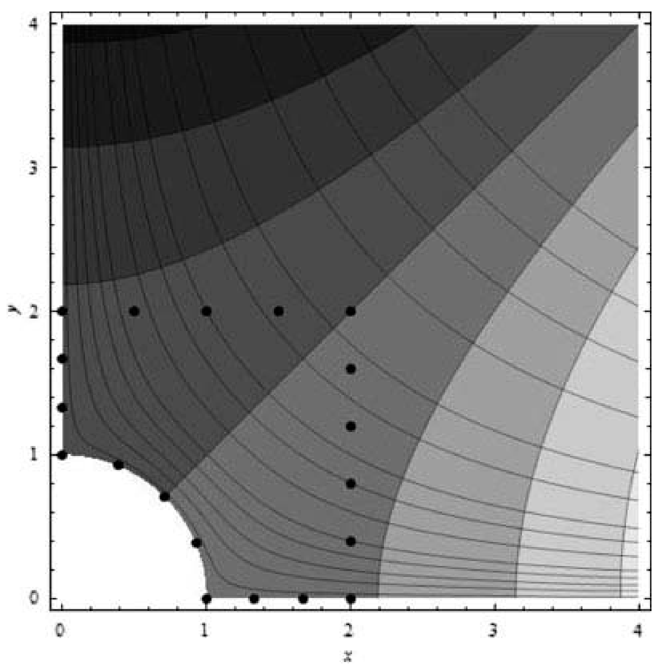

FIG. 4. The approximation solution for example problem 1.

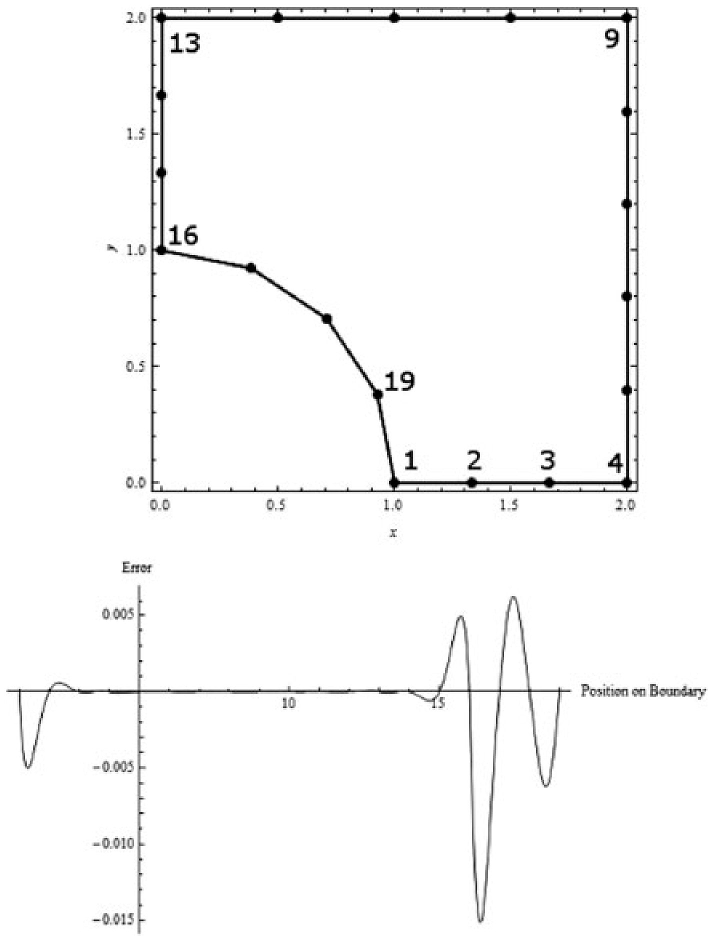

FIG. 5. Error along the problem domain. Note that error is largest closer to the pole.

Using the optimum pole location determined above, the partial sum of the Laurent series expansion is determined, resulting in the approximation function whose corresponding flow-net is shown in Fig. 4. Nine terms were used in this expansion based on the nineteen evaluation points used. The figure shows a flow net that extends beyond the problem domain to show the ability of the model to provide computational results outside of the problem domain union boundary.

Figure 5 provides a plot of the modeling estimates of the potential versus the exact solution potential values along the problem boundary as evaluated at the specified collocation points on the problem boundary. From Fig. 5, it is seen that using just a single pole in the above modeling procedure results in an excellent model of the subject problem.

The effect of the number of terms used in the expansion can be seen in Fig. 6. Because of the collocation technique used to fit the boundary values, the number of evaluations points must be changed along with the number of terms in the Laurent series. Squared error along the boundary consistently decreases as the number of terms increases; however, the computational complexity of the model also rapidly increases.

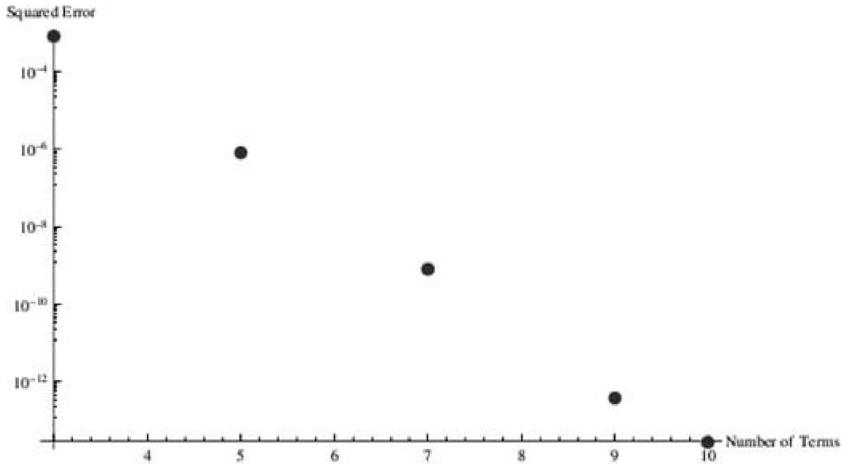

FIG. 6. Squared error along the problem boundary versus the number of terms used in the approximation.

B. Example Problem 2: Potential Flow Between a Source and a Sink

In this example problem, a source and a sink term, both of equal strength, are positioned equidistant from the origin and on the x-axis. See Ref. [4] for an more in-depth treatment of this problem. The problem domain is rectangular in shape, but with the boundary designed to locate the problem's source and sink points to be exterior of the problem domain (see Fig. 7). Because sources and sinks are important components of many potential problems in a variety of situations (e.g., electrostatics, ideal fluid flow, groundwater flow, torsion, among other potential problems), consideration of such a test problem is appropriate to consider. Because both components are of equal strength, the resulting plots of equipotentials and streamlines that form the flow net apply to other equal strength situations. The flow net corresponding to the exact solution as applied to the problem domain is also shown in the figure. Additionally, the exact solution to this problem is readily determined to be $\omega(z) = \mathrm{Ln}(z+1) - \mathrm{Ln}(z-1)$, and therefore, can be used to assess the goodness

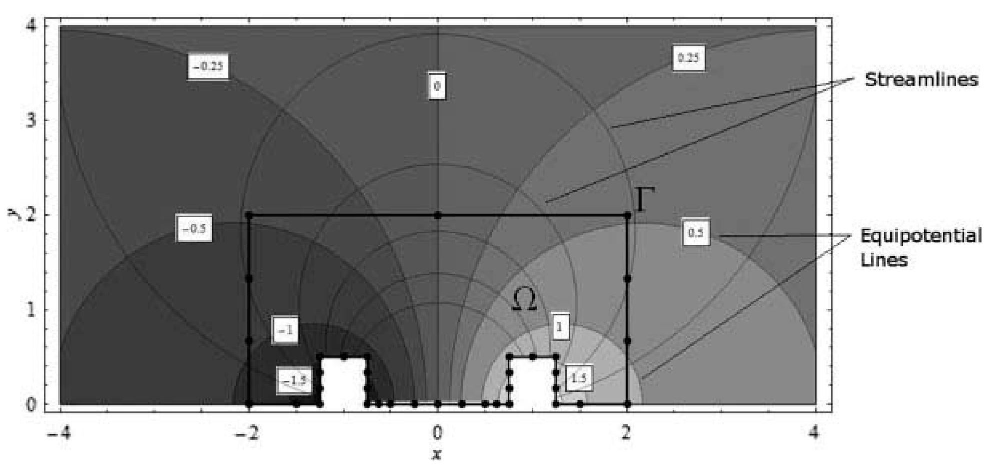

FIG. 7. The exact solution for example problem 2: potential flow between a source and a sink. Ω is the problem domain and Γ is the problem boundary. Dots represent the evaluation points.

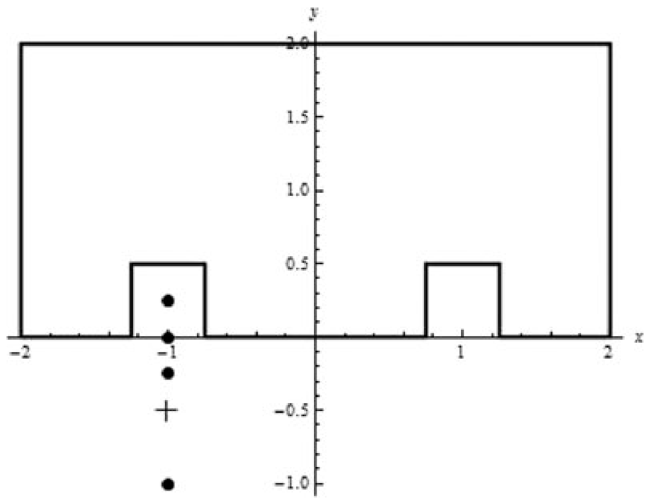

FIG. 8. Laurent series expansion test pole locations near source and sink.

of the Laurent series approximation within the problem domain. Using the exact solution, as done in the first example problem, boundary condition values of the potential function (for the considered Dirichlet problem) are determined from the real part of the exact solution. By specifying the boundary condition values at the collocations points set on the problem boundary (shown on boundary of Fig. 7), the Laurent series approximation method can proceed, by optimizing the location of the first introduced expansion point (or pole) as discussed previously. Holding the first pole location fixed, a second pole was introduced and its location optimized. The approximation error in matching the boundary condition values was found to be quite small with only the first two poles in the approximation, and no further poles were introduced for the purposes of this example problem. As before, the partial sum of the first 8 terms of the respective Laurent series expansions was used. The optimized locations of the two poles were determined to be immediately below the locations of the problem's source and sink locations (see Fig. 8). The flow net corresponding to the

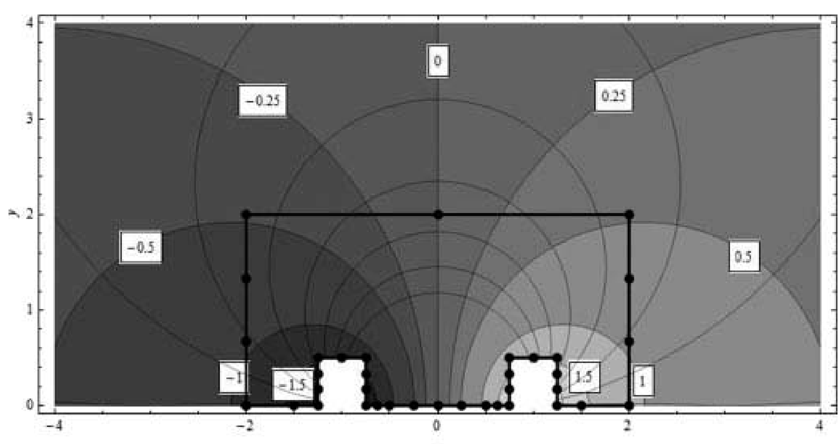

FIG. 9. The approximation solution for example problem 2.

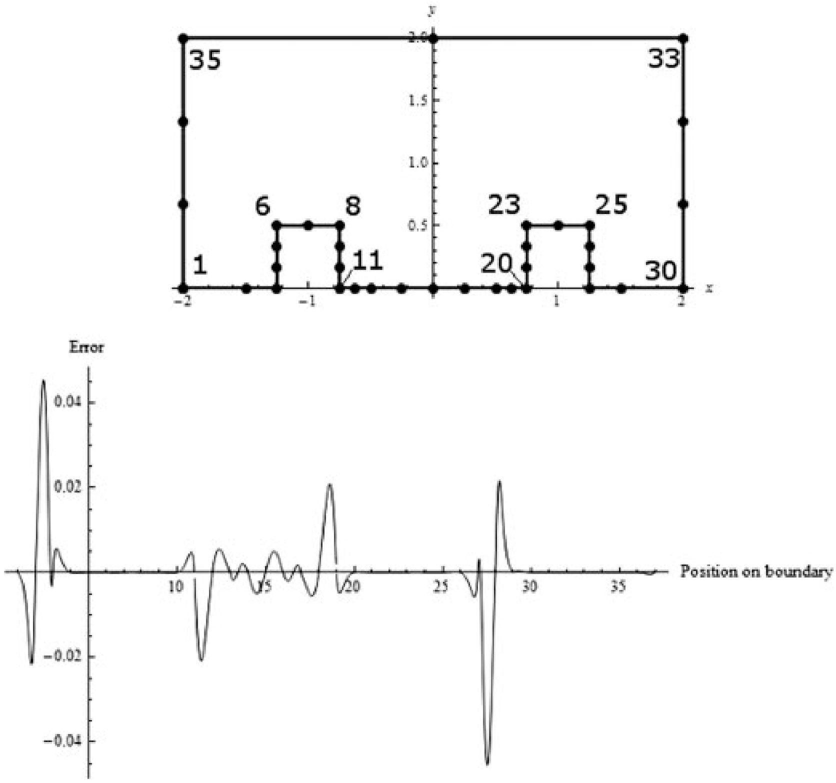

FIG. 10. Error along the problem domain. Error is largest on the x-axis, and decreases with distance from the poles.

two pole Laurent series approximation function is shown in Fig. 9. From this figures, the Laurent series approximation function is seen to provide good approximation results with just the use of two poles, when the location of the poles are optimized as discussed above, the numerical error can be seen in Fig. 10. It is noted that the resulting approximation function is analytic throughout the problem domain and so the real and also the imaginary parts of the approximation function

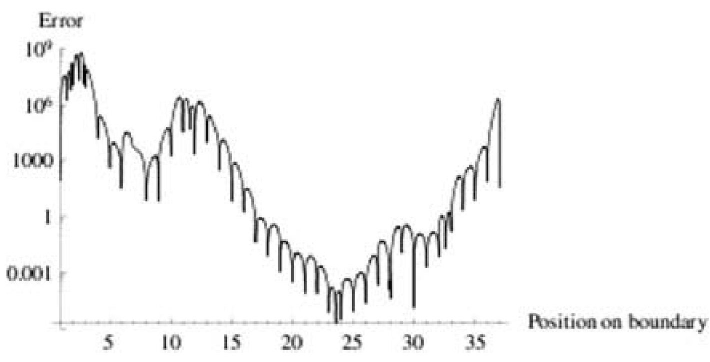

FIG. 11. Magnitude of error resulting from the modeling the example problem number two with only a single pole.

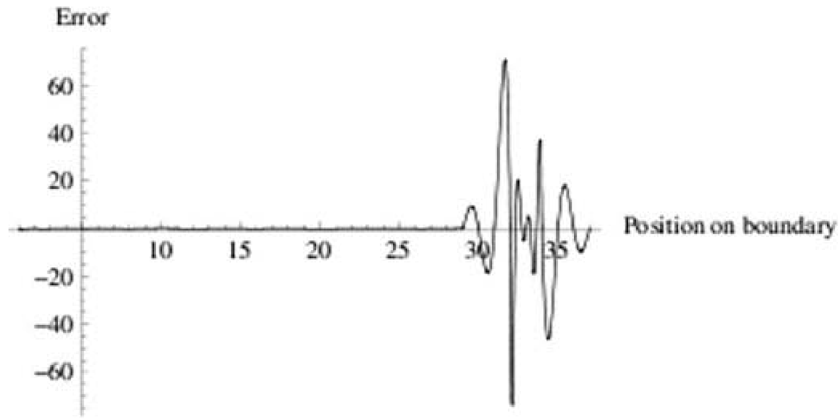

FIG. 12. Error along the boundary from modeling example problem two using a CPM expansion.

exactly solve the governing partial differential equation (i.e., the Laplace equation) throughout the problem domain.

This problem also demonstrates the technique of using multiple poles in the Laurent expansion. Figure 11 shows the computational results from using only a single pole to evaluate the same problem. Not only does the solution oscillate between collocation points but also the model becomes more complex as the model now requires much high-power complex monomials to be evaluated. These result show that modeling accuracy strongly depends on the location and number of Laurent series poles.

The Laurent series type of model can be compared other similar complex-variable-based modeling techniques such as the complex polynomial method (see Ref. [7]). Figure 12 shows the boundary error resulting from using the CPM to model the same demonstration problem with the same collocation points and number of expansion terms. For this demonstration problem, the CPM approximation resulted in a much larger magnitude of error.

VIII. TOPICS FOR FUTURE RESEARCH

Several topics for future research have been mentioned in the above development including but not limited to (i) development of more efficient methods for optimizing the location of Laurent series expansion points, (ii) development of more efficient methods to assess the extent of the Laurent series partial sum expansion, and (iii) fitting the model to the problem boundary conditions using a Hilbert-space setting to reduce Runge-type oscillations in the model.

IX. CONCLUSIONS

A new approach to modeling potential problems is presented that is based upon the theory of Laurent series expansions for complex variable analytic functions. Using poles located outside of the problem domain, partial sums of the Laurent series expansions about each pole are used as basis functions to be combined to form an approximation function. Because these partial sums are analytic functions within the problem region being studied, their real and imaginary function components exactly solve the well-known Laplace equation throughout the study region and also outside of the study region except at the actual pole locations themselves. Optimized locations for the poles are determined by pointwise testing and assessment of the goodness of it between the problem boundary conditions and the resulting approximation function values on the problem boundary. Two example problems are considered where exact solutions are known assess the efficiency and strength of the new approximation technique. Use of such Laurent series expansions can be combined with other families of basis functions to improve computational accuracy, with less computational effort, for both the complex variable boundary element method and the real variable boundary element method.

Acknowledgments are paid to the United States Military Academy at West Point, New York, the Department of Mathematical Sciences at West Point, and the several faculty and Cadets who have guided and encouraged the research effort leading to this work.

References

1. J. G. Simmonds, Analytic functions, ideal fluid flow, and Bernoulli's equation, SIAM Rev 38 (1996), 666–667.
2. D. Obrist, F. Boselli, and L. Kleiser, The method of fundamental solutions for predicting solutions for flow in semicircular canals, Schweizer Numerik Kolloquium, Fribourg, 2008.
3. J. J. Bertin and M. L. Smith, Aerodynamics for engineers, Prentice Hall, New Jersey, 1989.
4. T. V. Hromadka II and R. J. Whitley, Advances in the complex variable boundary element method, Springer-Verlag, New York, 1998.
5. T. V. Hromadka II, A multi-dimensional complex variable boundary element method, WIT Press, Southampton, England, 2002.
6. T. R. Dean and T. V. Hromadka II, A collocation CVBEM using program Mathematica, Eng Anal Boundary Elem 34 (2010), 417–422.
7. A. W. Bohannon and T. V. Hromadka II, The complex polynomial method with a least-squares fit to boundary conditions, Eng Anal Boundary Elem 33 (2009), 1100–1102.
8. J. H. Mathews and R. W. Howell, Complex analysis for mathematics and engineering, Jones and Bartlett, Massachusetts, 2008.