The work of Maryam Mirzakhani

18 August, 2014

Abstract

Maryam Mirzakhani has been awarded the Fields Medal for her outstanding work on the dynamics and geometry of Riemann surfaces and their moduli spaces.

1 Introduction

Mirzakhani has established a suite of powerful new results on orbit closures and invariant measures for dynamical systems on moduli spaces. She has also given a new proof of Witten's conjecture, which emerges naturally from a counting problem for simple closed geodesics on Riemann surfaces. This note gives a brief discussion of her main results and their ramifications, including the striking parallels between homogeneous spaces and moduli spaces that they suggest.

2 The setting

We begin with a résumé of background material, to set the stage.

Let $\mathcal{M}_g$ denote the moduli space of curves of genus $g \geq 2$. This space is both a complex variety, with $\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3$, and a symplectic orbifold. Its points are in bijection with the isomorphism classes of compact Riemann surfaces X of genus g.

The dimension of $\mathcal{M}_g$ was known already to Riemann. Rigorous constructions of moduli space were given in the 1960s, by Ahlfors and Bers in the setting of complex analysis and by Mumford in the setting of algebraic geometry. Today the theory of moduli spaces is a meeting ground for mathematical disciplines ranging from arithmetic geometry to string theory.

The symplectic form ω on $\mathcal{M}_g$ arises from the hyperbolic metric on X. As shown by Wolpert, in the length–twist coordinates coming from a pair of pants decomposition of X, one can write

$$\omega = \sum_1^{3g-3} d\ell_i \wedge d\tau_i.$$

The complex structure on $\mathcal{M}_g$ arises from the natural isomorphism

$$\mathrm{T}^*_X \mathcal{M}_g = Q(X) = \{\text{holomorphic forms } q = q(z)\,dz^2 \text{ on } X\}$$

between the cotangent space to $\mathcal{M}_g$ at X and the space of holomorphic quadratic differentials on X. The *Teichmüller metric* on $\mathcal{M}_g$ also emerges from its complex structure: on the one hand, it is dual to the L^1 norm

$$\|q\| = \int_X |q(z)|\,|dz|^2 = \mathrm{area}(X, |q|)$$

on $\mathrm{T}^*_X \mathcal{M}_g$; on the other hand, it agrees with the intrinsic Kobayashi metric on $\mathcal{M}_g$ (Royden).

Moduli space can be presented as the quotient $\mathcal{M}_g = \mathcal{T}_g / \operatorname{Mod}_g$ of Teichmüller space — its universal cover, a contractible bounded domain in $\mathbb{C}^{3g-3}$ — by the action of the mapping–class group of a surface.

One of the challenges of working with moduli space is that it is *totally inhomogeneous*: for example, the symmetry group of $\mathcal{T}_g$ (as a complex manifold) is simply the discrete group Mod_g (for $g > 2$). One of Mirzakhani's remarkable contributions is to show that, nevertheless, dynamics on moduli space displays many of the same rigidity properties as dynamics on homogeneous spaces (see §4).

3 From simple geodesics to Witten's conjecture

We begin with Mirzakhani's work on simple geodesics. In the 1940s, Delsarte, Huber and Selberg established the *prime number theorem* for hyperbolic surfaces, which states that the number of (oriented, primitive) closed geodesics on $X \in \mathcal{M}_g$ with length $\le L$ satisfies

$$\pi(X, L) \sim \frac{e^L}{L}.$$

(The usual prime number theorem says that the number of prime integers with $0 < \log p \le L$ is asymptotic to e^L/L.)

The number of *simple* closed geodesics $\sigma(X, L)$ behaves quite differently; it only has polynomial growth, and in 2004 Mirzakhani proved that

$$\sigma(X, L) \sim C_X L^{6g-6}.$$

In contrast to the prime number theorem, the right–hand side here depends on both the genus and geometry of X.

Although the statement above involves only a single Riemann surface X, Mirzakhani's proof involves integration over moduli space and leads to a cascade of new results, including a completely unexpected proof of the Witten conjecture. The latter conjecture, established by Kontsevich in 1992, relates the intersection numbers on moduli space defined by

$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(E_1)^{d_1} \cdots c_1(E_n)^{d_n}$$

to a power series solution to the KdV hierarchy (an infinite system of differential equations satisfying the Virasoro relations). Here $\overline{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of the moduli space of Riemann surfaces X with marked points $(p_1, \ldots, p_n)$, and $c_1(E_i)$ denotes the first Chern class of the line bundle $E_i \to \overline{\mathcal{M}}_{g,n}$ with fibers $\mathrm{T}^*_{p_i} X$.

Mirzakhani's investigation of $\sigma(X, L)$ also leads to formulas for the frequencies of different topological types of simple closed curves on X; for example, a random simple curve on a surface of genus 2 has probability 1/7 of cutting X into two pieces of genus 1. These frequencies are always rational numbers, and they depend only on g, not X.

At the core of these results is Mirzakhani's novel, recursive calculation of the volume of the moduli space of Riemann surfaces of genus g with n geodesic boundary components with lengths $(L_1, \ldots, L_n)$. This volume is defined by

$$P_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \omega^{3g-3+n};$$

for example, one can show that $P_{1,1}(L_1) = (1/24)(L_1^2 + 4\pi^2)$. In general, $P_{g,n}$ is a polynomial whose coefficients (which lie in $\mathbb{Q}(\pi)$) can be related to frequencies and characteristic classes, yielding the results discussed above. Previously only the values of $P_{g,n}(0, \ldots, 0)$ were known. The proofs depend on intricate formulas for dissections of surfaces along hyperbolic geodesics; see [Mir3], [Mir1] and [Mir2]. Mirzakhani has also studied the behavior of $\mathcal{M}_g$ as $g \to \infty$; see [Mir4],[Mir6].

4 Complex geodesics in moduli space

We now turn to Mirzakhani's work on moduli spaces and dynamics. Her contributions to this area include a prime number theorem for closed geodesics in $\mathcal{M}_g$, counting results for orbits of Mod_g on $\mathcal{T}_g$, and the classification of Mod_g–invariant measures on the space of measured laminations $\mathcal{ML}_g$. But perhaps her most striking work — which we will present here — is a version of Ratner's theorem for moduli spaces.

Complex geodesics. It has been known for some time that the Teichmüller geodesic flow is ergodic (Masur, Veech), and hence almost every geodesic $\gamma \subset \mathcal{M}_g$ is dense. It is difficult, however, to describe the behavior of *every single geodesic* γ; already on a hyperbolic surface, the closure of a geodesic can be a fractal cobweb, and matters only get worse in moduli space.

Teichmüller showed that moduli space is also abundantly populated by *complex* geodesics, these being holomorphic, isometric immersions

$$F : \mathbb{H} \to \mathcal{M}_g.$$

In fact there is a complex geodesic through every $X \in \mathcal{M}_g$ in every possible direction.

In principle, the closure of a complex geodesic might exhibit the same type of pathology as a real geodesic. But in fact, the opposite is true. In a major breakthrough, Mirzakhani and her coworkers have shown:

> *The closure of any complex geodesic is an algebraic subvariety* $V = \overline{F(\mathbb{H})} \subset \mathcal{M}_g$.

This long sought–after rigidity theorem was known previously only for $g = 2$, with some restrictions on F [Mc]. (In the case of genus two, V can be an isometrically immersed curve, a Hilbert modular surface, or the whole space $\mathcal{M}_2$.)

Dynamics over moduli space. The proof of this rigidity theorem involves the natural action of $\mathrm{SL}_2(\mathbb{R})$ on the sphere bundle

$$Q_1\mathcal{M}_g \to \mathcal{M}_g,$$

consisting of pairs (X, q) with $q \in Q(X)$ and $\|q\| = 1$.

To describe this action, consider a Riemann surface $X = P/\!\sim$ presented as the quotient of a polygon $P \subset \mathbb{C}$ under isometric edge identifications between pairs of parallel sides. Such identifications preserve the quadratic differential $dz^2|P$, so a polygonal model for X actually determines a pair

$(X, q) \in Q\mathcal{M}_g$ with $\|q\| = \text{area}(P)$. Conversely, every nonzero quadratic differential $(X, q) \in Q\mathcal{M}_g$ can be presented in this form.

Since $\text{SL}_2(\mathbb{R})$ acts linearly on $\mathbb{R}^2 \cong \mathbb{C}$, given $A \in \text{SL}_2(\mathbb{R})$ we can form a new polygon $A(P) \subset \mathbb{C}$, and use the corresponding edge identifications to define

$$A \cdot (X, q) = (X_A, q_A) = (A(P), dz^2)/\sim .$$

Note that $[X_A] = [X]$ if $A \in \text{SO}_2(\mathbb{R})$. Thus the map $A \mapsto X_A$ descends to give a map

$$F : \mathbb{H} \cong \text{SL}_2(\mathbb{R})/\,\text{SO}_2(\mathbb{R}) \to \mathcal{M}_g,$$

which is the complex geodesic *generated* by (X, q).

The proof that $\overline{F(\mathbb{H})} \subset \mathcal{M}_g$ is an algebraic variety involves the following three theorems, each of which a substantial work in its own right.

1. Measure classification (Eskin and Mirzakhani). *Every ergodic,* $\text{SL}_2(\mathbb{R})$*–invariant probability measure on* $Q_1\mathcal{M}_g$ *comes from Euclidean measure on a special complex–analytic subvariety* $A \subset Q\mathcal{M}_g$ (The variety A is linear in period coordinates).

   This is the deepest step in the proof; it uses a wide variety of techniques, including conditional measures and a random walk argument inspired by the work of Benoist and Quint [BQ].

2. Topological classification (Eskin, Mirzakhani and Mohammadi). *The closure of any* $\text{SL}_2(\mathbb{R})$ *orbit in* $Q_1\mathcal{M}_g$ *is given by* $A \cap Q_1\mathcal{M}_g$ *for some special analytic subvariety* A*.*

3. Algebraic structure (Filip). *Any special analytic subvariety* A *is in fact an algebraic subvariety of* $Q\mathcal{M}_g$*.* Thus its projection to $\mathcal{M}_g$, $V = \overline{F(\mathbb{H})}$, is an algebraic subvariety as well.

See [EM], [EMM] and [Fil] for these developments.

Ramifications: Beyond homogeneous spaces. This collection of results reveals that the theory of dynamics on homogeneous spaces, developed by Margulis, Ratner and others, has a *definite resonance* in the highly inhomogeneous, but equally important, world of moduli spaces.

The setting for homogeneous dynamics is the theory of Lie groups. Given a lattice Γ in a Lie group G, and a Lie subgroup H of G, one can consider the action

$$H \rightsquigarrow G/\Gamma$$

by left multiplication, just as in the setting of moduli spaces we have considered the action

$$\mathrm{SL}_2(\mathbb{R}) \rightsquigarrow Q_1\mathcal{T}_g / \mathrm{Mod}_g\,.$$

One of the most powerful results in homogeneous dynamics is *Ratner's theorem.* It implies that if H is generated by unipotent elements, then every orbit closure $\overline{Hx} \subset G/\Gamma$ is a *special submanifold* — in fact, it has the form

$$\overline{Hx} = Jx \subset G/\Gamma$$

for some Lie subgroup J with $H \subset J \subset G$. A similar statement holds for invariant measures. Since $\mathrm{SL}_2(\mathbb{R})$ is generated by unipotent elements (matrices such as $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$ and its transpose), one might hope for a version of Ratner's theorem to hold in moduli spaces. This is what Mirzakhani's work confirms.

Hodge theory versus geometry. For another perspective, recall that $\mathcal{M}_g$ embeds into the moduli space of Abelian varieties $\mathcal{A}_g = \mathfrak{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})$, a locally symmetric space amenable to the methods of homogeneous dynamics. But the complex geodesics in $\mathcal{M}_g$ become inhomogeneous when mapped into $\mathcal{A}_g$, so they cannot be analyzed by these methods. Mirzakhani's work shows that one can work effectively and directly with $\mathcal{M}_g$ rather than with $\mathcal{A}_g$, by geometric analysis on Riemann surfaces themselves.

Ramifications: Billiards. The $\mathrm{SL}_2(\mathbb{R})$ action on $Q_1\mathcal{M}_g$ is also connected with the theory of *billiards in polygons* — an elementary branch of dynamics in which difficult problems abound.

Let $T \subset \mathbb{C}$ be a connected polygon with angles in $\pi\mathbb{Q}$. The behavior of billiard paths in T is closely related to the behavior of the complex geodesic generated by a quadratic differential (X, q) obtained by 'unfolding' the table T.

Indeed, the first examples of complex geodesics such that $V = \overline{F(\mathbb{H})} \subset \mathcal{M}_g$ is an algebraic curve — i.e. the image of the complex geodesic is as small as possible — were constructed by Veech in his analysis of billiards in regular polygons. In this case the stabilizer of the corresponding quadratic differential is a lattice $\mathrm{SL}(X, q) \subset \mathrm{SL}_2(\mathbb{R})$, which serves as the *renormalization group* for the original billiard flow.

The work of Mirzakhani has bearing on several open conjectures in the field of billiard dynamics. For example, it provides progress on the open problem of showing that, for any table T, there is an algebraic number C_T such that the number $N(T, L)$ of types of *primitive, periodic* billiard paths in T of length $\leq L$ satisfies

$$N(T, L) \sim \frac{C_T L^2}{\pi \operatorname{area}(T)}.$$

Eskin and Mirzakhani have shown that an asymptotic equation of this form holds after averaging over L, and that C_T can assume only countably many values.

5 Dynamics of earthquakes

We conclude by discussing Mirzakhani's work on the earthquake flow, and a measurable bridge between the symplectic and holomorphic aspects of $\mathcal{M}_g$.

A classical construction of Fenchel and Nielsen associates to a simple closed geodesic $\gamma \subset X \in \mathcal{M}_g$ and $t \in \mathbb{R}$ a new Riemann surface

$$X_t = \operatorname{tw}_{t\gamma}(X) \in \mathcal{M}_g,$$

obtained by cutting X open along γ, twisting by length t to the right, and then regluing. The resulting *twist path* in $\mathcal{M}_g$ is periodic; if γ has length L, then $X_{t+L} = X_t$.

On the other hand, one can also twist along *limits* of weighted simple geodesics, called *measured laminations.* As shown by Thurston, the space of measured laminations forms a PL manifold $\mathcal{ML}_g \cong \mathbb{R}^{6g-6}$ with a natural volume form, and the limiting twists, called *earthquakes*, are defined for all time.

Earthquakes are a natural feature of the symplectic geometry of moduli space. While they can be defined geometrically by fracturing and regluing X along the (possibly fractal) support of $\lambda \in \mathcal{ML}_g$, they also arise more conventionally as the *Hamilton flows* associated to the functions $Y \mapsto \operatorname{length}(\lambda, Y)$.

The earthquake flow lives on the bundle $L_1\mathcal{M}_g$ of unit length laminations over $\mathcal{M}_g$. Mirzakhani has shown that, with respect to the natural measure on $L_1\mathcal{M}_g$:

> *Thurston's earthquake flow is ergodic.*

Prior to this result, the dynamics of earthquakes seemed completely opaque. Not a single example of a dense earthquake path in $\mathcal{M}_g$ was known; we can now assert that almost every earthquake path is dense and uniformly distributed.

Bridging the symplectic/holomorphic divide. The proof of ergodicity of the earthquake flow uses a remarkable bridge between the symplectic and holomorphic sides of moduli space.

In more detail, recall that the *horocycle flow* on $Q_1\mathcal{M}_g$ is defined by the action of the 1-parameter group $N = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{R}\} \subset \mathrm{SL}_2(\mathbb{R})$. Drawing on ideas from Thurston's work on stretch maps, Mirzakhani shows there is a measure–preserving map $\beta : L_1\mathcal{M}_g \to Q_1\mathcal{M}_g$ which transports the earthquake flow to the horocycle flow. In other words, we have a commutative diagram of the form

$$\begin{array}{ccccc} \text{earthquake flow} \circlearrowright & L_1\mathcal{M}_g & \stackrel{\beta}{\Longrightarrow} & Q_1\mathcal{M}_g & \circlearrowleft \text{horocycle flow}\ . \\ & \downarrow & & \downarrow & \\ & \mathcal{M}_g & & \mathcal{M}_g & \end{array}$$

But the horocycle flow on $Q_1\mathcal{M}_g$ is well–known to be ergodic (this is a formal corollary of ergodicity of the geodesic flow [Zim, Thm. 2.4.2]), so the same is true for the earthquake flow [Mir5]. (It is an open problem to establish Ratner–type rigidity for these flows.)

Summary. Mirzakhani's research has integrated, with great originality, a broad range of mathematical disciplines — including algebraic and symplectic geometry, low–dimensional topology, and random processes. Her breakthroughs have transformed our perspective on moduli spaces, and led the way to mathematical frontiers where striking developments are still unfolding.

Curtis T. McMullen, Cambridge, 2014.

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