TAYLOR AND MACLAURIN SERIES

1. Basics and examples

Consider a function f defined by a power series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{1}$$

with radius of convergence $R > 0$. If we write out the expansion of $f(x)$ as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^2 + c_4(x-a)^4 \dots,$$

we observe that $f(a) = c_0$. Moreover

$$\begin{aligned} f^{'}(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots, \\ f^{(2)}(x) &= 2c_2 + 2 \cdot 3 \cdot c_3(x-a) + 3 \cdot 4 \cdot c_4(x-a)^2 + \dots \\ f^{(3)}(x) &= 2 \cdot 3 \cdot c_3 + 2 \cdot 3 \cdot 4 \cdot c_4(x-a)^2 + \dots \end{aligned}$$

After computing the above derivatives we observe that

$$\begin{aligned} f(a) &= c_0, \\ f^{'}(a) &= c_1, \\ f^{(2)}(a) &= 2 \Longrightarrow c_2 = \frac{f^{(2)}(a)}{2!}, \\ f^{(3)}(a) &= 2 \cdot 3 \cdot c_3 \Longrightarrow c_3 = \frac{f^{(3)}(a)}{3!}. \end{aligned}$$

In general we have

$$f^{(n)}(a) = n! c_n \Longrightarrow c_n = \frac{f^{(n)}(a)}{n!},$$

We have shown the following

Theorem 1 (Taylor-Maclaurin series)**.** *Suppose that $f(x)$ has a power series expansion at $x = a$ with radius of convergence $R > 0$, then the series expansion of $f(x)$ takes the form*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{'}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots, \tag{2}$$

that is, the coefficient c_n *in the expansion of* $f(x)$ *centered at* $x = a$ *is precisely* $c_n = \dfrac{f^{(n)}(a)}{n!}$. *The expansion* (2) *is called* **Taylor series**. *If* $a = 0$, *the expansion*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f^{'}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \ldots,$$

is called **Maclaurin Series**.

Let us now consider several classical Taylor series expansions. For the following examples we will assume that all of the functions involved can be expanded into power series.

Example 1. The function $f(x) = e^x$ satisfies $f^{(n)}(x) = e^x$ for any integer $n \geq 1$ and in particular $f^{(n)}(0) = 1$ for all n and then the Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

observe that the radius of convergence of $f(x)$ is computed by noting that $c_n x^n = \dfrac{x^n}{n!}$ so that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0,$$

and the radius of convergence is $R = \infty$ since the above computation shows that the series converges absolutely for any x. Note that for any other center, say $x = a$ we have $f^{(n)}(a) = e^a$, so that the Taylor expansion of $f(x)$ is

$$e^x = \sum_{n=0}^{\infty} \frac{e^a (x-a)^n}{n!}.$$

and this series also has radius of convergence $R = \infty$.

Example 2. Compute the Maclaurin series of the function $f(x) = \cos(x)$. Note that $f(x)$ satisfies

$$\begin{cases} f^{'}(x) & = -\sin(x) \\ f^{(2)}(x) & = -\cos(x) \\ f^{(3)}(x) & = \sin(x) \\ f^{(4)}(x) & = \cos(x) \end{cases}$$

and the above pattern is periodic, in fact, we will have

$$f^{(2n)}(x) = (-1)^n \cos(x) \Longrightarrow f^{(2n)}(0) = (-1)^n$$
$$f^{(2n+1)}(x) = (-1)^n \sin(x) \Longrightarrow f^{(2n+1)}(0) = 0,$$

and therefore

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Note that $\cos(x)$ is an even function in the sense that $\cos(-x) = \cos(x)$ and this is reflected in its power series expansion that involves only even powers of x. The radius of convergence in this case is also $R = \infty$.

Example 3. Compute the Maclaurin series of $f(x) = \sin(x)$. For this case we note that

$$f^{(2n)}(x) = (-1)^n \sin(x) \Longrightarrow f^{(2n)}(0) = 0$$
$$f^{(2n+1)}(x) = (-1)^n \cos(x) \Longrightarrow f^{(2n+1)}(0) = (-1)^n,$$

and therefore

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

The radius of convergence is again $R = \infty$.

Example 4. Compute the Maclaurin series of the following functions

(1) $\dfrac{\sin(x)}{x}$
(2) $\dfrac{\sin(x^2)}{x^2}$
(3) $\displaystyle\int_0^x \frac{\sin(s^2)}{s^2} ds$

For (1) we use the the expansion $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ so that

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

For (2) we replace x by x^2 and obtain for $x > 0$ the series

$$\begin{aligned} \frac{\sin(x^2)}{x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{4n}}{(2n+1)!}. \end{aligned}$$

Finally, for (3) we integrate the Maclaurin series of $\dfrac{\sin(x^2)}{x^2}$

$$\begin{aligned} \int_0^x \frac{\sin(s^2)}{s^2} ds &= \sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{(s)^{4n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{4n+1}}{(4n+1) \cdot (2n+1)!}. \end{aligned}$$

Remark: For a function that has an even expansion like $f(x) = \dfrac{\sin(x)}{x}$, we can also expand $f(\sqrt{x})$ as a power series. As an **exercise**, compute the Maclaurin expansion of $\displaystyle\int_0^x \frac{\sin(\sqrt{s})}{\sqrt{s}}ds$.

1.1. **Taylor polynomials and Maclaurin polynomials.** The partial sums of Taylor (Maclaurin) series are called Taylor (Maclaurin) polynomials. More precisely, the Taylor polynomial of degree k of $f(x)$ at $x = a$ is the polynomial

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and the Maclaurin polynomial of degree k of $f(x)$ (at $x = 0$) is the polynomial

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}x^n$$

An important question about Taylor polynomials is how well they approximate the functions that generate them. In fact we have the following error estimate

Theorem 2. *Consider the interval (x_0, x_1) with $x_0 < a < x_1$ and suppose that $f(x)$ is differentiable to any order on (x_0, x_1) and continuous on $[x_0, x_1]$. Fix $k \geq 1$ and let $M > 0$ be a constant such that $\max_{[x_0,x_1]} |f^{(k+1)}(x)| \leq M$. Then for any x in (x_0, x_1) we have*

$$|f(x) - p_k(x)| \leq \frac{M|x-a|^{k+1}}{(k+1)!}.$$

On the other hand, when it comes to the practical computation of Taylor or Maclaurin polynomials it may not be necessary to compute all of the derivatives of $f(x)$.

Example 5. Compute the Maclaurin polynomial of degree 4 for the function $f(x) = \cos(x)\ln(1-x)$ for $-1 < x < 1$.

Idea: In order to compute the Maclaurin polynomial of degree 4 of $f(x)$ we will multiply out the series expansions of the functions $\cos(x)$ and $\ln(1-x)$ thus obtaining a new power series, however we will only keep those terms in the expansion of the new series that have degree **at most 4**. In other words, if after multiplying the power series expansions of $\cos(x)$ and $\ln(1-x)$ we manage to write out the power series expansion of $\cos(x)\ln(1-x)$ in the form

$$f(x) = \cos(x)\ln(1-x) = \underbrace{c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4} + c_5x^5 + \dots$$

then the Maclaurin polynomial p_4 of degree 4 of $f(x)$ is

$$p_4(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4.$$

Note that for $-1 < x < 1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$
$$\ln(1-x) = -\int_0^x \frac{ds}{(1-s)} = -\sum_{n=0}^{\infty} \int_0^x s^n ds$$
$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

on the other hand

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots, \tag{3}$$

let us use $(*)$ to denote the expansion in (3), meaning that $(*) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$, so that after multiplying both series we have

$$\begin{aligned}
\cos(x)\ln(1-x) &= \underbrace{(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots)}_{(*)}(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots) \\
&= -x(*) - \frac{x^2}{2}(*) - \frac{x^3}{3}(*) - \frac{x^4}{4}(*) - \dots \\
&\underbrace{\left(-x + \frac{x^3}{2} - \frac{x^5}{4!} + \dots\right)}_{-x(*)} + \underbrace{\left(-\frac{x^2}{2} + \frac{x^4}{2\cdot 2!} - \frac{x^6}{2\cdot 4!} + \dots\right)}_{-\frac{x^2}{2}(*)} \\
&+ \underbrace{\left(-\frac{x^3}{3} + \frac{x^5}{3\cdot 5!} - \dots\right)}_{-\frac{x^3}{3}(*)} + \underbrace{\left(-\frac{x^4}{4} + \frac{x^6}{4\cdot 2!} - \dots\right)}_{-\frac{x^4}{4}(*)} \\
&= \left(-x + \frac{x^3}{2}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{2\cdot 2!}\right) + \left(-\frac{x^3}{3}\right) + \left(-\frac{x^4}{4}\right) + \dots \\
&= \underbrace{-x - \frac{x^2}{2} + \frac{x^3}{6}}_{p_4(x)} + \dots
\end{aligned}$$

We have used the color blue to highlight those terms of degree **at most** 4 in the multiplication of the two series. It follows that the Maclaurin polynomial of order 4 of $f(x) = \cos(x)\ln(1-x)$ is

$$\boxed{p_4(x) = -x - \frac{x^2}{2} + \frac{1}{6}x^3}$$

Remark: The radius of convergence of $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ and this is also the case for $-\ln(1-x) = \sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$, however the interval of convergence of this last series is $[-1, 1)$ (closed on the left and open on the right) because for $x = -1$ the series is $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ which converges conditionally but for $x = 1$ the series is $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

Exercise: Compute the first four terms in the power series expansion of $f(x) = \dfrac{\ln(1+x)}{1+x}$.

Example 6. Compute the limit

$$\lim_{x\to 0} \frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}}.$$

Note that in this case using a L'Hospital rule is extremely tedious. An alternative approach is to expand $\cos(x^4) - 1 + \frac{1}{2}x^8$ as a power series

$$\cos(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n}}{(2n)!} = 1 - \frac{1}{2}x^8 + \frac{x^{16}}{4!} - \dots,$$

so that

$$\lim_{x\to 0} \left(\frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}} \right) = \frac{1}{4!}.$$

2. Intervals of convergence

The radius of convergence of a power series determines where the series is absolutely convergent but as we will see below there are points where the series may only be *conditionally convergent*. More precisely, if the radius of convergence of $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ is $R > 0$ then the series converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$ but it could still happen that the series converges at the points $x_0 - R$ or $x_0 + R$ (that is, at those points with $|x - x_0| = R$). Let us illustrate this with several examples

Example 7. The series

$$\ln(1+x) = \int_0^x \frac{ds}{1+s} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},$$

has radius of convergence equal to 1 so that x converges absolutely for $|x| < 1$. For $x = 1$ we obtain the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by the alternating series test. On the other hand, for $x = -1$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which diverges. The interval of convergence is then $(-1, 1]$ (closed on the right and open on the left).

Example 8. The series

$$\int_0^x s\ln(1+s^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{2(n+1)(n+2)},$$

has radius of convergence $R = 1$, and at either $x = 1$ or $x = -1$ the series equals

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+1)(n+2)},$$

which converges by the alternating series test. In this case the interval of convergence is the closed interval $[-1, 1]$.

Example 9. The series

$$\ln(1-x) = \int_0^x \frac{ds}{1-s} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},$$

has radius of converges $R = 1$. At $x = 1$ we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n+1},$$

which diverges and at $x = -1$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by the alternating series test. In this case the interval of convergence is the interval $[-1, 1)$ (closed on the left and open on the right).

Finally we have

Example 10. The series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} x^n,$$

has radius of convergence $R = 1$ but diverges at both $x = 1$ and $x = -1$. In this case the interval of convergence is only $(-1, 1)$.