Fractional Geometric Calculus: Toward A Unified Mathematical Language for Physics and Engineering ⋆

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Abstract: This paper discuss the longstanding problems of fractional calculus such as too many definitions while lacking physical or geometrical meanings, and try to extend fractional calculus to any dimension. First, some different definitions of fractional derivatives, such as the Riemann-Liouville derivative, the Caputo derivative, Kolwankar's local derivative and Jumarie's modified Riemann-Liouville derivative, are discussed and conclude that the very reason for introducing fractional derivative is to study nondifferentiable functions. Then, a concise and essentially local definition of fractional derivative for one dimension function is introduced and its geometrical interpretation is given. Based on this simple definition, the fractional calculus is extended to any dimension and the *Fractional Geometric Calculus* is proposed. Geometric algebra provided an powerful mathematical framework in which the most advanced concepts modern physic, such as quantum mechanics, relativity, electromagnetism, etc., can be expressed in this framework graciously. At the other hand, recent developments in nonlinear science and complex system suggest that scaling, fractal structures, and nondifferentiable functions occur much more naturally and abundantly in formulations of physical theories. In this paper, the extended framework namely the Fractional Geometric Calculus is proposed naturally, which aims to give a unifying language for mathematics, physics and science of complexity of the 21st century.

Keywords: geometric algebra, geometric calculus, fractional calculus, nondifferentiable function

1. INTRODUCTION

It seems a general pattern of our understanding of nature to evolve from integer to fraction. In number theory, we discovery integer number firstly, then we introduce fractional number in order to do division and finally we introduce real number to form a solid base for analysis.

In geometry, for a long time we only know the typical geometry objects like point, line, surface and volume with integer number dimensions. But recent discovery of fractal geometry has completely changed and enlarged people's understanding about the geometry. Fractal is even considered as the true geometry of nature, see Mandelbrot (1983).

In calculus, it also follow this path. At the beginning, the motivation of this extension from integer to fractional differential operator is just formal, but lack of substantial physical meaning. From the beauty of the mathematical formality, we hope that the semigroup of powers D^α will form a continuous semigroup with parameter α, inside which the original discrete semigroup of D^n for integer n can be recovered as a subgroup, see Samko et al. (1993); Miller and Ross (1993).

Moreover, for centuries we can only do calculus to a relatively rare and special kind of sufficiently well-behaved functions, i.e. the differentiable function or even smooth function. But later we come to realize that the typical and prevalent continuous functions are nowhere-differentiable functions like the Weierstrass function. The classical calculus with integer order is just powerless on this type of function.

In this paper, First, some different definitions of fractional derivatives, such as the Riemann-Liouville , the Caputo derivative, Kolwankar's local derivative Kolwankar and Gangal (1997); Kolwankar et al. (1996)and Jumarie's modified Riemann-Liouville derivative Jumarie (2006), are discussed. Then it's argued that the very reason for introducing fractional derivative is to study nondifferentiable functions. A concise and essentially local definition of fractional derivative for one dimension function is introduced and its geometrical interpretation is given. Based on this simple definition, the fractional calculus is extended to any dimension and the *Fractional Geometric Calculus* is proposed.

⋆ the paper is submitted for consideration of the best student paper award.

2. REVIEW AND REMARKS OF SOME DEFINITIONS

2.1 The Riemann-Liouville approach

Fractional integration According to the Riemann-Liouville approach to fractional calculus, the notion of fractional integral of order α ($\alpha > 0$) for a function $f(x)$, is a natural generalization of the well known Cauchy formula for repeated integration, which reduces the calculation of the $n-$fold primitive of a function $f(x)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$I^n f(x) := f^{(-n)}(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt,\ n \in N \tag{2.1}$$

where N is the set of positive integers.

Taking advantage of the *Gamma function*

$$\Gamma(z) := \int_0^\infty \mathrm{e}^{-u}\ u^{z-1}\,du\,, \mathrm{Re}\,\{z\} > 0 \tag{2.2}$$

which has the property

$$\Gamma(z+1) = z\,\Gamma(z)\,, (n-1)! = \Gamma(n)\,,$$

it's very natural to extend the above formula from positive integer values of the index to any positive real values and defines the *Fractional Integral of order* $\alpha > 0$:

Definition 1.

$$I^\alpha f(x) := f^{(-\alpha)}(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt. \tag{2.3}$$

where α is in the set of positive real numbers.

Fractional differentiation The definition of fractional differentiation is somehow more subtle than the fractional integration.

A straightforward way to define fractional differentiation is to do enough fractional integration firstly, then take integer differentiation (of course, one can do in another order, just as the Caputo approach). That is to introduce a positive integer m such that $m-1 < \alpha \le m$, then define the fractional derivative of order $\alpha > 0$:

Definition 2. $D^\alpha\, f(t) := D^m\, I^{m-\alpha}\, f(t)$, namely

$$f^{(\alpha)}(x) := \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha} f \tag{2.4}$$

where $\lceil\cdot\rceil$ denotes the *Ceiling function* which map a real number to the smallest following integer.

For complementation, differentiation of any real number order α could be defined as,

$$f^{(\alpha)}(x) := \begin{cases} \dfrac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha} f(x) & \alpha > 0 \\ f(x) & \alpha = 0 \\ I^{-\alpha} f(x) & \alpha < 0. \end{cases} \tag{2.5}$$

Remarks

(1) The fundamental relations hold

$$\frac{d}{dx} I^{\alpha+1} f(x) = I^\alpha f(x), \quad I^\alpha(I^\beta f) = I^\beta(I^\alpha f) = I^{\alpha+\beta} f, \tag{2.6}$$

the latter of which is a semigroup property fractional for integration operator. But unfortunately the derivative operator D^α is quite different and significantly more complex, for D^α is neither commutative nor additive in general.

(2) Take a constant function $f(x) = 1$ for example:

$$\begin{aligned} \frac{d}{dx}(I^{\frac{1}{2}} f)(x) &= \frac{d}{dx}\left(\frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{-\frac{1}{2}}.1dt\right) \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(\int_0^x (x-t)^{-\frac{1}{2}} dt\right) \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(\left[-2(x-t)^{\frac{1}{2}}\right]_0^x\right) \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dx}(2\sqrt{x}) \\ &= \frac{1}{\sqrt{\pi}\sqrt{x}} \end{aligned}$$

The fractional differentiation of constant functions is not necessary zero from this definition, which somehow obscures the physical meaning of these fractional derivatives as some kind of velocity.

(3) The domain and boundary conditions of the functions must be explicitly chosen. Information about the domain and boundary condition of the functions is needed for calculation, so the fractional derivative is not a local property of the function. Therefore, unlike the integral order derivative, the fractional order derivative at a point x is not determined by an arbitrarily small neighborhood of x. This obscures the geometric meaning of these fractional derivatives.

(4) To partially solve the above problem, many people use $-\infty$ as the base point, instead of 0:

$$I^\alpha f(x) := f^{(-\alpha)}(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1} f(t)\,dt. \tag{2.7}$$

2.2 The Caputo approach

Alternatively, we can take integer differentiation firstly, then do fractional integration. The Caputo fractional derivative:

Definition 3.

$$f^{(\alpha)}(x) := I^{\lceil\alpha\rceil-\alpha}\left(\frac{d^{\lceil\alpha\rceil} f}{dx^{\lceil\alpha\rceil}}\right) \tag{2.8}$$

Remarks

(1) This Caputo fractional derivative produces zero from constant functions.

(2) But it is easy to see that the Caputo definition for the fractional derivative is in a sense, more restrictive than the Riemann-Liouville definition because it require the function to have higher order differentiability, but it is often the case that $f^{(\alpha)}(x)$ may exist whilst $f^{'}(x)$ does not. These troublesome features greatly reduced its scope of application.

2.3 The Jumarie approach

Modified Riemann–Liouville Jumarie's modified Riemann–Liouville fractional derivative is defined by

Definition 4.

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}(f(t)-f(0))\,dt, \quad (2.9)$$

where α a real number on the interval $(0,1)$.

Remarks

1. If $f(0) = 0$, then $f^{(\alpha)}$ is equal to the Riemann–Liouville fractional derivative of f of order α.
2. has the advantages of both the standard Riemann–Liouville and Caputo fractional derivatives. The fractional derivative of a constant is zero, as desired.
3. It is defined for arbitrary continuous (nondifferentiable) functions.
4. But this definition is still an nonlocal definition. Information about the domain and boundary condition of the functions is still needed, thus has no good geometric interpretation.

2.4 Kolwankar-Gangal approach

Kolwankar-Gangal local fractional derivative Kolwankar and Gangal (henceforth K-G) has proposed a local fractional derivative by introducing

Definition 5. If, for a function $f : [0,1] \to I\!R$, the limit

$$I\!D^q f(y) = \lim_{x\to y}\frac{d^q(f(x)-f(y))}{d(x-y)^q} \quad (2.10)$$

exists and is finite, then we say that the *local fractional derivative* (LFD) of order q $(0 < q < 1)$, at $x = y$, exists.

Remarks

1. The operator in the RHS is the Riemann-Liouville derivative, so the K-G local fractional derivative can be equivalently expressed as
$$f^{(\alpha)}(x)|_{x=x_0} := \frac{1}{\Gamma(1-\alpha)}\lim_{x\to x_0}\frac{d}{dx}\int_{x_0}^x \frac{f(t)-f(x_0)}{(x-t)^\alpha}\,dt, \quad (2.11)$$
where α a real number on the interval $(0,1)$.
2. The fractional derivative of a constant is zero because the subtraction of $f(x_0)$.
3. The derivative is defined by the limiting process $x \to x_0$, so only local information near x_0 is needed.
4. Although this definition in local in nature, the geometric interpretation is still vague. One can hardly have any geometric intuition directly from this abstract definition.

2.5 Final remarks

We have reviewed several different definitions about fractional derivative, and each has some advantages and disadvantages. But we still hope to have a definition which can have all the following merits:

1. It should be local in nature, thus does not rely on information of domain and boundary conditions.
2. The derivative of constant function should be zero.
3. It should be applicable to a large class of functions (i.e. continuous but nondifferentiable, or even generalized function) which are not differentiable in the classical sense.
4. It should have reasonable geometric interpretation, like the classical one.
5. The calculation should be easy.

3. THE VERY REASON FOR INTRODUCING FRACTIONAL DERIVATIVE

3.1 Discussion on semigroup property

It is generally believed that the motivation for introducing fractional calculus is that we want the semigroup of powers D^a to form a continuous semigroup with parameter a, inside which the original discrete semigroup of D^n for integer n can be recovered as a subgroup. But we argue that this is due to we limit our scope of discussion in the very special *smooth function.*

Of course, we can calculate the fractional derivative of the common smooth function, like the following:

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x\right) &= \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}2\pi^{-\frac{1}{2}}x^{\frac{1}{2}}\\ &= 2\pi^{-\frac{1}{2}}\frac{\Gamma(1+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}}\\ &= 2\pi^{-\frac{1}{2}}\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}x^0\\ &= \frac{2\sqrt{\pi}x^0}{2\sqrt{\pi}0!} = 1,\end{aligned}$$

It's simple and elegant from the pure mathematical point of view, but the semigroup property is not always satisfied for derivative operator. Moreover, it's hard to give any geometrical interpretation of this kind $\frac{1}{2} - $*derivative.*

3.2 Differentiability class of function

The most basic functions $x^n, e^x, \sin x$ are smooth function. For smooth function we can use power series to approximate and express any smooth function.

In mathematical analysis, a differentiability class is a classification of functions according to the properties of their derivatives. Higher order differentiability classes correspond to the existence of more derivatives. Functions that have derivatives of all orders are called smooth.

Definition 6. The function f is said to be of class C^k if the derivatives $f', f'', ..., f^{(k)}$ exist and are continuous (except for a finite number of points or a measure zero set of points). The function f is said to be of class C^∞, or smooth, if it has derivatives of all orders.

Under the above definition, the class C^0 consists of all continuous functions. The class C^1 consists of all differentiable functions whose derivative is continuous; such functions are called continuously differentiable. Thus, a C^1 function is exactly a function whose derivative exists and is of class C^0.

The relation hold: $C^\infty \subset ... \subset C^n \subset ... \subset C^2 \subset C^1 \subset C^0 \subset G$, where G denote the generalized function. In particular, C^k is contained in C^{k-1} for every k, and there are examples to show that this containment is strict.

One interesting property is if $f \in C^m$ and $g \in C^n$, then $f+g \in C^{\min(m,n)}$.

3.3 Non-differentiable fuction

It's interesting that, in the above relation C^{k+1} is always a proper (in fact, a measure zero) subset of C^k. So could there be some non-trivial examples which are $\in C^0$ but not $\in C^1$ almost everywhere? The answer is yes, Weierstrass type of functions are typical such examples, see Berry and Lewis (1980).

What's more, the Weierstrass function is far from being an isolated or special example. In fact, it is the dominant and typical continuous functions, in the math sense.

In physics sense, it's also very important to study this kind of fractal curve. Observed path of a quantum mechanical particle are known to be fractals and are continuous but non-differentiable, see Abbott and Wise (1981); Sornette (1990); Cannata and Ferrari (1988).

A simple example of such a parametrization is the well known Weierstrass function given by

$$W_\lambda^s(t) = \sum_{k=1}^{\infty} \lambda^{(D-2)k} \sin \lambda^k t$$

where $\lambda > 1$ and $1 < D < 2$. The graph of $W_\lambda^s(t)$ is known to be a fractal curve with box-dimension D.

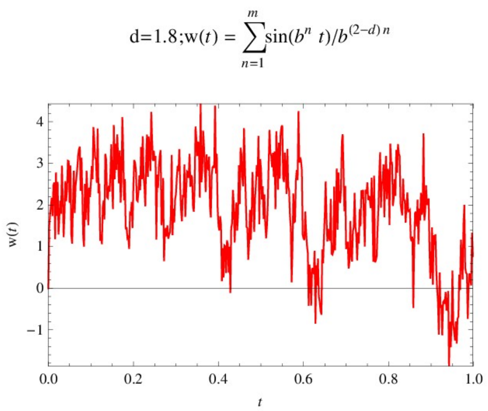

Fig. 1. Example of Weierstrass function with $D = 1.8$

3.4 Fractional differentiability class of function

So like the question of introducing fractional derivative, here we ask the following questions:

(1) Could there be fractional differentiability class of function?
(2) What's the structure of the space $C^{\frac{1}{2}}$, or generally the space of C^a?
(3) What's the representation of these kind of fractional differentiability function space?
(4) What's the relation between fractional differentiability function space and fractional derivative.

In general, the classes C^a for any non-negative real number a can be defined recursively as following:

(1) C^0 is the set of all continuous functions.
(2) C^α for $0 < \alpha < 1$ is the set functions which the $\alpha - derivatives$ exist and are continuous.
(3) C^k for any positive integer k is the set of all differentiable functions whose $1 - derivative$ is in C^{k-1}
(4) C^a for any positive real number $a = k + \alpha$ is the set of all functions whose $\alpha - derivative$ is in C^k
(5) C^∞ is the intersection of the sets C^k

So we argue that, the semigroup property may not be a necessary and essential condition when introducing fractional calculus.

The classical calculus is enough to deal with smooth function and the geometrical interpretation is natural and intuitive. So why do we need to introduce some other derivative but has no intuitive interpretation for smooth function? For smooth function, there is no need for something more. For nondifferentiable functions, things are quite different and classical calculus is not enough.

One of the main reason to introduce something new in math is when we want to enlarge our scope of discussion and find the old tool is not applicable. We argue that he very reason for introducing fractional derivative is to study nondifferentiable functions.

4. GEOMETRICAL MEANING OF FRACTIONAL DERIVATIVE

4.1 Geometrical meaning of ordinary derivative

The geometrical meaning of ordinary derivative is simple and intuitive:
For smooth function f which is differentiable at x, the local behavior of f around point x can be approximated by using the ordinary derivative:

$$f(x+h) \approx f(x) + f'(x)h \tag{4.1}$$

The ordinary derivative gives the linear approximation of smooth function.

Using Taylor series and higher order derivative, we have

$$f(x+h) \approx f(x)+f'(x)h+\frac{f''(x)}{2!}h^2+\frac{f^{(3)}(x)}{3!}h^3+... \tag{4.2}$$

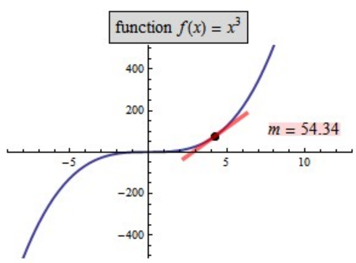

Fig. 2. The ordinary derivative gives the linear approximation.

Here we expect the fractional derivative to have the similar geometrical meaning. We hope for non-differentiable functions, the fractional derivative could give some kind approximation of its local behavior.

5. A SIMPLE DEFINITION DIRECTLY FROM GEOMETRICAL MEANING

We expect that **the fractional derivative could give nonlinear (power law) approximation of the local behavior of non-differentiable functions**:

$$f(x+h) \approx f(x) + \frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)} h^{\alpha} \tag{5.1}$$

in which the function f is not differentiable because $df \approx (dx)^{\alpha}$ so the classical derivative df/dx will diverge. Note that the purpose of adding the coefficient $\Gamma(1+\alpha)$ is just to make the formal consistency with the Taylor series.

We can give a very simple definition directly from the above meaning:

Definition 7. For function $f \in C^{\alpha}$, $0 < \alpha < 1$, the fractional derivative is defined as

$$f^{(\alpha)}(x) := \Gamma(1+\alpha) \lim_{h \to 0+} \frac{f(t+h) - f(t)}{h^{\alpha}} \tag{5.2}$$

Also the fractional integral can formally defined as:

Definition 8. For function f, the fractional integral of order $0 < \alpha < 1$ is a function $f^{(-\alpha)}(x) := g(x)$ which satisfy

$$g^{(\alpha)}(x) = \Gamma(1+\alpha) \lim_{h \to 0+} \frac{g(t+h) - g(t)}{h^{\alpha}} = f(x) \tag{5.3}$$

Remarks

(1) The derivative of constant function is zero.
(2) It could be applicable to a large class of functions C^{α} $0 < \alpha < 1$ which are not differentiable in the classical sense.
(3) It do have reasonable geometric interpretation, similar to the classical one. The fractional derivative could give nonlinear (power law) approximation of the local behavior of non-differentiable functions.
(4) The calculation of derivative is easy.
(5) Last but not least, this definition is *local* by nature. This property facilitate the generalization of one variable fractional derivative to *vector derivative*, which is very hard if the nonlocal domain and boundary condition of the function is needed for calculation.

6. GEOMETRIC CALCULUS

In order to generalize the one variable fractional derivative to vector derivative, we make use of the language of *geometric calculus* (GC) instead of Gibbs vector calculus. Geometric algebra provided an powerful mathematical framework in which the most advanced concepts modern physic, such as quantum mechanics, relativity, electromagnetism, etc., can be expressed in this framework graciously.

There are many great advantages of the GC formalism. For example, the cross product only applies in three dimensional space, because in four dimensions there is an infinity of perpendicular vectors. But in GC formalism, the inner product and outer product are unified and both have perfect generalization. Gibbs vector calculus is able to reduce Maxwell's twelve equations down to four, but in GC formalism Maxwell equation can be written in one single elegant equation with a spacetime ∇ operator which is reversible. Please see Hestenes and Sobczyk (1984); Hestenes (1966, 1999); Hestenes et al. (2001); Gull et al. (1993); Doran et al. (1993, 1996); Lasenby et al. (1998)for further information about *geometric calculus*.

The fundamental differential operator is the derivative with respect to the position vector x (any dimension). This is given the symbol ∇, and is defined in terms of its directional derivatives, with the derivative in the a direction of a general multivector M defined by

$$a \cdot \nabla M(x) = \lim_{\epsilon \to 0+} \frac{M(x+\epsilon a) - M(x)}{\epsilon}. \tag{6.1}$$

Similarly we can define the *Fractional Geometric Calculus*(FGC) as following:

Definition 9. The *Fractional Geometric Calculus* is given the symbol ∇^{α}, and is defined in terms of its fractional directional derivatives, with the fractional derivative in the a direction of a general multivector M defined by

$$a \cdot \nabla^{\alpha} M(x) = \lim_{\epsilon \to 0+} \frac{M(x+\epsilon a) - M(x)}{\epsilon^{\alpha}}. \tag{6.2}$$

The ∇^{α} acts algebraically as a vector, as well as inheriting a calculus from its directional derivatives.

As an explicit example, consider the $\{\gamma_{\mu}\}$ frame introduced above. In terms of this frame we can write the position vector x as $x^{\mu}\gamma_{\mu}$, with $x^0 = t$, $x^1 = \mathsf{x}$ *etc.* and $\{\mathsf{x}, \mathsf{y}, \mathsf{z}\}$ a usual set of Cartesian components for the rest-frame of the γ_0 vector. From the definition it is clear that

$$\gamma_{\mu} \cdot \nabla^{\alpha} = \frac{\partial^{\alpha}}{\partial x^{\mu}} \tag{6.3}$$

which we abbreviate to ∂_{μ}^{α}. From the definition we can now write

$$\nabla^{\alpha} = \gamma^{\mu}\partial_{\mu}^{\alpha} = \gamma^0 \partial_t^{\alpha} + \gamma^1 \partial_{\mathsf{x}}^{\alpha} + \gamma^2 \partial_{\mathsf{y}}^{\alpha} + \gamma^3 \partial_{\mathsf{z}}^{\alpha}. \tag{6.4}$$

7. CONCLUSION

In this paper, some different definitions of fractional derivatives are discussed and conclude that the very reason for introducing fractional derivative is to study non-differentiable functions. Then, a concise and essentially local definition of fractional derivative for one dimension function is introduced and its geometrical interpretation is given. Based on this simple definition, the fractional calculus is extended to any dimension and the *Fractional Geometric Calculus* is proposed.

There are still many open questions:

(1) Is it possible to give an explicit expression of $f^{(-\alpha)}(t)$ through $f(t)$ and α. That means a definition of some kind integration operator corresponding to the new kind of fractional differential operator.
(2) For Weierstrass function like the above, it has a fixed s, thus could do $\alpha - \mathit{derivative}$ with a fixed α. How to deal with nowhere differentiable function which has variable $\alpha(t)$?
(3) For what kind of $f(t)$, there could be non-trivial $f^{(\alpha)}(t)$ and $f^{(-\alpha)}(t)$?

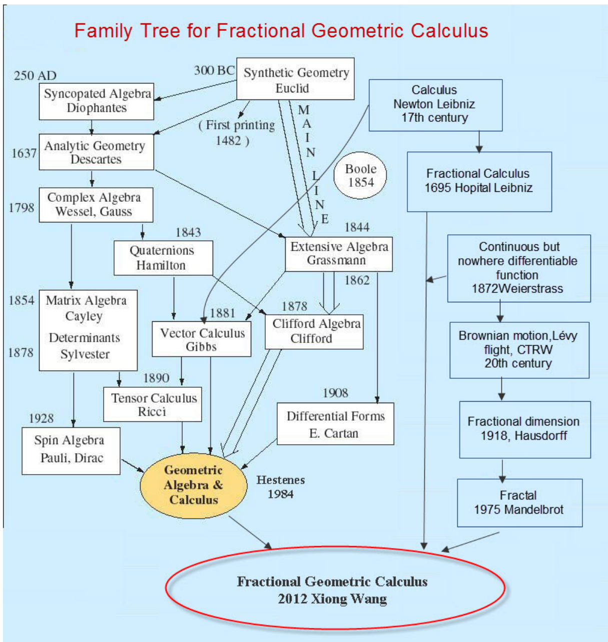

Fig. 3. Family tree for Fractional Geometric Calculus, extended from figure in http://geocalc.clas.asu.edu/html/Evolution.html

(4) Weierstrass function are $\in C^0$ but not $\in C^1$ almost everywhere. An interesting question is how to give explicit expression of all the continuous everywhere but nowhere differentiable functions, just like we can use power series to approximate and express any smooth function.
(5) Can we express some *basic* nowhere differentiable functions, and find the fractional derivative of these *basic* functions? Is it possible to give a fractional calculus table for nowhere differentiable functions, just like the classical calculus?
(6) Is it possible to give an example function $f \in C^1$ but not $\in C^2$ almost everywhere? Does the integral of Weierstrass function belong to this class? There should be such functions, but it's hard to image it. How it looks like?

Finally, we summarized the history of Fractional Geometric Calculus in the following figure 3.

ACKNOWLEDGEMENTS

Thanks my supervisor for his selfless support!

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