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CRITICAL POINT PHENOMENA — THE ROLE OF SERIES EXPANSIONS

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ABSTRACT. The use of series expansions to determine the nature of critical point singularities, especially the values of the critical exponents, is reviewed. Ratio techniques, based on extrapolation of the ratios of successive expansion coefficients, and Padé approximant methods, utilizing logarithmic derivatives, are explained, illustrated, and compared.

1. **Introduction.** In this lecture I will survey briefly the phenomena observed near the critical point of a system. The range of behavior is very varied and we shall only examine one or two typical examples. The main aim will be to establish the existence of the so-called "critical point exponents" which describe the nature of the mathematical singularities exhibited by the different thermodynamic functions as the critical point is approached. The theory of these singularities may be developed via suitable statistical-mechanical models which, while caricatures of a realistic physical system, retain such vital features as the dimensionality of space, the finite range of the interactions, and certain basic symmetry properties.

These models in turn are usually intractable as regards exact mathematical analysis. However, general methods can be developed which yield power series expansions for many functions of interest; frequently between 8 to 30 leading coefficients of such power series can be obtained. The problem is then a numerical one of extrapolating the series in such a way as to estimate the nature of the particular critical point singularity and the behavior of the function in its vicinity. I will show how this can, in favorable cases, be handled by studying the ratios of successive expansion coefficients. The method was pioneered by Domb and his coworkers, especially Sykes and Fisher [1], [2]. The reliability of the techniques can be tested against the relatively few closed form mathematical results, in particular, Onsager's renowned solution of the two-dimensional Ising models [3].

In less favorable examples this approach fails, and then, as first demonstrated by Baker [3], Padé approximants have a vital role to play. In a number of important cases the intelligent application of Padé approximants has been strikingly successful [3], [4]. Such techniques have thus established a central place for themselves in the

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