Mathematical Logic for Mathematicians, Part I

Joseph R. Mileti

September 20, 2007

Contents

Chapter 1

Introduction

1.1 The Nature of Mathematical Logic

Mathematical logic originated as an attempt to codify and formalize

1. The language of mathematics.
2. The basic assumptions of mathematics.
3. The permissible rules of proof.

One successful result of such a program is that we can study mathematical language and reasoning using mathematics. For example, we will eventually give a precise mathematical definition of a formal proof, and to avoid confusion with your current intuitive understand of what a proof is, we'll call these objects *deductions*. You should think of this as analogous to giving a precise mathematical definition of continuity to replace the fuzzy "a graph that can be drawn without lifting your pencil". Once we've codified the notion in this way, we'll have turned deductions into mathematical objects and this allows use to prove mathematical theorems about deductions using normal mathematical reasoning. Thus, we've opened up the possibility of proving that there is no deduction of a certain statement.

Some newcomers to the subject find the whole enterprise perplexing. For instance, if you come to the subject with the belief that the role of mathematical logic is to serve as a foundation to make mathematics more precise and secure, then the above description probably seems a little circular and this will almost certainly lead to a great deal of confusion. You may ask yourself:

> Okay, we've just given a decent definition of a deduction. However, instead of proving things about deductions following this formal definition, we're proving things about deductions using the usual informal proof style that I've grown accustomed to in other math courses. Why should I trust these informal proofs about deductions? How can we formally prove things (using deductions) about deductions? Isn't that circular? Is that why we're only giving informal proofs? I thought that I'd come away from this subject feeling better about the philosophical foundations of mathematics, but we've just added a new layer to mathematics and we have both informal proofs and deductions, making the whole thing even more dubious.

To others who begin a study of the subject, there is no problem. After all, mathematics is the most reliable method we have to establish truth, and there was never any serious question as to its validity. Such a person may react to the above thoughts as follows:

> We gave a mathematical definition of a deduction, so what's wrong with using mathematics to prove things about deductions? There's obviously a "real world" of true mathematics, and we're just working in that world to build a certain model of mathematical reasoning which is susceptible to mathematical analysis. It's quite cool, really, that we can subject mathematical proofs to a mathematical study by building this internal model. All of this philosophical speculation and worry about secure foundations is tiresome and in the end probably meaningless. Let's get on with the subject!

Should we be so dismissive of the first philosophically inclined student? The answer, or course, depends on your own philosophical views, but I'll try to give my own views as a mathematician specializing in logic with a definite interest in foundational questions. It is my firm belief that you should put all philosophical questions out of your mind during a first reading of the material (and perhaps forever, if you're so inclined), and come to the subject with a point of view which accepts an independent mathematical reality susceptible to the mathematical analysis you've grown accustomed to. In your mind, you should keep a careful distinction between normal "real" mathematical reasoning and the formal precise model of mathematical reasoning we are developing. Some people like to give this distinction a name by calling the normal mathematical realm we're working in the *metatheory*.

To those who are interested, we'll eventually be able to give reasonable answers to the first student and provide other respectable philosophical accounts of the nature of mathematics, but this should wait until we've developed the necessary framework. Once we've done so, we can give examples of formal theories, such as first-order set theory, which are able to support the entire enterprise of mathematics including mathematical logic. This is of great philosophical interest, because this makes it possible to carry out (nearly) all of mathematics inside this formal theory.

The ideas and techniques that were developed with philosophical goals in mind have now found application in other branches of mathematics and in computer science. The subject, like all mature areas of mathematics, has also developed its own very interesting internal questions which are often (for better or worse) divorced from its roots. Most of the subject developed after the 1930's has been concerned with these internal and tangential questions, along with applications to other areas, and now foundational work is just one small (but still important) part of mathematical logic. Thus, if you have no interest in the more philosophical aspects of the subject, there remains an impressive, beautiful, and mathematically applicable theory which is worth your attention.

1.2 The Language of Mathematics

Our first and probably most important task in providing a mathematical model of mathematics is to deal with the language of mathematics. In this section, we sketch the basic ideas and motivation for the development of a language, but we will leave precise detailed definitions until later.

The first important point is that we should not use English (or any other natural language) because it's constantly changing, often ambiguous, and allows the construction of statements that are certainly not mathematical and arguably express very subjective sentiments. Once we've thrown out natural language, our only choice is to invent our own formal language. This seems quite daunting. How could we possibly write down one formal language which encapsulates geometry, algebra, analysis, and every other field of mathematics, not to mention those we haven't developed yet, without using natural language? Our approach to this problem will be to avoid (consciously) doing it all at once.

Instead of starting from the bottom and trying to define primitive mathematical statements which can't be broken down further, let's first think about how to build new mathematical statements from old ones. The simplest way to do this is take already established mathematical statements and put them together using *and*, *or*, *not*, and *implies*. To keep a careful distinction between English and our language, we'll introduce symbols for each of these, and we'll call these symbols *connectives*.

1. $\wedge$ will denote *and*.

2. $\vee$ will denote *or*.

3. $\neg$ will denote *not*.

4. $\rightarrow$ will denote *implies*.

In order to ignore the nagging question of what constitutes a primitive statement, our first attempt will be to simply take an arbitrary set whose elements we think of as the primitive statements and put them together in all possible ways using the connectives.

For example, suppose we start with the set $P = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. We think of A, B, and C as our primitive statements, and we may or may not care what they might express. We now want to put together the elements of P using the connectives, perhaps repeatedly, but to avoid ambiguity we should be careful. Should the "meaning" of $\mathsf{A} \wedge \mathsf{B} \vee \mathsf{C}$ be "A holds, and either B holds or C holds", corresponding to $\mathsf{A} \wedge (\mathsf{B} \vee \mathsf{C})$, or should it be "Either both A and B holds, or C holds", corresponding to $(\mathsf{A} \wedge \mathsf{B}) \vee \mathsf{C}$? We need some way to avoid this ambiguity. Probably the most natural way to achieve this is to insert parentheses to make it clear how to group terms (although we'll see another natural way later). We now describe the *formulas* of our language, denoted by $Form_P$. First, we put every element of P in $Form_P$, and then we generate other formulas using the following rules.

1. If φ and ψ are in $Form_P$, then $(\varphi \wedge \psi)$ is in $Form_P$..

2. If φ and ψ are in $Form_P$, then $(\varphi \vee \psi)$ is in $Form_P$.

3. If φ is in $Form_P$, then $(\neg\varphi)$ is in $Form_P$.

4. If φ and ψ are in $Form_P$, then $(\varphi \rightarrow \psi)$ is in $Form_P$.

Thus, the following is an element of $Form_P$:

$$((\neg(\mathsf{B} \vee ((\neg\mathsf{A}) \rightarrow \mathsf{C}))) \vee \mathsf{A})$$

This simple setup, called *propositional logic*, is a drastic simplification of the language of mathematics, but there are already many interesting questions and theorems that arise from a careful study. We'll spend some time on it in Chapter 3.

Of course, mathematical language is much more rich and varied than what we can get using propositional logic. One important way to make more complicated and interesting mathematical statements is to make use of the quantifiers *for all* and *there exists* which we'll denote using the symbols $\forall$ and $\exists$. In order to do so, we will need *variables* to act as something to quantify over. We'll denote variables by letters like x, y, z, etc. Once we've come this far, however, we'll have have to refine our naive notion of primitive statements above because it's unclear how to interpret a statement like $\forall\mathsf{x}\mathsf{B}$ without knowledge of the role of x "inside" B.

Let's think a little about our primitive statements. As we mentioned above, it seems daunting to come up with primitive statements for all areas of mathematics at once, so let's think of the areas in isolation. For instance, take group theory. A group is a set G equipped with a binary operation $\cdot$ (that is, $\cdot$ takes in two elements $x, y \in G$ and produces a new element of G denoted by $x \cdot y$) and an element e such satisfying

1. Associativity: For all $x, y, z \in G$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

2. Identity: For all $x \in G$, we have $x \cdot e = x = e \cdot x$.

3. Inverses: For all $x \in G$, there exists $y \in G$ such that $x \cdot y = e = y \cdot x$.

Although it is customary and certainly easier on the eyes to put $\cdot$ between two elements of the group, let's instead use the standard function notation in order to make the mathematical notation uniform across fields. In this setting, a group is a set G equipped with a function $f\colon G \times G \to G$ and an element e satisfying

1. For all $x, y, z \in G$, we have $f(f(x, y), z) = f(x, f(y, z))$.
2. For all $x \in G$, we have $f(x, e) = x = f(e, x)$.
3. For all $x \in G$, there exists $y \in G$ such that $f(x, y) = e = f(y, x)$.

In order to allow our language to make statement about groups, we introduce a *function symbol* which we denote by f to represent the group operation, and a *constant symbol* which we denote by e to represent the group identity. Now the group operation is supposed to take in two elements of the group, so if x and y are variables, then we should allow the formation of $\mathsf{f}(\mathsf{x}, \mathsf{y})$ which should denote an element of the group (once we've assigned elements of the group to x and y). Also, we should allow the constant symbol to be used in this way, allowing us to form things like $\mathsf{f}(\mathsf{x}, \mathsf{e})$. Once we've formed these, we should be allowed to use them like variables in more complicated expressions such as $\mathsf{f}(\mathsf{f}(\mathsf{x}, \mathsf{e}), \mathsf{y})$. Each of these expressions formed by putting together, perhaps repeatedly, variables and the constant symbol e using the function symbol f is called a *term*. Intuitively, a term will name a certain element of the group once we've assigned elements to the variables.

With a way to name group elements in hand, we're now in position to say what out primitive statements are. The most basic thing that we can say about two group elements is whether or not they are equal, so we introduce a new *equality symbol*, which we will denote by the customary $=$. Given two terms t_1 and t_2, we call the expression $(t_1 = t_2)$ an *atomic formula*. These are our primitive statements.

With atomic formulas in hand, we can use the old connectives and the new quantifiers to make new statements. This puts us in a position to define *formulas*. First off, all atomic formulas are formulas. Given formulas we already know, we can put them together using the above connectives. Also, if φ is a formula and x is a variable then each of the following are formulas:

1. $\forall \mathsf{x} \varphi$
2. $\exists \mathsf{x} \varphi$

Perhaps without realizing it, we've described quite a powerful language which can make many nontrivial statements. For instance, we can write formulas in this language which express the axioms for a group:

1. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{f}(\mathsf{f}(\mathsf{x}, \mathsf{y}), \mathsf{z}) = \mathsf{f}(\mathsf{x}, \mathsf{f}(\mathsf{y}, \mathsf{z})))$
2. $\forall \mathsf{x}((\mathsf{f}(\mathsf{x}, \mathsf{e}) = \mathsf{x}) \wedge (\mathsf{f}(\mathsf{e}, \mathsf{x}) = \mathsf{x}))$
3. $\forall \mathsf{x} \exists \mathsf{y}((\mathsf{f}(\mathsf{x}, \mathsf{y}) = \mathsf{e}) \wedge (\mathsf{f}(\mathsf{y}, \mathsf{x}) = \mathsf{e}))$

We can also write a statement saying that the group is abelian:

$$\forall \mathsf{x} \forall \mathsf{y}(\mathsf{f}(\mathsf{x}, \mathsf{y}) = \mathsf{f}(\mathsf{y}, \mathsf{x}))$$

or that the center is nontrivial:

$$\exists \mathsf{x}(\neg(\mathsf{x} = \mathsf{e}) \wedge \forall \mathsf{y}(\mathsf{f}(\mathsf{x}, \mathsf{y}) = \mathsf{f}(\mathsf{y}, \mathsf{x})))$$

Perhaps unfortunately, we can write syntactically correct formulas which express things nobody would ever utter, such as:

$$\forall \mathsf{x} \exists \mathsf{y} \exists \mathsf{x}(\neg(\mathsf{e} = \mathsf{e}))$$

What if you want to do an area different from group theory? Commutative ring theory doesn't pose much of a problem, so long as we're allowed to alter the number of function symbols and constant symbols.

We can simply have two function symbols a and m which take two arguments (a to represent addition and m to represent multiplication) and two constant symbols $\mathsf{0}$ and $\mathsf{1}$ ($\mathsf{0}$ to represent the additive identity and $\mathsf{1}$ to represent the multiplicative identity). Writing the axioms for commutative rings in this language is fairly straightforward.

To take something fairly different, what about the theory of partially ordered sets? Recall that a partially ordered set is a set P equipped with a subset $\leq$ of $P \times P$, where we write $x \leq y$ to mean than (x, y) is an element of this subset, satisfying

1. Reflexive: For all $x \in P$, we have $x \leq x$.
2. Antisymmetric: If $x, y \in P$ are such that $x \leq y$ and $y \leq x$, then $x = y$.
3. Transitivity: If $x, y, z \in P$ are such that $x \leq y$ and $y \leq z$, then $x \leq z$.

Similar to how we handled the group operation, we'll use notation which puts the ordering in front of the two arguments. This may seem odd at this point, given how we're putting equality in the middle, but we'll see that this provides a unifying notation for other similar objects. We thus introduce a *relation symbol* R, and we keep the equality symbol $=$, but we no longer have a need for constant symbols or function symbols.

In this setting without constant or function symbols, the only terms that we have (i.e. the only names for elements of the partially ordered set) are the variables. However, our atomic formulas are more interesting because now there are two basic things we can say about elements of the partial ordering: whether they are equal and whether they are related by the ordering. Thus, our atomic formulas are things of the form $t_1 = t_2$ and $\mathsf{R}(t_1, t_2)$ where t_1 and t_2 are terms. From these atomic formulas, we build up all our formulas as above.

Similar to the situation for groups, we can know write formulas expressing the axioms of partial orderings:

1. $\forall \mathsf{x} \mathsf{R}(\mathsf{x}, \mathsf{x})$
2. $\forall \mathsf{x} \forall \mathsf{y}((\mathsf{R}(\mathsf{x}, \mathsf{y}) \wedge \mathsf{R}(\mathsf{y}, \mathsf{x})) \rightarrow (\mathsf{x} = \mathsf{y}))$
3. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}((\mathsf{R}(\mathsf{x}, \mathsf{y}) \wedge \mathsf{R}(\mathsf{y}, \mathsf{z})) \rightarrow \mathsf{R}(\mathsf{x}, \mathsf{z}))$

We can also write a statement saying that the partial ordering is a linear ordering:

$$\forall \mathsf{x} \forall \mathsf{y}(\mathsf{R}(\mathsf{x}, \mathsf{y}) \vee \mathsf{R}(\mathsf{y}, \mathsf{x}))$$

or that there exists a maximal element:

$$\exists \mathsf{x} \forall \mathsf{y}(\neg \mathsf{R}(\mathsf{x}, \mathsf{y}))$$

The general idea is that by leaving flexibility in the types and number of constant symbols, relation symbols, and function symbols, we'll be able to handle many areas of mathematics. We call this setup *first-order logic*. An analysis of first-order logic will consume the vast majority of our time.

Now we don't claim that first-order logic allows us to do and express everything in mathematics, nor do we claim that each of the setups above allow us to do and express everything of importance in that particular field. For example, take the group theory setting above. We can express that every nonidentity element has order two with:

$$\forall \mathsf{x}(\mathsf{f}(\mathsf{x}, \mathsf{x}) = \mathsf{e})$$

but it's unclear how to say that every element of the group has finite order. The natural guess is:

$$\forall \mathsf{x} \exists \mathsf{n}(\mathsf{x}^{\mathsf{n}} = \mathsf{e})$$

but this poses a problem for two reasons. The first is that our variables are supposed to quantify over elements of the group in question, not the natural numbers. The second is that we put no construction in our language to allow us to write something like x^{n}. For each fixed n, we can express it (for example, for

$n = 3$, we can write $\mathsf{f}(\mathsf{f}(\mathsf{x},\mathsf{x}),\mathsf{x})$ and for $n = 4$, we can write $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{x},\mathsf{x}),\mathsf{x}),\mathsf{x}))$, but it's not clear how to write it in a general way that would even allow quantification over the natural numbers.

Another example is trying to express that a group is simple (i.e. has no nontrivial normal subgroups). The natural instinct is to quantify over all subsets of the group H, and say that if it so happens that H is a normal subgroup, then H is either trivial or everything. However, we have no way to quantify over subsets. It's certainly possible to allow such constructions, and this gives *second-order logic*. If you allow quantifications over sets of subsets (for example one way of expressing that a ring is Noetherian is to say that every nonempty set of ideals has a minimal element), you get *third-order logic*, etc.

Newcomers to the field often find it strange that we focus primarily on first-order logic. There are many reasons to give special attention to first-order logic that will be developed throughout our study, but for now you should think of it as providing a simple example of a language which is capable of expressing many important aspects of various branches of mathematics.

1.3 Syntax and Semantics

Notice that in the above discussion, we introduced symbols to denote certain concepts (such as using $\wedge$ in place of "and", $\forall$ in place of "for all", and a function symbol f in place of the group operation f). Building and maintaining a careful distinction between formal symbols and how to interpret them is a fundamental aspect of mathematical logic.

The basic structure of the formal statements that we write down using the symbols, connectives, and quantifiers is known as the *syntax* of the logic that we're developing. This corresponds to the grammar of the language in question with no thought given to meaning. Imagine an English instructor who cared nothing for the content of your writings, but only that the it was grammatically correct. That is exactly what the syntax of a logic is all about. Syntax is combinatorial in nature and is based on rules which provide admissible ways to manipulate symbols devoid of meaning.

The manner in which we are permitted (or forced) to interpret the symbols, connectives, and quantifiers is known as the *semantics* of the the logic that we're developing. In a logic, some symbols are to be interpreted in only one way. For instance, in the above examples, we interpret the symbol $\wedge$ to mean *and*. In the propositional logic setting, this doesn't settle how to interpret a formula because we haven't said how to interpret the elements of P. We have some flexibility here, but once we assert that we should interpret certain elements of P as true and the others as false, our formulas express statements that are either true or false.

The first-order logic setting is more complicated. Since we have quantifiers, the first thing that must be done in order to interpret a formula is to fix a set X which will act as the set of objects over which the quantifiers will range. Once this is done, we can interpret any function symbol f taking k arguments as any actual function $f\colon X^k \to X$, any relation R symbol taking k arguments as a subset of X^k, and any constant symbol c as an element of X. Once we've fixed what we're talking about by provided such interpretations, we can view them as expressing something meaningful. For example, if we've fixed a group G interpreted f as the group operation and e as the identity, the formula

$$\forall \mathsf{x} \forall \mathsf{y}(\mathsf{f}(\mathsf{x},\mathsf{y}) = \mathsf{f}(\mathsf{y},\mathsf{x}))$$

is either true or false, according to whether G is abelian or not.

Always keep the distinction between syntax and semantics clear in your mind. The basic theorems of the subject involve the interplay between syntax and semantics. For example, in the logics we discuss, we will have two types of implication between formulas. Let Γ be a set of formulas and let φ be a formula. One way of saying that the formulas in Γ imply φ is semantic: whenever we provide an interpretation which makes all of the formulas of Γ true, it happens that φ is also true. For instance, if we're working in propositional logic and we have $\Gamma = \{((\mathsf{A} \wedge \mathsf{B}) \vee \mathsf{C})\}$ and $\varphi = (\mathsf{A} \vee \mathsf{C})$, then Γ implies φ in this sense because no matter how we assign true/false values to A, B, and C that make the formulas in Γ true, it happens that φ will also be true. Another approach that we'll develop is syntactic. We'll define deductions which are "formal proofs" built from certain permissible syntactic manipulations, and Γ will imply φ in this sense if there is a witnessing deduction. The Soundness Theorem and the Completeness Theorem for first-order logic (and propositional logic) says that the semantic version and syntactic version are the same. This result amazingly allows one to mimic mathematical reasoning with syntactic manipulations.

1.4 The Point of It All

One important aspect, often mistaken as the only aspect, of mathematical logic is that it allows us to study mathematical reasoning. A prime example of this is given by the last sentence of the previous section. The Completeness Theorem says that we can capture the idea of one mathematical statement following from other mathematical statements with nothing more than syntactic rules on symbols. This is certainly computationally, philosophically, and foundationally interesting, but it's much more than that. A simple consequence of this result is the Compactness Theorem, which says something very deep about mathematical reasoning and has many interesting applications in mathematics.

Although we've developed the above logics with modest goals of handling certain fields of mathematics, it's a wonderful and surprising fact that we can embed (nearly) all of mathematics in an elegant and natural first-order system: first-order set theory. This opens the door to the possibility of proving that certain mathematical statements are independent of our usual axioms. That is, that there are formulas φ such that there is no deduction from the usual axioms of either φ or $(\neg\varphi)$. Furthermore, the field of set theory has blossomed into an intricate field with its own deep and interesting questions.

Other very interesting and fundamental subjects arise when we ignore the foundational aspects and deductions altogether, and simply look at what we've accomplished by establishing a precise language to describe an area of mathematics. With a language in hand, we now have a way to say that certain objects are *definable* in that language. For instance, take the language of commutative rings mentioned above. If we fix a particular commutative ring, then the formula

$$\exists \mathsf{y}(\mathsf{m}(\mathsf{x},\mathsf{y}) = \mathsf{1})$$

has a free variable x and "defines" the set of units in the ring. With this point of view, we've opened up the possibility of proving lower bounds on the complexity of any definition of a certain object, or even of proving that no such definition exists in the language.

Another, closely related, way to take our definitions of precise languages and run with it is the subject of *model theory*. In group theory, you state some axioms and work from there in order to study all possible realizations of the axioms, i.e. groups. However, as we saw above, the group axioms arise in one possible language with one possible set of axioms. Instead, we can study all possible languages and all possible sets of axioms and see what we can prove in general and how the realizations compare to each other. In this sense, model theory is a kind of abstract abstract algebra.

Finally, although it's probably far from clear how it fits in at this point, *computability theory* is intimately related to the above subjects. To see the first glimmer of a connection, notice that computer programming languages are also formal languages with a precise grammar and a clear distinction between syntax and semantics. As we'll see in time, however, the connection is much deeper.

1.5 Some Basic Terminology and Notation

Definition 1.5.1. *We let* $\mathbb{N} = \{0, 1, 2, \dots\}$ *and we let* $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$.

Definition 1.5.2. *For each* $n \in \mathbb{N}$, *we let* $[n] = \{m \in \mathbb{N} : m < n\}$.

We will often find a need to work with finite sequences, so we establish notation here.

Definition 1.5.3. *Let X be a set. Given $n \in \mathbb{N}$, we call a function $\sigma\colon [n] \to X$ a* finite sequence from X of length n*. We denote the set of all finite sequences from X of length n by X^n. We use λ to denote the unique sequence of length 0, so $X^0 = \{\lambda\}$.*

Definition 1.5.4. *Let X be a set. We let $X^* = \bigcup_{n \in \mathbb{N}} X^n$, i.e. X^* is the set of all finite sequences from X.*

We denote finite sequences by simply listing the elements in order. For instance, if $X = \{a, b\}$, the sequence *aababbba* is an element of X^*. Sometimes for clarity, we'll insert commas and instead write a, a, b, a, b, b, b, a.

Definition 1.5.5. *If $\sigma, \tau \in X^*$, we say that σ is an* initial segment *of τ, and write $\sigma \subseteq \tau$, if $\sigma = \tau \upharpoonright [n]$ for some n. We say that σ is a* proper initial segment *of τ, and write $\sigma \subset \tau$, if $\sigma \subseteq \tau$ and $\sigma \neq \tau$.*

Definition 1.5.6. *If $\sigma, \tau \in X^*$, we denote the concatenation of σ and τ by $\sigma\tau$ or $\sigma * \tau$.*

Definition 1.5.7. *If $\sigma, \tau \in X^*$, we say that σ is a* substring *of τ if there exists $\theta, \rho \in X^*$ such that $\sigma = \theta\tau\rho$.*

Definition 1.5.8. *A set A is* countably infinite *if there exists a bijection $g\colon \mathbb{N} \to A$. A set A is* countable *if it is either finite or countably infinite.*

Definition 1.5.9. *Given a set A, we let $\mathcal{P}(A)$ be the set of all subsets of A, and we call $\mathcal{P}(A)$ the* power set *of A.*

Chapter 2

Induction and Recursion

The natural numbers are perhaps the only structure that you've had the pleasure of working with when doing proofs by induction or definitions by recursion, but there are many more arenas in which variants of induction and recursion apply. In fact, more delicate and exotic proofs by induction and definitions by recursion are two central tools in mathematical logic. Once we get to set theory, we'll see how to do transfinite induction and recursion, and this tool is invaluable in set theory and model theory. In this section, we develop the more modest tools of induction and recursion along structures which are generated by one-step processes, like the natural numbers.

2.1 Induction and Recursion on $\mathbb{N}$

We begin by compiling the basic facts about induction and recursion on the natural numbers. We don't seek to "prove" that proofs by induction or definitions by recursion on $\mathbb{N}$ are valid methods because these are "obvious" from the normal mathematical perspective which we are adopting. Besides, in order to do so, we would first have to fix a context in which we are defining $\mathbb{N}$, which we will do much later in the context of axiomatic set theory. Although you're no doubt familiar with the intuitive content of the results here, our goal here is simply to carefully codify these facts in more precise ways to ease the transition to more complicated types of induction and recursion.

Definition 2.1.1. *We define $S\colon \mathbb{N} \to \mathbb{N}$ by letting $S(n) = n + 1$ for all $n \in \mathbb{N}$.*

Induction is often stated in the form "If we know something holds of 0, and we know that it holds of $S(n)$ whenever it holds of n, then we know that it holds for all $n \in \mathbb{N}$". We state it in the following more precise set-theoretic fashion (avoiding explicit mention of "somethings" or "properties") because we can always form the set $X = \{n \in \mathbb{N} : \text{something holds of } n\}$.

Theorem 2.1.2 (Induction on $\mathbb{N}$ - Step Form)**.** *Suppose that $X \subseteq \mathbb{N}$ is such that $0 \in X$ and $S(n) \in X$ whenever $n \in X$. We then have $X = \mathbb{N}$.*

Definitions by recursion is usually referred to by saying that "When defining $f(S(n))$, you are allowed to refer to the value of $f(n)$". For instance, let $f\colon \mathbb{N} \to \mathbb{N}$ be the factorial function $f(n) = n!$. One usually sees this defined in the following manner:

$$f(0) = 1$$
$$f(S(n)) = S(n) \cdot f(n)$$

We aim to codify this idea a little more abstractly and rigorously in order to avoid the self-reference of f in the definition and allowable rules so that we can generalize it to other situations.

Suppose that X is a set and we're trying to define $f\colon \mathbb{N} \to X$ recursively. What do you need? Well, you need to know $f(0)$, and you need to have a method telling you how to define $f(S(n))$ from knowledge of n and the value of $f(n)$. If we want to avoid the self-referential reference to f in invoking the value of $f(n)$, what we need is a method which tells us what to do next regardless of the actual particular value of $f(n)$. That is, it needs to tell you what to do on any possible value, not just the one the ends up happening to be $f(n)$. Formally, this "method" can be given by a function $g\colon \mathbb{N} \times X \to X$ which tells you what to do at the next step. Intuitively, this function acts as an iterator. That is, it says if the the last thing you were working on was input n and it so happened that you set $f(n)$ to equal $x \in A$, then you should define $f(S(n))$ to be the value $g(n,x)$.

With all this setup, we now state the theorem which says that no matter what value you want to assign to $f(0)$, and no matter what iterating function $g\colon \mathbb{N} \times X \to X$ you give, there exists a unique function $f\colon \mathbb{N} \to X$ obeying the rules.

Theorem 2.1.3 (Recursion on $\mathbb{N}$ - Step Form)**.** *Let X be a set, let $y \in X$, and let $g\colon \mathbb{N} \times X \to X$. There exists a unique function $f\colon \mathbb{N} \to X$ such that*

1. *$f(0) = y$.*
2. *$f(S(n)) = g(n, f(n))$ for all $n \in \mathbb{N}$.*

In the case of the factorial function, we have $X = \mathbb{N}$, $y = 1$, and $g\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $g(n,x) = S(n)\cdot x$. The above theorem implies that there is a unique function $f\colon \mathbb{N} \to \mathbb{N}$ such that

1. $f(0) = y = 1$.
2. $f(S(n)) = g(n, f(n)) = S(n) \cdot f(n)$ for all $n \in \mathbb{N}$.

Notice how we've avoiding a self-reference in the definition. Our definition of g is given nonrecursively, and then the theorem states the existence and uniqueness of a function which behaves properly.

There is another version of induction on $\mathbb{N}$, sometimes called "strong induction", which appeals to the ordering of the natural numbers rather than the stepping of the successor function.

Theorem 2.1.4 (Induction on $\mathbb{N}$ - Order Form)**.** *Suppose that $X \subseteq \mathbb{N}$ is such that $n \in X$ whenever $m \in X$ for all $m < n$. We then have $X = \mathbb{N}$.*

Notice that there is no need to deal with the "base case" of $n = 0$, because this is handled vacuously due to the fact that there is no $m < 0$.

Theorem 2.1.5 (Recursion on $\mathbb{N}$ - Order Form)**.** *Let X be a set and let $g\colon X^* \to X$. There exists a unique function $f\colon \mathbb{N} \to X$ such that*

$$f(n) = g(f \upharpoonright [n])$$

for all $n \in \mathbb{N}$.

2.2 Generation

There are many situations throughout mathematics when we want to look at what a certain subset "generates". For instance, you have a subset of a group (vector space, ring), and you want to consider the subgroup (subspace, ideal) that they generate. Another example is you have a subset of a graph, and you want to consider the set of vertices in the graph reachable from the subset. In the introduction, we talked about generating all formulas from primitive formulas using certain connections. This situation will arise so frequently in what follows that it's a good idea to unify them all in a common framework.

Definition 2.2.1. *Let A be a set and let $k \in \mathbb{N}^+$. A function $h\colon A^k \to A$ is called a k-*ary function *on A. We call k the* arity *of h. A 1-ary function is sometimes called* unary *and a 2-ary function is sometimes called* binary*.*

Definition 2.2.2. *Suppose that A is a set, $B \subseteq A$ and $\mathcal{H}$ is a collection of functions such that each $h \in \mathcal{H}$ is a k-ary function on A for some $k \in \mathbb{N}^+$. We call $(A, B, \mathcal{H})$ a* simple generating system*. In such a situation, for each $k \in \mathbb{N}^+$, we denote the set of k-ary functions in $\mathcal{H}$ by $\mathcal{H}_k$.*

Examples.

1. Let G be a group and let $B \subseteq G$. We want the subgroup of G that B generates. The operations in question here are the group operation and inversion, so we let $\mathcal{H} = \{h_1, h_2\}$ where
   (a) $h_1\colon G^2 \to G$ is given by $h_1(x, y) = x \cdot y$ for all $x, y \in G$.
   (b) $h_2\colon G \to G$ is given by $h_2(x) = x^{-1}$ for all $x \in G$.

   $(G, B, \mathcal{H})$ is a simple generating system.

2. Let V be a vector space over $\mathbb{R}$ and let $B \subseteq V$. We want the subspace of V that B generates. The operations in question consist of vector addition and scalar multiplication, so we let $\mathcal{H} = \{g\} \cup \{h_\alpha : \alpha \in \mathbb{R}\}$ where
   (a) $g\colon V^2 \to V$ is given by $g(v, w) = v + w$ for all $v, w \in V$.
   (b) For each $\alpha \in \mathbb{R}$, $h_\alpha\colon V \to V$ is given by $h_\alpha(v) = \alpha \cdot v$ for all $v \in V$.

   $(V, B, \mathcal{H})$ is a simple generating system.

□

There are certain cases when the natural functions to put into $\mathcal{H}$ are not total or are "multi-valued". For instance, in the first example below, we'll talk about the subfield generated by a certain subset of a field, and we'll want to include multiplicative inverses for all nonzero elements. When putting a corresponding function in $\mathcal{H}$, there is no obvious way to define it on 0. Also, if generating the vertices reachable from a subset of a graph, we may want to throw many vertices in because a vertex can be linked to many others.

Definition 2.2.3. *Let A be a set and let $k \in \mathbb{N}^+$. A function $h\colon A^k \to \mathcal{P}(A)$ is called a* set-valued k-ary function *on A. We call k the* arity *of h. A 1-ary set-valued function is sometimes called* unary *and a 2-ary set-valued function is sometimes called* binary*.*

Definition 2.2.4. *Suppose that A is a set, $B \subseteq A$ and $\mathcal{H}$ is a collection of functions such that each $h \in \mathcal{H}$ is a set-valued k-ary function on A for some $k \in \mathbb{N}^+$. We call $(A, B, \mathcal{H})$ a* generating system*. In such a situation, for each $k \in \mathbb{N}^+$, we denote the set of multi-valued k-ary functions in $\mathcal{H}$ by $\mathcal{H}_k$.*

Examples.

1. Let K be a field and let $B \subseteq K$. We want the subfield of K that B generates. The operations in question here are addition, multiplication, and both additive and multiplicative inverses. We thus let $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ where
   (a) $h_1\colon K^2 \to \mathcal{P}(K)$ is given by $h_1(a, b) = \{a + b\}$ for all $a, b \in K$.
   (b) $h_2\colon K^2 \to \mathcal{P}(K)$ is given by $h_2(a, b) = \{a \cdot b\}$ for all $a, b \in K$.
   (c) $h_3\colon K \to \mathcal{P}(K)$ is given by $h_3(a) = \{-a\}$ for all $a \in K$.

(d) $h_4\colon K \to \mathcal{P}(K)$ is given by

$$h_4(a) = \begin{cases} \{a^{-1}\} & \text{if } a \neq 0 \\ \emptyset & \text{if } a = 0 \end{cases}$$

$(K, B, \mathcal{H})$ is a generating system.

2. Let G be a graph with vertex set V and edge set E, and let $B \subseteq V$. We want to consider the subset of V reachable from B using edges from E. Thus, we want to say that if we've generated $v \in V$, and $w \in V$ is connected to v via some edge, then we should generate w. We thus let $\mathcal{H} = \{h\}$ where $h\colon V \to V$ is defined as follows:

$$h(v) = \{u \in V : (v, u) \in E\}$$

$(V, B, \mathcal{H})$ is a generating system.

□

Notice that if we have a simple generating system $(A, B, \mathcal{H})$, then we can associate to it the generating system $(A, B, \mathcal{H}')$ where $\mathcal{H}' = \{h' : h \in \mathcal{H}\}$ where if $h\colon A^k \to A$ is an element of $\mathcal{H}_k$, then $h'\colon A^k \to \mathcal{P}(A)$ is defined by letting $h'(a_1, a_2, \dots, a_k) = \{h(a_1, a_2, \dots, a_k)\}$.

Given a generating system $(A, B, \mathcal{H})$, we want to define the set of elements of A generated from B using the functions in $\mathcal{H}$. There are many natural ways of doing this. We discuss three different ways which divide into approaches "from above" and approaches "from below". Each of these descriptions can be slightly simplified for simple generating systems, but it's not much harder to handle the more general case.

2.2.1 From Above

Our first approach is a "top-down" approach.

Definition 2.2.5. *Let $(A, B, \mathcal{H})$ be a generating system, and let $J \subseteq A$. We say that J is* inductive *if*

1. *$B \subseteq J$.*
2. *If $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \dots, a_k \in J$, then $h(a_1, a_2, \dots, a_k) \subseteq J$.*

Given a generating system $(A, B, \mathcal{H})$, we certainly have a candidate for an inductive set, namely A itself. However, this set may be too big. For instance, consider the generating system $A = \mathbb{R}$, $B = \{7\}$, and $\mathcal{H} = \{h\}$ where $h\colon \mathbb{R} \to \mathbb{R}$ is the function $h(x) = 2x$. In this situation, each of the sets $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\{n \in \mathbb{N} : n \text{ is a multiple of } 7\}$ is inductive, but they're not what we want. The idea is to consider the *smallest* inductive subset of A containing B. Of course, we need to prove that such a set exists.

Proposition 2.2.6. *Let $(A, B, \mathcal{H})$ be a generating system. There exists a unique inductive set I such that $I \subseteq J$ for every inductive set J.*

Proof. We first prove existence. Let I be the intersection of all inductive sets, i.e. $I = \{a \in A : a \in J \text{ for every inductive set } J\}$. By definition, we have $I \subseteq J$ for every inductive set J, so we need only show that I is inductive. Since $B \subseteq J$ for every inductive set J (by definition of inductive), it follows that $B \subseteq I$. Suppose that $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$ and $a_1, a_2, \dots, a_k \in I$. For any inductive set J, we have $a_1, a_2, \dots, a_k \in J$, hence $h(a_1, a_2, \dots, a_k) \subseteq J$ because J is inductive. Therefore, $h(a_1, a_2, \dots, a_k) \subseteq J$ for every inductive set J, hence $h(a_1, a_2, \dots, a_k) \subseteq I$. It follows that I is inductive.

To see uniqueness suppose that both I_1 and I_2 are inductive sets such that $I_1 \subseteq J$ and $I_2 \subseteq J$ for every inductive set J. We then have $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$, hence $I_1 = I_2$. □

Definition 2.2.7. *Let $(A, B, \mathcal{H})$ be a generating system. We denote the unique set of the previous proposition by $I(A, B, \mathcal{H})$, or simply by I when the context is clear.*

2.2.2 From Below: Building by Levels

The second idea is to make a system of levels, at each new level adding elements of A which are reachable from elements already accumulated by applying an element of $\mathcal{H}$.

Definition 2.2.8. *Let $(A, B, \mathcal{H})$ be a generating system. We define a sequence $V_n(A, B, \mathcal{H})$, or simply V_n, recursively as follows.*

$$V_0 = B$$

$$V_{n+1} = V_n \cup \{c \in A : \textit{There exists } k \in \mathbb{N}^+, h \in \mathcal{H}_k, \textit{ and } a_1, a_2, \ldots, a_k \in V_n \textit{ such that } c \in h(a_1, a_2, \ldots, a_k)\}$$

Let $V(A, B, \mathcal{H}) = V = \bigcup_{n\in\mathbb{N}} V_n$.

The following remarks are immediate from our definition.

Remark 2.2.9. *Let $(A, B, \mathcal{H})$ be a generating system.*

1. *If $m \leq n$, then $V_m \subseteq V_n$.*
2. *For all $c \in V$, either $c \in B$ or there exists $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \ldots, a_k \in V$ with $c \in h(a_1, a_2, \ldots, a_k)$.*

2.2.3 From Below: Witnessing Sequences

The third method is to consider those elements of A which you are forced to put in because you see a witnessing construction.

Definition 2.2.10. *Let $(A, B, \mathcal{H})$ be a generating system. A* witnessing sequence *is an element $\sigma \in A^* \backslash \{\lambda\}$ such that for all $j < |\sigma|$, either*

1. *$\sigma(j) \in B$*
2. *There exists $k \in \mathbb{N}$, $h \in \mathcal{H}_k$ and $i_1, i_2, \ldots, i_k < j$ such that $\sigma(j) \in h(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$.*

If σ is a witnessing sequence, we call it a witnessing sequence for $\sigma(|\sigma| - 1)$.

Definition 2.2.11. *Let $(A, B, \mathcal{H})$ be a generating system. Set*

$$W(A, B, \mathcal{H}) = W = \{a \in A : \textit{ there exists a witnessing sequence for } a\}.$$

It sometimes useful to look only at those elements reachable which are witnessed by sequences of a bounded length, so for each $n \in \mathbb{N}^+$, set

$$W_n = \{a \in A : \textit{ there exists a witnessing sequence for } a \textit{ of length } n\}.$$

The first simple observation is that if we truncate a witnessing sequence, what remains is a witnessing sequence.

Remark 2.2.12. *If σ is a witnessing sequence and $|\sigma| = n$, then for all $m \in \mathbb{N}^+$ with $m < n$ we have that $\sigma \upharpoonright [m]$ is a witnessing sequence.*

Another straightforward observation is that if we concatenate two witnessing sequences, the result is a witnessing sequence.

Proposition 2.2.13. *If σ and τ are witnessing sequences, then so is $\sigma\tau$.*

Finally, since we can always insert "dummy" elements from B (assuming it's nonempty because otherwise the result is trivial), we have the following observation.

Proposition 2.2.14. *Let $(A, B, \mathcal{H})$ be a generating system. If $m \leq n$, then $W_m \subseteq W_n$.*

2.2.4 Equivalence of the Definitions

Theorem 2.2.15. *Let $(A, B, \mathcal{H})$ be a generating system. We then have*

$$I(A, B, \mathcal{H}) = V(A, B, \mathcal{H}) = W(A, B, \mathcal{H})$$

Proof. Let $I = I(A, B, \mathcal{H})$, $V = V(A, B, \mathcal{H})$, and $W = W(A, B, \mathcal{H})$.

We first show that V is inductive, hence $I \subseteq V$. Notice first that $B = V_0 \subseteq V$. Suppose now that $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$ and $a_1, a_2, \dots, a_k \in V$. For each i, fix n_i such that $a_i \in V_{n_i}$. Let $m = \max\{n_1, n_2, \dots, n_k\}$. We then have $a_i \in V_m$ for all i, hence $h(a_1, a_2, \dots, a_k) \subseteq V_{m+1} \subseteq V$. It follows that V is inductive.

We next show that W is inductive, hence $I \subseteq W$. Notice first that for every $b \in B$, the sequence b is a witnessing sequence, so $b \in W_1 \subseteq W$. Suppose now that $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \dots, a_k \in W$. Let $c \in h(a_1, a_2, \dots, a_k)$. For each i, fix a witnessing sequence σ_i for a_i. The sequence $\sigma_1\sigma_2 \cdots \sigma_k c$ is a witnessing sequence for c. Therefore. $h(a_1, a_2, \dots, a_k) \subseteq W$. It follows that W is inductive.

We next show that $V_n \subseteq I$ by induction on n, and hence $V \subseteq I$. Notice first that $V_0 = B \subseteq I$. Suppose now that $n \in \mathbb{N}$ and $V_n \subseteq I$. Fix $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \dots, a_k \in V_n$. Since $V_n \subseteq I$, we have $a_1, a_2, \dots, a_k \in I$, hence $h(a_1, a_2, \dots, a_k) \subseteq I$ because I is inductive. It follows that $V_{n+1} \subseteq I$. By induction, $V_n \subseteq I$ for every $n \in \mathbb{N}$, hence $V \subseteq I$.

We next show that $W_n \subseteq I$ by induction on $n \in \mathbb{N}^+$, and hence $W \subseteq I$. Notice first that $W_1 = B \subseteq I$. Suppose now that $n \in \mathbb{N}^+$ and $W_n \subseteq I$. Let σ be an witnessing sequence of length $n+1$. We then have that that $\sigma \upharpoonright [m+1]$ is a witnessing sequence of length $m+1$ for all $m < n$, hence $\sigma(m) \in W_m \subseteq W_n \subseteq I$ for all $m < n$. Now either $\sigma(n) \in B$ or there exists $i_1, i_2, \dots, i_k < n$ such that $\sigma(n) = h(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k))$. In either case, $\sigma(n) \in I$ because I is inductive. It follows that $W_{n+1} \subseteq I$. By induction, $W_n \subseteq I$ for every $n \in \mathbb{N}^+$, hence $W \subseteq I$. □

Definition 2.2.16. *Let $(A, B, \mathcal{H})$ be a generating system. We denote the common value of I, V, W by $G(A, B, \mathcal{H})$ or simply G.*

The nice thing about having multiple equivalent definitions for the same concept is that we can use the most convenient one when proving a theorem. For example, using (2) of Remark 2.2.9, we get the following corollary.

Corollary 2.2.17. *Let $(A, B, \mathcal{H})$ be a generating system. For all $c \in G$, either $c \in B$ or there exists $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \dots, a_k \in G$ with $c \in h(a_1, a_2, \dots, a_k)$.*

2.3 Step Induction

Here's a simple example of using the I definition to prove that we can argue by induction.

Proposition 2.3.1 (Step Induction)**.** *Let $(A, B, \mathcal{H})$ be a generating system. Suppose that $X \subseteq A$ satisfies*

1. *$B \subseteq X$.*
2. *$h(a_1, a_2, \dots, a_k) \subseteq X$ whenever $k \in \mathbb{N}^+$, $h \in \mathcal{H}_k$, and $a_1, a_2, \dots, a_k \in X$.*

We then have that $G \subseteq X$. Thus, if $X \subseteq G$, we have $X = G$.

Proof. Our assumption simply asserts that X is inductive, hence $G = I \subseteq X$. □

The next example illustrates how we can sometimes identify G explicitly. Notice that we use 2 different types of induction in the argument. One direction uses induction on $\mathbb{N}$ and the other uses induction on G as just described.

Example 2.3.2. *Consider the following simple generating system. Let $A = \mathbb{R}$, $B = \{7\}$, and $\mathcal{H} = \{h\}$ where $h\colon \mathbb{R} \to \mathbb{R}$ is the function $h(x) = 2x$. Determine G explicitly.*

Proof. Intuitively, we want the set $\{7, 14, 28, 56, \dots\}$, which we can write more formally as $\{7 \cdot 2^n : n \in \mathbb{N}\}$. Let $X = \{7 \cdot 2^n : n \in \mathbb{N}\}$

We first show that $X \subseteq G$ by showing that $7 \cdot 2^n \in G$ for all $n \in \mathbb{N}$ by induction (on $\mathbb{N}$). We have $7 \cdot 2^0 = 7 \cdot 1 = 7 \in G$ because $B \subseteq G$ as G is inductive. Suppose that $n \in \mathbb{N}$ is such that $7 \cdot 2^n \in G$. Since G is inductive, it follows that $h(7 \cdot 2^n) = 2 \cdot 7 \cdot 2^n = 7 \cdot 2^{n+1} \in G$. Therefore, $7 \cdot 2^n \in G$ for all $n \in \mathbb{N}$ by induction, hence $X \subseteq G$.

We now show that $G \subseteq X$ by induction (on G). Notice that $B \subseteq X$ because $7 = 7 \cdot 1 = 7 \cdot 2^0 \in X$. Suppose now that $x \in X$ and fix $n \in \mathbb{N}$ with $x = 7 \cdot 2^n$. We then have $h(x) = 2 \cdot x = 7 \cdot 2^{n+1} \in X$. Therefore $G \subseteq X$ by induction.

It follows that $X = G$. □

In many cases, it's very hard to give a simple explicit description of the set G. This is where induction really shines, because it allows us to prove something about all elements of G despite the fact that we have a hard time getting a handle on what exactly the elements of G look like. Here's an example.

Example 2.3.3. *Consider the following simple generating system. Let $A = \mathbb{Z}$, $B = \{6, 183\}$, and $\mathcal{H} = \{h\}$ where $h\colon A^3 \to A$ is given by $h(k, m, n) = k \cdot m + n$. Every element of G is divisible by 3.*

Proof. Let $X = \{n \in \mathbb{Z} : n \text{ is divisible by } 3\}$. We prove by induction that $G \subseteq X$. We first handle the bases case. Notice that $6 = 3 \cdot 2$ and $183 = 3 \cdot 61$, so $B \subseteq X$.

We now do the inductive step. Suppose that $k, m, n \in X$, and fix $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}$ with $k = 3\ell_1$, $m = 3\ell_2$, and $n = 3\ell_3$. We then have

$$\begin{aligned} h(k, m, n) &= k \cdot m + n \\ &= (3\ell_1) \cdot (3\ell_2) + 3\ell_3 \\ &= 9\ell_1\ell_2 + 3\ell_3 \\ &= 3(3\ell_1\ell_2 + \ell_3) \end{aligned}$$

hence $h(k, m, n) \in X$.

It follows by induction that $G \subseteq X$, i.e. that every element of G is divisible by 3. □

2.4 Step Recursion

In this section, we restrict attention to simple generating systems for simplicity (and also because all examples that we'll need which support definition by recursion will be simple). Naively, one might expect that a straightforward analogue of Step Form of Recursion on $\mathbb{N}$ will carry over to recursion on generated sets. The hope would be the following.

Hope 2.4.1. *Suppose that $(A, B, \mathcal{H})$ is a simple generating system and X is a set. Suppose also that $\iota\colon B \to X$ and that for every $h \in \mathcal{H}_k$, we have a function $g_h\colon (A \times X)^k \to X$. There exists a unique function $f\colon G \to X$ such that*

1. *$f(b) = \iota(b)$ for all $b \in B$.*
2. *$f(h(a_1, a_2, \dots, a_k)) = g_h(a_1, f(a_1), a_2, f(a_2), \dots, a_k, f(a_k))$ for all $a_1, a_2, \dots, a_k \in G$.*

Unfortunately, this hope is too good to be true. Intuitively, we may generate an element a of A in many very different ways, and our different iterating functions conflict on what values we should assign to a. Here's a simple example to see what can go wrong.

Example 2.4.2. *Consider the following simple generating system. Let $A = \{1,2\}$, $B = \{1\}$, and $\mathcal{H} = \{h\}$ where $h\colon A \to A$ is given by $h(1) = 2$ and $h(2) = 1$. Let $X = \mathbb{N}$. Define $\iota\colon B \to \mathbb{N}$ by letting $\iota(1) = 1$ and define $g_h\colon A \times \mathbb{N} \to \mathbb{N}$ by letting $g_h(a,n) = n+1$. There is no function $f\colon G \to \mathbb{N}$ such that*

1. *$f(b) = \iota(b)$ for all $b \in B$.*
2. *$f(h(a)) = g_h(a, f(a))$ for all $a \in G$.*

Proof. Notice first that $G = \{1,2\}$. Suppose that $f\colon G \to \mathbb{N}$ satisfies (1) and (2) above. Since f satisfies (1), we must have $f(1) = \iota(1) = 1$. By (2), we then have that

$$f(2) = f(h(1)) = g_h(1, f(1)) = f(1) + 1 = 1 + 1 = 2.$$

By (2) again, it follows that

$$f(1) = f(h(2)) = g_h(2, f(2)) = f(2) + 2 = 1 + 2 = 3,$$

contradicting the fact that $f(1) = 1$. □

To get around this problem, we want a definition of a "nice" simple generating system. Intuitively, we want to say something like "every element of G is generated in a unique way". The following definition is a relatively straightforward way to formulate this.

Definition 2.4.3. *A simple generating system $(A, B, \mathcal{H})$ is* free *if*

1. *$ran(h \restriction G^k) \cap B = \emptyset$ whenever $h \in \mathcal{H}_k$.*
2. *$h \restriction G^k$ is injective for every $h \in \mathcal{H}_k$.*
3. *$ran(h_1 \restriction G^k) \cap ran(h_2 \restriction G^\ell) = \emptyset$ whenever $h_1 \in \mathcal{H}_k$ and $h_2 \in \mathcal{H}_\ell$ with $h_1 \neq h_2$.*

Here's a simple example which will play a role for us in Section 2.5. We'll see more subtle and important examples when we come to Propositional Logic and First-Order Logic.

Example 2.4.4. *Let X be a set. Consider the following simple generating system. Let $A = X^*$, let $B = X$, and let $\mathcal{H} = \{h_x : x \in X\}$ where $h_x\colon X^* \to X^*$ is the function $h_x(\sigma) = x\sigma$. We then have that $G = X^* \backslash \{\lambda\}$ and that $(A, B, \mathcal{H})$ is free.*

Proof. First notice that $X^* \backslash \{\lambda\}$ is inductive because $\lambda \notin B$ and $h_x(\sigma) \neq \lambda$ for all $\sigma \in X^*$. Next, a simple induction on n shows that $X^n \subseteq G$ for all $n \in \mathbb{N}^+$. It follows that $G = X^* \backslash \{\lambda\}$.

We now show that $(A, B, \mathcal{H})$ is free. First notice that for any $x \in X$, we have that $\mathrm{ran}(h_x \restriction G) \cap X = \emptyset$ because every elemnts of $\mathrm{ran}(h_x \restriction G)$ has length at least 2 (because $\lambda \notin G$).

Now for any $x \in X$, we have that $h_x \restriction G$ is injective because if $h_x(\sigma) = h_x(\tau)$, then $x\sigma = x\tau$, and hence $\sigma = \tau$.

Finally, notice that if $x, y \in X$ with $x \neq y$, we have that $\mathrm{ran}(h_x \restriction G) \cap \mathrm{ran}(h_y \restriction G) = \emptyset$ because every elements of $\mathrm{ran}(h_x \restriction G)$ begins with x while every element of $\mathrm{ran}(h_y \restriction G)$ begins with y. □

On to the theorem which says that if a simple generating system is free, then we can perform recursive definitions on it.

Theorem 2.4.5. *Suppose that the simple generating system $(A, B, \mathcal{H})$ is free and X is a set. Suppose also that $\iota\colon B \to X$ and that for every $h \in \mathcal{H}_k$, we have a function $g_h\colon (A \times X)^k \to X$. There exists a unique function $f\colon G \to X$ such that*

1. *$f(b) = \iota(b)$ for all $b \in B$.*

2. *$f(h(a_1, a_2, \ldots, a_k)) = g_h(a_1, f(a_1), a_2, f(a_2), \ldots, a_k, f(a_k))$ for all $h \in \mathcal{H}_k$ and all $a_1, a_2, \ldots, a_k \in G$.*

Proof. We first prove the existence of an f by using a fairly slick argument. The basic idea is to build a new simple generating system whose elements are pairs (a, x) where $a \in A$ and $x \in X$. Intuitively, we want to generate the pair (a, x) if something (either ι or one the g_h functions) tells us that we'd better set $f(a) = x$ if we want to satisfy the above the conditions. We then go on to prove (by induction on G) that for every $a \in A$, there exists a unique $x \in X$ such that (a, x) is in our new generating system. Thus, there are no conflicts, so we can use this to define our function.

Now for the details. Let $A' = A \times X$, $B' = \{(b, \iota(b)) : b \in B\} \subseteq A'$, and $\mathcal{H}' = \{g'_h : h \in \mathcal{H}\}$ where for each $h \in \mathcal{H}_k$, the function $g'_h \colon (A \times X)^k \to A \times X$ is given by

$$g'_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k) = (h(a_1, a_2, \ldots, a_k), g_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)).$$

Let $G' = G(A', B', \mathcal{H}')$. A simple induction (on G') shows that if $(a, x) \in G'$, then $a \in G$. Let

$$Z = \{a \in G : \text{ there exists a unique } x \in X \text{ such that } (a, x) \in G'\}$$

We prove by induction (on G) that $Z = G$.

Base Case: Notice that for each $b \in B$, we have $(b, \iota(b)) \in B' \subseteq G'$, hence there exists $x \in X$ such that $(b, x) \in G'$. Fix $b \in B$ and suppose that $y \in X$ is such that $(b, y) \in G'$ and $y \neq \iota(b)$. We then have $(b, y) \notin B'$, hence by Corollary 2.2.17 there exists $h \in \mathcal{H}_k$ and $(a_1, x_1), (a_2, x_2), \ldots, (a_k, x_k) \in G'$ such that

$$\begin{aligned}(b, y) &= g'_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)\\ &= (h(a_1, a_2, \ldots, a_k), g_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)).\end{aligned}$$

Since $a_1, a_2, \ldots, a_k \in G$, this contradicts the fact that $\text{ran}(h \upharpoonright G^k) \cap B = \emptyset$. Therefore, for every $b \in B$, there exists a unique $x \in X$, namely $\iota(b)$, such that $(b, x) \in G'$. Thus, $B \subseteq Z$.

Inductive Step: Fix $h \in \mathcal{H}_k$, and suppose that $a_1, a_2, \ldots, a_k \in Z$. For each i, let x_i be the unique element of X with $(a_i, x_i) \in G'$. Notice that

$$(h(a_1, a_2, \ldots, a_k), g_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)) = g'_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k) \in G'$$

hence there exists $x \in X$ such that $(h(a_1, a_2, \ldots, a_k), x) \in G'$. Suppose now that $y \in X$ is such that $(h(a_1, a_2, \ldots, a_k), y) \in G'$. We have $(h(a_1, a_2, \ldots, a_k), y) \notin B'$ because $\text{ran}(h \upharpoonright G^k) \cap B = \emptyset$, so there exists $\hat{h} \in \mathcal{H}_\ell$ together with $(c_1, z_1), (c_2, z_2), \ldots, (c_\ell, z_\ell) \in G'$ such that

$$(h(a_1, a_2, \ldots, a_k), y) = (\hat{h}(c_1, c_2, \ldots, c_\ell), g_{\hat{h}}(c_1, z_1, c_2, z_2, \ldots, c_\ell, z_\ell)).$$

Since $c_1, c_2, \ldots, c_\ell \in G$, it follows that $h = \hat{h}$ because $\text{ran}(h \upharpoonright G^k) \cap \text{ran}(\hat{h} \upharpoonright G^\ell) = \emptyset$ if $h \neq \hat{h}$, and hence $k = \ell$. Also, since $h \upharpoonright G^k$ is injective, it follows that $a_i = c_i$ for all i. We therefore have $y = g_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)$. Therefore, there exists a unique $x \in X$, namely $g_h(a_1, x_1, a_2, x_2, \ldots, a_k, x_k)$, such that $(h(a_1, a_2, \ldots, a_k), x) \in G'$. It now follows by induction that $Z = G$.

Define $f \colon G \to X$ by letting $f(a)$ be the unique $x \in X$ such that $(a, x) \in G'$. We need to check that f satisfies the needed conditions. As stated above, for each $b \in B$, we have $(b, \iota(b)) \in G'$, so $f(b) = \iota(b)$. Thus, f satisfies condition (1). Suppose now that $h \in \mathcal{H}_k$ and all $a_1, a_2, \ldots, a_k \in G$. We have $(a_i, f(a_i)) \in G'$ for all i, hence

$$(h(a_1, a_2, \ldots, a_k), g_h(a_1, f(a_1), a_2, f(a_2), \ldots, a_k, f(a_k))) \in G'$$

by the above comments. It follows that $f(h(a_1, a_2, \ldots, a_k)) = g_h(a_1, f(a_1), a_2, f(a_2), \ldots, a_k, f(a_k))$. Thus, f also satisfies condition (2).

Finally, we need show that f is unique. Suppose that $f_1, f_2 \colon G \to X$ satisfy the conditions (1) and (2). Let $Y = \{a \in G : f_1(a) = f_2(a)\}$ We show that $Y = G$ by induction on G. First notice that for any $b \in B$ we have

$$f_1(b) = \iota(b) = f_2(b)$$

hence $b \in Y$. It follows that $B \subseteq Y$. Suppose now that $h \in \mathcal{H}_k$ and $a_1, a_2, \dots, a_k \in Y$. Since $a_i \in Y$ for each i, we have $f_1(a_i) = f_2(a_i)$ for each i, and hence

$$\begin{aligned} f_1(h(a_1, a_2, \dots, a_k)) &= g_h(a_1, f_1(a_1), a_2, f_1(a_2), \dots, a_k, f_1(a_k)) \\ &= g_h(a_1, f_2(a_1), a_2, f_2(a_2), \dots, a_k, f_2(a_k)) \\ &= f_2(h(a_1, a_2, \dots, a_k)) \end{aligned}$$

Thus, $h(a_1, a_2, \dots, a_k) \in Y$. It follows by induction that $Y = G$, i.e. $f_1(a) = f_2(a)$ for all $a \in G$. □

2.5 An Illustrative Example

We now embark on a careful formulation and proof of the statement: If $f\colon A^2 \to A$ is associative, i.e. $f(a, f(b, c)) = f(f(a, b), c)$ for all $a, b, c \in A$, then any "grouping" of terms which preserves the ordering of the elements inside the grouping gives the same value. In particular, if we are working in a group A, then we can write things like *acabba* without parentheses because any allowable insertion of parentheses gives the same value.

Throughout this section, let A be a set not containing the symbols [,], or $\star$. Let $Sym_A = A \cup \{[,], \star\}$.

Definition 2.5.1. *Define a binary function* $h\colon (Sym_A^*)^2 \to Sym_A^*$ *by letting* $h(\sigma, \tau)$ *be the sequence* $[\sigma \star \tau]$. *Let* $ValidExp_A = G(Sym_A^*, A, \{h\})$ *(viewed as a simple generating system).*

For example, suppose that $A = \{a, b, c\}$. Typical elements of $G(Sym_A^*, A, \{h\})$ are c, $[b \star [a \star c]]$ and $[c \star [[c \star b] \star a]$. The idea now is that if we have a particular function $f\colon A^2 \to A$, we can intrepret $\star$ as application of the function, and then this should give us a way to "make sense of", that is evaluate, any element of $ValidExp_A$.

2.5.1 Proving Freeness

Proposition 2.5.2. *The simple generating system* $(Sym_A^*, A, \{h\})$ *is free.*

Definition 2.5.3. *Define* $K\colon Sym_A^* \to \mathbb{Z}$ *as follows. We first define* $w\colon Sym_A \to \mathbb{Z}$ *by letting* $w(a) = 0$ *for all* $a \in A$*, letting* $w([) = -1$*, and letting* $w(]) = 1$*. We then define* $K\colon Sym_A^* \to \mathbb{Z}$ *by letting* $K(\lambda) = 0$ *and letting* $K(\sigma) = \sum_{i<|\sigma|} w(\sigma(i))$ *for all* $\sigma \in Sym_A^* \backslash \{\lambda\}$.

Remark 2.5.4. *If* $\sigma, \tau \in Sym_A^*$*, then* $K(\sigma\tau) = K(\sigma) + K(\tau)$.

Proposition 2.5.5. *If* $\varphi \in ValidExp_A$*, then* $K(\varphi) = 0$.

Proof. The proof is by induction on φ. In other words, we let $X = \{\varphi \in ValidExp_A : K(\varphi) = 0\}$, and we prove by induction that $X = ValidExp_A$. Notice that for every $a \in A$, we have that $K(a) = 0$. Suppose that $\varphi, \psi \in ValidExp_A$ are such that $K(\varphi) = 0 = K(\psi)$. We then have that

$$\begin{aligned} K([\varphi \star \psi]) &= K([) + K(\varphi) + +K(\star) + K(\psi) + K(]) \\ &= -1 + 0 + 0 + 0 + 1 \\ &= 0. \end{aligned}$$

The result follows by induction. □

Proposition 2.5.6. *If* $\varphi \in ValidExp_A$ *and* $\sigma \subset \varphi$ *with* $\sigma \neq \lambda$*, then* $K(\sigma) \leq -1$.

Proof. In other words, we let $X = \{\varphi \in ValidExp_A : \text{whenever } \sigma \subset \varphi \text{ and } \sigma \neq \lambda, \text{ we have that } K(\sigma) \leq -1\}$, and we prove by induction that $X = ValidExp_A$.

For every $a \in A$, this is trivial because there is no $\sigma \neq \lambda$ with $\sigma \subset a$.

Suppose that $\varphi, \psi \in ValidExp_A$ and the result holds for φ and ψ. We prove the result for $[\varphi \star \psi]$. Suppose that $\sigma \subset [\varphi \star \psi]$ and $\sigma \neq \lambda$. If σ is $[$, then $K(\sigma) = -1$. If σ is $[\tau$ where $\tau \neq \lambda$ and $\tau \subset \varphi$, then

$$\begin{aligned} K(\sigma) &= -1 + K(\tau) \\ &\leq -1 - 1 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

If σ is $[\varphi$ or $[\varphi\star$, then

$$\begin{aligned} K(\sigma) &= -1 + K(\varphi) \\ &= -1 + 0 && \text{(by Proposition 2.5.5)} \\ &= -1. \end{aligned}$$

If σ is $[\varphi \star \tau$, where $\tau \neq \lambda$ and $\tau \subset \varphi$, then

$$\begin{aligned} K(\sigma) &= -1 + K(\varphi) + K(\tau) \\ &= -1 + 0 + K(\tau) && \text{(by Proposition 2.5.5)} \\ &\leq -1 + 0 - 1 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

Otherwise, σ is $[\varphi \star \psi$, and

$$\begin{aligned} K(\sigma) &= -1 + K(\varphi) + K(\psi) \\ &= -1 + 0 + 0 && \text{(by Proposition 2.5.5)} \\ &= -1. \end{aligned}$$

Thus, the result holds for $[\varphi \star \psi]$. □

Corollary 2.5.7. *If $\varphi, \psi \in ValidExp_A$, then $\varphi \not\subset \psi$.*

Proof. This follows by combining Proposition 2.5.5 and Proposition 2.5.6, along with noting that $\lambda \notin ValidExp_A$ (which follows by a trivial induction). □

Theorem 2.5.8. *The generating system $(Sym_A^*, A, \{h\})$ is free.*

Proof. First notice that $\text{ran}(h \upharpoonright (ValidExp_A)^2) \cap A = \emptyset$ because all elements of $\text{ran}(h)$ begin with $[$.

Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in ValidExp_A$ and $h(\varphi_1, \psi_1) = h(\varphi_2, \psi_2)$. We then have $[\varphi_1 \star \psi_1] = [\varphi_2 \star \psi_2]$, hence $\varphi_1 \star \psi_1 = \varphi_2 \star \psi_2$. Since $\varphi_1 \subset \varphi_2$ and $\varphi_2 \subset \varphi_1$ are both impossible by Corollary 2.5.7, it follows that $\varphi_1 = \varphi_2$. Therefore, $\star\psi_1 = \star\psi_2$, and so $\psi_1 = \psi_2$. It follows that $h \upharpoonright (ValidExp_A)^2$ is injective. □

2.5.2 The Result

Since we have established freeness, we can define functions recursively. The first such function we define is the "evaluation" function.

Definition 2.5.9. *We define a function $Ev_f : ValidExp_A \to A$ recursively by letting*

- *$Ev_f(a) = a$ for all $a \in A$.*

- $Ev_f([\varphi \star \psi]) = f(Ev_f(\varphi), Ev_f(\psi))$ *for all* $\varphi, \psi \in ValidExp_A$.

Formally, we use freeness to justify this definition as follows. Let $\iota\colon A \to A$ be the identity map, and let $g_h\colon (Sym_A^* \times A)^2 \to A$ be the function defined by letting $g_h((\varphi, a), (\psi, b)) = f(a, b)$. By freeness, there is a unique function $Ev_f\colon ValidExp_A \to A$ such that

1. $Ev_f(a) = \iota(a)$ for all $a \in A$.
2. $Ev_f(h(\varphi, \psi)) = g_h((\varphi, Ev_f(\varphi)), (\psi, Ev_f(\psi)))$ for all $\varphi, \psi \in ValidExp_A$.

which, unravelling definitions, is exactly what we wrote above.

We now define the function which elimanates all mention of parentheses and $\star$. Thus, it produces the sequence of elements of A occuring in the given sequence in order.

Definition 2.5.10. *Define a function* $D\colon ValidExp_A \to A^*$ *recursively by letting*

- $D(a) = a$ *for all* $a \in A$.
- $D([\varphi \star \psi]) = D(\varphi) * D(\psi)$ *for all* $\varphi, \psi \in ValidExp_A$.

With these definitions in hand, we can now precisely state our theorem.

Theorem 2.5.11. *Suppose that* $f\colon A^2 \to A$ *is associative, i.e.* $f(a, f(b, c)) = f(f(a, b), c)$ *for all* $a, b, c \in A$. *For all* $\varphi, \psi \in ValidExp_A$ *with* $D(\varphi) = D(\psi)$, *we have* $Ev_f(\varphi) = Ev_f(\psi)$.

In order to prove our theorem, we'll make use of the following function. Intuitively, it takes a sequence such as *cabc* and "associates to the right" to produce $[c \star [a \star [b \star c]]]$. Thus, it provides a canonical way to put together the elements of the sequence into something we can evaluate.

To make the recursive definition precise, consider the simple generating system $(A^*, A, \{h_a : a \in A\})$ where $h_a\colon A^* \to A^*$ is defined by $h_a(\sigma) = a\sigma$. As shown in Example 2.4.4, we know that $(A^*, A, \{h_a : a \in A\})$ is free and we have that $G = A^* \backslash \{\lambda\}$.

Definition 2.5.12. *We define* $R\colon A^*\backslash\{\lambda\} \to Sym_A^*$ *recursively by letting* $R(a) = a$ *for all* $a \in A$, *and letting* $R(a\sigma) = [a \star R(\sigma)]$ *for all* $a \in A$ *and all* $\sigma \in A^*\backslash\{\lambda\}$.

In order to prove our theorem, we will show that $Ev_f(\varphi) = Ev_f(R(D(\varphi)))$ for all $\varphi \in ValidExp_A$, i.e. that we can take any $\varphi \in ValidExp_A$, rip it apart so that we see the elements of A in order, and then associate to the right, without affecting the result of the evaluation. We first need the following lemma.

Lemma 2.5.13. $Ev_f([R(\sigma) \star R(\tau)]) = Ev_f(R(\sigma\tau))$ *for all* $\sigma, \tau \in A^*\backslash\{\lambda\}$.

Proof. Fix $\tau \in A^*\backslash\{\lambda\}$. We prove the result for this fixed τ by induction on $A^*\backslash\{\lambda\}$. That is, we let $X = \{\sigma \in A^*\backslash\{\lambda\} : Ev_f([R(\sigma) \star R(\tau)]) = Ev_f(R(\sigma\tau))\}$ and prove by induction on $(A^*, A, \{h_a : a \in A\})$ that $X = A^*\backslash\{\lambda\}$. Suppose first that $a \in A$. We then have

$$\begin{aligned} Ev_f([R(a) \star R(\tau)]) &= Ev_f([a \star R(\tau)]) && \text{(by definition of } R) \\ &= Ev_f(R(a\tau)) && \text{(by definition of } R) \end{aligned}$$

so $a \in X$. Suppose now that $\sigma \in X$ and that $a \in A$. We show that $a\sigma \in X$. We have

$$\begin{aligned} Ev_f([R(a\sigma) \star R(\tau)]) &= Ev_f([[a \star R(\sigma)] \star R(\tau)]) && \text{(by definition of } R) \\ &= f(Ev_f([a \star R(\sigma)]), Ev_f(R(\tau))) && \text{(by definition of } Ev_f) \\ &= f(f(a, Ev_f(R(\sigma))), Ev_f(R(\tau))) && \text{(by definition of } Ev_f \text{ using } Ev_f(a) = a) \\ &= f(a, f(Ev_f(R(\sigma)), Ev_f(R(\tau)))) && \text{(since } f \text{ is associative)} \\ &= f(a, Ev_f([R(\sigma) \star R(\tau)])) && \text{(by definition of } Ev_f) \\ &= f(a, Ev_f(R(\sigma\tau))) && \text{(since } \sigma \in X) \\ &= Ev_f([a \star R(\sigma\tau)]) && \text{(by definition of } Ev_f \text{ using } Ev_f(a) = a) \\ &= Ev_f(R(a\sigma\tau)) && \text{(by definition of } R) \end{aligned}$$

so $a\sigma \in X$. The result follows by induction. □

Lemma 2.5.14. *$Ev_f(\varphi) = Ev_f(R(D(\varphi)))$ for all $\varphi \in ValidExp_A$.*

Proof. By induction on $ValidExp_A$. If $a \in A$, this is trivial because $R(D(a)) = R(a) = a$. Suppose that $\varphi, \psi \in ValidExp_A$ and the result holds for φ and ψ.

$$\begin{aligned} Ev_f([\varphi \star \psi]) &= f(Ev_f(\varphi), Ev_f(\psi)) && \text{(by definition of } Ev_f\text{)} \\ &= f(Ev_f(R(D(\varphi))), Ev_f(R(D(\psi)))) && \text{(by induction)} \\ &= Ev_f([R(D(\varphi)) \star R(D(\psi))]) && \text{(by definition of } Ev_f\text{)} \\ &= Ev_f(R(D(\varphi) * D(\psi))) && \text{(by Lemma 2.5.13)} \\ &= Ev_f(R(D([\varphi \star \psi]))) && \text{(by definition of } D\text{)} \end{aligned}$$

□

Proof of Theorem 2.5.11. Suppose that $\varphi, \psi \in ValidExp_A$ are such that $D(\varphi) = D(\psi)$. We then have that

$$\begin{aligned} Ev_f(\sigma) &= Ev_f(R(D(\sigma))) && \text{(by Lemma 2.5.14)} \\ &= Ev_f(R(D(\tau))) && \text{(since } D(\sigma) = D(\tau)\text{)} \\ &= Ev_f(\tau) && \text{(by Lemma 2.5.14)} \end{aligned}$$

□

2.5.3 An Alternate Syntax - Polish Notation

Throughout this section, let A be a set not containing the symbol $\star$. Let $Sym_A = A \cup \{\star\}$.

Definition 2.5.15. *Define a binary function $h\colon (Sym_A^*)^2 \to Sym_A^*$ by letting $h(\sigma, \tau)$ be the sequence $\star\sigma\tau$. Let $PolishExp_A = G(Sym_A^*, A, \{h\})$ (viewed as a simple generating system).*

Proposition 2.5.16. *The simple generating system $(Sym_A^*, A, \{h\})$ is free.*

Definition 2.5.17. *Define $K\colon Sym_A^* \to \mathbb{Z}$ as follows. We first define $w\colon Sym_A \to \mathbb{Z}$ by letting $w(a) = 1$ for all $a \in A$ and letting $w(\star) = -1$. We then define $K\colon Sym_A^* \to \mathbb{Z}$ by letting $K(\lambda) = 0$ and letting $K(\sigma) = \sum_{i<|\sigma|} w(\sigma(i))$ for all $\sigma \in Sym_A^* \backslash \{\lambda\}$.*

Remark 2.5.18. *If $\sigma, \tau \in Sym_A^*$, then $K(\sigma\tau) = K(\sigma) + K(\tau)$.*

Proposition 2.5.19. *If $\varphi \in PolishExp_A$, then $K(\varphi) = 1$.*

Proof. The proof is by induction on φ. Notice that for every $a \in A$, we have that $K(a) = 1$. Suppose that $\varphi, \psi \in PolishExp_A$ are such that $K(\varphi) = 1 = K(\psi)$. We then have that

$$\begin{aligned} K(\star\varphi\psi) &= K(\star) + K(\varphi) + K(\psi) \\ &= K(\varphi) \\ &= 1. \end{aligned}$$

The result follows by induction. □

Proposition 2.5.20. *If $\varphi \in PolishExp_A$ and $\sigma \subset \varphi$, then $K(\sigma) \leq 0$.*

Proof. The proof is by induction on φ. For every $a \in A$, this is trivial because the only $\sigma \subset A$ is $\sigma = \lambda$, and we have $K(\lambda) = 0$.

Suppose that $\varphi, \psi \in PolishExp_A$ and the result holds for φ and ψ. We prove the result for $\star\varphi\psi$. Suppose that $\sigma \subset \star\varphi\psi$. If $\sigma = \lambda$, then $K(\sigma) = 0$. If σ is $\star\tau$ for some $\tau \subset \varphi$, then

$$\begin{aligned} K(\sigma) &= K(\star) + K(\tau) \\ &\leq -1 + 0 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

Otherwise, σ is $\star\varphi\tau$ for some $\tau \subset \psi$, in which case

$$\begin{aligned} K(\sigma) &= K(\star) + K(\varphi) + K(\tau) \\ &= -1 + 0 + K(\tau) && \text{(by Proposition 2.5.19)} \\ &\leq -1 + 0 + 0 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

Thus, the result holds for $\star\varphi\psi$. □

Corollary 2.5.21. *If $\varphi, \psi \in PolishExp_A$, then $\varphi \not\subset \psi$.*

Proof. This follows by combining Proposition 2.5.19 and Proposition 2.5.20. □

Theorem 2.5.22. *The generating system $(Sym_A^*, A, \mathcal{H})$ is free.*

Proof. First notice that $\mathrm{ran}(h \upharpoonright (PolishExp_A)^2) \cap A = \emptyset$ because all elements of $\mathrm{ran}(h)$ begin with $\star$.

Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in PolishExp_A$ and that $h(\varphi_1, \psi_1) = h(\varphi_2, \psi_2)$. We then have $\star\varphi_1\psi_1 = \star\varphi_2\psi_2$, hence $\varphi_1\psi_1 = \varphi_2\psi_2$. Since $\varphi_1 \subset \varphi_2$ and $\varphi_2 \subset \varphi_1$ are both impossible by Corollary 2.5.21, it follows that $\varphi_1 = \varphi_2$. Therefore, $\psi_1 = \psi_2$. It follows that $h \upharpoonright (PolishExp_A)^2$ is injective. □

Chapter 3

Propositional Logic

3.1 The Syntax of Propositional Logic

3.1.1 Standard Syntax

Definition 3.1.1. *Let P be a nonempty set not containing the symbols $(,), \neg, \wedge, \vee$, and $\rightarrow$. Let $Sym_P = P \cup \{(,), \neg, \wedge, \vee, \rightarrow\}$. Define a unary function $h_\neg$ and binary functions $h_\wedge, h_\vee$, and $h_\rightarrow$ on Sym_P^* as follows.*

$$\begin{aligned} h_\neg(\sigma) &= (\neg\sigma) \\ h_\wedge(\sigma, \tau) &= (\sigma \wedge \tau) \\ h_\vee(\sigma, \tau) &= (\sigma \vee \tau) \\ h_\rightarrow(\sigma, \tau) &= (\sigma \rightarrow \tau) \end{aligned}$$

Definition 3.1.2. *Fix P. Let $Form_P = G(Sym_P^*, P, \mathcal{H})$ where $\mathcal{H} = \{h_\neg, h_\wedge, h_\vee, h_\rightarrow\}$.*

Definition 3.1.3. *Define $K \colon Sym_P^* \to \mathbb{Z}$ as follows. We first define $w \colon Sym_P \to \mathbb{Z}$ by letting $w(A) = 0$ for all $A \in P$, letting $w(\Diamond) = 0$ for all $\Diamond \in \{\neg, \wedge, \vee, \rightarrow\}$, letting $w(() = -1$, and letting $w()) = 1$. We then define $K \colon Sym_P^* \to \mathbb{Z}$ by letting $K(\lambda) = 0$ and letting $K(\sigma) = \sum_{i<|\sigma|} w(\sigma(i))$ for all $\sigma \in Sym_P^* \backslash \{\lambda\}$.*

Remark 3.1.4. *If $\sigma, \tau \in Sym_P^*$, then $K(\sigma\tau) = K(\sigma) + K(\tau)$.*

Proposition 3.1.5. *If $\varphi \in Form_P$, then $K(\varphi) = 0$.*

Proof. A simple induction as above. □

Proposition 3.1.6. *If $\varphi \in Form_P$ and $\sigma \subset \varphi$ with $\sigma \neq \lambda$, then $K(\sigma) \leq -1$.*

Proof. A simple induction as above. □

Corollary 3.1.7. *If $\varphi, \psi \in Form_P$, then $\varphi \not\subset \psi$.*

Proof. This follows by combining Proposition 3.1.5 and Proposition 3.1.6, along with noting that $\lambda \notin Form_P$ (which follows by a simple induction). □

Theorem 3.1.8. *The generating system $(Sym_P^*, P, \mathcal{H})$ is free.*

Proof. First notice that $\text{ran}(h_\neg \upharpoonright Form_P) \cap P = \emptyset$ because all elements of $\text{ran}(h_\neg)$ begin with (. Similarly, for any $\Diamond \in \{\wedge, \vee, \rightarrow\}$, we have $\text{ran}(h_\Diamond \upharpoonright Form_P^2) \cap P = \emptyset$ since all elements of $\text{ran}(h_\Diamond)$ begin with (.

Suppose that $\varphi, \psi \in Form_P$ and $h_\neg(\varphi) = h_\neg(\psi)$. We then have $(\neg\varphi) = (\neg\psi)$, hence $\varphi = \psi$. Therefore, $h_\neg \upharpoonright Form_P$ is injective. Fix $\Diamond \in \{\wedge, \vee, \rightarrow\}$. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in Form_P$ and that $h_\Diamond(\varphi_1, \psi_1) = h_\Diamond(\varphi_2, \psi_2)$. We then have $(\varphi_1 \Diamond \psi_1) = (\varphi_2 \Diamond \psi_2)$, hence $\varphi_1 \Diamond \psi_1 = \varphi_2 \Diamond \psi_2$. Since $\varphi_1 \subset \varphi_2$ and $\varphi_2 \subset \varphi_1$ are both impossible by Corollary 3.1.7, it follows that $\varphi_1 = \varphi_2$. Therefore, $\Diamond\psi_1 = \Diamond\psi_2$, and so $\psi_1 = \psi_2$. It follows that $h_\Diamond \upharpoonright Form_P^2$ is injective.

Let $\Diamond \in \{\wedge, \vee, \rightarrow\}$. Suppose that $\varphi, \psi_1, \psi_2 \in Form_P$ and $h_\neg(\varphi) = h_\Diamond(\psi_1, \psi_2)$. We then have $(\neg\varphi) = (\psi_1 \Diamond \psi_2)$, hence $\neg\varphi = \psi_1 \Diamond \psi_2$, contradicting the fact that no element of $Form_P$ begins with $\neg$ (by a simple induction). Therefore, $\text{ran}(h_\neg \upharpoonright Form_P) \cap \text{ran}(h_\Diamond \upharpoonright Form_P^2) = \emptyset$.

Suppose now that $\Diamond_1, \Diamond_2 \in \{\wedge, \vee, \rightarrow\}$ with $\Diamond_1 \neq \Diamond_2$. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in Form_P$ and $h_{\Diamond_1}(\varphi_1, \psi_1) = h_{\Diamond_2}(\varphi_2, \psi_2)$. We then have $(\varphi_1 \Diamond_1 \psi_1) = (\varphi_2 \Diamond_2 \psi_2)$, hence $\varphi_1 \Diamond_1 \psi_1 = \varphi_2 \Diamond_2 \psi_2$. Since $\varphi_1 \subset \varphi_2$ and $\varphi_2 \subset \varphi_1$ are both impossible by Corollary 3.1.7, it follows that $\varphi_1 = \varphi_2$. Therefore, $\Diamond_1 = \Diamond_2$, a contradiction. It follows that $\text{ran}(h_{\Diamond_1} \upharpoonright Form_P^2) \cap \text{ran}(h_{\Diamond_2} \upharpoonright Form_P^2) = \emptyset$. □

3.1.2 Polish Notation

Definition 3.1.9. *Let P be a set not containing the symbols $\neg, \wedge, \vee$, and $\rightarrow$. Let $Sym_P = P \cup \{\neg, \wedge, \vee, \rightarrow\}$. Define a unary function $h_\neg$ and binary functions $h_\wedge, h_\vee$, and $h_\rightarrow$ on Sym_P^* as follows.*

$$\begin{aligned} h_\neg(\sigma) &= \neg\sigma \\ h_\wedge(\sigma, \tau) &= \wedge\sigma\tau \\ h_\vee(\sigma, \tau) &= \vee\sigma\tau \\ h_\rightarrow(\sigma, \tau) &= \rightarrow \sigma\tau \end{aligned}$$

Definition 3.1.10. *Fix P. Let $Form_P = G(Sym_P^*, P, \mathcal{H})$ where $\mathcal{H} = \{h_\neg, h_\wedge, h_\vee, h_\rightarrow\}$.*

Definition 3.1.11. *Define $K\colon Sym_P^* \to \mathbb{Z}$ as follows. We first define $w\colon Sym_P \to \mathbb{Z}$ by letting $w(A) = 1$ for all $A \in P$, letting $w(\neg) = 0$, and letting $w(\Diamond) = -1$ for all $\Diamond \in \{\neg, \wedge, \vee, \rightarrow\}$. We then define $K\colon Sym_P^* \to \mathbb{Z}$ by letting $K(\lambda) = 0$ and letting $K(\sigma) = \sum_{i<|\sigma|} w(\sigma(i))$ for all $\sigma \in Sym_P^* \backslash \{\lambda\}$.*

Remark 3.1.12. *If $\sigma, \tau \in Sym_P^*$, then $K(\sigma\tau) = K(\sigma) + K(\tau)$.*

Proposition 3.1.13. *If $\varphi \in Form_P$, then $K(\varphi) = 1$.*

Proof. The proof is by induction on φ. Notice that for every $A \in P$, we have that $K(A) = 1$. Suppose that $\varphi \in Form_P$ is such that $K(\varphi) = 1$. We then have that

$$\begin{aligned} K(\neg\varphi) &= 0 + K(\varphi) \\ &= K(\varphi) \\ &= 1. \end{aligned}$$

Suppose now that $\varphi, \psi \in Form_P$ are such that $K(\varphi) = 1 = K(\psi)$, and $\Diamond \in \{\wedge, \vee, \rightarrow\}$. We then have that

$$\begin{aligned} K(\Diamond\varphi\psi) &= -1 + K(\varphi) + K(\psi) \\ &= -1 + 1 + 1 \\ &= 1. \end{aligned}$$

The result follows by induction. □

Proposition 3.1.14. *If $\varphi \in Form_P$ and $\sigma \subset \varphi$, then $K(\sigma) \leq 0$.*

Proof. The proof is by induction on φ. For every $A \in P$, this is trivial because the only $\sigma \subset A$ is $\sigma = \lambda$, and we have $K(\lambda) = 0$.

Suppose that $\varphi \in Form_P$ and the result holds for φ. We prove the result for $\neg\varphi$. Suppose that $\sigma \subset \neg\varphi$. If $\sigma = \lambda$, then $K(\sigma) = 0$. Otherwise, σ is $\neg\tau$ for some $\tau \subset \varphi$, in which case

$$\begin{aligned} K(\sigma) &= 0 + K(\tau) \\ &\leq 0 + 0 && \text{(by induction)} \\ &\leq 0. \end{aligned}$$

Thus, the result holds for $\neg\varphi$.

Suppose that $\varphi, \psi \in Form_P$ and the result holds for φ and ψ. Let $\Diamond \in \{\wedge, \vee, \rightarrow\}$. We prove the result for $\Diamond\varphi\psi$. Suppose that $\sigma \subset \Diamond\varphi\psi$. If $\sigma = \lambda$, then $K(\sigma) = 0$. If σ is $\Diamond\tau$ for some $\tau \subset \varphi$, then

$$\begin{aligned} K(\sigma) &= -1 + K(\tau) \\ &\leq -1 + 0 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

Otherwise, σ is $\Diamond\varphi\tau$ for some $\tau \subset \psi$, in which case

$$\begin{aligned} K(\sigma) &= -1 + K(\varphi) + K(\tau) \\ &= -1 + 0 + K(\tau) && \text{(by Proposition 3.1.13)} \\ &\leq -1 + 0 + 0 && \text{(by induction)} \\ &\leq -1. \end{aligned}$$

Thus, the result holds for $\Diamond\varphi\psi$. □

Corollary 3.1.15. *If $\varphi, \psi \in Form_P$, then $\varphi \not\subset \psi$.*

Proof. This follows by combining Proposition 3.1.13 and Proposition 3.1.14. □

Theorem 3.1.16. *The generating system $(Sym_P^*, P, \mathcal{H})$ is free.*

Proof. First notice that $\text{ran}(h_\neg \upharpoonright Form_P) \cap P = \emptyset$ because all elements of $\text{ran}(h_\neg)$ begin with $\neg$. Similarly, for any $\Diamond \in \{\wedge, \vee, \rightarrow\}$, we have $\text{ran}(h_\Diamond \upharpoonright Form_P^2) \cap P = \emptyset$ since all elements of $\text{ran}(h_\Diamond)$ begin with $\Diamond$.

Suppose that $\varphi, \psi \in Form_P$ and $h_\neg(\varphi) = h_\neg(\psi)$. We then have $\neg\varphi = \neg\psi$, hence $\varphi = \psi$. Therefore, $h_\neg \upharpoonright Form_P$ is injective. Fix $\Diamond \in \{\wedge, \vee, \rightarrow\}$. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in Form_P$ and that $h_\Diamond(\varphi_1, \psi_1) = h_\Diamond(\varphi_2, \psi_2)$. We then have $\Diamond\varphi_1\psi_1 = \Diamond\varphi_2\psi_2$, hence $\varphi_1\psi_1 = \varphi_2\psi_2$. Since $\varphi_1 \subset \varphi_2$ and $\varphi_2 \subset \varphi_1$ are both impossible by Corollary 3.1.15, it follows that $\varphi_1 = \varphi_2$. Therefore, $\psi_1 = \psi_2$. It follows that $h_\Diamond \upharpoonright Form_P^2$ is injective.

For any $\Diamond \in \{\wedge, \vee, \rightarrow\}$, we have $\text{ran}(h_\neg \upharpoonright Form_P) \cap \text{ran}(h_\Diamond \upharpoonright Form_P^2) = \emptyset$ because all elements of $\text{ran}(h_\neg)$ begin with $\neg$ and all elements of $\text{ran}(h_\Diamond)$ begin with $\Diamond$. Similarly, if $\Diamond_1, \Diamond_2 \in \{\wedge, \vee, \rightarrow\}$ with $\Diamond_1 \neq \Diamond_2$, we have $\text{ran}(h_{\Diamond_1} \upharpoonright Form_P^2) \cap \text{ran}(h_{\Diamond_2} \upharpoonright Form_P^2) = \emptyset$ because all elements of $\text{ran}(h_{\Diamond_1})$ begin with $\Diamond_1$ and all elements of $\text{ran}(h_{\Diamond_2})$ begin with $\Diamond_2$. □

3.1.3 Official Syntax and Our Abuses of It

Since we should probably fix an official syntax, let's agree to use Polish notation because it's simpler in many aspects and it will be natural to generalize when we talk about the possibility of other connectives and when we discuss first-order logic. However, as with many official definitions in mathematics, we'll ignore and abuse this convention constantly in the interest of readability. For example, we'll often write things in standard syntax or in more abbreviated forms. For example, we'll write $\mathsf{A} \wedge \mathsf{B}$ instead of $\wedge\mathsf{AB}$ (or $(\mathsf{A} \wedge \mathsf{B})$ in the original syntax) . We'll also write something like

$$\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \cdots \wedge \mathsf{A}_{n-1} \wedge \mathsf{A}_n$$

or

$$\bigwedge_{i=1}^{n} \mathsf{A}_i$$

instead of $(\mathsf{A}_1 \wedge (\mathsf{A}_2 \wedge (\cdots (\mathsf{A}_{n-1} \wedge \mathsf{A}_n) \cdots)))$ in standard syntax or $\wedge \mathsf{A}_1 \wedge \mathsf{A}_2 \cdots \wedge \mathsf{A}_{n-1}\mathsf{A}_n$ in Polish notation (which can be precisely defined in a similar manner as R in Section 2.5). In general, when we string together multiple applications of an operation (such as $\wedge$) occur in order, we always associate to the right.

When it comes to mixing symbols, let's agree to the following conventions about "binding" in a similar fashion to how we think of $\cdot$ as more binding than $+$ (so that $3 \cdot 5 + 2$ is read as $(3 \cdot 5) + 2$). We think of $\neg$ as the most binding, so we read $\neg \mathsf{A} \wedge \mathsf{B}$ as $((\neg \mathsf{A}) \wedge \mathsf{B})$. After that, we consider $\wedge$ and $\vee$ as the next most binding, and $\rightarrow$ has the least binding. We'll insert parentheses when we wish to override this binding. For example, $\mathsf{A} \wedge \neg \mathsf{B} \rightarrow \mathsf{C} \vee \mathsf{D}$ is really $((\mathsf{A} \wedge (\neg \mathsf{B})) \rightarrow (\mathsf{C} \vee \mathsf{D}))$ while $\mathsf{A} \wedge (\neg \mathsf{B} \rightarrow \mathsf{C} \vee \mathsf{D})$ is really $(\mathsf{A} \wedge ((\neg \mathsf{B}) \rightarrow (\mathsf{C} \vee \mathsf{D})))$.

3.1.4 Recursive Definitions

Since we've shown that our generating system is free, we can define functions recursively. It is possible to avoid using recursion on $Form_P$ to define some of functions. In such cases, you may wonder why we bother. Since our only powerful way to prove things about the set $Form_P$ is by induction, and definitions of functions by recursion are well-suited to induction, it's simply the easiest way ahead.

Definition 3.1.17. *If X is a set, we denote by $\mathcal{P}(X)$ the set of all subsets of X. Thus $\mathcal{P}(X) = \{Z : Z \subseteq X\}$. We call $\mathcal{P}(X)$ the* power set *of X.*

Definition 3.1.18. *We define a function $OccurProp\colon Form_P \rightarrow \mathcal{P}(P)$ recursively as follows.*

- *$OccurProp(\mathsf{A}) = \{\mathsf{A}\}$ for all $\mathsf{A} \in P$.*
- *$OccurProp(\neg\varphi) = OccurProp(\varphi)$.*
- *$OccurProp(\diamond\varphi\psi) = OccurProp(\varphi) \cup OccurProp(\psi)$ for each $\diamond \in \{\wedge, \vee, \rightarrow\}$.*

If you want to be precise in the previous definition, we're defining functions $\iota\colon P \rightarrow \mathcal{P}(P)$, $g_{h_\neg}\colon Sym_P^* \times \mathcal{P}(P) \rightarrow \mathcal{P}(P)$ and $g_{h_\diamond}\colon (Sym_P^* \times \mathcal{P}(P))^2 \rightarrow \mathcal{P}(P)$ for each $\diamond \in \{\wedge, \vee, \rightarrow\}$ as follows.

- $\iota(\mathsf{A}) = \{\mathsf{A}\}$ for all $\mathsf{A} \in P$.
- $g_{h_\neg}(\sigma, Z) = Z$.
- $g_{h_\diamond}(\sigma_1, Z_1, \sigma_2, Z_2) = Z_1 \cup Z_2$ for each $\diamond \in \{\wedge, \vee, \rightarrow\}$.

and we're using our result on freeness to assure that there is a unique function $OccurProp\colon Form_P \rightarrow \mathcal{P}(P)$ which satisfy the associated requirements. Of course, this method is more precise, but it's hardly more intuitive to use. It's a good exercise to make sure that you can translate a few more informal recursive definitions in this way, but once you understand how it works you can safely keep the formalism in the back of your mind.

Here's a somewhat trivial example of using induction to prove a result based on a recursive definition.

Proposition 3.1.19. *Suppose that $Q \subseteq P$. We then have that $Form_Q \subseteq Form_P$.*

Proof. A trivial induction on $\varphi \in Form_Q$. □

Proposition 3.1.20. *Fix P. For any $\varphi \in Form_P$, we have $\varphi \in Form_{OccurProp(\varphi)}$.*

Proof. The proof is by induction on $\varphi \in Form_P$. Suppose first that $\mathsf{A} \in P$. Since $OccurProp(\mathsf{A}) = \{\mathsf{A}\}$ and $\mathsf{A} \in Form_{\mathsf{A}}$, we have $\mathsf{A} \in Form_{OccurProp(\mathsf{A})}$.

Suppose that $\varphi \in Form_P$ and that the result holds for φ, i.e. we have $\varphi \in Form_{OccurProp(\varphi)}$. Since $OccurProp(\neg\varphi) = OccurProp(\varphi)$, it follows that $\varphi \in Form_{OccurProp(\neg\varphi)}$. Hence, $\neg\varphi \in Form_{OccurProp(\neg\varphi)}$.

Suppose that $\varphi, \psi \in Form_P$, that $\Diamond \in \{\wedge, \vee, \rightarrow\}$, and that the result holds for φ and ψ, i.e. we have $\varphi \in Form_{OccurProp(\varphi)}$ and $\psi \in Form_{OccurProp(\psi)}$. Since

$$OccurProp(\varphi) \subseteq OccurProp(\Diamond\varphi\psi) \text{ and } OccurProp(\psi) \subseteq OccurProp(\Diamond\varphi\psi)$$

it follows from Proposition 3.1.19 that $\varphi, \psi \in Form_{OccurProp(\Diamond\varphi\psi)}$. Therefore, $\Diamond\varphi\psi \in Form_{OccurProp(\Diamond\varphi\psi)}$. □

On to some more important recursive definitions.

Definition 3.1.21. *We define a function $Depth\colon Form_P \to \mathbb{N}$ recursively as follows.*

- $Depth(\mathsf{A}) = 0$ *for all* $\mathsf{A} \in P$.
- $Depth(\neg\varphi) = Depth(\varphi) + 1$.
- $Depth(\Diamond\varphi\psi) = \max\{Depth(\varphi), Depth(\psi)\} + 1$ *for each* $\Diamond \in \{\wedge, \vee, \rightarrow\}$.

Example 3.1.22. $Depth(\vee \wedge \mathsf{AB} \vee \mathsf{CD}) = 2$.

Definition 3.1.23. *We define a function $Subform\colon Form_P \to \mathcal{P}(Form_P)$ recursively as follows.*

- $Subform(\mathsf{A}) = \{\mathsf{A}\}$ *for all* $\mathsf{A} \in P$.
- $Subform(\neg\varphi) = \{\neg\varphi\} \cup Subform(\varphi)$.
- $Subform(\Diamond\varphi\psi) = \{\Diamond\varphi\psi\} \cup Subform(\varphi) \cup Subform(\psi)$ *for each* $\Diamond \in \{\wedge, \vee, \rightarrow\}$.

Example 3.1.24. $Subform(\wedge\neg\mathsf{AB}) = \{\wedge\neg\mathsf{AB}, \neg\mathsf{A}, \mathsf{A}, \mathsf{B}\}$.

Definition 3.1.25. *Let $\theta, \gamma \in Form_P$. We define a function $Subst_\gamma^\theta\colon Form_P \to Form_P$ recursively as follows.*

- $Subst_\gamma^\theta(\mathsf{A}) = \begin{cases} \theta & \text{if } \gamma = \mathsf{A} \\ \mathsf{A} & \text{otherwise} \end{cases}$
- $Subst_\gamma^\theta(\neg\varphi) = \begin{cases} \theta & \text{if } \gamma = \neg\varphi \\ \neg Subst_\gamma^\theta(\varphi) & \text{otherwise} \end{cases}$
- $Subst_\gamma^\theta(\Diamond\varphi\psi) = \begin{cases} \theta & \text{if } \gamma = \Diamond\varphi\psi \\ \Diamond Subst_\gamma^\theta(\varphi) Subst_\gamma^\theta(\psi) & \text{otherwise} \end{cases}$

  for each $\Diamond \in \{\wedge, \vee, \rightarrow\}$.

Example 3.1.26. $Subst_{\neg\mathsf{C}}^{\wedge\mathsf{AB}}(\rightarrow \vee\neg\mathsf{CA}\neg\mathsf{C}) = \rightarrow \vee \wedge \mathsf{ABA} \wedge \mathsf{AB}$.

3.2 Truth Assignments and Semantic Implication

Definition 3.2.1. *A function $v \colon P \to \{0,1\}$ is called a* truth assignment *on P.*

Definition 3.2.2. *Let $v \colon P \to \{0,1\}$ be a truth assignment. We denote by $\overline{v}$ the unique function $\overline{v} \colon Form_P \to \{0,1\}$ such that*

- $\overline{v}(\mathsf{A}) = v(\mathsf{A})$ *for all* $\mathsf{A} \in P$.
- $\overline{v}(\neg\varphi) = \begin{cases} 1 & \text{if } \overline{v}(\varphi) = 0 \\ 0 & \text{if } \overline{v}(\varphi) = 1 \end{cases}$
- $\overline{v}(\wedge\varphi\psi) = \begin{cases} 0 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 0 \\ 0 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 1 \\ 0 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 1 \end{cases}$
- $\overline{v}(\vee\varphi\psi) = \begin{cases} 0 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 1 \\ 1 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 1 \end{cases}$
- $\overline{v}(\to\varphi\psi) = \begin{cases} 1 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 1 \\ 0 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 1 \end{cases}$

Before moving on, we should a couple of things about what happens when we shrink/enlarge the set P. Intuitively, if $\varphi \in Form_Q$ and $Q \subseteq P$, then we can extend the truth assigment from Q to P arbitrarily without affecting the value of $\overline{v}(\varphi)$. Here is the precise statement.

Proposition 3.2.3. *Suppose that $Q \subseteq P$ and that $v \colon P \to \{0,1\}$ is a truth assignment on P. We then have that $\overline{v}(\varphi) = \overline{(v \upharpoonright Q)}(\varphi)$ for all $\varphi \in Form_Q$.*

Proof. A trivial induction on $\varphi \in Form_Q$. □

Proposition 3.2.4. *Suppose $\varphi \in Form_P$. Whenever v_1 and v_2 are truth assignments on P such that $v_1(\mathsf{A}) = v_2(\mathsf{A})$ for all $\mathsf{A} \in OccurProp(\varphi)$, we have $\overline{v}_1(\varphi) = \overline{v}_2(\varphi)$.*

Proof. Let $Q = OccurProp(\varphi)$. We then have that $\varphi \in Form_Q$ by Proposition 3.1.20. Since $v_1 \upharpoonright Q = v_2 \upharpoonright Q$, we have

$$\overline{v}_1(\varphi) = \overline{(v_1 \upharpoonright Q)}(\varphi) = \overline{(v_2 \upharpoonright Q)}(\varphi) = \overline{v}_2(\varphi)$$

□

With a method of assigning true/false values to formulas in hand (once we've assigned them to P), we're now in position to use our semantic definitions to given a precise meaning to "The set of formulas Γ implies the formula φ".

Definition 3.2.5. *Let P be given Let $\Gamma \subseteq Form_P$ and let $\varphi \in Form_P$. We write $\Gamma \vDash_P \varphi$, or simply $\Gamma \vDash \varphi$ if P is clear, to mean that whenever v is a truth assignment on P such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, we have $\overline{v}(\varphi) = 1$. We pronounce $\Gamma \vDash \varphi$ as "Γ semantically implies φ".*

We also have a semantic way to say that a set of formulas is not contradictory.

Definition 3.2.6. Γ *is* satisfiable *if there exists a truth assignment* $v\colon P \to \{0,1\}$ *such that* $\overline{v}(\gamma) = 1$ *for all* $\gamma \in \Gamma$. *Otherwise, we say that* Γ *is* unsatisfiable.

Example 3.2.7. *Let* $P = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. *We have* $\{\mathsf{A} \vee \mathsf{B}, \neg(\mathsf{A} \wedge (\neg\mathsf{C}))\} \vDash \mathsf{B} \vee \mathsf{C}$.

Proof. Let $v\colon P \to \{0,1\}$ be a truth assignment such that $\overline{v}(\mathsf{A} \vee \mathsf{B}) = 1$ and $\overline{v}(\neg(\mathsf{A} \wedge (\neg\mathsf{C}))) = 1$. We need to show that $\overline{v}(\mathsf{B} \vee \mathsf{C}) = 1$. Suppose not. We would then have that $\overline{v}(\mathsf{B}) = 0$ and $\overline{v}(\mathsf{C}) = 0$. Since $\overline{v}(\mathsf{A} \vee \mathsf{B}) = 1$, this implies that $\overline{v}(\mathsf{A}) = 1$. Therefore, $\overline{v}(\mathsf{A} \wedge (\neg\mathsf{C})) = 1$, so $\overline{v}(\neg(\mathsf{A} \wedge (\neg\mathsf{C}))) = 0$, a contradiction. □

Example 3.2.8. *Let* P *be given. For any* $\varphi, \psi \in Form_P$, *we have* $\{\varphi \to \psi, \varphi\} \vDash \psi$

Proof. Let $v\colon P \to \{0,1\}$ be a truth assignment and suppose that $\overline{v}(\varphi \to \psi) = 1$ and $\overline{v}(\varphi) = 1$. If $\overline{v}(\psi) = 0$, it would follows that $\overline{v}(\varphi \to \psi) = 0$, a contradiction. Thus, $\overline{v}(\psi) = 1$. □

Notation 3.2.9.

1. *If* $\Gamma = \emptyset$, *we write* $\vDash \varphi$ *instead of* $\emptyset \vDash \varphi$.
2. *If* $\Gamma = \{\gamma\}$, *we write* $\gamma \vDash \varphi$ *instead of* $\{\gamma\} \vDash \varphi$.

Definition 3.2.10.

1. *Let* $\varphi \in Form_P$. *We say that* φ *is a* tautology *if* $\vDash \varphi$.
2. *If* $\varphi \vDash \psi$ *and* $\psi \vDash \varphi$, *we say that* φ *and* ψ *are* semantically equivalent.

Remark 3.2.11. *Notice that* φ *and* ψ *are semantically equivalent if and only if for all truth assigments* $v\colon P \to \{0,1\}$, *we have* $\overline{v}(\varphi) = \overline{v}(\psi)$.

Example 3.2.12. $\varphi \vee \neg\varphi$ *is a tautology for any* $\varphi \in Form_P$.

Proof. Fix $\varphi \in Form_P$. Let $v\colon P \to \{0,1\}$ be a truth assignment. If $\overline{v}(\varphi) = 1$, then $\overline{v}(\varphi \vee \neg\varphi) = 1$. Otherwise, we have $\overline{v}(\varphi) = 0$, in which case $\overline{v}(\neg\varphi) = 1$, and hence $\overline{v}(\varphi \vee \neg\varphi) = 1$. Therefore, $\overline{v}(\varphi \vee \neg\varphi) = 1$ for all truth assignments $v\colon P \to \{0,1\}$, hence $\varphi \vee \neg\varphi$ is a tautology. □

Example 3.2.13. φ *is semantically equivalent to* $\neg\neg\varphi$ *for any* $\varphi \in Form_P$.

Proof. Fix $\varphi \in Form_P$. We need to show that for any truth assignment $v\colon P \to \{0,1\}$, we have $\overline{v}(\varphi) = 1$ if and only if $\overline{v}(\neg\neg\varphi) = 1$. We have

$$\begin{aligned} \overline{v}(\varphi) = 1 &\Longleftrightarrow \overline{v}(\neg\varphi) = 0 \\ &\Longleftrightarrow \overline{v}(\neg\neg\varphi) = 1 \end{aligned}$$

□

Example 3.2.14. $\varphi \to \psi$ *and* $\neg\varphi \vee \psi$ *are semantically equivalent for any* $\varphi, \psi \in Form_P$.

Proof. Fix $\varphi, \psi \in Form_P$. We need to show that for any truth assignment $v\colon P \to \{0,1\}$, we have $\overline{v}(\varphi \to \psi) = 1$ if and only if $\overline{v}(\neg\varphi \vee \psi) = 1$. Let $v\colon P \to \{0,1\}$ be a truth assignment. We have

$$\begin{aligned} \overline{v}(\varphi \to \psi) = 1 &\Longleftrightarrow \overline{v}(\varphi) = 0 \text{ or } \overline{v}(\psi) = 1 \\ &\Longleftrightarrow \overline{v}(\neg\varphi) = 1 \text{ or } \overline{v}(\psi) = 1 \\ &\Longleftrightarrow \overline{v}(\neg\varphi \vee \psi) = 1 \end{aligned}$$

□

Definition 3.2.15. *Define* $OccurProp\colon \mathcal{P}(Form_P) \to \mathcal{P}(P)$ *by letting*

$$OccurProp(\Gamma) = \{\mathsf{A} \in P : \mathsf{A} \in OccurProp(\gamma) \text{ for some } \gamma \in \Gamma\}.$$

Proposition 3.2.16. *Suppose that* $Q \subseteq P$, *that* $\Gamma \subseteq Form_Q$, *and that* $\varphi \in Form_Q$. *We then have that* $\Gamma \vDash_P \varphi$ *if and only if* $\Gamma \vDash_Q \varphi$.

Proof. First notice that $\Gamma \subseteq Form_P$ and $\varphi \in Form_P$ by Proposition 3.1.19 and Proposition 3.1.20.

Suppose first that $\Gamma \vDash_Q \varphi$. Let $v\colon P \to \{0,1\}$ be a truth assignment such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. We then have that $\overline{(v \restriction Q)}(\gamma) = 1$ for all $\gamma \in \Gamma$, hence $\overline{(v \restriction Q)}(\varphi) = 1$ because $\Gamma \vDash_Q \varphi$. Therefore, $\overline{v}(\varphi) = 1$. It follows that $\Gamma \vDash_P \varphi$.

Suppose then that $\Gamma \vDash_P \varphi$. Let $v\colon Q \to \{0,1\}$ be a truth assignment such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. Define a truth assignment $w\colon P \to \{0,1\}$ by letting $w(\mathsf{A}) = v(\mathsf{A})$ for all $\mathsf{A} \in Q$ and letting $w(\mathsf{A}) = 0$ for all $\mathsf{A} \in P \backslash Q$. Since $w \restriction Q = v$, we have $\overline{w}(\gamma) = \overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. Since $\Gamma \vDash_P \varphi$, it follows that $\overline{w}(\varphi) = 1$, hence $\overline{v}(\varphi) = 1$. Therefore, $\Gamma \vDash_Q \varphi$. ☐

Suppose that Γ is finite, and we want to determine whether or not $\Gamma \vDash \varphi$. By the previous proposition, instead of examining all truth assignments $v\colon P \to \{0,1\}$ on P, we need only consider truth assignments $v\colon OccurProp(\Gamma \cup \{\varphi\}) \to \{0,1\}$. Now $OccurProp(\Gamma \cup \{\varphi\})$ is a finite set, so there are only finitely many possibilities. Thus, one way of determining whether $\Gamma \vDash \varphi$ is simply to check all of them. If $|OccurProp(\Gamma \cup \{\varphi\})| = n$, then there are 2^n different truth assignments. We can systematically arrange them in a table like below, where we ensure we put the the elements of $OccurProp(\Gamma \cup \{\varphi\})$ in the first columns, and put all elements of $\Gamma \cup \{\varphi\}$ in later columns. We also ensure that if ψ is in a column, then all subformulas of ψ appear in an earlier column. This allows us to fill in the table one column at a time. This simple-minded exhaustive technique is called the method of *truth tables*.

Example 3.2.17. *Show that* $\{(\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C}, \mathsf{A} \to (\neg\mathsf{C})\} \vDash (\neg\mathsf{C}) \to \mathsf{B}$.

Proof.

A	B	C	$\mathsf{A} \vee \mathsf{B}$	$(\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C}$	$\neg\mathsf{C}$	$\mathsf{A} \to (\neg\mathsf{C})$	$(\neg\mathsf{C}) \to \mathsf{B}$
0	0	0	0	0	1	1	0
0	0	1	0	0	0	1	1
0	1	0	1	0	1	1	1
0	1	1	1	1	0	1	1
1	0	0	1	0	1	1	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	1	1
1	1	1	1	1	0	0	1

Notice that every row in which both of the $(\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C}$ column and the $\mathsf{A} \to (\neg\mathsf{C})$ column have a 1, namely just the row beginning with 011, we have that the entry under the $(\neg\mathsf{C}) \to \mathsf{B}$ column is a 1. Therefore, $\{(\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C}, \mathsf{A} \to (\neg\mathsf{C})\} \vDash (\neg\mathsf{C}) \to \mathsf{B}$. ☐

Example 3.2.18. *Show that* $\neg(\mathsf{A} \wedge \mathsf{B})$ *is semantically equivalent to* $\neg\mathsf{A} \vee \neg\mathsf{B}$.

Proof.

A	B	$\mathsf{A} \wedge \mathsf{B}$	$\neg(\mathsf{A} \wedge \mathsf{B})$	$\neg\mathsf{A}$	$\neg\mathsf{B}$	$\neg\mathsf{A} \vee \neg\mathsf{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Notice that the rows in which the $\neg(\mathsf{A} \wedge \mathsf{B})$ column has a 1 are exactly the same as the rows in which the $\neg\mathsf{A} \vee \neg\mathsf{B}$ column has a 1. Therefore, $\neg(\mathsf{A} \wedge \mathsf{B})$ is semantically equivalent to $\neg\mathsf{A} \vee \neg\mathsf{B}$. □

3.3 Boolean Functions and Connectives

It's natural to wonder if our choice of connectives is the "right" one. For example, why didn't we introduce a new connective $\leftrightarrow$, allowing ourselves to form the formulas $\varphi \leftrightarrow \psi$ (or $\leftrightarrow \varphi\psi$ in Polish notation) and extend our definition of $\overline{v}$ so that

$$\overline{v}(\leftrightarrow \varphi\psi) = \begin{cases} 1 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 0 \\ 0 & \text{if } \overline{v}(\varphi) = 0 \text{ and } \overline{v}(\psi) = 1 \\ 0 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 0 \\ 1 & \text{if } \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 1 \end{cases}$$

The idea is that there's no real need to introduce this connective because for any $\varphi, \psi \in Form_P$ we would have that $\varphi \leftrightarrow \psi$ is semantically equivalent to $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Perhaps we could be more exotic and introduce a new connective $\Box$ with takes three formulas allowing us to form the formulas $\Box\varphi\psi\theta$ (here's an instance when Polish notation becomes important), and extend our definition of $\overline{v}$ so that

$$\overline{v}(\Box\varphi\psi\theta) = \begin{cases} 1 & \text{if at least two of } \overline{v}(\varphi) = 1, \overline{v}(\psi) = 1, \overline{v}(\theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

It's not hard (and a good exercise) to show that for any $\varphi, \psi, \theta \in Form_P$, there exists $\alpha \in Form_P$ such that $\Box\varphi\psi\theta$ is semantically equivalent to α. We want a general theorem which says that no matter how exotic a connective one invents, it's always possible to find an element of $Form_P$ which is semantically equivalent, and thus our choice of connectives is sufficient to express everything we'd ever want.

Rather than deal with arbitrary connectives, the real issue here is whether we can express any possible function taking k true/false values to true/false values.

Definition 3.3.1. *Let $k \in \mathbb{N}^+$. A function $f\colon \{0,1\}^k \to \{0,1\}$ is called a* boolean function of arity k.

Definition 3.3.2. *Suppose that $P = \{\mathsf{A_0}, \mathsf{A_1}, \dots, \mathsf{A_{k-1}}\}$. Given $\varphi \in Form_P$, we define a boolean function $B_\varphi\colon \{0,1\}^k \to \{0,1\}$ as follows. Given $\sigma \in \{0,1\}^k$, define a truth assignment $v\colon P \to \{0,1\}$ by letting $v(\mathsf{A_i}) = \sigma(i)$ for all i, and set $B_\varphi(\sigma) = \overline{v}(\varphi)$.*

Theorem 3.3.3. *Fix $k \in \mathbb{N}^+$, and let $P = \{\mathsf{A_0}, \mathsf{A_1}, \dots, \mathsf{A_{k-1}}\}$. For any boolean function $f\colon \{0,1\}^k \to \{0,1\}$ of arity k, there exists $\varphi \in Form_P$ such that $f = B_\varphi$.*

In fact, we'll prove a stronger theorem below which says that we may assume that our formula φ is in a particularly simple form.

Let's look at an example before we do the proof. Suppose that $f\colon \{0,1\}^3 \to \{0,1\}$ is given by

0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose we wanted to come up with a formula φ such that $f = B_\varphi$. One option is to use a lot of thought to come up with an elegant solution. Another is simply to think as follows. Since $f(000) = 1$, perhaps we should put

$$\neg \mathsf{A}_0 \wedge \neg \mathsf{A}_1 \wedge \neg \mathsf{A}_2$$

into the formula somewhere. Similarly, since $f(010) = 1$, perhaps we should put

$$\neg \mathsf{A}_0 \wedge \mathsf{A}_1 \wedge \neg \mathsf{A}_2$$

into the formula somewhere. If we do the same to the other lines which have value 1, we can put all of these pieces together in a manner which makes them all play nice by connecting them with $\vee$. Thus, our formula is

$$(\neg \mathsf{A}_0 \wedge \neg \mathsf{A}_1 \wedge \neg \mathsf{A}_2) \vee (\neg \mathsf{A}_0 \wedge \mathsf{A}_1 \wedge \neg \mathsf{A}_2) \vee (\mathsf{A}_0 \wedge \mathsf{A}_1 \wedge \neg \mathsf{A}_2) \vee (\mathsf{A}_0 \wedge \mathsf{A}_1 \wedge \mathsf{A}_2)$$

We now give the general proof.

Definition 3.3.4. *A* literal *is a element of $P \cup \{\neg \mathsf{A} : \mathsf{A} \in P\}$. We denote the set of literals by Lit_P.*

Definition 3.3.5.

- *Let $Conj_P = G(Sym_P^*, Lit_P, \{h_\wedge\})$. We call the elements of $Conj_P$* conjunctive formulas*.*
- *Let $Disj_P = G(Sym_P^*, Lit_P, \{h_\vee\})$. We call the elements of $Disj_P$* disjunctive formulas*.*

Definition 3.3.6.

- *Let $DNF_P = G(Sym_P^*, Conj_P, \{h_\vee\})$. We say that an element of DNF_P is in* disjunctive normal form*.*
- *Let $CNF_P = G(Sym_P^*, Disj_P, \{h_\wedge\})$. We say that an element of CNF_P is in* conjunctive normal form*.*

Theorem 3.3.7. *Fix $k \in \mathbb{N}^+$, and let $P = \{\mathsf{A}_0, \mathsf{A}_1, \dots, \mathsf{A}_{k-1}\}$. For any boolean function $f\colon \{0,1\}^k \to \{0,1\}$ of arity k, there exists $\varphi \in DNF_P$ such that $f = B_\varphi$.*

Proof. Let $T = \{\sigma \in \{0,1\}^k : f(\sigma) = 1\}$. If $T = \emptyset$, we may let φ be $\mathsf{A}_0 \wedge (\neg \mathsf{A}_0)$. Suppose then that $T \neq \emptyset$. For each $\sigma \in T$, let

$$\psi_\sigma = \bigwedge_{i=0}^{k-1} \theta_i$$

where

$$\theta_i = \begin{cases} \mathsf{A}_i & \text{if } \sigma(i) = 1 \\ \neg \mathsf{A}_i & \text{if } \sigma(i) = 0 \end{cases}$$

For each $\sigma \in T$, notice that $\psi_\sigma \in Conj_P$ because $\theta_i \in Lit_P$ for all i. Finally, let

$$\varphi = \bigvee_{\sigma \in T} \psi_\sigma$$

and notice that $\varphi \in DNF_P$. □

Since DNF_P formulas suffice, we don't even need $\rightarrow$ if all we want to do is have the ability to express all boolean functions. In fact we can also get rid of one of $\wedge$ or $\vee$ as well (think about why).

3.4 Syntactic Implication

3.4.1 Motivation

We now seek to define a different notion of implication which is based on syntactic manipulations instead of a detour through truth assignments and other semantic notions. We will do this by setting up a "proof system" which gives rules on how to transform certain implications to other implications. There are many many ways to do this. Some approaches pride themselves on being minimalistic by using a minimal number of axioms and rules, often at the expense of making the system unnatural to actually work with. We'll take a different approach and set down our rules and axioms based on the types of steps in a proof that are used naturally throughout mathematics.

We begin with a somewhat informal description of what we plan to do. The objects that will manipulate are pairs, where the first component is a set of formulas and the second is a formula. Given $\Gamma \subseteq Form_P$ and $\varphi \in Form_P$, we write $\Gamma \vdash \varphi$ to intuitively mean that there is a proof of φ from the assumptions Γ. We begin with the most basic proofs. If $\varphi \in \Gamma$, i.e. if φ is one of your assumptions, then you're permitted to assert that $\Gamma \vdash \varphi$.

Basic Proofs: $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$.

Rules for $\wedge$: We have two rules for $\wedge$-elimination and one for $\wedge$-introduction.

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ (\wedge EL) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ (\wedge ER) \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ (\wedge I)$$

Rules for $\vee$: We have two rules for introducing $\vee$.

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee IL) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ (\vee IR)$$

Rules for $\rightarrow$:

$$\frac{\Gamma \vdash \varphi \rightarrow \psi}{\Gamma \cup \{\varphi\} \vdash \psi}\ (\rightarrow E) \qquad \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}\ (\rightarrow I)$$

Rules for proofs by cases:

$$\frac{\Gamma \cup \{\varphi\} \vdash \theta \quad \Gamma \cup \{\psi\} \vdash \theta}{\Gamma \cup \{\varphi \vee \psi\} \vdash \theta}\ (\vee PC) \qquad \frac{\Gamma \cup \{\psi\} \vdash \varphi \quad \Gamma \cup \{\neg\psi\} \vdash \varphi}{\Gamma \vdash \varphi}\ (\neg PC)$$

Rule for proof by contradiction:

$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \psi \quad \Gamma \cup \{\neg\varphi\} \vdash \neg\psi}{\Gamma \vdash \varphi}\ (Contr)$$

3.4.2 Official Definitions

Definition 3.4.1. *Let* $Line_P = \mathcal{P}(Form_P) \times Form_P$.

Definition 3.4.2. *Let* $Assume_P = \{(\Gamma, \varphi) \in Line_P : \varphi \in \Gamma\}$.

We need to define functions corresponding to various rules. For example, we define $h_{\wedge EL} \colon Line_P \rightarrow \mathcal{P}(Line_P)$ by letting

$$h_{\wedge EL}(\Gamma, \alpha) = \begin{cases} \{(\Gamma, \varphi)\} & \text{if } \alpha = \varphi \wedge \psi \text{ where } \varphi, \psi \in Form_P \\ \emptyset & \text{otherwise} \end{cases}$$

$h_{\wedge EL}$ is similar, and we define $h_{\wedge I}\colon (Line_P)^2 \to \mathcal{P}(Line_P)$ by

$$h_{\wedge I}((\Gamma_1,\varphi_1),(\Gamma_2,\varphi_2)) = \begin{cases} \{(\Gamma_1,\varphi_1 \wedge \varphi_2)\} & \text{if } \Gamma_1 = \Gamma_2 \\ \emptyset & \text{otherwise} \end{cases}$$

For the $\vee IL$ rule, we have the function $h_{\vee IL}\colon Line_P \to \mathcal{P}(Line_P)$ by

$$h_{\vee IL}(\Gamma,\varphi) = \{(\Gamma,\varphi \vee \psi) : \psi \in Form_P\}$$

Similarly, we define functions $h_{\vee IR}$.

We let $\mathcal{H}$ be the collection of all of these functions.

Definition 3.4.3. *A* deduction *is a witnessing sequence in* $(Line_P, Assume_P, \mathcal{H})$.

Definition 3.4.4. *Let* $\Gamma \subseteq Form_P$ *and let* $\varphi \in Form_P$. *We write* $\Gamma \vdash_P \varphi$, *or simply* $\Gamma \vdash \varphi$ *if* P *is clear, to mean that*

$$(\Gamma,\varphi) \in G(Line_P, Assume_P, \mathcal{H}).$$

We pronounce $\Gamma \vdash \varphi$ *as "*Γ *syntactically implies* φ*".*

Notation 3.4.5.

1. *If* $\Gamma = \emptyset$, *we write* $\vdash \varphi$ *instead of* $\emptyset \vdash \varphi$.
2. *If* $\Gamma = \{\gamma\}$, *we write* $\gamma \vdash \varphi$ *instead of* $\{\gamma\} \vdash \varphi$.

Definition 3.4.6. Γ *is* inconsistent *if there exists* $\theta \in Form_P$ *such that* $\Gamma \vdash \theta$ *and* $\Gamma \vdash \neg\theta$. *Otherwise, we say that* Γ *is* consistent.

3.4.3 Examples Of Deductions

Proposition 3.4.7. $\mathsf{A} \wedge \mathsf{B} \vdash \mathsf{A} \vee \mathsf{B}$.

Proof.

$$\begin{array}{lrr} \{\mathsf{A}\wedge\mathsf{B}\} \vdash \mathsf{A}\wedge\mathsf{B} & (Assume_P) & (1) \\ \{\mathsf{A}\wedge\mathsf{B}\} \vdash \mathsf{A} & (\wedge EL \text{ on } 1) & (2) \\ \{\mathsf{A}\wedge\mathsf{B}\} \vdash \mathsf{A}\vee\mathsf{B} & (\vee I \text{ on } 2) & (3) \end{array}$$

□

Proposition 3.4.8. $\neg\neg\varphi \vdash \varphi$ *for all* $\varphi \in Form_P$.

Proof.

$$\begin{array}{rrr} \{\neg\neg\varphi,\neg\varphi\} \vdash \neg\varphi & (Assume_P) & (1) \\ \{\neg\neg\varphi,\neg\varphi\} \vdash \neg\neg\varphi & (Assume_P) & (2) \\ \{\neg\neg\varphi\} \vdash \varphi & (Contr \text{ on 1 and 2}) & (3) \end{array}$$

□

Proposition 3.4.9. $\vdash \varphi \vee \neg\varphi$ *for all* $\varphi \in Form_P$.

Proof.

$$\begin{aligned}
\{\varphi\} &\vdash \varphi && (Assume_P) && (1)\\
\{\varphi\} &\vdash \varphi \vee \neg\varphi && (\vee IL \text{ on } 1) && (2)\\
\{\neg\varphi\} &\vdash \neg\varphi && (Assume_P) && (3)\\
\{\neg\varphi\} &\vdash \varphi \vee \neg\varphi && (\vee IR \text{ on } 3) && (4)\\
\emptyset &\vdash \varphi \vee \neg\varphi && (\neg PC \text{ on 2 and } 4) && (5)
\end{aligned}$$

□

Proposition 3.4.10. *$\{\neg\varphi, \varphi \vee \psi\} \vdash \psi$ for all $\varphi, \psi \in Form_P$.*

Proof.

$$\begin{aligned}
\{\neg\varphi, \varphi, \neg\psi\} &\vdash \varphi && (Assume_P) && (1)\\
\{\neg\varphi, \varphi, \neg\psi\} &\vdash \neg\varphi && (Assume_P) && (2)\\
\{\neg\varphi, \varphi\} &\vdash \psi && (Contr \text{ on 1 and } 2) && (3)\\
\{\neg\varphi, \psi\} &\vdash \psi && (Assume_P) && (4)\\
\{\neg\varphi, \varphi \vee \psi\} &\vdash \psi && (\vee PC \text{ on 3 and } 4) && (5)
\end{aligned}$$

□

3.4.4 Theorems about $\vdash$

Proposition 3.4.11. *If $\Gamma \vdash \varphi$ and $\Gamma' \supseteq \Gamma$, then $\Gamma' \vdash \varphi$.*

Proof. The proof is by induction. We let $X = \{(\Gamma, \varphi) \in G : \Gamma' \vdash \varphi \text{ for all } \Gamma' \supseteq \Gamma\}$ and we show by induction on G that $X = G$. We begin by noting that if $\varphi \in \Gamma$, then for every $\Gamma' \supseteq \Gamma$, we have $\varphi \in \Gamma'$ and hence $\Gamma' \vdash \varphi$. Therefore, $(\Gamma, \varphi) \in X$ for all $(\Gamma, \varphi) \in Assume_P$.

We first handle the $\wedge EL$ rule. Suppose that $(\Gamma, \varphi \wedge \psi) \in X$. We need to show that $(\Gamma, \varphi) \in X$. Suppose that $\Gamma' \supseteq \Gamma$. We then have that $\Gamma' \vdash \varphi \wedge \psi$ by induction (i.e. since $(\Gamma, \varphi \wedge \psi) \in X$), hence $\Gamma' \vdash \varphi$ by the $\wedge EL$ rule. Therefore, $(\Gamma, \varphi) \in X$. The other $\wedge$ rules and the $\vee$ rules are similar.

We now handle $\rightarrow E$ rule. Suppose that $(\Gamma, \varphi \rightarrow \psi) \in X$. We need to show that $(\Gamma \cup \{\varphi\}, \psi)$. Suppose that $\Gamma' \supseteq \Gamma \cup \{\varphi\}$. We then have that $\Gamma' \supseteq \Gamma$, hence $\Gamma' \vdash \varphi \rightarrow \psi$ by induction, and so $\Gamma' \cup \{\varphi\} \vdash \psi$ by the $\rightarrow E$ rule. However, $\Gamma' \cup \{\varphi\} = \Gamma'$ because $\varphi \in \Gamma'$, so $\Gamma' \vdash \psi$. Therefore, $(\Gamma \cup \{\varphi\}, \psi) \in X$.

We now handle $\rightarrow I$ rule. Suppose that $(\Gamma \cup \{\varphi\}, \psi) \in X$. We need to show that $(\Gamma, \varphi \rightarrow \psi)$. Suppose that $\Gamma' \supseteq \Gamma$. We then have that $\Gamma' \cup \{\varphi\} \supseteq \Gamma \cup \{\varphi\}$, hence $\Gamma' \cup \{\varphi\} \vdash \psi$ by induction, and so $\Gamma' \vdash \varphi \rightarrow \psi$ by the $\rightarrow I$ rule. Therefore, $(\Gamma, \varphi \rightarrow \psi) \in X$.

Let's go for the $\vee PC$ rule. Suppose that $(\Gamma \cup \{\varphi\}, \theta) \in X$ and $(\Gamma \cup \{\psi\}, \theta) \in X$. We need to show that $(\Gamma \cup \{\varphi \vee \psi\}, \theta) \in X$. Suppose that $\Gamma' \supseteq \Gamma \cup \{\varphi \vee \psi\}$. We then have that $\Gamma' \cup \{\varphi\} \supseteq \Gamma \cup \{\varphi\}$ and $\Gamma' \cup \{\psi\} \supseteq \Gamma \cup \{\psi\}$, hence $\Gamma' \cup \{\varphi\} \vdash \theta$ and $\Gamma' \cup \{\psi\} \vdash \theta$ by induction, and so $\Gamma' \cup \{\varphi \vee \psi\} \vdash \theta$ by the $\neg PC$ rule. However, $\Gamma' \cup \{\varphi \vee \psi\} = \Gamma'$ because $\varphi \vee \psi \in \Gamma'$, so $\Gamma' \vdash \theta$. Therefore, $(\Gamma \cup \{\varphi \vee \psi\}, \theta) \in X$.

Let's next attack the $\neg PC$ rule. Suppose that $(\Gamma \cup \{\psi\}, \varphi) \in X$ and $(\Gamma \cup \{\neg\psi\}, \varphi) \in X$. We need to show that $(\Gamma, \varphi) \in X$. Suppose that $\Gamma' \supseteq \Gamma$. We then have that $\Gamma' \cup \{\psi\} \supseteq \Gamma \cup \{\psi\}$ and $\Gamma' \cup \{\neg\psi\} \supseteq \Gamma \cup \{\neg\psi\}$, hence $\Gamma' \cup \{\psi\} \vdash \varphi$ and $\Gamma' \cup \{\neg\psi\} \vdash \varphi$ by induction, and so $\Gamma' \vdash \varphi$ by the $\neg PC$ rule. Therefore, $(\Gamma, \varphi) \in X$.

We finish off with the $Contr$ rule. Suppose that $(\Gamma \cup \{\neg\varphi\}, \psi) \in X$ and $(\Gamma \cup \{\neg\varphi\}, \neg\psi) \in X$. We need to show that $(\Gamma, \varphi) \in X$. Suppose that $\Gamma' \supseteq \Gamma$. We then have that $\Gamma' \cup \{\neg\varphi\} \supseteq \Gamma \cup \{\neg\varphi\}$, hence $\Gamma' \cup \{\neg\varphi\} \vdash \psi$ and $\Gamma' \cup \{\neg\varphi\} \vdash \neg\psi$ by induction, and so $\Gamma' \vdash \varphi$ by the $Contr$ rule. Therefore, $(\Gamma, \varphi) \in X$.

The result follows by induction. □

Proposition 3.4.12. *If Γ is inconsistent, then $\Gamma \vdash \varphi$ for all $\varphi \in Form_P$.*

Proof. Fix θ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$, and fix $\varphi \in Form_P$. We have that $\Gamma \cup \{\neg\varphi\} \vdash \theta$ and $\Gamma \cup \{\neg\varphi\} \vdash \neg\theta$ by Proposition 3.4.11. Therefore, $\Gamma \vdash \varphi$ by using the *Contr* rule. □

Proposition 3.4.13.

1. *If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$.*

2. *If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \vdash \varphi$.*

Proof.

1. Since $\Gamma \cup \{\varphi\}$ is inconsistent, we know that $\Gamma \cup \{\varphi\} \vdash \neg\varphi$ by Proposition 3.4.12. Since we also have that $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by $Assume_P$, it follows that $\Gamma \vdash \neg\varphi$ by the $\neg PC$ rule.

2. Since $\Gamma \cup \{\neg\varphi\}$ is inconsistent, we know that $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition 3.4.12. Since we also have that $\Gamma \cup \{\varphi\} \vdash \varphi$ by $Assume_P$, it follows that $\Gamma \vdash \varphi$ by the $\neg PC$ rule.

□

Corollary 3.4.14. *If $\Gamma \subseteq Form_P$ is consistent and $\varphi \in Form_P$, then either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{\neg\varphi\}$ is consistent.*

Proof. If both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent, then both $\Gamma \vdash \neg\varphi$ and $\Gamma \vdash \varphi$ by Proposition 3.4.13, so Γ is inconsistent. □

Proposition 3.4.15.

1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \psi$.*

2. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.*

Proof.

1. Since $\Gamma \vdash \varphi$, it follows from Proposition 3.4.11 that $\Gamma \cup \{\neg\varphi\} \vdash \varphi$. Since we also have $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by $Assume_P$, we may conclude that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. Therefore, by Proposition 3.4.12, we have that $\Gamma \cup \{\neg\varphi\} \vdash \psi$. Now we also have $\Gamma \cup \{\varphi\} \vdash \psi$ by assumption, so the $\neg PC$ rule gives that $\Gamma \vdash \psi$.

2. Since $\Gamma \vdash \varphi \to \psi$, we can conclude that $\Gamma \cup \{\varphi\} \vdash \psi$ by rule $\to E$. The result follows from part 1.

□

Proposition 3.4.16. *$\Gamma \vdash \varphi$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

Proof. The proof is by induction. We let $X = \{(\Gamma, \varphi) \in G : \text{there exists a finite } \Gamma_0 \subseteq \Gamma \text{ such that } \Gamma_0 \vdash \varphi\}$ and we show by induction on G that $X = G$. We begin by noting that if $\varphi \in \Gamma$, then we have $\{\varphi\}$ is a finite subset of Γ and $\{\varphi\} \vdash \varphi$. Therefore, $(\Gamma, \varphi) \in X$ for all $(\Gamma, \varphi) \in Assume_P$.

We first handle the $\wedge EL$ rule. Suppose that $(\Gamma, \varphi \wedge \psi) \in X$. We need to show that $(\Gamma, \varphi) \in X$. By induction (i.e. since $(\Gamma, \varphi \wedge \psi) \in X$), we may fix a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi \wedge \psi$. We then have that $\Gamma_0 \vdash \varphi$ by the $\wedge EL$ rule. Therefore, $(\Gamma, \varphi) \in X$. The other $\wedge ER$ rule and the $\vee$ rules are similar.

We now handle the $\wedge I$ rule. Suppose that $(\Gamma, \varphi) \in X$ and $(\Gamma, \psi) \in X$. We need to show that $(\Gamma, \varphi \wedge \psi) \in X$. By induction, we may fix a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$ and we may fix a finite $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 \vdash \psi$. We then have that $\Gamma_0 \cup \Gamma_1 \vdash \varphi$ and $\Gamma_0 \cup \Gamma_1 \vdash \psi$ by Proposition 3.4.11, hence $\Gamma_0 \cup \Gamma_1 \vdash \varphi \wedge \psi$ by the $\wedge I$ rule. Therefore, $(\Gamma, \varphi \wedge \psi) \in X$ because $\Gamma_0 \cup \Gamma_1$ is a finite subset of Γ.

We now handle $\to E$ rule. Suppose that $(\Gamma, \varphi \to \psi) \in X$. We need to show that $(\Gamma \cup \{\varphi\}, \psi) \in X$. By induction, we may fix a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi \to \psi$. We then have that $\Gamma_0 \cup \{\varphi\} \vdash \psi$ by the $\to E$ rule. Therefore, $(\Gamma \cup \{\varphi\}, \psi) \in X$ because $\Gamma_0 \cup \{\varphi\}$ is a finite subset of $\Gamma \cup \{\varphi\}$.

We now handle $\to I$ rule. Suppose that $(\Gamma \cup \{\varphi\}, \psi) \in X$. We need to show that $(\Gamma, \varphi \to \psi) \in X$. By induction, we may fix a finite $\Gamma_0 \subseteq \Gamma \cup \{\varphi\}$ such that $\Gamma_0 \vdash \psi$. Let $\Gamma_0' = \Gamma_0 - \{\varphi\}$. We then have that $\Gamma_0 \subseteq \Gamma_0' \cup \{\varphi\}$, hence $\Gamma_0' \cup \{\varphi\} \vdash \psi$ by Proposition 3.4.11, and so $\Gamma_0' \vdash \varphi \to \psi$ by the $\to I$ rule. Therefore, $(\Gamma, \varphi \to \psi) \in X$ because Γ_0' is a finite subset of Γ.

The other rules are exercises. The result follows by induction. □

Corollary 3.4.17. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. Suppose that Γ is inconsistent, and fix $\theta \in Form_P$ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. By Proposition 3.4.16, there exists finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_0 \vdash \theta$ and $\Gamma_1 \vdash \neg\theta$. Using Proposition 3.4.11, it follows that $\Gamma_0 \cup \Gamma_1 \vdash \theta$ and $\Gamma_0 \cup \Gamma_1 \vdash \neg\theta$, so $\Gamma_0 \cup \Gamma_1$ is a finite inconsistent subset of Γ. □

3.5 Soundness and Completeness

3.5.1 The Soundness Theorem

Theorem 3.5.1 (Soundness Theorem)**.**

1. *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*
2. *Every satisfiable set of formulas is consistent.*

Proof.

1. The proof is by induction. We let $X = \{(\Gamma, \varphi) \in G : \Gamma \vDash \varphi\}$ and we show by induction on G that $X = G$. We begin by noting that if $\varphi \in \Gamma$, then $\Gamma \vDash \varphi$ because if $v\colon P \to \{0,1\}$ is such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\overline{v}(\varphi) = 1$ simply because $\varphi \in \Gamma$. Therefore, $(\Gamma, \varphi) \in X$ for all $X \in Assume_P$.

   We first handle the $\wedge EL$ rule. Suppose that $\Gamma \vDash \varphi \wedge \psi$. We need to show that $\Gamma \vDash \varphi$. However, this is straightforward because if $v\colon P \to \{0,1\}$ is such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\overline{v}(\varphi \wedge \psi) = 1$ because $\Gamma \vDash \varphi \wedge \psi$, hence $\overline{v}(\varphi) = 1$. Therefore, $\Gamma \vDash \varphi$. The other $\wedge$ rules and the $\vee$ rules are similar.

   We now handle $\to E$ rule. Suppose that $\Gamma \vDash \varphi \to \psi$. We need to show that $\Gamma \cup \{\varphi\} \vDash \psi$. Let $v\colon P \to \{0,1\}$ be such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma \cup \{\varphi\}$. Since $\Gamma \vDash \varphi \to \psi$, we have $\overline{v}(\varphi \to \psi) = 1$. Since $\overline{v}(\varphi) = 1$, it follows that $\overline{v}(\psi) = 1$. Therefore, $\Gamma \cup \{\varphi\} \vDash \psi$. The $\to I$ rule is similar.

   Let's next attack the $\neg PC$ rule. Suppose that $\Gamma \cup \{\psi\} \vDash \varphi$ and $\Gamma \cup \{\neg\psi\} \vDash \varphi$. We need to show that $\Gamma \vDash \varphi$. Let $v\colon P \to \{0,1\}$ be such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. If $\overline{v}(\psi) = 1$, then $\overline{v}(\varphi) = 1$ because $\Gamma \cup \{\psi\} \vDash \varphi$. Otherwise, we have $\overline{v}(\psi) = 0$, hence $\overline{v}(\neg\psi) = 1$, and thus $\overline{v}(\varphi) = 1$ because $\Gamma \cup \{\neg\psi\} \vDash \varphi$. Therefore, $\Gamma \vDash \varphi$. The $\vee PC$ rule is similar.

   We finish off with the *Contr* rule. Suppose that $\Gamma \cup \{\neg\varphi\} \vDash \psi$ and $\Gamma \cup \{\neg\varphi\} \vDash \neg\psi$. We need to show that $\Gamma \vDash \varphi$. Let $v\colon P \to \{0,1\}$ be such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. Suppose that $\overline{v}(\varphi) = 0$. We then have $\overline{v}(\neg\varphi) = 1$, and so both $\overline{v}(\psi) = 1$ and $\overline{v}(\neg\psi) = 1$ because $\Gamma \cup \{\neg\varphi\} \vDash \psi$ and $\Gamma \cup \{\neg\varphi\} \vDash \neg\psi$. This is a contradiction, so we may conclude that $\overline{v}(\varphi) = 1$. Therefore, $\Gamma \vDash \varphi$.

   The result follows by induction.

2. Let Γ be a satisfiable set of formulas. Fix a truth assignment $v\colon P \to \{0,1\}$ such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. Suppose that Γ is inconsistent, and fix $\theta \in Form_P$ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. We then have $\Gamma \vDash \theta$ and $\Gamma \vDash \neg\theta$ by part 1, hence $\overline{v}(\theta) = 1$ and $\overline{v}(\neg\theta) = 1$, a contradiction. It follows that Γ is consistent.

□

3.5.2 The Completeness Theorem

Our first task is to show that every consistent set of formulas is satisfiable. From here, the Completeness Theorem follows easily, as we'll see below.

Definition 3.5.2. Γ *is* complete *if for all* $\varphi \in Form_P$*, either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*.*

Proposition 3.5.3. *Suppose that P is countable. If Γ is consistent, then there exists a set $\Delta \supseteq \Gamma$ which is consistent and complete.*

Proof. Since P is countable, it follows that $Form_P$ is countable. List $Form_P$ as $\psi_1, \psi_2, \psi_3, \dots$. We define a sequence of sets $\Gamma_0, \Gamma_1, \Gamma_2, \dots$ recursively as follows. Let $\Gamma_0 = \Gamma$. Suppose that $n \in \mathbb{N}$ and we have defined Γ_n. Let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Gamma_n \cup \{\neg\psi_n\} & \text{otherwise} \end{cases}$$

Using induction and Corollary 3.4.14, it follows that Γ_n is consistent for all $n \in \mathbb{N}$. Let $\Delta = \bigcup_{n\in\mathbb{N}} \Gamma_n$.

We first argue that Δ is consistent. For any finite subset Δ_0 of Δ, there exists an $n \in \mathbb{N}$ such that $\Delta_0 \subseteq \Gamma_n$, and so Δ_0 is consistent because every Γ_n is consistent. Therefore, Δ is consistent by Proposition 3.4.17. We end by arguing that Δ is complete. Fix $\varphi \in Form_P$, and fix $n \in \mathbb{N}^+$ such that $\varphi = \psi_n$. By construction, we either have $\varphi \in \Gamma_n \subseteq \Delta$ or $\neg\varphi \in \Gamma_n \subseteq \Delta$. Therefore, Δ is complete. □

Definition 3.5.4. Δ *is* maximal consistent *if Δ is consistent and there is no $\Delta' \supset \Delta$ which is consistent.*

Proposition 3.5.5. *Δ is maximal consistent if and only if Δ is consistent and complete.*

Proof. Suppose that Δ is maximal consistent. We certainly have that Δ is consistent. Fix $\varphi \in Form_P$. By Corollary 3.4.14, either $\Delta \cup \{\varphi\}$ is consistent or $\Delta \cup \{\neg\varphi\}$ is consistent. If $\Delta \cup \{\varphi\}$ is consistent, then $\varphi \in \Delta$ because Δ is maximal consistent. Similarly, If $\Delta \cup \{\neg\varphi\}$ is consistent, then $\neg\varphi \in \Delta$ because Δ is maximal consistent. Therefore, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

Suppose that Δ is consistent and complete. Suppose that $\Delta' \supset \Delta$ and fix $\varphi \in \Delta' - \Delta$. Since Δ is complete and $\varphi \notin \Delta$, we have $\neg\varphi \in \Delta$. Therefore, $\Delta' \vdash \varphi$ and $\Delta' \vdash \neg\varphi$, so Δ' is inconsistent. It follows that Δ is maximal consistent. □

Proposition 3.5.6. *If Γ is consistent, then there exists a set $\Delta \supseteq \Gamma$ which is consistent and complete.*

Proof. Let $\mathcal{S} = \{\Phi \subseteq Form_P : \Gamma \subseteq \Phi \text{ and } \Phi \text{ is consistent}\}$, and order $\mathcal{S}$ be $\subseteq$. Notice that $\mathcal{S}$ is nonempty because $\Gamma \in \mathcal{S}$. Suppose that $\mathcal{C} \subseteq \mathcal{S}$ is a chain in $\mathcal{S}$. Let $\Psi = \bigcup \mathcal{C} = \{\psi \in Form_P : \psi \in \Phi \text{ for some } \Phi \in \mathcal{C}\}$. We need to argue that Ψ is consistent. Suppose that Ψ_0 is a finite subset of Ψ, say $\Psi_0 = \{\psi_1, \psi_2, \dots, \psi_n\}$. For each ψ_i, fix $\Phi_i \in \mathcal{C}$ with $\psi_i \in \Phi_i$. Since $\mathcal{C}$ is a chain, there exists j such that $\Phi_j \supseteq \Phi_i$ for all i. Now $\Phi_j \in \mathcal{C} \subseteq \mathcal{S}$, so Φ_j is consistent, and hence Ψ_0 is consistent. Therefore, Ψ is consistent by Proposition 3.4.17. It follows that $\Psi \in \mathcal{S}$ and using the fact that $\Phi \subseteq \Psi$ for all $\Phi \in \mathcal{C}$, we may conclude that $\mathcal{C}$ has an upper bound.

Therefore, by Zorn's Lemma, $\mathcal{S}$ has a maximal element Δ. Notice that Δ is maximal consistent, hence Δ is complete and consistent by Proposition 3.5.5. □

Lemma 3.5.7. *Suppose that Δ is consistent and complete. If $\Delta \vdash \varphi$, then $\varphi \in \Delta$.*

Proof. Suppose that $\Delta \vdash \varphi$. Since Δ is complete, we have that either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$. Now if $\neg\varphi \in \Delta$, then $\Delta \vdash \neg\varphi$, hence Δ is inconsistent contradicting our assumption. It follows that $\varphi \in \Delta$. □

Lemma 3.5.8. *Suppose that Δ is consistent and complete. We have*

1. *$\neg\varphi \in \Delta$ if and only if $\varphi \notin \Delta$.*

2. $\varphi \wedge \psi \in \Delta$ *if and only if* $\varphi \in \Delta$ *and* $\psi \in \Delta$.

3. $\varphi \vee \psi \in \Delta$ *if and only if either* $\varphi \in \Delta$ *or* $\psi \in \Delta$.

4. $\varphi \to \psi \in \Delta$ *if and only if either* $\varphi \notin \Delta$ *or* $\psi \in \Delta$.

Proof.

1. If $\neg\varphi \in \Delta$, then $\varphi \notin \Delta$ because otherwise $\Delta \vdash \varphi$ and so Δ would be inconsistent.

   Conversely, if $\varphi \notin \Delta$, then $\neg\varphi \in \Delta$ because Δ is complete.

2. Suppose first that $\varphi \wedge \psi \in \Delta$. We then have that $\Delta \vdash \varphi \wedge \psi$, hence $\Delta \vdash \varphi$ by the $\wedge EL$ rule and $\Delta \vdash \psi$ by the $\wedge ER$ rule. Therefore, $\varphi \in \Delta$ and $\psi \in \Delta$ by Lemma 3.5.7.

   Conversely, suppose that $\varphi \in \Delta$ and $\psi \in \Delta$. We then have $\Delta \vdash \varphi$ and $\Delta \vdash \psi$, hence $\Delta \vdash \varphi \wedge \psi$ by the $\wedge I$ rule. Therefore, $\varphi \wedge \psi \in \Delta$ by Lemma 3.5.7.

3. Suppose first that $\varphi \vee \psi \in \Delta$. Suppose that $\varphi \notin \Delta$. Since Δ is complete, we have that $\neg\varphi \in \Delta$. From Proposition 3.4.10, we know that $\{\neg\varphi, \varphi \vee \psi\} \vdash \psi$, hence $\Delta \vdash \psi$ by Proposition 3.4.11. Therefore, $\psi \in \Delta$ by Lemma 3.5.7. It follows that either $\varphi \in \Delta$ or $\psi \in \Delta$.

   Conversely, suppose that either $\varphi \in \Delta$ or $\psi \in \Delta$.

   Case 1: Suppose that $\varphi \in \Delta$. We have $\Delta \vdash \varphi$, hence $\Delta \vdash \varphi \vee \psi$ by the $\vee IL$ rule. Therefore, $\varphi \vee \psi \in \Delta$ by Lemma 3.5.7.

   Case 2: Suppose that $\psi \in \Delta$. We have $\Delta \vdash \psi$, hence $\Delta \vdash \varphi \vee \psi$ by the $\vee IR$ rule. Therefore, $\varphi \vee \psi \in \Delta$ by Lemma 3.5.7.

4. Suppose first that $\varphi \to \psi \in \Delta$. Suppose that $\varphi \in \Delta$. We then have that $\Delta \vdash \varphi$ and $\Delta \vdash \varphi \to \psi$, hence $\Delta \vdash \psi$ by Proposition 3.4.15. Therefore, $\psi \in \Delta$ by Lemma 3.5.7. It follows that either $\varphi \notin \Delta$ or $\psi \in \Delta$.

   Conversely, suppose that either $\varphi \notin \Delta$ or $\psi \in \Delta$.

   Case 1: Suppose that $\varphi \notin \Delta$. We have $\neg\varphi \in \Delta$ because Δ is complete, hence $\Delta \cup \{\varphi\}$ is inconsistent (as $\Delta \cup \{\varphi\} \vdash \varphi$ and $\Delta \cup \{\neg\varphi\} \vdash \neg\varphi$). It follows that $\Delta \cup \{\varphi\} \vdash \psi$ by Proposition 3.4.12, hence $\Delta \vdash \varphi \to \psi$ by the $\to I$ rule. Therefore, $\varphi \to \psi \in \Delta$ by Lemma 3.5.7.

   Case 2: Suppose that $\psi \in \Delta$. We have $\psi \in \Delta \cup \{\varphi\}$, hence $\Delta \cup \{\varphi\} \vdash \psi$, and so $\Delta \vdash \varphi \to \psi$ by the $\to I$ rule. Therefore, $\varphi \to \psi \in \Delta$ by Lemma 3.5.7.

□

Proposition 3.5.9. *If* Δ *is consistent and complete, then* Δ *is satisfiable.*

Proof. Suppose that Δ is complete and consistent. Define $v \colon P \to \{0, 1\}$ by

$$v(\mathsf{A}) = \begin{cases} 1 & \text{if } \mathsf{A} \in \Delta \\ 0 & \text{if } \mathsf{A} \notin \Delta \end{cases}$$

We prove by induction on φ that $\varphi \in \Delta$ if and only if $\overline{v}(\varphi) = 1$. For any $\mathsf{A} \in P$, we have

$$\mathsf{A} \in \Delta \Leftrightarrow v(\mathsf{A}) = 1 \Leftrightarrow \overline{v}(\mathsf{A}) = 1$$

by our definition of v.

Suppose that the result holds for φ. We have

$$\begin{aligned}\neg\varphi \in \Delta &\Leftrightarrow \varphi \notin \Delta && \text{(by Lemma 3.5.8)}\\ &\Leftrightarrow \overline{v}(\varphi) = 0 && \text{(by induction)}\\ &\Leftrightarrow \overline{v}(\neg\varphi) = 1\end{aligned}$$

Suppose that the result holds for φ and ψ. We have

$$\begin{aligned}\varphi \wedge \psi \in \Delta &\Leftrightarrow \varphi \in \Delta \text{ and } \psi \in \Delta && \text{(by Lemma 3.5.8)}\\ &\Leftrightarrow \overline{v}(\varphi) = 1 \text{ and } \overline{v}(\psi) = 1 && \text{(by induction)}\\ &\Leftrightarrow \overline{v}(\varphi \wedge \psi) = 1\end{aligned}$$

and

$$\begin{aligned}\varphi \vee \psi \in \Delta &\Leftrightarrow \varphi \in \Delta \text{ or } \psi \in \Delta && \text{(by Lemma 3.5.8)}\\ &\Leftrightarrow \overline{v}(\varphi) = 1 \text{ or } \overline{v}(\psi) = 1 && \text{(by induction)}\\ &\Leftrightarrow \overline{v}(\varphi \vee \psi) = 1\end{aligned}$$

and finally

$$\begin{aligned}\varphi \to \psi \in \Delta &\Leftrightarrow \varphi \notin \Delta \text{ or } \psi \in \Delta && \text{(by Lemma 3.5.8)}\\ &\Leftrightarrow \overline{v}(\varphi) = 0 \text{ or } \overline{v}(\psi) = 1 && \text{(by induction)}\\ &\Leftrightarrow \overline{v}(\varphi \to \psi) = 1\end{aligned}$$

Therefore, by induction, we have $\varphi \in \Delta$ if and only if $\overline{v}(\varphi) = 1$. In particular, we have $\overline{v}(\varphi) = 1$ for all $\varphi \in \Delta$, hence Δ is satisfiable. □

Theorem 3.5.10 (Completeness Theorem)**.** *(Suppose that P is countable.)*

1. *Every consistent set of formulas is satisfiable.*
2. *If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.*

Proof.

1. Suppose that Γ is consistent. By Proposition 3.5.6, we may fix $\Delta \supseteq \Gamma$ which is consistent and complete. Now Δ is satisfiable by Proposition 3.5.9, so we may fix $v\colon P \to \{0, 1\}$ such that $\overline{v}(\delta) = 1$ for all $\delta \in \Delta$. Since $\Gamma \subseteq \Delta$, it follows that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, hence Γ is satisfiable.
2. Suppose that $\Gamma \vDash \varphi$. We then have that $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable, hence $\Gamma \cup \{\neg\varphi\}$ is inconsistent by part 1. It follows from Proposition 3.4.13 that $\Gamma \vdash \varphi$.

□

3.6 Compactness and Applications

3.6.1 The Compactness Theorem

Corollary 3.6.1 (Compactness Theorem)**.**

1. *If $\Gamma \vDash \varphi$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi$.*

2. *If every finite subset of* Γ *is satisfiable, then* Γ *is satisfiable.*

Proof. We first prove 1. Suppose that $\Gamma \vDash \varphi$. By the Completeness Theorem, we have $\Gamma \vdash \varphi$. Using Proposition 3.4.16, we may fix a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$. By the Soundness Theorem, we have $\Gamma_0 \vDash \varphi$.

We now prove 2. If every finite subset of Γ is satisfiable, then every finite subset of Γ is consistent by the Soundness Theorem, hence Γ is consistent by Corollary 3.4.17, and so Γ is satisfiable by the Completeness Theorem. □

3.6.2 Combinatorial Applications

Definition 3.6.2. *Let $G = (V, E)$ be a graph, and let $k \in \mathbb{N}^+$. A k-coloring of G is a function $f \colon V \to [k]$ such that for all $u, v \in V$ which are linked by an edge in E, we have $f(u) \neq f(v)$.*

Proposition 3.6.3. *Let $G = (V, E)$ be a (possibly infinite) graph and let $k \in \mathbb{N}^+$. If every finite subgraph of G is k-colorable, then G is k-colorable.*

Proof. Let $P = \{\mathsf{A}_{u,i} : u \in V \text{ and } i \in [k]\}$. Let

$$\Gamma = \{\bigvee_{i=0}^{k-1} \mathsf{A}_{u,i} : u \in V\} \cup \{\neg(\mathsf{A}_{u,i} \wedge \mathsf{A}_{u,j}) : u \in V \text{ and } i, j \in [k] \text{ with } i \neq j\}$$
$$\cup \{\neg(\mathsf{A}_{u,i} \wedge \mathsf{A}_{w,i}) : u \text{ and } w \text{ are linked by an edge in } E \text{ and } i \in [k]\}\}$$

We use the Compactness Theorem to show that Γ is satisfiable. Suppose that $\Gamma_0 \subseteq \Gamma$ is finite. Let $\{u_1, u_2, \dots, u_n\}$ be all of the elements $u \in V$ such that $\mathsf{A}_{u,i}$ occurs in some element of Γ_0 for some i. Since every finite subgraph of G is k-colorable, we may fix a k-coloring $f \colon \{u_1, u_2, \dots, u_n\} \to [k]$ such that whenever u_ℓ and u_m are linked by an edge of E, we have $f(u_\ell) \neq f(u_m)$. If we define a truth assignment $v \colon P \to \{0, 1\}$ by

$$v(\mathsf{A}_{w,i}) = \begin{cases} 1 & \text{if } w = u_\ell \text{ and } f(u_\ell) = i \\ 0 & \text{otherwise} \end{cases}$$

we see that $\overline{v}(\varphi) = 1$ for all $\varphi \in \Gamma_0$. Thus, Γ_0 is satisfiable. Therefore, Γ is satisfiable by the Compactness Theorem.

Fix a truth assignment $v \colon P \to \{0, 1\}$ such that $\overline{v}(\varphi) = 1$ for all $\varphi \in \Gamma$. Notice that for each $u \in V$, there exists a unique i such that $v(\mathsf{A}_{u,i}) = 1$ because of the first two sets in the definition of Γ. If we define $f \colon V \to \{1, 2, \dots, k\}$ by letting $f(u)$ be the unique i such that $v(\mathsf{A}_{u,i}) = 1$, then for all $u, w \in V$ linked by an edge in E we have that $f(u) \neq f(w)$ (because of the third set in the definition of Γ). Therefore, G is k-colorable. □

Corollary 3.6.4. *Every (possibly infinite) planar graph is 4-colorable.*

Definition 3.6.5. *A set $T \subseteq \{0, 1\}^*$ is called a* tree *if whenever $\sigma \in T$ and $\tau \subseteq \sigma$, we have $\tau \in T$.*

Theorem 3.6.6 (Weak König's Lemma)**.** *Suppose that $T \subseteq \{0, 1\}^*$ is a tree which is infinite. There exists an $f \colon \mathbb{N} \to \{0, 1\}$ such that $f \upharpoonright [n] \in T$ for all $n \in \mathbb{N}$.*

Proof. Let $P = \{\mathsf{A}_\sigma : \sigma \in T\}$. For each $n \in \mathbb{N}$, let $T_n = \{\sigma \in T : |\sigma| = n\}$ and notice that $T_n \neq \emptyset$ for all $n \in \mathbb{N}$ because T is infinite. Let

$$\Gamma = \{\bigvee_{\sigma \in T_n} \mathsf{A}_\sigma : n \in \mathbb{N}\} \cup \{\neg(\mathsf{A}_\sigma \wedge \mathsf{A}_\tau) : \sigma, \tau \in T_n \text{ and } \sigma \neq \tau\}$$
$$\cup \{(\mathsf{A}_\sigma \to \mathsf{A}_\tau) : \sigma, \tau \in T, \tau \subseteq \sigma\}$$

We use the Compactness Theorem to show that Γ is satisfiable. Suppose that $\Gamma_0 \subseteq \Gamma$ is finite. Let $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ be all of the elements $\sigma \in \{0,1\}^*$ such that A_σ occurs in some element of Γ_0. Let $n = \max\{|\sigma_1|, |\sigma_2|, \dots, |\sigma_k|\}$. Since $T_n \neq \emptyset$, we may fix $\tau \in T_n$. If we define a truth assignment $v \colon P \to \{0,1\}$ by

$$v(\mathsf{A}_\sigma) = \begin{cases} 1 & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise} \end{cases}$$

we see that $\overline{v}(\varphi) = 1$ for all $\varphi \in \Gamma_0$. Thus, Γ_0 is satisfiable. Therefore, Γ is satisfiable by the Compactness Theorem.

Fix a truth assignment $v \colon P \to \{0,1\}$ such that $\overline{v}(\varphi) = 1$ for all $\varphi \in \Gamma$. Notice that for each $n \in \mathbb{N}^+$, there exists a unique $\sigma \in T_n$ such that $v(\mathsf{A}_\sigma) = 1$ because of the first two sets in the definition of Γ. For each n, denote the unique such σ by ρ_n and notice that $\rho_m \subseteq \rho_n$ whenver $m \leq n$. Define $f \colon \mathbb{N} \to \{0,1\}$ by letting $f(n) = \rho_{n+1}(n)$. We then have that $f \upharpoonright [n] = \rho_n \in T$ for all $n \in \mathbb{N}$. ☐

3.6.3 An Algebraic Application

Definition 3.6.7. *An* ordered abelian group *is an abelian group $(A, +, 0)$ together with a relation $\leq$ on A^2 such that*

1. *$\leq$ is a linear ordering on A, i.e. we have*
   - *For all $a \in A$, we have $a \leq a$.*
   - *For all $a, b \in A$, either $a \leq b$ or $b \leq a$.*
   - *If $a \leq b$ and $b \leq a$, then $a = b$.*
   - *If $a \leq b$ and $b \leq c$, then $a \leq c$.*
2. *If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.*

Example 3.6.8. $(\mathbb{Z}, +, 0)$ *with its usual order is an ordered abelian group.*

Example 3.6.9. *Define $\leq$ on $\mathbb{Z}^n$ using the lexicographic order. In other words, given distinct elements $\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$ in $\mathbb{Z}^n$, let i be least such that $a_i \neq b_i$, and set $\vec{a} < \vec{b}$ if $a_i <_{\mathbb{Z}} b_i$, and $\vec{b} < \vec{a}$ if $b_i <_{\mathbb{Z}} a_i$. With this order, $(\mathbb{Z}^n, +, 0)$ is an ordered abelian group.*

Proposition 3.6.10. *Suppose that $(A, +, 0)$ is an ordered abelian group, and define $<$ be letting $a < b$ if $a \leq b$ and $a \neq b$. We the have*

1. *For all $a, b \in A$, exactly one of $a < b$, $a = b$, or $b < a$ holds.*
2. *If $a < b$ and $b \leq c$, then $a < c$.*
3. *If $a \leq b$ and $b < c$, then $a < c$.*

Proof.

1. Let $a, b \in A$. We first show that at least one happens. Suppose then that $a \neq b$. We either have $a \leq b$ or $b \leq a$. If $a \leq b$, we then have $a < b$, while if $b \leq a$, we then have $b < a$.

   We now show that at most one occurs. Clearly, we can't have both $a < b$ and $a = b$, nor can we have both $a = b$ and $b < a$. Suppose then that we have both $a < b$ and $b < a$. We would then have both $a \leq b$ and $b \leq a$, hence $a = b$, a contradiction.

2. Since $a \leq b$ and $b \leq c$, we have $a \leq c$. If $a = c$, it would follow that $a \leq b$ and $b \leq a$, hence $a = b$, a contradiction.

3. Since $a \leq b$ and $b \leq c$, we have $a \leq c$. If $a = c$, it would follow that $c \leq b$ and $b \leq c$, hence $b = c$, a contradiction.

□

Proposition 3.6.11. *Suppose that $(A, +, 0)$ is an ordered abelian group.*

1. *If $a < b$ and $c \in A$, then $a + c < b + c$.*
2. *If $a < b$ and $c \leq d$, then $a + c < b + d$.*
3. *If $a > 0$, then $-a < 0$.*
4. *If $a < 0$, then $-a > 0$.*

Proof.

1. Since $a < b$ and $c \leq c$, we have $a+c \leq b+c$. If $a+c = b+c$, then we would have $a = b$, a contradiction. Therefore, $a + c < b + c$.
2. We have $a + c < b + c$ and $b + c \leq b + d$, hence $a + c < b + d$ by the previous proposition.
3. We have $a \neq 0$, hence $-a \neq 0$. Suppose that $-a > 0$. We would then have $a + (-a) > 0$, hence $0 > 0$, a contradiction.
4. Similar to 3.

□

Definition 3.6.12. *An abelian group $(A, +, 0)$ is torsion-free if every nonzero element of A has infinite order.*

Proposition 3.6.13. *Every ordered abelian group is torsion-free.*

Proof. Let $(A, +, 0)$ be an ordered abelian group. Let $a \in A$. If $a > 0$, then we have $n \cdot a > 0$ for every $n \in \mathbb{N}^+$ by induction. If $a < 0$, then we have $n \cdot a < 0$ for every $n \in \mathbb{N}^+$ by induction. □

Theorem 3.6.14. *Every torsion-free abelian group can be ordered.*

Proof. First notice that every finitely generated torsion-free abelian group is isomorphic to $\mathbb{Z}^n$ for some n, which we can order lexicographically from above. We can transer this ordering across the isomorphism to order our finitely generated abelian group.

Suppose now that A is an arbitrary torsion-free abelian group. Let P be the set $\{\mathsf{L}_{a,b} : a, b \in A\}$ and let Γ be the union of the sets

- $\{\mathsf{L}_{a,a} : a \in A\}$.
- $\{\mathsf{L}_{a,b} \vee \mathsf{L}_{b,a} : a, b \in A\}$.
- $\{\neg(\mathsf{L}_{a,b} \wedge \mathsf{L}_{b,a}) : a, b \in A, a \neq b\}$
- $\{(\mathsf{L}_{a,b} \wedge \mathsf{L}_{b,c}) \rightarrow \mathsf{L}_{a,c} : a, b, c \in A\}$.
- $\{(\mathsf{L}_{a,b} \wedge \mathsf{L}_{c,d}) \rightarrow \mathsf{L}_{a+c,b+d} : a, b, c, d \in A\}$

We show that Γ is satisfiable. By Compactness, it suffices to show that any finite subset of Γ is satisfiable. Suppose that $\Gamma_0 \subseteq \Gamma$ is finite, and let S the finite subset of A consisting of all elements of A which appear as a subscript of a symbol occuring in Γ_0. Let B be the subgroup of A generated by S. We then have that B is a finitely generated torsion-free abelian group, so from above we may fix an order $\leq$ on it. If we define a truth assignment $v\colon P \to \{0,1\}$ by

$$v(\mathsf{L}_{a,b}) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

we see that $\overline{v}(\varphi) = 1$ for all $\varphi \in \Gamma_0$. Thus, Γ_0 is satisfiable. Therefore, Γ is satisfiable by the Compactness Theorem.

Fix a truth assignment $v\colon P \to \{0,1\}$ such that $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$. Define $\leq$ on A^2 by letting $a \leq b$ if and only if $v(\mathsf{L}_{a,b}) = 1$. We then have that $\leq$ orders A. Therefore, A can be ordered. $\square$

Chapter 4

First-Order Logic : Syntax and Semantics

Now that we've succeeded in giving a decent analysis of propositional logic, together with proving a few nontrivial theorems, it's time to move on to a much more substantial and important logic: first-order logic. As summarized in the introduction, the general idea is as follows. Many areas of mathematics deal with mathematical structures consisting of special constants, relations, and functions, together with certain axioms that these objects obey. We want our logic to be able to handle many different types of situations, so we allow ourselves to vary the number and types of these symbols. Any such choice gives rise to a *language*, and once we've fixed such a language, we can build up formulas which will express something meaningful once we've decided on an interpretation of the symbols.

4.1 The Syntax of First-Order Logic

Since our logic will have quantifiers, the first thing that we need is a collection of variables.

Definition 4.1.1. *Fix a countably infinite set Var called* variables*.*

Definition 4.1.2. *A* first-order language*, or simply a* language*, consists of the following:*

1. *A set $\mathcal{C}$ of* constant symbols.
2. *A set $\mathcal{F}$ of* function symbols *together with a function $Arity_{\mathcal{F}}\colon \mathcal{F} \to \mathbb{N}^+$.*
3. *A set $\mathcal{R}$ of* relation symbols *together with a function $Arity_{\mathcal{R}}\colon \mathcal{R} \to \mathbb{N}^+$.*

We also assume that $\mathcal{C}$, $\mathcal{R}$, $\mathcal{F}$, Var, and $\{\forall, \exists, =, \neg, \wedge, \vee, \rightarrow\}$ are pairwise disjoint. For each $k \in \mathbb{N}^+$, we let

$$\mathcal{F}_k = \{\mathsf{f} \in \mathcal{F} : Arity_{\mathcal{F}}(\mathsf{f}) = k\}$$

and we let

$$\mathcal{R}_k = \{\mathsf{R} \in \mathcal{R} : Arity_{\mathcal{R}}(\mathsf{R}) = k\}$$

Definition 4.1.3. *Let $\mathcal{L}$ be a language. We let $Sym_{\mathcal{L}} = \mathcal{C} \cup \mathcal{R} \cup \mathcal{F} \cup Var \cup \{\forall, \exists, =, \neg, \wedge, \vee, \rightarrow\}$.*

Now that we've described all of the symbols that are available once we've fixed a language, we need to talk about how to build up formulas. Before doing this, however, we need a way to name objects. Intuitively, our constant symbols and variables name objects once we've fixed an interpretation. From here, we can get new objects by applying, perhaps repeatedly, interpretations of function symbols. This is starting to sound like a recursive definition.

Definition 4.1.4. *Let $\mathcal{L}$ be a language. For each $\mathsf{f} \in \mathcal{F}_k$, define $h_\mathsf{f} \colon (Sym_\mathcal{L}^*)^k \to Sym_\mathcal{L}^*$ by letting*

$$h_\mathsf{f}(\sigma_1, \sigma_2, \dots, \sigma_k) = \mathsf{f}\sigma_1\sigma_2 \cdots \sigma_k$$

Let

$$Term_\mathcal{L} = G(Sym_\mathcal{L}^*, \mathcal{C} \cup Var, \{h_\mathsf{f} : \mathsf{f} \in \mathcal{F}\})$$

Now that we have terms which intuitively name elements once we've fixed an interpretation, we need to say what our atomic formulas are. The idea is that the most basic things we can say are whether or not two objects are equal or whether or not a k-tuple is in the interpretation of some relation symbol $\mathsf{R} \in \mathcal{R}_k$.

Definition 4.1.5. *Let $\mathcal{L}$ be a language. We let*

$$AtomicForm_\mathcal{L} = \{\mathsf{R}t_1t_2 \cdots t_k : k \in \mathbb{N}^+, \mathsf{R} \in \mathcal{R}_k, \text{ and } t_1, t_2, \dots, t_k \in Term_\mathcal{L}\} \cup \{= t_1t_2 : t_1, t_2 \in Term_\mathcal{L}\}$$

From here, we can build up all formulas.

Definition 4.1.6. *Let $\mathcal{L}$ be a language. Define a unary function $h_\neg$ and binary functions $h_\wedge, h_\vee$, and $h_\rightarrow$ on $Sym_\mathcal{L}^*$ as follows.*

$$h_\neg(\sigma) = \neg\sigma$$
$$h_\wedge(\sigma, \tau) = \wedge\sigma\tau$$
$$h_\vee(\sigma, \tau) = \vee\sigma\tau$$
$$h_\rightarrow(\sigma, \tau) = \rightarrow \sigma\tau$$

Also, for each $\mathsf{x} \in Var$, define two unary functions $h_{\forall,\mathsf{x}}$ and $h_{\exists,\mathsf{x}}$ on $Sym_\mathcal{L}^*$ as follows

$$h_{\forall,\mathsf{x}}(\sigma) = \forall\mathsf{x}\sigma$$
$$h_{\exists,\mathsf{x}}(\sigma) = \exists\mathsf{x}\sigma$$

Let

$$Form_\mathcal{L} = G(Sym_\mathcal{L}^*, AtomicForm_\mathcal{L}, \{h_\neg, h_\wedge, h_\vee, h_\rightarrow\} \cup \{h_{\forall,\mathsf{x}}, h_{\exists,\mathsf{x}} : \mathsf{x} \in Var\})$$

As with propositional logic, we'd like to be able to define things recursively, so we need to check that our generating systems are free. Notice that in the construction of formulas, we have two generating systems around. We first generate all terms. With terms taken care of, we next describe the atomic formulas, and from them we generate all formulas. Thus, we'll need to prove that two generating systems are free. The general idea is to make use of the insights gained by proving the corresponding result for Polish notation in propositional logic.

Definition 4.1.7. *Let $\mathcal{L}$ be a language. Define $K \colon Sym_\mathcal{L}^* \to \mathbb{Z}$ as follows. We first define $w \colon Sym_\mathcal{L} \to \mathbb{Z}$ as follows.*

$w(\mathsf{c}) = 1$	*for all* $\mathsf{c} \in \mathcal{C}$
$w(\mathsf{f}) = 1 - k$	*for all* $\mathsf{f} \in \mathcal{F}_k$
$w(\mathsf{R}) = 1 - k$	*for all* $\mathsf{R} \in \mathcal{R}_k$
$w(\mathsf{x}) = 1$	*for all* $\mathsf{x} \in Var$
$w(=) = -1$	
$w(\mathsf{Q}) = -1$	*for all* $\mathsf{Q} \in \{\forall, \exists\}$
$w(\neg) = 0$	
$w(\diamond) = -1$	*for all* $\diamond \in \{\wedge, \vee, \rightarrow\}$

We then define K on all of $Sym_\mathcal{L}^*$ by letting $K(\lambda) = 0$ and letting $K(\sigma) = \sum_{i<|\sigma|} w(\sigma(i))$ for all $\sigma \in Sym_\mathcal{L}^* \backslash \{\lambda\}$.

Remark 4.1.8. *If $\sigma, \tau \in Sym_{\mathcal{L}}^*$, then $K(\sigma\tau) = K(\sigma) + K(\tau)$.*

Proposition 4.1.9. *If $t \in Term_{\mathcal{L}}$, then $K(t) = 1$.*

Proof. The proof is by induction on t. Notice first that $K(\mathsf{c}) = 1$ for all $\mathsf{c} \in \mathcal{C}$ and $K(\mathsf{x}) = 1$ for all $\mathsf{x} \in Var$. Suppose that $k \in \mathbb{N}^+$, $\mathsf{f} \in \mathcal{F}_k$, and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$ are such that $K(t_i) = 1$ for all i. We then have that

$$\begin{aligned} K(\mathsf{f}t_1t_2\cdots t_k) &= K(\mathsf{f}) + K(t_1) + K(t_2) + \cdots + K(t_k) \\ &= (1-k) + 1 + 1 + \cdots + 1 && \text{(by induction)} \\ &= 1. \end{aligned}$$

The result follows by induction. □

Proposition 4.1.10. *If $t \in Term_{\mathcal{L}}$ and $\sigma \subset t$, then $K(\sigma) \leq 0$.*

Proof. The proof is by induction on t. For every $\mathsf{c} \in \mathcal{C}$, this is trivial because the only $\sigma \subset \mathsf{c}$ is $\sigma = \lambda$ and we have $K(\lambda) = 0$. Similarly, for every $\mathsf{x} \in Var$, the only $\sigma \subset \mathsf{x}$ is $\sigma = \lambda$ and we have $K(\lambda) = 0$.

Suppose that $k \in \mathbb{N}^+$, $\mathsf{f} \in \mathcal{F}_k$, and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$ are such that the result holds for each t_i. We prove the result for $\mathsf{f}t_1t_2\cdots t_k$. Suppose that $\sigma \subset \mathsf{f}t_1t_2\cdots t_k$. If $\sigma = \lambda$, then $K(\sigma) = 0$. Otherwise, there exists $i < k$ and $\tau \subset t_i$ such that $\sigma = \mathsf{f}t_1t_2\cdots t_{i-1}\tau$, in which case

$$\begin{aligned} K(\sigma) &= K(\mathsf{f}) + K(t_1) + K(t_2) + \cdots + K(t_{i-1}) + K(\tau) \\ &= (1-k) + 1 + 1 + \cdots + 1 + K(\tau) && \text{(by Proposition 4.1.9)} \\ &= (1-k) + i + K(\tau) \\ &\leq (1-k) + i + 0 && \text{(by induction)} \\ &= 1 + (i-k) \\ &\leq 0. && \text{(since } i < k\text{)} \end{aligned}$$

Thus, the result holds for $\mathsf{f}t_1t_2\cdots t_k$. □

Corollary 4.1.11. *If $t, u \in Term_{\mathcal{L}}$, then $t \not\subset u$.*

Proof. This follows by combining Proposition 4.1.9 and Proposition 4.1.10. □

Theorem 4.1.12. *The generating system $(Sym_{\mathcal{L}}^*, \mathcal{C} \cup Var, \{h_{\mathsf{f}} : \mathsf{f} \in \mathcal{F}\})$ is free.*

Proof. First notice that for all $\mathsf{f} \in \mathcal{F}$, we have that $\text{ran}(h_{\mathsf{f}} \upharpoonright (Term_{\mathcal{L}})^k) \cap (\mathcal{C} \cup Var) = \emptyset$ because all elements of $\text{ran}(h_{\mathsf{f}})$ begin with f and we know that $\mathsf{f} \notin \mathcal{C} \cup Var$.

Fix $\mathsf{f} \in \mathcal{F}_k$. Suppose that $t_1, t_2, \dots, t_k, u_1, u_2, \dots, u_k \in Term_{\mathcal{L}}$ and $h_{\mathsf{f}}(t_1, t_2, \dots, t_k) = h_{\mathsf{f}}(u_1, u_2, \dots, u_k)$. We then have $\mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1u_2\cdots u_k$, hence $t_1t_2\cdots t_k = u_1u_2\cdots u_k$. Since $t_1 \subset u_1$ and $u_1 \subset t_1$ are both impossible by Corollary 4.1.11, it follows that $t_1 = u_1$. Thus, $t_2\cdots t_k = u_2\cdots u_k$, and so $t_2 = u_2$ for the same reason. Continuing in this fashion, we conclude that $t_i = u_i$ for all i. It follows that $h_{\mathsf{f}} \upharpoonright (Term_{\mathcal{L}})^k$ is injective.

Finally notice that for any $\mathsf{f} \in \mathcal{F}_k$ and any $\mathsf{g} \in \mathcal{F}_\ell$ with $\mathsf{f} \neq \mathsf{g}$, we have that $\text{ran}(h_{\mathsf{f}} \upharpoonright (Term_{\mathcal{L}})^k) \cap \text{ran}(h_{\mathsf{g}} \upharpoonright (Term_{\mathcal{L}})^\ell) = \emptyset$ because all elements of $\text{ran}(h_{\mathsf{f}} \upharpoonright (Term_{\mathcal{L}})^k)$ begin with f while all elements of $\text{ran}(h_{\mathsf{g}} \upharpoonright (Term_{\mathcal{L}})^\ell)$ begin with g. □

Proposition 4.1.13. *If $\varphi \in Form_{\mathcal{L}}$, then $K(\varphi) = 1$.*

Proof. The proof is by induction on φ. We first show that $K(\varphi) = 1$ for all $\varphi \in AtomicForm_{\mathcal{L}}$. Suppose that φ is $\mathsf{R}t_1t_2\cdots t_k$ where $\mathsf{R} \in \mathcal{R}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. We then have

$$\begin{aligned} K(\mathsf{R}t_1t_2\cdots t_k) &= K(\mathsf{R}) + K(t_1) + K(t_2) + \cdots + K(t_k) \\ &= (1-k) + 1 + 1 + \cdots + 1 && \text{(by Proposition 4.1.9)} \\ &= 1. \end{aligned}$$

Suppose that φ is $= t_1t_2$ where $t_1, t_2 \in Term_{\mathcal{L}}$. We then have

$$\begin{aligned} K(= t_1t_2) &= K(=) + K(t_1) + K(t_2) \\ &= -1 + 1 + 1 && \text{(by Proposition 4.1.9)} \\ &= 1. \end{aligned}$$

Thus, $K(\varphi) = 1$ for all $\varphi \in AtomicForm_{\mathcal{L}}$.

Suppose that $\varphi \in Form_{\mathcal{L}}$ is such that $K(\varphi) = 1$. We then have that

$$\begin{aligned} K(\neg\varphi) &= K(\neg) + K(\varphi) \\ &= 0 + 1 \\ &= 1. \end{aligned}$$

For any $\mathsf{Q} \in \{\forall, \exists\}$ and any $\mathsf{x} \in Var$ we also have

$$\begin{aligned} K(\mathsf{Qx}\varphi) &= K(\mathsf{Q}) + K(\mathsf{x}) + K(\varphi) \\ &= -1 + 1 + 1 \\ &= 1. \end{aligned}$$

Suppose now that $\varphi, \psi \in Form_{\mathcal{L}}$ are such that $K(\varphi) = 1 = K(\psi)$, and $\Diamond \in \{\wedge, \vee, \rightarrow\}$. We then have that

$$\begin{aligned} K(\Diamond\varphi\psi) &= -1 + K(\varphi) + K(\psi) \\ &= -1 + 1 + 1 \\ &= 1. \end{aligned}$$

The result follows by induction. □

Proposition 4.1.14. *If $\varphi \in Form_{\mathcal{L}}$ and $\sigma \subset \varphi$, then $K(\sigma) \leq 0$.*

Proof. The proof is by induction on φ. We first show that the results holds for all $\varphi \in AtomicForm_{\mathcal{L}}$. Suppose that φ is $\mathsf{R}t_1t_2\cdots t_k$ where $\mathsf{R} \in \mathcal{R}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. Suppose that $\sigma \subset \mathsf{R}t_1t_2\cdots t_k$. If $\sigma = \lambda$, then $K(\sigma) = 0$. Otherwise, there exists $i < k$ and $\tau \subset t_i$ such that σ is $\mathsf{R}t_1t_2\cdots t_{i-1}\tau$, in which case

$$\begin{aligned} K(\sigma) &= K(\mathsf{R}) + K(t_1) + K(t_2) + \cdots + K(t_{i-1}) + K(\tau) \\ &= (1-k) + 1 + 1 + \cdots + 1 + K(\tau) && \text{(by Proposition 4.1.9)} \\ &= (1-k) + i + K(\tau) \\ &\leq (1-k) + i + 0 && \text{(by induction)} \\ &= 1 + (i-k) \\ &\leq 0. && \text{(since } i < k\text{)} \end{aligned}$$

Thus, the result holds for $\mathsf{R}t_1t_2\cdots t_k$. The same argument works for $= t_1t_2$ where $t_1, t_2 \in Term_{\mathcal{L}}$, so the result holds for all $\varphi \in AtomicForm_{\mathcal{L}}$.

Suppose that the result holds for $\varphi \in Form_{\mathcal{L}}$. Suppose that $\sigma \subset \neg\varphi$. If $\sigma = \lambda$, then $K(\sigma) = 0$. Otherwise, $\sigma = \neg\tau$ for some $\tau \subset \varphi$, in which case

$$\begin{aligned} K(\sigma) &= K(\neg) + K(\tau) \\ &= 0 + K(\tau) \\ &\leq 0. && \text{(by induction)} \end{aligned}$$

Suppose now that $\mathsf{Q} \in \{\forall, \exists\}$, that $\mathsf{x} \in Var$, and that $\sigma \subset \mathsf{Q}\mathsf{x}\varphi$. If $\sigma = \lambda$, then $K(\sigma) = 0$, and if $\sigma = Q$, then $K(\sigma) = -1$. Otherwise, $\sigma = \mathsf{Q}\mathsf{x}\tau$ for some $\tau \subset \varphi$, in which case

$$\begin{aligned} K(\sigma) &= K(\mathsf{Q}) + K(\mathsf{x}) + K(\tau) \\ &= -1 + 1 + K(\tau) \\ &= 0 && \text{(by induction)} \end{aligned}$$

Suppose now that the result holds for $\varphi, \psi \in Form_{\mathcal{L}}$, and $\Diamond \in \{\wedge, \vee, \rightarrow\}$. Suppose that $\sigma \subset \Diamond\varphi\psi$. If $\sigma = \lambda$, then $K(\sigma) = 0$. If σ is $\Diamond\tau$ for some $\tau \subset \varphi$, then

$$\begin{aligned} K(\sigma) &= K(\Diamond) + K(\tau) \\ &= -1 + K(\tau) \\ &\leq -1. && \text{(by induction)} \end{aligned}$$

Otherwise, σ is $\Diamond\varphi\tau$ for some $\tau \subset \psi$, in which case

$$\begin{aligned} K(\sigma) &= K(\Diamond) + K(\varphi) + K(\tau) \\ &= -1 + 0 + K(\tau) && \text{(by Proposition 3.1.13)} \\ &\leq -1. && \text{(by induction)} \end{aligned}$$

Thus, the result holds for $\Diamond\varphi\psi$. □

Corollary 4.1.15. *If $\varphi, \psi \in Form_{\mathcal{L}}$, then $\varphi \not\subset \psi$.*

Proof. This follows by combining Proposition 4.1.13 and Proposition 4.1.14. □

Theorem 4.1.16. *The generating system $(Sym^*_{\mathcal{L}}, AtomicForm_{\mathcal{L}}, \{h_{\neg}, h_{\wedge}, h_{\vee}, h_{\rightarrow}\} \cup \{h_{\forall,\mathsf{x}}, h_{\exists,\mathsf{x}} : \mathsf{x} \in V\})$ is free.*

Proof. Similar to the others. □

With these freeness results, we are now able to define functions recursively on $Term_{\mathcal{L}}$ and $Form_{\mathcal{L}}$. Since we use terms in our definition of atomic formulas, which are the basic formulas, we will often need to make two recursive definitions (on terms first, then on formulas) in order to define a function on formulas. Here's an example.

Definition 4.1.17. *Let $\mathcal{L}$ be a language.*

1. *We define a function $OccurVar \colon Term_{\mathcal{L}} \rightarrow \mathcal{P}(Var)$ recursively as follows.*
   - *$OccurVar(\mathsf{c}) = \emptyset$ for all $\mathsf{x} \in \mathcal{C}$.*
   - *$OccurVar(\mathsf{x}) = \{\mathsf{x}\}$ for all $\mathsf{x} \in Var$.*
   - *$OccurVar(\mathsf{f}t_1t_2\cdots t_k) = OccurVar(t_1) \cup OccurVar(t_2) \cup \cdots \cup OccurVar(t_k)$ for all $\mathsf{f} \in \mathcal{F}_k$ and $t_1, t_2, \ldots, t_k \in Term_{\mathcal{L}}$.*

2. *We define a function* $OccurVar\colon Form_{\mathcal{L}} \to \mathcal{P}(Var)$ *recursively as follows.*
   - $OccurVar(\mathsf{R}t_1t_2\cdots t_k) = OccurVar(t_1) \cup OccurVar(t_2) \cup \cdots \cup OccurVar(t_k)$ *for all* $\mathsf{R} \in \mathcal{R}_k$ *and* $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$.
   - $OccurVar(= t_1t_2) = OccurVar(t_1) \cup OccurVar(t_2)$ *for all* $t_1, t_2 \in Term_{\mathcal{L}}$.
   - $OccurVar(\neg\varphi) = OccurVar(\varphi)$ *for all* $\varphi \in Form_{\mathcal{L}}$.
   - $OccurVar(\Diamond\varphi\psi) = OccurVar(\varphi) \cup OccurVar(\psi)$ *for each* $\Diamond \in \{\wedge, \vee, \rightarrow\}$ *and* $\varphi, \psi \in Form_{\mathcal{L}}$.
   - $OccurVar(\mathsf{Q}\mathsf{x}\varphi) = OccurVar(\varphi) \cup \{\mathsf{x}\}$ *for each* $\mathsf{Q} \in \{\forall, \exists\}$, $\mathsf{x} \in Var$, *and* $\varphi \in Form_{\mathcal{L}}$.

Definition 4.1.18. *Let* $\mathcal{L}$ *be a language.*

1. *We define a function* $FreeVar\colon Form_{\mathcal{L}} \to \mathcal{P}(Var)$ *recursively as follows.*
   - $FreeVar(\mathsf{R}t_1t_2\cdots t_k) = OccurVar(t_1) \cup OccurVar(t_2) \cup \cdots \cup OccurVar(t_k)$ *for all* $\mathsf{R} \in \mathcal{R}_k$ *and* $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$.
   - $FreeVar(= t_1t_2) = OccurVar(t_1) \cup OccurVar(t_2)$ *for all* $t_1, t_2 \in Term_{\mathcal{L}}$.
   - $FreeVar(\neg\varphi) = FreeVar(\varphi)$ *for all* $\varphi \in Form_{\mathcal{L}}$.
   - $FreeVar(\Diamond\varphi\psi) = FreeVar(\varphi) \cup FreeVar(\psi)$ *for each* $\Diamond \in \{\wedge, \vee, \rightarrow\}$ *and* $\varphi, \psi \in Form_{\mathcal{L}}$.
   - $FreeVar(\mathsf{Q}\mathsf{x}\varphi) = FreeVar(\varphi) \backslash \{\mathsf{x}\}$ *for each* $\mathsf{Q} \in \{\forall, \exists\}$, $\mathsf{x} \in Var$, *and* $\varphi \in Form_{\mathcal{L}}$.

2. *We define a function* $BoundVar\colon Form_{\mathcal{L}} \to \mathcal{P}(Var)$ *recursively as follows.*
   - $BoundVar(\mathsf{R}t_1t_2\cdots t_k) = \emptyset$ *for all* $\mathsf{R} \in \mathcal{R}_k$ *and* $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$.
   - $BoundVar(= t_1t_2) = \emptyset$ *for all* $t_1, t_2 \in Term_{\mathcal{L}}$.
   - $BoundVar(\neg\varphi) = BoundVar(\varphi)$ *for all* $\varphi \in Form_{\mathcal{L}}$.
   - $BoundVar(\Diamond\varphi\psi) = BoundVar(\varphi) \cup BoundVar(\psi)$ *for each* $\Diamond \in \{\wedge, \vee, \rightarrow\}$ *and* $\varphi, \psi \in Form_{\mathcal{L}}$.
   - $BoundVar(\mathsf{Q}\mathsf{x}\varphi) = BoundVar(\varphi) \cup \{\mathsf{x}\}$ *for each* $\mathsf{Q} \in \{\forall, \exists\}$, $\mathsf{x} \in Var$, *and* $\varphi \in Form_{\mathcal{L}}$.

Definition 4.1.19. *Let* $\mathcal{L}$ *be a language and let* $\varphi \in Form_{\mathcal{L}}$. *We say that* φ *is an* $\mathcal{L}$-sentence, *or simply a* sentence, *if* $FreeVar(\varphi) = \emptyset$. *We let* $Sent_{\mathcal{L}}$ *be the set of sentences.*

4.2 Structures: The Semantics of First-Order Logic

4.2.1 Structures: Definition and Satisfaction

Up until this point, all that we've dealt with are sequences of symbols without meaning. Sure, our motivation was to capture meaningful situations with our languages and the way we've described formulas, but all we've done so far is describe the grammar. If we want our formulas to actually express something, we need to set up a context in which to interpret them. Since we have quantifiers, the first thing we'll need is a nonempty set M to serve as the domain of objects that the quantifiers range over. Once we've fixed that, we need to interpret the symbols of language as actual elements of our set (in the case of constant symbols), actual k-ary relations on M (in the case of $\mathsf{R} \in \mathcal{R}_k$), and actual k-ary functions on M (in the case of $\mathsf{f} \in \mathcal{F}_k$).

Definition 4.2.1. *Let* $\mathcal{L}$ *be a language. An* $\mathcal{L}$-structure, *or simply a* structure, *is a set* $\mathcal{M} = (M, g_{\mathcal{C}}, g_{\mathcal{F}}, g_{\mathcal{R}})$ *where*

1. M *is a nonempty set called the* universe *of* $\mathcal{M}$.
2. $g_{\mathcal{C}}\colon \mathcal{C} \to M$.

3. *$g_{\mathcal{R}}$ is a function on $\mathcal{R}$ such that $g_{\mathcal{R}}(\mathsf{R})$ is a subset of M^k for all $\mathsf{R} \in \mathcal{R}_k$.*

4. *$g_{\mathcal{F}}$ is a function on $\mathcal{F}$ such that $g_{\mathcal{F}}(\mathsf{f})$ is a k-ary function on M for all $\mathsf{f} \in \mathcal{F}_k$.*

We use the following notation.

1. *For each $\mathsf{c} \in \mathcal{C}$, we use $\mathsf{c}^{\mathcal{M}}$ to denote $g_{\mathcal{C}}(\mathsf{c})$.*

2. *For each $\mathsf{R} \in \mathcal{R}_k$, we use $\mathsf{R}^{\mathcal{M}}$ to denote $g_{\mathcal{R}}(\mathsf{R})$.*

3. *For each $\mathsf{f} \in \mathcal{F}_k$, we use $\mathsf{f}^{\mathcal{M}}$ to denote $g_{\mathcal{F}}(\mathsf{f})$.*

Example 4.2.2. *Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Here are some examples of $\mathcal{L}$-structures.*

1. *$M = \mathbb{N}$ and $\mathsf{R}^{\mathcal{M}} = \{(m, n) \in M^2 : m \mid n\}$.*

2. *$M = \{0, 1\}^*$ and $\mathsf{R}^{\mathcal{M}} = \{(\sigma, \tau) \in M^2 : \sigma \subseteq \tau\}$.*

3. *$M = \mathbb{R}^2$ and $\mathsf{R}^{\mathcal{M}} = \{((a_1, b_1), (a_2, b_2)) \in M^2 : a_1 = a_2\}$.*

4. *$M = \{0, 1, 2, 3, 4\}$ and $\mathsf{R}^{\mathcal{M}} = \{(0, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$.*

Example 4.2.3. *Let $\mathcal{L} = \{\mathsf{c}, \mathsf{f}\}$ where c is a constant symbol and f is a binary function symbol. Here are some examples of $\mathcal{L}$-structures.*

1. *$M = \mathbb{Z}$, $\mathsf{c}^{\mathcal{M}} = 3$ and $\mathsf{f}^{\mathcal{M}}$ is the addition function $(m, n) \mapsto m - n$.*

2. *$M = \mathbb{R}$, $\mathsf{c}^{\mathcal{M}} = \pi$ and $\mathsf{f}^{\mathcal{M}}$ is the function $(a, b) \mapsto \sin(a \cdot b)$.*

3. *For any group $(G, e, \cdot)$, we get an $\mathcal{L}$-structure by letting $M = G$, $\mathsf{c}^{\mathcal{M}} = e$, and letting $\mathsf{f}^{\mathcal{M}}$ be the group operation.*

At first, it may appear than an $\mathcal{L}$-structure provides a means to make sense out of any formula. However, this is not the case, as you can see by looking at the formula $\mathsf{x} = \mathsf{y}$ where $\mathsf{x}, \mathsf{y} \in Var$. Until we provide a way to interpret the elements of Var, this formula is meaningless. This motivates the following definition.

Definition 4.2.4. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A function $s \colon Var \to M$ is called a* variable assignment *on $\mathcal{M}$.*

Recall in propositional logic that every truth assignment $v \colon P \to \{0, 1\}$ gave rise to an extension $\overline{v} \colon Form_P \to \{0, 1\}$ telling us how to interpret every formula. In our case, every variable assignment $s \colon Var \to M$ gives rise to an extension $\overline{s} \colon Term_{\mathcal{L}} \to M$ telling us which element of M to assign to each term.

Definition 4.2.5. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $s \colon Var \to M$ be a variable assignment. By freeness, there exists a unique $\overline{s} \colon Term_{\mathcal{L}} \to M$ such that*

- *$\overline{s}(\mathsf{x}) = s(\mathsf{x})$ for all $\mathsf{v} \in Var$.*
- *$\overline{s}(\mathsf{c}) = \mathsf{c}^{\mathcal{M}}$ for all $\mathsf{c} \in \mathcal{C}$.*
- *$\overline{s}(\mathsf{f}t_1 t_2 \cdots t_k) = \mathsf{f}^{\mathcal{M}}(\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k))$*

We're now in position to define the intuitive statement "φ holds in the $\mathcal{L}$-structure $\mathcal{M}$ with variable assignment s" recursively. We need the following definition.

Definition 4.2.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $s\colon Var \to M$ be a variable assignment. Given $\mathsf{x} \in Var$ and $a \in M$, we let $s[\mathsf{x} \Rightarrow a]$ denote the variable assignment*

$$s[\mathsf{x} \Rightarrow a](\mathsf{y}) = \begin{cases} a & \textit{if } \mathsf{y} = \mathsf{x} \\ s(\mathsf{y}) & \textit{otherwise} \end{cases}$$

Definition 4.2.7. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We define a relation $(\mathcal{M}, s) \vDash \varphi$ (pronounced "φ holds in $(\mathcal{M}, s)$", or "φ is true in $(\mathcal{M}, s)$", or "$(\mathcal{M}, s)$ models φ") for all $\varphi \in Form_{\mathcal{L}}$ and all variable assignments s by induction on φ.*

- *Suppose first that φ is an atomic formula.*
  - *If φ is $\mathsf{R}t_1t_2\cdots t_k$,, we have $(\mathcal{M}, s) \vDash \varphi$ if and only if $(\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k)) \in \mathsf{R}^{\mathcal{M}}$.*
  - *If φ is $= t_1t_2$, we have $(\mathcal{M}, s) \vDash \varphi$ if and only if $\overline{s}(t_1) = \overline{s}(t_2)$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \neg\varphi$ if and only if $(\mathcal{M}, s) \nvDash \varphi$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \varphi \wedge \psi$ if and only if $(\mathcal{M}, s) \vDash \varphi$ and $(\mathcal{M}, s) \vDash \psi$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \varphi \vee \psi$ if and only if either $(\mathcal{M}, s) \vDash \varphi$ or $(\mathcal{M}, s) \vDash \psi$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \varphi \to \psi$ if and only if either $(\mathcal{M}, s) \nvDash \varphi$ or $(\mathcal{M}, s) \vDash \psi$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \exists\mathsf{x}\varphi$ if and only if there exists $a \in M$ such that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$.*
- *For any s, we have $(\mathcal{M}, s) \vDash \forall\mathsf{x}\varphi$ if and only if for all $a \in M$, we have $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$.*

Comments. The above recursive definition takes a little explanation, because some recursive "calls" change the variable assignment. Thus, we are *not* fixing an $\mathcal{L}$-structurve $\mathcal{M}$ and a variable assignment s on $\mathcal{M}$, and then doing a recursive definition on $\varphi \in Form_{\mathcal{L}}$. We can make the definition formal as follows. Fix an $\mathcal{L}$-structure $\mathcal{M}$. Let $VarAssign_{\mathcal{M}}$ be the set of all variable assignments on $\mathcal{M}$. We then define a function $g_{\mathcal{M}}\colon Form_{\mathcal{P}} \to VarAssign_{\mathcal{M}}$ recursively using the above rules as guides, and we write $(\mathcal{M}, s) \vDash \varphi$ to mean that $s \in g_{\mathcal{M}}(\varphi)$. □

Example. Let $\mathcal{L} = \{\mathsf{R}, \mathsf{f}\}$ where R is a unary relation symbol and f is a unary function symbol. Let $\mathcal{M}$ be the following $\mathcal{L}$-structure. We have $M = \{0, 1, 2, 3\}$, $\mathsf{R}^{\mathcal{M}} = \{1, 3\}$, and $\mathsf{f}^{\mathcal{M}}\colon M \to M$ is the function defined by

$$\mathsf{f}^{\mathcal{M}}(0) = 3 \qquad \mathsf{f}^{\mathcal{M}}(1) = 1 \qquad \mathsf{f}^{\mathcal{M}}(2) = 0 \qquad \mathsf{f}^{\mathcal{M}}(3) = 3$$

1. For every $s\colon Var \to M$ with $s(\mathsf{x}) = 2$, we have $(\mathcal{M}, s) \vDash \neg\mathsf{R}\mathsf{x}$ because

$$\begin{aligned} (\mathcal{M}, s) \vDash \neg\mathsf{R}\mathsf{x} &\Leftrightarrow (\mathcal{M}, s) \nvDash \mathsf{R}\mathsf{x} \\ &\Leftrightarrow s(\mathsf{x}) \notin \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow 2 \notin \mathsf{R}^{\mathcal{M}} \end{aligned}$$

   which is true.

2. For every s, we have $(\mathcal{M}, s) \vDash \exists\mathsf{x}\mathsf{R}\mathsf{x}$. To see this, fix $s\colon Var \to M$. We have

$$\begin{aligned} (\mathcal{M}, s) \vDash \exists\mathsf{x}\mathsf{R}\mathsf{x} &\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \mathsf{R}\mathsf{x} \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } \overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{x}) \in \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } s[\mathsf{x} \Rightarrow a](\mathsf{x}) \in \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } a \in \mathsf{R}^{\mathcal{M}} \end{aligned}$$

   with is true since $1 \in \mathsf{R}^{\mathcal{M}}$.

3. For every s, we have $(\mathcal{M}, s) \vDash \forall \mathsf{x}(\mathsf{Rx} \to (\mathsf{fx} = \mathsf{x}))$. To see this, fix $s\colon Var \to M$. We have

$$\begin{aligned}(\mathcal{M}, s) \vDash \forall \mathsf{x}(\mathsf{Rx} \to (\mathsf{f}(\mathsf{x}) = \mathsf{x})) &\Leftrightarrow \text{For all } a \in M, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash (\mathsf{Rx} \to (\mathsf{fx} = \mathsf{x})) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have either} \\ &\qquad (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \nvDash \mathsf{Rx} \text{ or } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash (\mathsf{fx} = \mathsf{x}) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have either} \\ &\qquad \overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{x}) \notin \mathsf{R}^{\mathcal{M}} \text{ or } \overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{f}x) = \overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{x}) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have either} \\ &\qquad s[\mathsf{x} \Rightarrow a](\mathsf{x}) \notin \mathsf{R}^{\mathcal{M}} \text{ or } \mathsf{f}^{\mathcal{M}}(\overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{x})) = \overline{s[\mathsf{x} \Rightarrow a]}(\mathsf{x}) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have either} \\ &\qquad s[\mathsf{x} \Rightarrow a](\mathsf{x}) \notin \mathsf{R}^{\mathcal{M}} \text{ or } \mathsf{f}^{\mathcal{M}}(s[\mathsf{x} \Rightarrow a](\mathsf{x})) = s[\mathsf{x} \Rightarrow a](\mathsf{x}) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have either } a \notin \mathsf{R}^{\mathcal{M}} \text{ or } \mathsf{f}^{\mathcal{M}}(a) = a\end{aligned}$$

which is true because $0 \notin R^{\mathcal{M}}$, $\mathsf{f}^{\mathcal{M}}(1) = 1$, $2 \notin R^{\mathcal{M}}$, and $\mathsf{f}^{\mathcal{M}}(3) = 3$.

□

In the above examples, it's clear that only the value of s on the free variables in φ affect whether or not $(\mathcal{M}, s) \vDash \varphi$. The following precise statement of this fact follows by a straightforward induction.

Proposition 4.2.8. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that $t \in Term_{\mathcal{L}}$ and $s_1, s_2\colon Var \to M$ are two variable assignments such that $s_1(\mathsf{x}) = s_2(\mathsf{x})$ for all $\mathsf{x} \in OccurVar(t)$. We then have $\overline{s}_1(t) = \overline{s}_2(t)$.*

Proposition 4.2.9. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that $\varphi \in Form_{\mathcal{L}}$ and $s_1, s_2\colon Var \to M$ are two variable assignments such that $s_1(\mathsf{x}) = s_2(\mathsf{x})$ for all $\mathsf{x} \in FreeVar(\varphi)$. We then have*

$$(\mathcal{M}, s_1) \vDash \varphi \textit{ if and only if } (\mathcal{M}, s_2) \vDash \varphi$$

Notation 4.2.10. *Let $\mathcal{L}$ be a language.*

1. *If $\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_\mathsf{n} \in Var$ are distinct, and we refer to a formula $\varphi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_\mathsf{n}) \in Form_{\mathcal{L}}$ we mean that $\varphi \in Form_{\mathcal{L}}$ and $FreeVar(\varphi) \subseteq \{\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_\mathsf{n}\}$.*
2. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure, $\varphi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_\mathsf{n}) \in Form_{\mathcal{L}}$, and $a_1, a_2, \dots, a_n \in M$. We write $(\mathcal{M}, a_1, a_2, \dots, a_n) \vDash \varphi$ to mean that $(\mathcal{M}, s) \vDash \varphi$ for some (any) $s\colon Var \to M$ with $s(\mathsf{x}_\mathsf{i}) = a_i$ for all i.*
3. *As a special case of 2, we have the following. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $\sigma \in Sent_{\mathcal{L}}$. We write $\mathcal{M} \vDash \sigma$ to mean that $(\mathcal{M}, s) \vDash \sigma$ for some (any) $s\colon Var \to M$.*

4.2.2 Elementary Classes of Structures

As we've seen, once you fix a language $\mathcal{L}$, an $\mathcal{L}$-structure can fix any set M at all, interpret the elements of $\mathcal{C}$ as arbitrary elements of M, interpret the elements of $\mathcal{R}_k$ as arbitrary subsets of M^k, and interpret the elements $\mathcal{F}_k$ as arbitrary k-ary functions on M. However, since we have a precise languague in hand, we now carve out classes of structures which satisfy certain sentences of our language.

Definition 4.2.11. *Let $\mathcal{L}$ be a language, and let $\Sigma \subseteq Sent_{\mathcal{L}}$. We let $Mod(\Sigma)$ be the class of all $\mathcal{L}$-structures $\mathcal{M}$ such that $\mathcal{M} \vDash \sigma$ for all $\sigma \in \Sigma$. If $\sigma \in Sent_{\mathcal{L}}$, we write $Mod(\sigma)$ instead of $Mod(\Sigma)$.*

Definition 4.2.12. *Let $\mathcal{L}$ be a language and let $\mathcal{K}$ be a class of $\mathcal{L}$-structures.*

1. *$\mathcal{K}$ is an* elementary class *if there exists $\sigma \in Sent_{\mathcal{L}}$ such that $\mathcal{K} = Mod(\sigma)$.*

2. *$\mathcal{K}$ is a* weak elementary class *if there exists $\Sigma \subseteq Sent_{\mathcal{L}}$ such that $\mathcal{K} = Mod(\Sigma)$.*

By taking conjunctions, we have the following simple proposition.

Proposition 4.2.13. *Let $\mathcal{L}$ be a language and let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. $\mathcal{K}$ is an elementary class if and only if there exists a finite $\Sigma \subseteq Sent_{\mathcal{L}}$ such that $\mathcal{K} = Mod(\Sigma)$.*

Examples. Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol.

1. The class of partially ordered sets is an elementary class as we saw in the introduction. We may let Σ be the following collection of sentences:
   (a) $\forall x Rxx$
   (b) $\forall x \forall y ((Rxy \wedge Ryx) \rightarrow (x = y))$
   (c) $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$

2. The class of equivalence relations is an elementary class. We may let Σ be the following collection of sentences:
   (a) $\forall x Rxx$
   (b) $\forall x \forall y (Rxy \rightarrow Ryx)$
   (c) $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$

3. The class of graphs is an elementary class. We may let Σ be the following collection of sentences:
   (a) $\forall x (\neg Rxx)$
   (b) $\forall x \forall y (Rxy \rightarrow Ryx)$

□

Example. Let $\mathcal{L}$ be any language whatsoever, and let $n \in \mathbb{N}^+$. The class of $\mathcal{L}$-structures of cardinality at least n is an elementary class as witnessed by the formula:

$$\exists x_1 \exists x_2 \cdots \exists x_n (\bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j))$$

Furthermore, the class of $\mathcal{L}$-structures of cardinality equal to n is an elementary class. Letting σ_n be the above formula for n, we can see this by considering $\sigma_n \wedge (\neg \sigma_{n+1})$. □

Examples. Let $\mathcal{L} = \{0, 1, +, \cdot\}$ where $0, 1$ are constant symbols and $+, \cdot$ are binary function symbols.

1. The class of fields is an elementary class. We may let Σ be the following collection of sentences:
   (a) $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$
   (b) $\forall x ((x + 0 = x) \wedge (0 + x = x))$
   (c) $\forall x \exists y ((x + y = 0) \wedge (y + x = 0))$
   (d) $\forall x \forall y (x + y = y + x)$
   (e) $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$
   (f) $\forall x ((x \cdot 1 = x) \wedge (1 \cdot x = x))$
   (g) $\forall x (x \neq 0 \rightarrow \exists y ((x \cdot y = 1) \wedge (y \cdot x = 0)))$

(h) $\forall x \forall y (x \cdot y = y \cdot x)$

(i) $\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

2. For each prime $p > 0$, the class of fields of characteristic p is an elementary class. Fix a prime $p > 0$, and let Σ_p be the above sentences togheter with the sentence $1 + 1 + \cdots + 1 = 0$ (where there are p 1's in the sum).

3. The class of fields of characteristic 0 is a weak elementary class. Let Σ be the above sentences together with $\{\tau_n : n \in \mathbb{N}^+\}$ where for each $n \in \mathbb{N}^+$, we have $\tau_n = \neg(1 + 1 + \cdots + 1 = 0)$ (where there are n 1's in the sum).

□

Example. Let F be a field, and let $\mathcal{L}_F = \{0, +\} \cup \{h_\alpha : \alpha \in F\}$ where 0 is a constant symbol, $+$ is binary function symbol, and each h_α is a unary function symbol. The class of vector spaces over F is a weak elementary class. We may let Σ be the following collection of sentences:

1. $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$

2. $\forall x ((x + 0 = x) \wedge (0 + x = x))$

3. $\forall x \exists y ((x + y = 0) \wedge (y + x = 0))$

4. $\forall x \forall y (x + y = y + x)$

5. $\forall x \forall y (h_\alpha(x + y) = h_\alpha(x) + h_\alpha(y))$ for each $\alpha \in F$.

6. $\forall x (h_{\alpha+\beta}(x) = (h_\alpha(x) + h_\beta(x)))$ for each $\alpha \in F$.

7. $\forall x (h_{\alpha \cdot \beta}(x) = (h_\alpha(h_\beta(x)))$ for each $\alpha, \beta \in F$.

8. $\forall x (h_1(x) = x)$

□

At this point, it's often clear how to show that a certain class of structures is a (weak) elementary class: simply exhibit the correct sentences. However, it may seem very difficult to show that a class is not a (weak) elementary class. For example, is the class of fields of characteristic 0 an elementary class? Is the class of finite groups a weak elementary class? There are no obvious ways to answer these questions affirmatively. We'll develop some tools later which will allow us to resolve these questions.

Another interesting case is that of Dedekind-complete ordered fields. Now the ordered field axioms are easily written down in the first-order language $\mathcal{L} = \{0, 1, <, +, \cdot\}$. In contrast, the Dedekind-completeness axiom, which says that every nonempty subset which is bounded above has a least upper bound, can not be directly translated in the language $\mathcal{L}$ because it involves quantifying over subsets instead of elements. However, we are unable to immediately conclude that this isn't due to a lack of cleverness on our part. Perhaps there is an alternative approach which captures Dedekind-complete ordered fields in a first-order way (by finding a clever equivalent first-order expression of Dedekind-completeness). More formally, the precise question is whether the complete ordered fields are a (weak) elementary class in the language $\mathcal{L}$. We'll be able to answer this question later.

4.2.3 Definability in Structures

Another wonderful side-effect of developing a formal language is the ability to talk about what objects we can define using that language.

Definition 4.2.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that $k \in \mathbb{N}^+$ and $X \subseteq M^k$. We say that X is* definable in $\mathcal{M}$ *if there exists $\varphi(\mathsf{x_1}, \mathsf{x_2}, \dots, \mathsf{x_k}) \in Form_{\mathcal{L}}$ such that*

$$X = \{(a_1, a_2, \dots, a_k) \in M^k : (\mathcal{M}, a_1, a_2, \dots, a_k) \vDash \varphi\}$$

Examples. Let $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +, \cdot\}$ where $\mathsf{0}$ and $\mathsf{1}$ are constant symbols and $+$ and $\cdot$ are binary function symbols.

1. The set $X = \{(m, n) \in \mathbb{N}^2 : m < n\}$ is definable in $(\mathbb{N}, 0, 1, +, \cdot)$ as witnessed by the formula

$$\exists \mathsf{z}(\mathsf{z} \neq \mathsf{0} \wedge (\mathsf{x} + \mathsf{z} = \mathsf{y}))$$

2. The set $X = \{n \in \mathbb{N} : n \text{ is prime}\}$ is definable in $(\mathbb{N}, 0, 1, +, \cdot)$ as witnessed by the formula

$$\neg(\mathsf{x} = \mathsf{1}) \wedge \forall \mathsf{y} \forall \mathsf{z}(\mathsf{x} = \mathsf{y} \cdot \mathsf{z} \rightarrow (\mathsf{y} = \mathsf{1} \vee \mathsf{z} = \mathsf{1}))$$

3. The set $X = \{r \in \mathbb{R} : r \geq 0\}$ is definable in $(\mathbb{R}, 0, 1, +, \cdot)$ as witnessed by the formula

$$\exists \mathsf{y}(\mathsf{y} \cdot \mathsf{y} = \mathsf{x})$$

□

Example.

1. Let $\mathcal{L} = \{<\}$ where $<$ is a binary relation symbol. For every $n \in \mathbb{N}$, the set $\{n\}$ is definable in $(\mathbb{N}, <)$. To see this, first define $\varphi_n(\mathsf{x})$ to be the formula

$$\exists \mathsf{y_1} \exists \mathsf{y_2} \cdots \exists \mathsf{y_n}\Big(\bigwedge_{1 \leq \mathsf{i} < \mathsf{j} \leq \mathsf{n}} (\mathsf{y_i} \neq \mathsf{y_j}) \wedge \bigwedge_{\mathsf{i}=1}^{\mathsf{n}} (\mathsf{y_i} < \mathsf{x})\Big)$$

Now notice that $\{0\}$ is definable as witnessed by the formula

$$\neg \exists \mathsf{y}(\mathsf{y} < \mathsf{x})$$

and for each $n \in \mathbb{N}^+$, the set $\{n\}$ is definable as witnessed by the formula

$$\varphi_n(\mathsf{x}) \wedge \neg \varphi_{n+1}(\mathsf{x})$$

2. Let $\mathcal{L} = \{\mathsf{e}, \mathsf{f}\}$ where e is a constant symbol and f is a binary function symbol. Let $(G, e, \cdot)$ be a group interpreted as an $\mathcal{L}$-stucture. The center of G is definable in $(G, e, \cdot)$ as witnessed by the formula

$$\forall \mathsf{y}(\mathsf{f}(\mathsf{x}, \mathsf{y}) = \mathsf{f}(\mathsf{y}, \mathsf{x}))$$

□

Sometimes, there isn't an obvious way to show that a set is definable, but some cleverness really pays off.

Examples. Let $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +, \cdot\}$ where $\mathsf{0}$ and $\mathsf{1}$ are constant symbols and $+$ and $\cdot$ are binary function symbols.

1. The set $\mathbb{N}$ is definable in $(\mathbb{Z}, 0, 1, +, \cdot)$ as witnessed by the formula:

$$\exists \mathsf{y}_1 \exists \mathsf{y}_2 \exists \mathsf{y}_3 \exists \mathsf{y}_4 (\mathsf{x} = \mathsf{y}_1 \cdot \mathsf{y}_1 + \mathsf{y}_2 \cdot \mathsf{y}_2 + \mathsf{y}_3 \cdot \mathsf{y}_3 + \mathsf{y}_4 \cdot \mathsf{y}_4)$$

because by Lagrange's Theorem, every natural number is the sum of four squares.

2. The set $X = \{(k, m, n) \in \mathbb{N}^3 : k^m = n\}$ is definable in $(\mathbb{N}, 0, 1, +, \cdot)$, as is the set $\{(m, n) \in \mathbb{N}^2 : m$ is the n^{th} digit in the decimal expansion of $\pi\}$. These are nontrivial result we'll prove later.

3. The set $\mathbb{Z}$ is definable in $(\mathbb{Q}, 0, 1, +, \cdot)$. This is a deep result of Julia Robinson.

4. Let $(R, 0, 1, +, \cdot)$ be a commutative ring. The Jacobson radical of R is definable in $(R, 0, 1, +, \cdot)$ as witnessed by the formula

$$\forall \mathsf{y} \exists \mathsf{z} ((\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z} = \mathsf{z} + 1)$$

□

As for elementary classes, it's clear how to attempt to show that something is definable (although as we've seen this may require a great deal of cleverness). However, it's not at all obvious how one could show that a set is not definable. We'll develop a few tools to do this in time.

4.2.4 Substitution

In time, we will see the need to "substitute" terms for variables. Roughly, you would think that if $\forall \mathsf{x} \varphi$ was true in some structure, then when you take any term and substitute it in for x in the formula φ, the resulting formula would be true. We need a way to relate truth before substituting with truth after substituting. The hope would be the following, where we use the notation φ^t_{x} to intuitively mean that you substitute t for x:

Hope 4.2.15. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $s \colon Var \to M$, let $t \in Term_{\mathcal{L}}$, and let $\mathsf{x} \in Var$. For all $\varphi \in Form_{\mathcal{L}}$, we have*

$$(\mathcal{M}, s) \vDash \varphi^t_{\mathsf{x}} \text{ if and only if } (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi$$

In order to make this precise, we first need to define substitition. Even with the "correct" definition of substitution, however, the above statement is not true. Let's first define substitution for terms and show that it behaves well.

Definition 4.2.16. *Let $\mathsf{x} \in Var$ and let $t \in Term_{\mathcal{L}}$. We define a function $Subst^t_{\mathsf{x}} \colon Term_{\mathcal{L}} \to Term_{\mathcal{L}}$ denoted by u^t_{x} as follows.*

1. $\mathsf{c}^t_{\mathsf{x}} = \mathsf{c}$ *for all* $\mathsf{c} \in \mathcal{C}$.

2. $\mathsf{y}^t_{\mathsf{x}} = \begin{cases} t & \text{if } \mathsf{y} = \mathsf{x} \\ \mathsf{y} & \text{otherwise} \end{cases}$

   for all $\mathsf{y} \in Var$.

3. $(\mathsf{f} u_1 u_2 \dots u_k)^t_{\mathsf{x}} = \mathsf{f}(u_1)^t_{\mathsf{x}}(u_2)^t_{\mathsf{x}} \cdots (u_k)^t_{\mathsf{x}}$ *for all* $\mathsf{f} \in \mathcal{F}_k$ *and all* $u_1, u_2, \dots, u_k \in Term_{\mathcal{L}}$.

Here's the key lemma that relates how to interpret a term before and after substitition.

Lemma 4.2.17. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $s \colon Var \to M$, let $t \in Term_{\mathcal{L}}$, and let $\mathsf{x} \in Var$. For all $u \in Term_{\mathcal{L}}$, we have*

$$\overline{s}(u^t_{\mathsf{x}}) = \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u)$$

Proof. The proof is by induction on $Term_{\mathcal{L}}$. For any $\mathsf{c} \in \mathcal{C}$, we have

$$\begin{aligned}\overline{s}(\mathsf{c}_{\mathsf{x}}^{t}) &= \overline{s}(\mathsf{c}) \\ &= \mathsf{c}^{\mathcal{M}} \\ &= s[\mathsf{x} \Rightarrow \overline{s}(t)](\mathsf{c}) \\ &= \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(\mathsf{c})\end{aligned}$$

Suppose that $u = \mathsf{x}$. We then have

$$\begin{aligned}\overline{s}(\mathsf{x}_{\mathsf{x}}^{t}) &= \overline{s}(t) \\ &= s[\mathsf{x} \Rightarrow \overline{s}(t)](\mathsf{x}) \\ &= \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(\mathsf{x})\end{aligned}$$

Suppose that $u = \mathsf{y} \in Var$ and that $\mathsf{y} \neq \mathsf{x}$. We then have

$$\begin{aligned}\overline{s}(\mathsf{y}_{\mathsf{x}}^{t}) &= \mathsf{y} \\ &= s[\mathsf{x} \Rightarrow \overline{s}(t)](\mathsf{y}) \\ &= \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(\mathsf{y})\end{aligned}$$

Finally, suppose that $\mathsf{f} \in \mathcal{F}_k$ and that the result holds for $u_1, u_2, \dots, u_k \in Term_{\mathcal{L}}$. We then have

$$\begin{aligned}\overline{s}((\mathsf{f}u_1u_2\cdots u_k)_{\mathsf{x}}^{t}) &= \overline{s}(\mathsf{f}(u_1)_{\mathsf{x}}^{t}(u_2)_{\mathsf{x}}^{t}\cdots(u_k)_{\mathsf{x}}^{t}) \\ &= \mathsf{f}^{\mathcal{M}}(\overline{s}((u_1)_{\mathsf{x}}^{t}), \overline{s}((u_2)_{\mathsf{x}}^{t}), \dots, \overline{s}((u_k)_{\mathsf{x}}^{t})) \\ &= \mathsf{f}^{\mathcal{M}}(\overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_1), \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_2), \dots, \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_k)) \qquad \text{(by induction)} \\ &= \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(\mathsf{f}u_1u_2\cdots u_k)\end{aligned}$$

□

With subsitution in terms defined, we now move to define substitution in formals. The key fact about this definition is that we only replace x by the term t for the free occurances of x because we certainly don't want to change $\forall\mathsf{x}\varphi$ into $\forall t\varphi$, nor do we want to mess with an x inside the scope of such a quantifier. We thus make the following recursive definition.

Definition 4.2.18. *We now define $FreeSubst_{t,\mathsf{x}} \colon Form_{\mathcal{L}} \to Form_{\mathcal{L}}$, again denoted φ_{x}^{t}, as follows.*

1. $(\mathsf{R}u_1u_2\cdots u_k)_{\mathsf{x}}^{t} = \mathsf{R}(u_1)_{\mathsf{x}}^{t}(u_2)_{\mathsf{x}}^{t}\cdots(u_k)_{\mathsf{x}}^{t}$ *for all* $\mathsf{R} \in \mathcal{R}_k$ *and all* $u_1, u_2, \dots, u_k \in Term_{\mathcal{L}}$.

2. $(= u_1u_2)_{\mathsf{x}}^{t} =$ "$= (u_1)_{\mathsf{x}}^{t}(u_2)_{\mathsf{x}}^{t}$" *for all* $u_1, u_2 \in Term_{\mathcal{L}}$.

3. $(\neg\varphi)_{\mathsf{x}}^{t} = \neg(\varphi_{\mathsf{x}}^{t})$ *for all* $\varphi \in Form_{\mathcal{L}}$.

4. $(\Diamond\varphi\psi)_{\mathsf{x}}^{t} = \Diamond\varphi_{\mathsf{x}}^{t}\psi_{\mathsf{x}}^{t}$ *for all* $\varphi, \psi \in Form_{\mathcal{L}}$ *and all* $\Diamond \in \{\wedge, \vee, \to\}$.

5. $(\mathsf{Q}\mathsf{y}\varphi)_{\mathsf{x}}^{t} = \begin{cases} \mathsf{Q}\mathsf{y}\varphi & \text{if } \mathsf{x} = \mathsf{y} \\ \mathsf{Q}\mathsf{y}(\varphi_{\mathsf{x}}^{t}) & \text{otherwise} \end{cases}$

   for all $\varphi \in Form_{\mathcal{L}}$, $\mathsf{y} \in Var$, *and* $\mathsf{Q} \in \{\exists, \forall\}$.

With the definition in hand, let's analyze the above hope. Suppose that $\mathcal{L} = \emptyset$, and consider the formula $\varphi(\mathsf{y}) \in Form_{\mathcal{L}}$ given by

$$\exists \mathsf{x} \neg (\mathsf{x} = \mathsf{y})$$

For any $\mathcal{L}$-structure $\mathcal{M}$ and any $s \colon Var \to M$, we have $(\mathcal{M}, s) \vDash \varphi$ if and only if $|M| \geq 2$. Now notice that the formula $\varphi^{\mathsf{x}}_{\mathsf{y}}$ is

$$\exists \mathsf{x} \neg (\mathsf{x} = \mathsf{x})$$

so for any $\mathcal{L}$-structure $\mathcal{M}$ and any $s \colon Var \to M$, we have $(\mathcal{M}, s) \nvDash \varphi^{\mathsf{x}}_{\mathsf{y}}$. Therefore, the above hope fails whenever $\mathcal{M}$ is an $\mathcal{L}$-stucture with $|M| \geq 2$. The problem is that the term we substituted (in this case x) had a variable which became "captured" by a quantifier, and thus the "meaning" of the formula became transformed. We thus defined a function which indicates with this does not happen.

Definition 4.2.19. *Let $t \in Term_{\mathcal{L}}$ and let $\mathsf{x} \in Var$. We define a function $ValidSubst^t_{\mathsf{x}} \colon Form_{\mathcal{L}} \to \{0, 1\}$ as follows.*

1. $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$ *for all* $\varphi \in AtomicForm_{\mathcal{L}}$.
2. $ValidSubst^t_{\mathsf{x}}(\neg\varphi) = ValidSubst^t_{\mathsf{x}}(\varphi)$ *for all* $\varphi \in Form_{\mathcal{L}}$.
3. $ValidSubst^t_{\mathsf{x}}(\Diamond\varphi\psi) = \begin{cases} 1 & \text{if } ValidSubst^t_{\mathsf{x}}(\varphi) = 1 \text{ and } ValidSubst^t_{\mathsf{x}}(\psi) = 1 \\ 0 & \text{otherwise} \end{cases}$

   for all $\varphi, \psi \in Form_{\mathcal{L}}$ *and all* $\Diamond \in \{\wedge, \vee, \rightarrow\}$.
4. $ValidSubst^t_{\mathsf{x}}(\mathsf{Q}\mathsf{y}\varphi) = \begin{cases} 1 & \text{if } \mathsf{x} \notin FreeVar(\mathsf{Q}\mathsf{y}\varphi) \\ 1 & \text{if } \mathsf{y} \notin OccurVar(t) \text{ and } ValidSubst^t_{\mathsf{x}}(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$

   for all $\varphi \in Form_{\mathcal{L}}$, $\mathsf{x}, \mathsf{y} \in Var$, *and* $\mathsf{Q} \in \{\forall, \exists\}$.

Theorem 4.2.20 (Substitution Theorem)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $s \colon Var \to M$, let $t \in Term_{\mathcal{L}}$, and let $\mathsf{x} \in Var$. For all $\varphi \in Form_{\mathcal{L}}$ with $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$, we have*

$$(\mathcal{M}, s) \vDash \varphi^t_x \text{ if and only if } (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi$$

Proof. The proof is by induction on φ. We first handle the case when $\varphi \in AtomicForm_{\mathcal{L}}$. Suppose that $\mathsf{R} \in \mathcal{R}_k$ and that $u_1, u_2, \dots, u_k \in Term_{\mathcal{L}}$. We then have

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\mathsf{R}u_1u_2\cdots u_k)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \mathsf{R}(u_1)^t_{\mathsf{x}}(u_2)^t_{\mathsf{x}}\cdots(u_k)^t_{\mathsf{x}} \\
&\Leftrightarrow (\overline{s}((u_1)^t_{\mathsf{x}}), \overline{s}((u_2)^t_{\mathsf{x}}), \cdots, \overline{s}((u_k)^t_{\mathsf{x}})) \in \mathsf{R}^{\mathcal{M}} \\
&\Leftrightarrow (\overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_1), \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_2), \cdots, \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_k)) \in \mathsf{R}^{\mathcal{M}} \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \mathsf{R}u_1u_2\cdots u_k
\end{aligned}$$

If $u_1, u_2 \in Term_{\mathcal{L}}$, we have

$$\begin{aligned}
(\mathcal{M}, s) \vDash (= u_1u_2)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash = (u_1)^t_{\mathsf{x}}(u_2)^t_{\mathsf{x}} \\
&\Leftrightarrow \overline{s}((u_1)^t_{\mathsf{x}}) = \overline{s}((u_2)^t_{\mathsf{x}}) \\
&\Leftrightarrow \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_1) = \overline{s[\mathsf{x} \Rightarrow \overline{s}(t)]}(u_2) \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash = u_1u_2
\end{aligned}$$

Suppose that the results holds for φ and that $ValidSubst^t_{\mathsf{x}}(\neg\varphi) = 1$. We then have that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$, and hence

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\neg\varphi)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \neg(\varphi^t_{\mathsf{x}}) \\
&\Leftrightarrow (\mathcal{M}, s) \nvDash \varphi^t_{\mathsf{x}} \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \nvDash \varphi && \text{(by induction)} \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \neg\varphi
\end{aligned}$$

The connectives $\wedge$,$\vee$, and $\rightarrow$ are similarly uninteresting.

Suppose that the result holds for φ and that $ValidSubst^t_{\mathsf{x}}(\exists\mathsf{y}\varphi) = 1$. First, if $\mathsf{x} \notin FreeVar(\exists\mathsf{y}\varphi)$, we have

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\exists\mathsf{y}\varphi)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \exists\mathsf{y}\varphi \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \exists\mathsf{y}\varphi
\end{aligned}$$

Suppose then that $\mathsf{x} \in FreeVar(\exists\mathsf{y}\varphi)$, so in particular $\mathsf{x} \neq \mathsf{y}$. Since $ValidSubst^t_{\mathsf{x}}(\exists\mathsf{y}\varphi) = 1$, we have that $\mathsf{y} \notin OccurVar(t)$, and also that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$. Now using the fact that $\mathsf{y} \notin OccurVar(t)$, it follows that $\overline{s[\mathsf{y} \Rightarrow a]}(t) = \overline{s}(t)$ for every $a \in M$. Therefore,

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\exists\mathsf{y}\varphi)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \exists\mathsf{y}(\varphi^t_{\mathsf{x}}) \\
&\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^t_{\mathsf{x}} \\
&\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s[\mathsf{y} \Rightarrow a]}(t)]) \vDash \varphi \quad \text{(by induction)} \\
&\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi \\
&\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, (s[\mathsf{x} \Rightarrow \overline{s}(t)])[\mathsf{y} \Rightarrow a]) \vDash \varphi \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \exists\mathsf{y}\varphi
\end{aligned}$$

Suppose that the result holds for φ and that $ValidSubst^t_{\mathsf{x}}(\forall\mathsf{y}\varphi) = 1$. First, if $\mathsf{x} \notin FreeVar(\exists\mathsf{y}\varphi)$, we have

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\forall\mathsf{y}\varphi)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \forall\mathsf{y}\varphi \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \forall\mathsf{y}\varphi
\end{aligned}$$

Suppose then that $\mathsf{x} \in FreeVar(\forall\mathsf{y}\varphi)$, so in particular $\mathsf{x} \neq \mathsf{y}$. Since $ValidSubst^t_{\mathsf{x}}(\forall\mathsf{y}\varphi) = 1$, we have that $\mathsf{y} \notin OccurVar(t)$ and also that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$. Now using the fact that $\mathsf{y} \notin OccurVar(t)$, it follows that $\overline{s[\mathsf{y} \Rightarrow a]}(t) = \overline{s}(t)$ for every $a \in M$. Therefore,

$$\begin{aligned}
(\mathcal{M}, s) \vDash (\forall\mathsf{y}\varphi)^t_{\mathsf{x}} &\Leftrightarrow (\mathcal{M}, s) \vDash \forall\mathsf{y}(\varphi^t_{\mathsf{x}}) \\
&\Leftrightarrow \text{For all } a \in M\text{, we have } (\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^t_{\mathsf{x}} \\
&\Leftrightarrow \text{For all } a \in M\text{, we have } (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s[\mathsf{y} \Rightarrow a]}(t)]) \vDash \varphi \\
&\Leftrightarrow \text{For all } a \in M\text{, we have } (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi \\
&\Leftrightarrow \text{For all } a \in M\text{, we have } (\mathcal{M}, (s[\mathsf{x} \Rightarrow \overline{s}(t)])[\mathsf{y} \Rightarrow a]) \vDash \varphi \\
&\Leftrightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \forall\mathsf{y}\varphi
\end{aligned}$$

□

4.3 Relationships Between Structures

4.3.1 Homomorphisms and Embeddings

Definition 4.3.1. *Let $\mathcal{L}$ be a language, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures.*

1. *A function* $h\colon M \to N$ *is called a* homomorphism *if*
   (a) *For all* $\mathsf{c} \in \mathcal{C}$*, we have* $h(\mathsf{c}^{\mathcal{M}}) = \mathsf{c}^{\mathcal{N}}$
   (b) *For all* $\mathsf{R} \in \mathcal{R}_k$ *and all* $a_1, a_2, \dots, a_k \in \mathcal{M}$*, we have*
   $$(a_1, a_2, \dots, a_k) \in \mathsf{R}^{\mathcal{M}} \text{ if and only if } (h(a_1), h(a_2), \dots, h(a_k)) \in \mathsf{R}^{\mathcal{N}}$$
   (c) *For all* $f \in \mathcal{F}_k$ *and all* $a_1, a_2, \dots, a_k \in \mathcal{M}$*, we have*
   $$h(\mathsf{f}^{\mathcal{M}}(a_1, a_2, \dots, a_k)) = \mathsf{f}^{\mathcal{N}}(h(a_1), h(a_2), \dots, h(a_k))$$
2. *A function* $h\colon M \to N$ *is called an* embedding *if it is an injective homomorphism.*
3. *A function* $h\colon M \to N$ *is called an* isomorphism *if it is a bijective homomorphism.*

Notation 4.3.2. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be* $\mathcal{L}$*-structures. If* $\mathcal{M}$ *and* $\mathcal{N}$ *are isomorphic, i.e. if there exists an isomorphism* $h\colon \mathcal{M} \to \mathcal{N}$*, then we write* $\mathcal{M} \cong \mathcal{N}$*.*

Definition 4.3.3. *Let* $\mathcal{M}$ *be an* $\mathcal{L}$*-structure. An isomorphism* $h\colon M \to M$ *is called an* automorphism.

Theorem 4.3.4. *Let* $\mathcal{L}$ *be a language, and let* $\mathcal{M}$ *and* $\mathcal{N}$ *be* $\mathcal{L}$*-structures. Suppose that* $h\colon M \to N$ *is a homomorphism, and suppose that* $s\colon Var \to M$ *is a variable assignment.*

1. *For any* $t \in Term_{\mathcal{L}}$*, we have* $h(\overline{s}(t)) = \overline{h \circ s}(t)$*.*
2. *For every quantifier-free* $\varphi \in Form_{\mathcal{L}}$ *not containing the equality symbol, we have*
   $$(\mathcal{M}, s) \models \varphi \text{ if and only if } (\mathcal{N}, h \circ s) \models \varphi$$
3. *If* h *is an embedding, then for every quantifier-free* $\varphi \in Form_{\mathcal{L}}$*, we have*
   $$(\mathcal{M}, s) \models \varphi \text{ if and only if } (\mathcal{N}, h \circ s) \models \varphi$$
4. *If* h *is an isomorphism, then for every* $\varphi \in Form_{\mathcal{L}}$*, we have*
   $$(\mathcal{M}, s) \models \varphi \text{ if and only if } (\mathcal{N}, h \circ s) \models \varphi$$

Proof.

1. We prove this by induction on t. First, for any $\mathsf{c} \in \mathcal{C}$, we have
   $$\begin{aligned} h(\overline{s}(\mathsf{c})) &= h(\mathsf{c}^{\mathcal{M}}) \\ &= \mathsf{c}^{\mathcal{N}} && \text{(since } h \text{ is a homomorphism)} \\ &= \overline{h \circ s}(\mathsf{c}). \end{aligned}$$
   Now for $\mathsf{x} \in Var$, we have
   $$\begin{aligned} h(\overline{s}(\mathsf{x})) &= h(s(\mathsf{x}))) \\ &= (h \circ s)(\mathsf{x}) \\ &= \overline{h \circ s}(\mathsf{x}) \end{aligned}$$

Suppose now that $\mathsf{f} \in \mathcal{F}_k$, that $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$, and the result holds for each t_i. We then have

$$\begin{aligned} h(\overline{s}(\mathsf{f}t_1t_2\cdots t_k)) &= h(\mathsf{f}^{\mathcal{M}}(\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k))) \\ &= \mathsf{f}^{\mathcal{M}}(h(\overline{s}(t_1)), h(\overline{s}(t_2)), \dots, h(\overline{s}(t_k))) && \text{(since } h \text{ is a homomorphism)} \\ &= \mathsf{f}^{\mathcal{N}}(\overline{h \circ s}(t_1), \overline{h \circ s}(t_2), \dots, \overline{h \circ s}(t_k)) && \text{(by induction)} \\ &= \overline{h \circ s}(\mathsf{f}t_1t_2\cdots t_k) \end{aligned}$$

The result follows by induction

2. Suppose that h is an embedding. We prove the result by induction on φ. Suppose first that $\mathsf{R} \in \mathcal{R}_k$ and that $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. We then have

$$\begin{aligned} (\mathcal{M}, s) \vDash \mathsf{R}t_1t_2\cdots t_k &\Leftrightarrow (\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k)) \in \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow (h(\overline{s}(t_1)), h(\overline{s}(t_2)), \dots, h(\overline{s}(t_k))) \in \mathsf{R}^{\mathcal{N}} && \text{(since } h \text{ is a homomorphism)} \\ &\Leftrightarrow (\overline{h \circ s}(t_1), \overline{h \circ s}(t_2), \dots, \overline{h \circ s}(t_k)) \in \mathsf{R}^{\mathcal{N}} && \text{(by part 1)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \mathsf{R}t_1t_2\cdots t_k \end{aligned}$$

Suppose that the result holds for φ. We prove it for $\neg\varphi$. We have

$$\begin{aligned} (\mathcal{M}, s) \vDash \neg\varphi &\Leftrightarrow (\mathcal{M}, s) \nvDash \varphi \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \nvDash \varphi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \neg\varphi \end{aligned}$$

Suppose that the result holds for φ and ψ. We have

$$\begin{aligned} (\mathcal{M}, s) \vDash \varphi \wedge \psi &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \text{ and } (\mathcal{M}, s) \vDash \psi \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \varphi \text{ and } (\mathcal{N}, h \circ s) \vDash \varphi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \varphi \wedge \psi \end{aligned}$$

and similarly for $\vee$ and $\rightarrow$. The result follows by induction.

3. In light of the proof of 2, we need only show that if φ is $= t_1t_2$ where $t_1, t_2 \in Term_{\mathcal{L}}$, then $(\mathcal{M}, s) \vDash \varphi$ if and only if $(\mathcal{N}, h \circ s) \vDash \varphi$. For any $t_1, t_2 \in Term_{\mathcal{L}}$, we have

$$\begin{aligned} (\mathcal{M}, s) \vDash = t_1t_2 &\Leftrightarrow \overline{s}(t_1) = \overline{s}(t_2) \\ &\Leftrightarrow h(\overline{s}(t_1)) = h(\overline{s}(t_2)) && \text{(since } h \text{ is injective)} \\ &\Leftrightarrow \overline{h \circ s}(t_1) = \overline{h \circ s}(t_2) && \text{(by part 1)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash = t_1t_2 \end{aligned}$$

4. Suppose that the result holds for φ and $\mathsf{x} \in Var$. We have

$$\begin{aligned} (\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi &\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{N}, h \circ (s[\mathsf{x} \Rightarrow a])) \vDash \varphi \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{N}, (h \circ s)[x \Rightarrow h(a)] \vDash \varphi \\ &\Leftrightarrow \text{There exists } b \in N \text{ such that } (\mathcal{N}, (h \circ s)[x \Rightarrow b]) \vDash \varphi && \text{(since } h \text{ is bijective)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \exists \mathsf{x}\varphi \end{aligned}$$

and also

$$\begin{aligned}(\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi &\Leftrightarrow \text{For all } a \in M, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have } (\mathcal{N}, h \circ (s[\mathsf{x} \Rightarrow a])) \vDash \varphi \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have } (\mathcal{N}, (h \circ s)[x \Rightarrow h(a)] \vDash \varphi \\ &\Leftrightarrow \text{For all } b \in N, \text{ we have } (\mathcal{N}, (h \circ s)[x \Rightarrow b]) \vDash \varphi \qquad \text{(since } h \text{ is bijective)} \\ &\Leftrightarrow (\mathcal{N}, h \circ s) \vDash \forall \mathsf{x}\varphi\end{aligned}$$

□

Definition 4.3.5. *Let $\mathcal{L}$ be a language, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We write $\mathcal{M} \equiv \mathcal{N}$, and say that $\mathcal{M}$ and $\mathcal{N}$ are* elementarily equivalent*, if for all $\sigma \in Sent_{\mathcal{L}}$, we have $\mathcal{M} \vDash \sigma$ if and only if $\mathcal{N} \vDash \sigma$.*

Corollary 4.3.6. *Let $\mathcal{L}$ be a language, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

4.3.2 An Application To Definability

Proposition 4.3.7. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $k \in \mathbb{N}^+$. Suppose also that $X \subseteq M^k$ is definable in $\mathcal{M}$ and that $h\colon M \to M$ is an automorphism. For every $a_1, a_2, \dots, a_k \in M$, we have*

$$(a_1, a_2, \dots, a_k) \in X \textit{ if and only if } (h(a_1), h(a_2), \dots, h(a_k)) \in X$$

Proof. Fix $\varphi(x_1, x_2, \dots, x_k) \in Form_{\mathcal{L}}$ such that

$$X = \{(a_1, a_2, \dots, a_k) \in M^k : (\mathcal{M}, a_1, a_2, \dots, a_k) \vDash \varphi\}$$

By part 4 of Theorem 4.3.4, we know that for every $a_1, a_2, \dots, a_k \in M$, we have

$$(\mathcal{M}, a_1, a_2, \dots, a_k) \vDash \varphi \text{ if and only if } (\mathcal{M}, h(a_1), h(a_2), \dots, h(a_k)) \vDash \varphi$$

Therefore, for every $a_1, a_2, \dots, a_k \in M$, we have

$$(a_1, a_2, \dots, a_k) \in X \text{ if and only if } (h(a_1), h(a_2), \dots, h(a_k)) \in X$$

□

Corollary 4.3.8. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $k \in \mathbb{N}^+$. Suppose also that $X \subseteq M^k$ is and that $h\colon M \to M$ is an automorphism. If there exists $a_1, a_2, \dots, a_k \in M$ such that exactly one of the following holds:*

- $(a_1, a_2, \dots, a_k) \in X$
- $(h(a_1), h(a_2), \dots, h(a_k)) \in X$

then X is not definiable in $\mathcal{M}$.

Example. Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol, and let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{Z}$ and $\mathsf{R}^{\mathcal{M}} = \{(a, b) \in \mathbb{Z}^2 : a < b\}$. We show that a set $X \subseteq M$ is definable in $\mathcal{M}$ if and only if either $X = \emptyset$ or $X = \mathbb{Z}$. First notice that $\emptyset$ is definable as witnessed by $\neg(\mathsf{x} = \mathsf{x})$ and $\mathbb{Z}$ as witnessed by $\mathsf{x} = \mathsf{x}$. Suppose now that $X \subseteq \mathbb{Z}$ is such that $X \neq \emptyset$ and $X \neq \mathbb{Z}$. Fix $a, b \in \mathbb{Z}$ such that $a \in X$ and $b \notin X$. Define $h\colon M \to M$

by letting $h(c) = c + (b - a)$ for all $c \in M$. Notice that h is automorphism of $\mathcal{M}$ because it is bijective (the map $g(c) = c - (b - a)$ is clearly an inverse) and a homomorphism because if $c_1, c_2 \in \mathbb{Z}$, then have have

$$\begin{aligned}(c_1, c_2) \in \mathsf{R}^{\mathcal{M}} &\Leftrightarrow c_1 < c_2 \\ &\Leftrightarrow c_1 + (b - a) < c_2 + (b - a) \\ &\Leftrightarrow h(c_1) < h(c_2) \\ &\Leftrightarrow (h(c_1), h(c_2)) \in \mathsf{R}^{\mathcal{M}}\end{aligned}$$

Notice also that $h(a) = a + (b - a) = b$, so $a \in X$ but $h(a) \notin X$. It follows from the proposition that X is not definable in $\mathcal{M}$. □

4.3.3 Substructures

Definition 4.3.9. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{A}$ be $\mathcal{L}$-structures. We say that $\mathcal{A}$ is a* substructure *of $\mathcal{M}$, and we write $\mathcal{A} \subseteq \mathcal{M}$ if*

1. *$A \subseteq M$.*
2. *$\mathsf{c}^{\mathcal{A}} = \mathsf{c}^{\mathcal{M}}$ for all $\mathsf{c} \in \mathcal{C}$.*
3. *$\mathsf{R}^{\mathcal{A}} = \mathsf{R}^{\mathcal{M}} \cap A^k$ for all $\mathsf{R} \in \mathcal{R}_k$.*
4. *$\mathsf{f}^{\mathcal{A}} = \mathsf{f}^{\mathcal{M}} \upharpoonright A^k$ for all $\mathsf{f} \in \mathcal{F}_k$.*

Remark 4.3.10. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{A}$ be $\mathcal{L}$-structures with $A \subseteq M$. We then have that $\mathcal{A} \subseteq \mathcal{M}$ if and only if the identity map $\iota \colon A \to M$ is a homomorphism.*

Remark 4.3.11. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that $A \subseteq M$. A is the universe of a substructure of M if and only if $\{\mathsf{c}^{\mathcal{M}} : \mathsf{c} \in \mathcal{C}\} \subseteq A$ and $\mathsf{f}^{\mathcal{M}}(a_1, a_2, \dots, a_k) \in A$ for all $\mathsf{f} \in \mathcal{F}_k$ and all $a_1, a_2, \dots, a_k \in A$.*

Proposition 4.3.12. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $B \subseteq M$. Suppose either that $B \neq \emptyset$ or $\mathcal{C} \neq \emptyset$. If we let $A = G(M, B \cup \{\mathsf{c}^{\mathcal{M}} : \mathsf{c} \in \mathcal{C}\}, \{\mathsf{f}^{\mathcal{M}} : \mathsf{f} \in \mathcal{F}\})$, then A is the universe of a substructure of $\mathcal{M}$. Moreover, if $\mathcal{N} \subseteq \mathcal{M}$ with $B \subseteq N$, then $A \subseteq N$.*

Proposition 4.3.13. *Let $\mathcal{L}$ be a language.*

1. *A Σ_1-formula is an element of $G(Sym_{\mathcal{L}}^*, QuantFreeForm_{\mathcal{L}}, \{h_{\exists,\mathsf{x}} : \mathsf{x} \in Var\})$.*
2. *A Π_1-formula is an element of $G(Sym_{\mathcal{L}}^*, QuantFreeForm_{\mathcal{L}}, \{h_{\forall,\mathsf{x}} : \mathsf{x} \in Var\})$.*

Proposition 4.3.14. *Suppose that $\mathcal{A} \subseteq \mathcal{M}$.*

1. *For any $\varphi \in QuantFreeForm_{\mathcal{L}}$ and any $s \colon Var \to A$, we have*

$$(\mathcal{A}, s) \vDash \varphi \text{ if and only if } (\mathcal{M}, s) \vDash \varphi$$

2. *For any Σ_1-formula $\varphi \in Form_{\mathcal{L}}$ and any $s \colon Var \to A$, we have*

$$\text{If } (\mathcal{A}, s) \vDash \varphi, \text{ then } (\mathcal{M}, s) \vDash \varphi$$

3. *For any Π_1-formula $\varphi \in Form_{\mathcal{L}}$ and any $s \colon Var \to A$, we have*

$$\text{If } (\mathcal{M}, s) \vDash \varphi, \text{ then } (\mathcal{A}, s) \vDash \varphi$$

Proof.

1. This follows from Remark 4.3.10 and Theorem 4.3.4.

2. We prove this by induction. If φ is quantifier-free, this follows from part 1. Suppose that we know the result for φ, and suppose that $(\mathcal{A}, s) \vDash \exists \mathsf{x}\varphi$. Fix $a \in A$ such that $(\mathcal{A}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$. By induction, we know that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$, hence $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$.

3. We prove this by induction. If φ is quantifier-free, this follows from part 1. Suppose that we know the result for φ, and suppose that $(\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi$. For every $a \in A$, we then have $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$, and hence $(\mathcal{A}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$ by induction. It follows that $(\mathcal{A}, s) \vDash \forall \mathsf{x}\varphi$.

□

4.3.4 Elementary Substructures

Definition 4.3.15. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{A}$ be $\mathcal{L}$-structures. We say that $\mathcal{A}$ is an* elementary substructure *of $\mathcal{M}$ if $\mathcal{A} \subseteq \mathcal{M}$ and for all $\varphi \in Form_{\mathcal{L}}$ and all $s\colon Var \to A$, we have*

$$(\mathcal{A}, s) \vDash \varphi \textit{ if and only if } (\mathcal{M}, s) \vDash \varphi$$

We write $\mathcal{A} \preceq \mathcal{M}$ to mean that $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$.

Example. Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a unary function symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure with $M = \mathbb{N}$ and $\mathsf{f}^{\mathcal{M}}(n) = n + 1$. Let $\mathcal{A}$ be $\mathcal{L}$-structure with $A = \mathbb{N}^+$ and $\mathsf{f}^{\mathcal{A}}(n) = n + 1$. We then have that $\mathcal{A} \subseteq \mathcal{M}$. Furthermore, we have $\mathcal{M} \cong \mathcal{A}$, hence for all $\sigma \in Sent_{\mathcal{L}}$ we have

$$\mathcal{A} \vDash \sigma \text{ if and only if } \mathcal{M} \vDash \sigma$$

However, notice that $\mathcal{A} \not\preceq \mathcal{M}$ because if $\varphi(\mathsf{x})$ is the formula $\neg\exists \mathsf{y}(\mathsf{fy} = \mathsf{x})$, we then have that $(\mathcal{A}, 1) \vDash \varphi$ but $(\mathcal{M}, 1) \nvDash \varphi$. □

Theorem 4.3.16 (Tarski-Vaught Test)**.** *Suppose that $\mathcal{A} \subseteq \mathcal{M}$. The following are equivalent.*

1. *$\mathcal{A} \preceq \mathcal{M}$.*

2. *Whenever $\varphi \in Form_{\mathcal{L}}$, $\mathsf{x} \in Var$, and $s\colon Var \to A$ satisfy $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$, there exists $a \in A$ such that*

$$(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$$

Proof. We first prove that 1 implies 2. Suppose then that $\mathcal{A} \preceq \mathcal{M}$. Let $\varphi \in Form_{\mathcal{L}}$ and $s\colon Var \to A$ be such that $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$. Using the fact that $\mathcal{A} \preceq \mathcal{M}$, it follows that $(\mathcal{A}, s) \vDash \exists \mathsf{x}\varphi$. Fix $a \in A$ such that $(\mathcal{A}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$. Using again the fact that $\mathcal{A} \preceq \mathcal{M}$, we have $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$.

We now prove that 2 implies 1. We prove by induction on $\varphi \in Form_{\mathcal{L}}$ that for all $s\colon Var \to A$, we have $(\mathcal{A}, s) \vDash \varphi$ if and only if $(\mathcal{M}, s) \vDash \varphi$. That is, we let

$$X = \{\varphi \in Form_{\mathcal{L}} : \text{ For all } s\colon Var \to A \text{ we have } (\mathcal{A}, s) \vDash \varphi \text{ if and only if } (\mathcal{M}, s) \vDash \varphi\}$$

and prove that $X = Form_{\mathcal{L}}$ by induction. First notice that $\varphi \in X$ for all quantifier-free φ because $\mathcal{A} \subseteq \mathcal{M}$.

Suppose now that $\varphi \in X$. For any $s\colon Var \to A$, we have

$$\begin{aligned} (\mathcal{A}, s) \vDash \neg\varphi &\Leftrightarrow (\mathcal{A}, s) \nvDash \varphi \\ &\Leftrightarrow (\mathcal{M}, s) \nvDash \varphi && \text{(since } \varphi \in X\text{)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \neg\varphi \end{aligned}$$

Therefore, $\neg\varphi \in X$.

Suppose now that $\varphi, \psi \in X$. For any $s\colon Var \to A$, we have

$$\begin{aligned}(\mathcal{A}, s) \vDash \varphi \wedge \psi &\Leftrightarrow (\mathcal{A}, s) \vDash \varphi \text{ and } (\mathcal{A}, s) \vDash \psi \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \text{ and } (\mathcal{M}, s) \vDash \psi && (\text{since } \varphi, \psi \in X) \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \wedge \psi\end{aligned}$$

Therefore, $\varphi \wedge \psi \in X$. Similarly, we have $\varphi \vee \psi \in X$ and $\varphi \to \psi \in X$.

Suppose now that $\varphi \in X$ and $\mathsf{x} \in Var$. For any $s\colon Var \to A$, we have

$$\begin{aligned}(\mathcal{A}, s) \vDash \exists \mathsf{x}\varphi &\Leftrightarrow \text{There exists } a \in A \text{ such that } (\mathcal{A}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Leftrightarrow \text{There exists } a \in A \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi && (\text{since } \varphi \in X) \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi && (\text{by our assumption 2})\end{aligned}$$

Therefore, $\exists \mathsf{x}\varphi \in X$.

Suppose now that $\varphi \in X$ and $\mathsf{x} \in Var$. We then have that $\neg\varphi \in X$ from above, hence $\exists \mathsf{x}\neg\varphi \in X$ from above, hence $\neg\exists \mathsf{x}\neg\varphi \in X$ again from above. Thus, for any $s\colon Var \to A$, we have

$$\begin{aligned}(\mathcal{A}, s) \vDash \forall \mathsf{x}\varphi &\Leftrightarrow (\mathcal{A}, s) \vDash \neg\exists \mathsf{x}\neg\varphi \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \neg\exists \mathsf{x}\neg\varphi && (\text{since } \neg\exists \mathsf{x}\neg\varphi \in X) \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi\end{aligned}$$

Therefore, $\forall \mathsf{x}\varphi \in X$. □

Theorem 4.3.17 (Countable Lowenheim-Skolem-Tarski Theorem)**.** *Suppose that $\mathcal{L}$ is countable, that $\mathcal{M}$ is an $\mathcal{L}$-structure, and that $X \subseteq M$ is countable. There exists a countable $\mathcal{A} \preceq \mathcal{M}$ such that $X \subseteq A$.*

Proof. Fix an element $d \in M$ (this will be used as a "dummy" element of M to ensure that we always have something to go to when all else fails).

For each $\varphi \in Form_{\mathcal{L}}$ and $\mathsf{x} \in Var$ such that $FreeVar(\varphi) = \{\mathsf{x}\}$, we define an element $n_{\varphi,\mathsf{x}} \in M$ as follows. If $\mathcal{M} \vDash \exists \mathsf{x}\varphi$, fix an arbitrary $m \in M$ such that $(\mathcal{M}, m) \vDash \varphi$, and let $n_{\varphi,\mathsf{x}} = m$. Otherwise, let $n_{\varphi,\mathsf{x}} = d$.

Now for each $\varphi \in Form_{\mathcal{L}}$ and $\mathsf{x} \in Var$ such that $\{\mathsf{x}\} \subsetneq FreeVar(\varphi)$, we define a function. Suppose that $FreeVar(\varphi) = \{\mathsf{y}_1, \mathsf{y}_2, \dots, \mathsf{y}_k, \mathsf{x}\}$. We define a function $h_{\varphi,\mathsf{x}}\colon M^k \to M$ as follows. Let $b_1, b_2, \dots, b_k \in M$. If $(\mathcal{M}, b_1, b_2, \dots, b_k) \vDash \exists \mathsf{x}\varphi$, fix an arbitrary $a \in M$ such that $(\mathcal{M}, b_1, b_2, \dots, b_k, a) \vDash \varphi$, and let $h_{\varphi,\mathsf{x}}(b_1, b_2, \dots, b_k) = a$. Otherwise, let $h_{\varphi,\mathsf{x}}(b_1, b_2, \dots, b_k) = d$.

We now let

$$B = X \cup \{d\} \cup \{\mathsf{c}^{\mathcal{M}} : \mathsf{c} \in \mathcal{C}\} \cup \{n_{\varphi,\mathsf{x}} : \mathsf{x} \in Var, \varphi \in Form_{\mathcal{L}}, \text{ and } FreeVar(\varphi) = \{\mathsf{x}\}\}$$

and we let

$$A = G(M, B, \{\mathsf{f}^{\mathcal{M}} : \mathsf{f} \in \mathcal{F}_k\} \cup \{h_{\varphi,\mathsf{x}} : \varphi \in Form_P, \mathsf{x} \in Var\})$$

We then have that A is the universe of a substructure $\mathcal{A}$ of $\mathcal{M}$. Notice that $X \subseteq A$ and that by a problem on Homework 1, we have that $\mathcal{A}$ is countable. Thus, we need only show that $\mathcal{A} \preceq \mathcal{M}$, which we do by the Tarski-Vaught test. Suppose that $\varphi \in Form_{\mathcal{L}}$, $\mathsf{x} \in Var$, and $s\colon Var \to A$ are such that $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$.

Suppose first that $\mathsf{x} \notin FreeVar(\varphi)$. Since $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$, we may fix $m \in \mathcal{M}$ such that $(\mathcal{M}, s[\mathsf{x} \Rightarrow m]) \vDash \varphi$. Now using the fact that $\mathsf{x} \notin FreeVar(\varphi)$, it follows that $(\mathcal{M}, s[\mathsf{x} \Rightarrow d]) \vDash \varphi$.

Suppose now that $FreeVar(\varphi) = \{\mathsf{x}\}$, and let $a = n_{\varphi,\mathsf{x}} \in A$. Since $\mathcal{M} \vDash \exists \mathsf{x}\varphi$, hence $(\mathcal{M}, a) \vDash \varphi$ by definition of $n_{\varphi,\mathsf{x}}$. It follows that there exists $a \in A$ such that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$.

Finally, suppose that $FreeVar(\varphi) = \{\mathsf{y}_1, \mathsf{y}_2, \dots, \mathsf{y}_k, \mathsf{x}\}$. For each i with $1 \le i \le k$, let $b_i = s(\mathsf{y}_i)$, and let $a = h_{\varphi,\mathsf{x}}(b_1, b_2, \dots, b_k) \in A$. Since $(\mathcal{M}, b_1, b_2, \dots, b_k) \vDash \exists \mathsf{x}\varphi$, hence $(\mathcal{M}, b_1, b_2, \dots, b_k, a) \vDash \varphi$ by definition of $h_{\varphi,\mathsf{x}}$. It follows that there exists $a \in A$ such that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$. □

Corollary 4.3.18. *Suppose that $\mathcal{L}$ is countable and that $\mathcal{M}$ is an $\mathcal{L}$-structure. There exists a countable $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \equiv \mathcal{M}$.*

Proof. Let $\mathcal{N}$ be a countable elementary substructure of $\mathcal{M}$. For any $\sigma \in Sent_{\mathcal{L}}$, we then have that $\mathcal{N} \vDash \sigma$ if and only if $\mathcal{M} \vDash \sigma$, so $\mathcal{N} \equiv \mathcal{M}$. □

This is our first indication that first-order logic is not powerful enough to distinguish certain aspects of cardinality, and we'll see more examples of this phenomenon after the Compactness Theorem (for first-order logic) and once we talk about infinite cardinalities and extend the Lowenheim-Skolem-Tarski result.

This restriction already has some interesting consequences. For example, you may be familiar with the result that $(\mathbb{R}, 0, 1, <, +, \cdot)$ is the unique (up to isomorphism) Dedekind-complete ordered field.

Corollary 4.3.19. *The Dedekind-complete ordered fields are not a weak elementary class in the language $\mathcal{L} = \{0, 1, <, +, \cdot\}$.*

Proof. Let $\mathcal{K}$ be the class of all Dedekind-complete ordered fields. Suppose that $\Sigma \subseteq Sent_{\mathcal{L}}$ is such that $\mathcal{K} = Mod(\Sigma)$. By the Countable Lowenheim-Skolem-Tarski Theorem, there exists a countable $\mathcal{N}$ such that $\mathcal{N} \equiv (\mathbb{R}, 0, 1, <, +, \cdot)$. Since $(\mathbb{R}, 0, 1, <, +, \cdot) \in \mathcal{K}$, we have $(\mathbb{R}, 0, 1, <, +, \cdot) \vDash \sigma$ for all $\sigma \in \Sigma$, so $\mathcal{N} \vDash \sigma$ for all $\sigma \in \Sigma$, and hence $\mathcal{N} \in \mathcal{K}$. However, this is a contradiction because all Dedekind-complete ordered fields are isomorphic to $(\mathbb{R}, 0, 1, <, +, \cdot)$, hence are uncountable. □

Definition 4.3.20. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Suppose that $h \colon N \to M$. We say that h is an* elementary embedding *if h is an embedding and for all $\varphi \in Form_{\mathcal{L}}$ and all $s \colon Var \to N$, we have*

$$(\mathcal{N}, s) \vDash \varphi \text{ if and only if } (\mathcal{M}, h \circ s) \vDash \varphi$$

4.4 Changing the Language

4.4.1 Expansions and Restrictions

Definition 4.4.1. *Let $\mathcal{L} \subseteq \mathcal{L}'$ be languages, let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $\mathcal{M}'$ be an $\mathcal{L}'$-structure. We say that $\mathcal{M}$ is the* restriction *of $\mathcal{M}'$ to $\mathcal{L}$, and that $\mathcal{M}'$ is an* expansion *of $\mathcal{M}$ to $\mathcal{L}'$, if*

- $M = M'$.
- $\mathsf{c}^{\mathcal{M}} = \mathsf{c}^{\mathcal{M}'}$ *for all* $\mathsf{c} \in \mathcal{C}$.
- $\mathsf{R}^{\mathcal{M}} = \mathsf{R}^{\mathcal{M}'}$ *for all* $\mathsf{R} \in \mathcal{R}$.
- $\mathsf{f}^{\mathcal{M}} = \mathsf{f}^{\mathcal{M}'}$ *for all* $\mathsf{f} \in \mathcal{F}$.

Proposition 4.4.2. *Let $\mathcal{L} \subseteq \mathcal{L}'$ be languages, let $\mathcal{M}'$ be an $\mathcal{L}'$-structure, and let $\mathcal{M}$ be the restriction of $\mathcal{M}'$ to $\mathcal{L}$. For all $\varphi \in Form_{\mathcal{L}}$ and all $s \colon Var \to M$, we have $(\mathcal{M}, s) \vDash \varphi$ if and only if $(\mathcal{M}', s) \vDash \varphi$.*

Proof. By induction. □

4.4.2 Adding Constants to Name Elements

Definition 4.4.3. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. For each $a \in M$, introduce a new constant c_a (not appearing in the original language $\mathcal{L}$ and all distinct). Let $\mathcal{L}_{\mathcal{M}} = \mathcal{L} \cup \{\mathsf{c}_a : a \in M\}$ and let $\mathcal{M}_{exp}$ be the $\mathcal{L}_{\mathcal{M}}$-structure which is the expansion of $\mathcal{M}$ in which $\mathsf{c}_a^{\mathcal{M}_{exp}} = a$ for all $a \in M$. We call $\mathcal{M}_{exp}$ the* expansion of $\mathcal{M}$ obtained by adding names for elements of M.

Definition 4.4.4. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $s\colon Var \to M$ be a variable assignment. Define a function $Name_s\colon Term_{\mathcal{L}} \to Term_{\mathcal{L}_{\mathcal{M}}}$ by plugging in names for free variables according to s. Define a function $Name_s\colon Form_{\mathcal{L}} \to Sent_{\mathcal{L}_{\mathcal{M}}}$ again by plugging in names for free variables according to s.*

Proposition 4.4.5. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $s\colon Var \to M$ be a variable assignment. For every $\varphi \in Form_{\mathcal{L}}$, we have*

$$(\mathcal{M}, s) \vDash \varphi \textit{ if and only if } \mathcal{M}_{exp} \vDash Name_s(\varphi)$$

Definition 4.4.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure.*

- *We let $AtomicDiag(\mathcal{M}) = \{\sigma \in Sent_{\mathcal{L}_{\mathcal{M}}} \cap AtomicForm_{\mathcal{L}_{\mathcal{M}}} : \mathcal{M}_{exp} \vDash \sigma\}$.*
- *We let $Diag(\mathcal{M}) = \{\sigma \in Sent_{\mathcal{L}_{\mathcal{M}}} : \mathcal{M}_{exp} \vDash \sigma\}$.*

Proposition 4.4.7. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be the $\mathcal{L}$-structures. The following are equivalent:*

- *There exists an embedding h from $\mathcal{M}$ to $\mathcal{N}$.*
- *There exists an expansion of $\mathcal{N}$ to an $\mathcal{L}_{\mathcal{M}}$-structure which is a model of $AtomicDiag(\mathcal{M})$.*

Proposition 4.4.8. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be the $\mathcal{L}$-structures. The following are equivalent:*

- *There exists an elementary embedding h from $\mathcal{M}$ to $\mathcal{N}$.*
- *There exists an expansion of $\mathcal{N}$ to an $\mathcal{L}_{\mathcal{M}}$-structure which is a model of $Diag(\mathcal{M})$.*

Chapter 5

Semantic and Syntactic Implication

5.1 Semantic Implication and Theories

5.1.1 Definitions

Definition 5.1.1. *Let $\mathcal{L}$ be a language and let $\Gamma \subseteq Form_{\mathcal{L}}$. A* model of Γ *is a pair $(\mathcal{M}, s)$ where*

- *$\mathcal{M}$ is an $\mathcal{L}$-structure.*
- *$s\colon Var \to M$ is a variable assignment.*
- *$(\mathcal{M}, s) \vDash \gamma$ for all $\gamma \in \Gamma$.*

Definition 5.1.2. *Let $\mathcal{L}$ be a language. Let $\Gamma \subseteq Form_{\mathcal{L}}$ and let $\varphi \in Form_{\mathcal{L}}$. We write $\Gamma \vDash \varphi$ to mean that whenever $(\mathcal{M}, s)$ is a model of Γ, we have that $(\mathcal{M}, s) \vDash \varphi$. We pronounce $\Gamma \vDash \varphi$ as Γ* semantically implies *φ.*

Definition 5.1.3. *Let $\mathcal{L}$ be a language and let $\Gamma \subseteq Form_{\mathcal{L}}$. We say that Γ is* satisfiable *if there exists a model of Γ.*

Definition 5.1.4. *Let $\mathcal{L}$ be a language. A set $\Sigma \subseteq Sent_{\mathcal{L}}$ is an $\mathcal{L}$-*theory *if*

- *Σ is a satisfiable.*
- *For every $\tau \in Sent_{\mathcal{L}}$ with $\Sigma \vDash \tau$, we have $\tau \in \Sigma$.*

There are two standard ways to get theories. One is to take a stucture, and consider all of the sentences that are true in that structure.

Definition 5.1.5. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We let $Th(\mathcal{M}) = \{\sigma \in Sent_{\mathcal{L}} : \mathcal{M} \vDash \sigma\}$. We call $Th(\mathcal{M})$ the* theory of *$\mathcal{M}$.*

Proposition 5.1.6. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. $Th(\mathcal{M})$ is an $\mathcal{L}$-theory.*

Proof. First notice that $Th(\mathcal{M})$ is satisfiable because $\mathcal{M}$ is a model of $Th(\mathcal{M})$ (since $\mathcal{M} \vDash \sigma$ for all $\sigma \in Th(\mathcal{M})$ by definition. Suppose now that $\tau \in Sent_{\mathcal{L}}$ is such that $Th(\mathcal{M}) \vDash \tau$. Since $\mathcal{M}$ is a model of $Th(\mathcal{M})$, it follows that $\mathcal{M} \vDash \tau$, and hence $\tau \in Th(\mathcal{M})$. □

Another standard way to get a theory is to take an arbitrary satisfiable set of sentences, and close it off under semantic implication.

Definition 5.1.7. *Let $\mathcal{L}$ be a language and let $\Sigma \subseteq Sent_{\mathcal{L}}$. We let $Cn(\Sigma) = \{\tau \in Sent_{\mathcal{L}} : \Sigma \vDash \tau\}$. We call $Cn(\Sigma)$ the set of* consequences *of Σ.*

Proposition 5.1.8. *Let $\mathcal{L}$ be a language and let $\Sigma \subseteq Sent_{\mathcal{L}}$ be satisfiable. We then have that $Cn(\Sigma)$ is an $\mathcal{L}$-theory.*

Proof. We first show that $Cn(\Sigma)$ is satisfiable. Since Σ is satsfiable, we may fix a model $\mathcal{M}$ of Σ. Let $\tau \in Cn(\Sigma)$. We then have that $\Sigma \vDash \tau$, so using the fact that $\mathcal{M}$ is a model of Σ we conclude that $\mathcal{M} \vDash \tau$. Therefore, $\mathcal{M}$ is a model of $Cn(\Sigma)$, hence $Cn(\Sigma)$ is satisfiable.

Suppose now that $\tau \in Sent_{\mathcal{L}}$ and that $Cn(\Sigma) \vDash \tau$. We need to show that $\tau \in Cn(\Sigma)$, i.e. that $\Sigma \vDash \tau$. Let $\mathcal{M}$ be a model of Σ. Since $\Sigma \vDash \sigma$ for all $\sigma \in Cn(\Sigma)$, it follows that $\mathcal{M} \vDash \sigma$ for all $\sigma \in Cn(\Sigma)$. Thus, $\mathcal{M}$ is a model of $Cn(\Sigma)$. Since $Cn(\Sigma) \vDash \tau$, it follows that $\mathcal{M} \vDash \tau$. Thus, $\Sigma \vDash \tau$, and so $\tau \in Cn(\Sigma)$. □

Definition 5.1.9. *An $\mathcal{L}$-theory Σ is* complete *if for all $\tau \in Sent_{\mathcal{L}}$, either $\tau \in \Sigma$ or $\neg\tau \in \Sigma$.*

Proposition 5.1.10. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-theory. $Th(\mathcal{M})$ is a complete $\mathcal{L}$-theory.*

Proof. We've already seen that $Th(\mathcal{M})$ is a theory. Suppose now that $\sigma \in Sent_{\mathcal{L}}$. If $\mathcal{M} \vDash \sigma$, we then have that $\sigma \in Th(\mathcal{M})$. Otherwise, we have $\mathcal{M} \not\vDash \sigma$, so by definition $\mathcal{M} \vDash \neg\sigma$, and hence $\neg\sigma \in Th(\mathcal{M})$. □

Example. Let $\mathcal{L} = \{\mathsf{f}, \mathsf{e}\}$ where f is a binary function symbol and e is a constant symbol. Consider the following sentences.

$$\varphi_1 = \forall x \forall y \forall z (f(f(x, y), z) = f(x, f(x, y)))$$
$$\varphi_2 = \forall x (f(x, e) = x \wedge f(e, x) = x)$$
$$\varphi_3 = \forall x \exists y (f(x, y) = e \wedge f(y, x) = e)$$

The theory $T = Cn(\{\varphi_1, \varphi_2, \varphi_3\})$ is the theory of groups. T is not complete because it neither contains $\forall x \forall y (f(x, y) = f(y, x))$ nor its negation because there are both abelian groups and nonabelian groups. □

Definition 5.1.11. *Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Consider the following sentences*

$$\varphi_1 = \forall x \neg Rxx$$
$$\varphi_2 = \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\varphi_3 = \forall x \forall y \neg (Rxy \wedge Ryx)$$
$$\varphi_4 = \forall x \forall y (x = y \vee Rxy \vee Ryx)$$

and let $LO = Cn(\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\})$. LO is called the theory of (strict) linear orderings. LO is not complete because it neither contains $\exists y \forall x (x = y \vee x < y)$ nor its negation because there are linear ordering with greatest elements and linear orderings without greatest elements.

Definition 5.1.12. *Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Consider the following sentences*

$$\varphi_1 = \forall x \neg Rxx$$
$$\varphi_2 = \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\varphi_3 = \forall x \forall y \neg (Rxy \wedge Ryx)$$
$$\varphi_4 = \forall x \forall y (x = y \vee Rxy \vee Ryx)$$
$$\varphi_5 = \forall x \forall y (Rxy \rightarrow \exists z (Rxz \wedge Rzy))$$
$$\varphi_6 = \forall x \exists y Rxy$$
$$\varphi_7 = \forall x \exists y Ryx$$

and let $DLO = Cn(\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7\})$. DLO is called the theory of dense (strict) linear orderings without endpoints. DLO is complete as we'll see below.

Theorem 5.1.13 (Countable Lowenheim-Skolem Theorem)**.** *Suppose that $\mathcal{L}$ is countable and that $\Gamma \subseteq Form_{\mathcal{L}}$ is satisfiable. There exists a countable model $(\mathcal{M}, s)$ of Γ.*

Proof. Since Γ is satisfiable, we may fix a model $(\mathcal{N}, s)$ of Γ. Let $X = \text{ran}(s) \subseteq N$ and notice that X is countable. By the Countable Lowenheim-Skolem-Tarski Theorem, there exists a countable elementary substructure $\mathcal{M} \preceq \mathcal{N}$ such that $X \subseteq M$. Notice that s is also a variable assigment on M. Now for any $\gamma \in \Gamma$, we have that $(\mathcal{N}, s) \vDash \gamma$ because $(\mathcal{N}, s)$ is a model of Γ, hence $(\mathcal{M}, s) \vDash \gamma$ because $\mathcal{M} \preceq \mathcal{N}$. It follows that $(\mathcal{M}, s)$ is a model of Γ. □

5.1.2 Finite Models of Theories

Given a theory T and an $n \in \mathbb{N}^+$, we want to count the number of models of T of cardinality n up to isomorphism. There are some technical set-theoretic difficulties here which will be elaborated upon later, but the key fact that limits the number of isomorphism classes is the following result.

Proposition 5.1.14. *Let $\mathcal{L}$ be a language and let $n \in \mathbb{N}^+$. For every $\mathcal{L}$-structure $\mathcal{M}$ with $|M| = n$, there exists an $\mathcal{L}$-stucture $\mathcal{N}$ with $N = [n]$ such that $\mathcal{M} \cong \mathcal{N}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{L}$-structure with $|M| = n$. Fix a bijection $h \colon M \to [n]$. Let $\mathcal{N}$ be the $\mathcal{L}$-structure where

- $N = [n]$.
- $\mathsf{c}^{\mathcal{N}} = h(\mathsf{c}^{\mathcal{M}})$ for all $\mathsf{c} \in \mathcal{C}$.
- $\mathsf{R}^{\mathcal{N}} = \{(b_1, b_2, \dots, b_k) \in N^k : (h^{-1}(b_1), h^{-1}(b_2), \dots, h^{-1}(b_k)) \in \mathsf{R}^{\mathcal{N}}\}$ for all $\mathsf{R} \in \mathcal{R}_k$.
- $\mathsf{f}^{\mathcal{N}}$ is the function from N^k to N defined by $\mathsf{f}^{\mathcal{N}}(b_1, b_2, \dots, b_k) = h(\mathsf{f}^{\mathcal{M}}(h^{-1}(b_1), h^{-1}(b_2), \dots, h^{-1}(b_k)))$ for all $\mathsf{f} \in \mathcal{F}^k$.

We then have that h is an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. □

Proposition 5.1.15. *If $\mathcal{L}$ is finite and $n \in \mathbb{N}^+$, then there are only finitely many $\mathcal{L}$-structures with universe $[n]$.*

Definition 5.1.16. *Let $\mathcal{L}$ be a finite language and let T be an $\mathcal{L}$-theory. For each $n \in \mathbb{N}^+$, let $I(T, n)$ be the number of models of T of cardinality n up to isomorphism. Formally, we consider the set of all $\mathcal{L}$-structures with universe $[n]$, and count the number of equivalence classes under the equivalence relation of isomorphism.*

Example 5.1.17. *If T is the theory of groups, then $I(T, n)$ is a very interesting function that you study in algebra courses. For example, you show that $I(T, p) = 1$ for all primes p, that $I(T, 6) = 2$, and that $I(T, 8) = 5$.*

Example 5.1.18. *Let $\mathcal{L} = \emptyset$ and let $T = Cn(\emptyset)$. We have $I(T, n) = 1$ for all $n \in \mathbb{N}^+$.*

Proof. First notice that for every $n \in \mathbb{N}^+$, the $\mathcal{L}$-structure $\mathcal{M}$ with universe $[n]$ is a model of T of cardinality n, so $I(T, n) \geq 1$. Now notice that if $\mathcal{M}$ and $\mathcal{N}$ are models of T of cardinality n, then any bijection $h \colon M \to N$ is an isomorphism (because $\mathcal{L} = \emptyset$), so $I(T, n) \leq 1$. It follows that $I(T, n) = 1$ for all $n \in \mathbb{N}$. □

Example 5.1.19. *$I(LO, n) = 1$ for all $n \in \mathbb{N}^+$.*

Proof. First notice that for every $n \in \mathbb{N}$, the $\mathcal{L}$-strucutre $\mathcal{M}$ where $M = [n]$ and $\mathsf{R}^{\mathcal{M}} = \{(k, \ell) \in [n]^2 : k < \ell\}$ is a model of LO of cardinality n, so $I(LO, n) \geq 1$. Next notice that any two linear orderings of cardinality n are isomorphic. Intuitively, this works as follows. Notice (by induction on the number of elements) that every finite linear ordering has a least element. Let $\mathcal{M}$ and $\mathcal{N}$ be two linear orderings of cardinality n. Each must have a least element, so map the least element of $\mathcal{M}$ to that of $\mathcal{N}$. Remove these elements, then map the least element remaining in $\mathcal{M}$ to the least element remaining in $\mathcal{N}$, and continue. This gives an isomorphism. Formally, you can turn this into a proof by induction on n. □

Example 5.1.20. *Let* $\mathcal{L} = \{\mathsf{f}\}$ *where* f *is a unary function symbol, and let* $T = Cn(\{\forall \mathsf{x}(\mathsf{ffx} = \mathsf{x})\})$. *We have* $I(T, n) = \lfloor \frac{n}{2} \rfloor + 1$ *for all* $n \in \mathbb{N}^+$.

Proof. Let's first analyze the finite models of T. Suppose that $\mathcal{M}$ is a model of T of cardinality n. For every $a \in M$, we then have $\mathsf{f}^{\mathcal{M}}(\mathsf{f}^{\mathcal{M}}(a)) = a$. There are now two cases. Either $\mathsf{f}^{\mathcal{M}}(a) = a$, or $\mathsf{f}^{\mathcal{M}}(a) = b \neq a$ in which case $\mathsf{f}^{\mathcal{M}}(b) = a$. Let

- $Fix_{\mathcal{M}} = \{a \in M : f^{\mathcal{M}}(a) = a\}$.
- $Move_{\mathcal{M}} = \{a \in M : f^{\mathcal{M}}(a) \neq a\}$.

From above, we then have that $|Move_{\mathcal{M}}|$ is even and that $|Fix_{\mathcal{M}}| + |Move_{\mathcal{M}}| = n$. Now the idea is that two models $\mathcal{M}$ and $\mathcal{N}$ of T of cardinality n are isomorphic if and only if they have the same number of fixed points, because then we can match up the fixed points and then match up the "pairings" left over to get an isomorphism. Here's a more formal argument.

We know show that if $\mathcal{M}$ and $\mathcal{N}$ are models of T of cardinality n, then $\mathcal{M} \cong \mathcal{N}$ if and only if $|Fix_{\mathcal{M}}| = |Fix_{\mathcal{N}}|$. Clearly, if $\mathcal{M} \cong \mathcal{N}$, then $|Fix_{\mathcal{M}}| = |Fix_{\mathcal{N}}|$. Suppose conversely that $|Fix_{\mathcal{M}}| = |Fix_{\mathcal{N}}|$. We then must have $|Move_{\mathcal{M}}| = |Move_{\mathcal{N}}|$. Let $X_{\mathcal{M}} \subseteq Move_{\mathcal{M}}$ be a set of cardinality $\frac{|Move_{\mathcal{M}}|}{2}$ such that $f^{\mathcal{M}}(x) \neq y$ for all $x, y \in X$ (that is, we pick out one member from each pairing given by $f^{\mathcal{M}}$), and let $X_{\mathcal{N}}$ be such a set for $\mathcal{N}$. Define a function $h \colon M \to N$. Fix a bijection from $\alpha \colon Fix_{\mathcal{M}} \to Fix_{\mathcal{N}}$ and a bijection $\beta \colon X_{\mathcal{M}} \to X_{\mathcal{N}}$. Define h by letting $h(a) = \alpha(a)$ for all $a \in Fix_{\mathcal{M}}$, letting $h(x) = \beta(x)$ for all $x \in X_{\mathcal{M}}$, and letting $h(y) = f^{\mathcal{N}}(\beta(f^{\mathcal{M}}(y)))$ for all $y \in Move_{\mathcal{M}} \backslash X$. We then have that h is an isomophism from $\mathcal{M}$ to $\mathcal{N}$.

Now we need only count how many possible values there are for $|Fix_{\mathcal{M}}|$. Let $n \in \mathbb{N}^+$. Suppose first that n is even. Since $|Move_{\mathcal{M}}|$ must be even, it follows that $|Fix_{\mathcal{M}}|$ must be even. Thus, $|Fix_{\mathcal{M}}| \in \{0, 2, 4, \dots, n\}$, so there are $\frac{n}{2} + 1$ many possibilities, and it's easy to construct models in which each of these possibilities occurs. Suppose now that n is odd. Since $|Move_{\mathcal{M}}|$ must be even, it follows that $|Fix_{\mathcal{M}}|$ must be odd. Thus, $|Fix_{\mathcal{M}}| \in \{1, 3, 5, \dots, n\}$, so there are $\frac{n-1}{2} + 1$ many possibilities, and it's easy to construct models in which each of these possibilities occurs. Thus, in either case, we have $I(T, n) = \lfloor \frac{n}{2} \rfloor + 1$. □

Example 5.1.21. $I(DLO, n) = 0$ *for all* $n \in \mathbb{N}^+$.

Proof. As mentioned in the LO example, every finite linear ordering has a least element. □

Definition 5.1.22. *Suppose that* $\mathcal{L}$ *is a finite language and* $\sigma \in Sent_{\mathcal{L}}$. *Let*

$$Spec(\sigma) = \{n \in \mathbb{N}^+ : I(Cn(\sigma), n) > 0\}$$

Proposition 5.1.23. *There exists a finite language* $\mathcal{L}$ *and a* $\sigma \in Sent_{\mathcal{L}}$ *such that* $Spec(\sigma) = \{2n : n \in \mathbb{N}^+\}$.

Proof. We'll give two separate arguments. First, let $\mathcal{L} = \{\mathsf{e}, \mathsf{f}\}$ be the language of group theory. Let σ be the conjunction of the group axioms with the sentence $\exists \mathsf{x}(\neg(\mathsf{x} = \mathsf{e}) \wedge \mathsf{fxx} = \mathsf{e})$ expressing that there is an element of order 2. Now for every $n \in \mathbb{N}^+$, the group $\mathbb{Z}/(2n)\mathbb{Z}$ is a model of σ of cardinality $2n$ because $\overline{n}$ is an element of order 2. Thus, $\{2n : n \in \mathbb{N}^+\} \subseteq Spec(\sigma)$. Suppose now that $k \in Spec(\sigma)$, and fix a model $\mathcal{M}$ of σ of order k. We then have that $\mathcal{M}$ is a group with an element of order 2, so by Lagrange's Theorem it follows that $2 \mid k$, so $k \in \{2n : n \in \mathbb{N}^+\}$. It follows that $Spec(\sigma) = \{2n : n \in \mathbb{N}^+\}$.

For a second example, let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Let σ be the conjunction of the following sentences:

- $\forall \mathsf{x} \mathsf{Rxx}$.
- $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \to \mathsf{Ryx})$.
- $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}((\mathsf{Rxy} \wedge \mathsf{Ryz}) \to \mathsf{Rxz})$.

- $\forall x \exists y(\neg(y = x) \wedge Rxy \wedge \forall z(Rxz \rightarrow (z = x \vee z = y)))$.

Notice that a model of σ is simply an equivalence relation in which every equivalence class has exactly 2 elements. It is now straightforward to show that $Spec(\sigma) = \{2n : n \in \mathbb{N}^+\}$. ☐

Proposition 5.1.24. *There exists a finite language $\mathcal{L}$ and a $\sigma \in Sent_{\mathcal{L}}$ such that $Spec(\sigma) = \{2^n : n \in \mathbb{N}^+\}$.*

Proof. Again, let's give two separate arguments. First, let $\mathcal{L} = \{e, f\}$ be the language of group theory. Let σ be the conjunction of the group axioms with the sentences $\exists x \neg(x = e)$ and $\forall x(fxx = e)$ expressing that the group is nontrivial and that there every nonidentity element has order 2. Now for every $n \in \mathbb{N}^+$, the group $(\mathbb{Z}/2\mathbb{Z})^n$ is a model of σ of cardinality 2^n. Thus, $\{2^n : n \in \mathbb{N}^+\} \subseteq Spec(\sigma)$. Suppose now that $k \in Spec(\sigma)$, and fix a model $\mathcal{M}$ of σ of order k. We then have that $k > 1$ and that $\mathcal{M}$ is a group such that every nonidentity element has order 2. Now for any prime $p \neq 2$, it is not the case that p divides k because otherwise $\mathcal{M}$ would have to have an element of order p by Cauchy's Theorem. Thus, the only prime that divides k is 2, and so $k \in \{2n : n \in \mathbb{N}^+\}$. It follows that $Spec(\sigma) = \{2^n : n \in \mathbb{N}^+\}$.

For a second example, let $\mathcal{L} = \{0, 1, +, \cdot\}$ be the language where $0, 1$ are constant symbols and $+, \cdot$ are binary function symbols. Let σ be the conjunction of the field axioms together with $1 + 1 = 0$. Thus, the models of σ are exactly the fields of characteristic 2. By results in algebra, there is a finite field of characteristic 2 of order k if and only if k is a power of 2. ☐

5.1.3 Countable Models of Theories

Theorem 5.1.25. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two countably infinite models of DLO. We then have that $\mathcal{M} \cong \mathcal{N}$.*

Proof. Back-and-forth construction. See Damir's carefully written proof. ☐

Corollary 5.1.26 (Countable Los-Vaught Test)**.** *Let $\mathcal{L}$ be a countable language. Suppose that T is an $\mathcal{L}$-theory such that all models of T are infinite, and suppose also that every two countably infinite models of T are isomorphic. We then have that T is complete.*

Proof. Suppose that T is not complete and fix $\sigma \in Sent_{\mathcal{L}}$ such that $\sigma \notin T$ and $\neg\sigma \notin T$. We then have that $T \cup \{\sigma\}$ and $T \cup \{\neg\sigma\}$ are both satisfiable by infinite models (because all models of T are infinite), so by the Countable Lowenheim-Skolem Theorem we may fix countably infinite models $\mathcal{M}_1$ of $T \cup \{\sigma\}$ and $\mathcal{M}_2$ of $T \cup \{\neg\sigma\}$. We then have that $\mathcal{M}_1$ and $\mathcal{M}_2$ are countably infinite models of T which are not isomorphic (because they are not elementarily equivalent), a contradiction. ☐

Corollary 5.1.27. *DLO is complete.*

Proposition 5.1.28. *Suppose that T is a complete $\mathcal{L}$-theory. If $\mathcal{M}$ and $\mathcal{N}$ are models of T, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $\sigma \in Sent_{\mathcal{L}}$. If $\sigma \in T$, we then have that both $\mathcal{M} \vDash \sigma$ and $\mathcal{N} \vDash \sigma$. Suppose that $\sigma \notin T$. Since T is complete, we then have that $\neg\sigma \in T$, hence $\mathcal{M} \vDash \neg\sigma$ and $\mathcal{N} \vDash \neg\sigma$. It follows that both $\mathcal{M} \nvDash \sigma$ and $\mathcal{N} \nvDash \sigma$. Therefore, for all $\sigma \in Sent_{\mathcal{L}}$, we have that $\mathcal{M} \vDash \sigma$ if and only if $\mathcal{N} \vDash \sigma$, so $\mathcal{M} \equiv \mathcal{N}$. ☐

Corollary 5.1.29. *In the language $\mathcal{L} = \{R\}$ where R is a binary relation symbol, we have $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$.*

5.2 Syntactic Implication

5.2.1 Definitions

Basic Proofs:

- $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ $\quad$ $(Assume_{\mathcal{L}})$

- $\Gamma \vdash t = t$ for all $t \in Term_{\mathcal{L}}$ $\quad (EqRefl)$

Proof Rules:

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \ (\wedge EL) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \ (\wedge ER) \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \ (\wedge I)$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \ (\vee IL) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \ (\vee IR)$$

$$\frac{\Gamma \vdash \varphi \to \psi}{\Gamma \cup \{\varphi\} \vdash \psi} \ (\to E) \qquad \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to I)$$

$$\frac{\Gamma \cup \{\varphi\} \vdash \theta \quad \Gamma \cup \{\psi\} \vdash \theta}{\Gamma \cup \{\varphi \vee \psi\} \vdash \theta} \ (\vee PC) \qquad \frac{\Gamma \cup \{\psi\} \vdash \varphi \quad \Gamma \cup \{\neg\psi\} \vdash \varphi}{\Gamma \vdash \varphi} \ (\neg PC)$$

$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \psi \quad \Gamma \cup \{\neg\varphi\} \vdash \neg\psi}{\Gamma \vdash \varphi} \ (Contr)$$

Equality Rules:

$$\frac{\Gamma \vdash \varphi^t_{\mathsf{x}} \quad \Gamma \vdash t = u}{\Gamma \vdash \varphi^u_{\mathsf{x}}} \quad \text{if } ValidSubst^t_{\mathsf{x}}(\varphi) = 1 = ValidSubst^u_{\mathsf{x}}(\varphi) \quad (= Sub)$$

Existential Rules:

$$\frac{\Gamma \vdash \varphi^t_{\mathsf{x}}}{\Gamma \vdash \exists \mathsf{x} \varphi} \quad \text{if } ValidSubst^t_{\mathsf{x}}(\varphi) = 1 \quad (\exists I)$$

$$\frac{\Gamma \cup \{\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vdash \psi}{\Gamma \cup \{\exists \mathsf{x} \varphi\} \vdash \psi} \quad \text{if } \mathsf{y} \notin FreeVar(\Gamma \cup \{\exists \mathsf{x}\varphi, \psi\}) \text{ and } ValidSubst^{\mathsf{y}}_{\mathsf{x}}(\varphi) = 1 \quad (\exists P)$$

Universal Rules:

$$\frac{\Gamma \vdash \forall \mathsf{x} \varphi}{\Gamma \vdash \varphi^t_{\mathsf{x}}} \quad \text{if } ValidSubst^t_{\mathsf{x}}(\varphi) = 1 \quad (\forall E)$$

$$\frac{\Gamma \vdash \varphi^{\mathsf{y}}_{\mathsf{x}}}{\Gamma \vdash \forall \mathsf{x} \varphi} \quad \text{if } \mathsf{y} \notin FreeVar(\Gamma \cup \{\forall \mathsf{x}\varphi\}) \text{ and } ValidSubst^{\mathsf{y}}_{\mathsf{x}}(\varphi) = 1 \quad (\forall I)$$

Superset Rule:

$$\frac{\Gamma \vdash \varphi}{\Gamma' \vdash \varphi} \quad \text{if } \Gamma' \supseteq \Gamma \quad (Super)$$

Definition 5.2.1. *A* deduction *is a witnessing sequence in* $(\mathcal{P}(Form_{\mathcal{L}}) \times Form_{\mathcal{L}}, Assume_{\mathcal{L}} \cup EqRefl, \mathcal{H})$.

Definition 5.2.2. *Let* $\Gamma \subseteq Form_P$ *and let* $\varphi \in Form_P$. *We write* $\Gamma \vdash \varphi$ *to mean that*

$$(\Gamma, \varphi) \in (\mathcal{P}(Form_{\mathcal{L}}) \times Form_{\mathcal{L}}, Assume_{\mathcal{L}} \cup EqRefl, \mathcal{H})$$

We pronounce $\Gamma \vdash \varphi$ *as "*Γ *syntactically implies* φ*".*

Notation 5.2.3.

1. *If* $\Gamma = \emptyset$, *we write* $\vdash \varphi$ *instead of* $\emptyset \vdash \varphi$.
2. *If* $\Gamma = \{\gamma\}$, *we write* $\gamma \vdash \varphi$ *instead of* $\{\gamma\} \vdash \varphi$.

Definition 5.2.4. Γ *is* inconsistent *if there exists* $\theta \in Form_P$ *such that* $\Gamma \vdash \theta$ *and* $\Gamma \vdash \neg\theta$. *Otherwise, we say that* Γ *is* consistent.

5.2.2 Some Fundamental Deductions

Proposition 5.2.5. *For any $t, u \in Term_{\mathcal{L}}$, we have $t = u \vdash u = t$.*

Proof. Fix $t, u \in Term_{\mathcal{L}}$.

$$\begin{array}{lr r}
\{t = u\} \vdash t = t & (EqRefl) & (1) \\
\{t = u\} \vdash t = u & (Assume_{\mathcal{L}}) & (2) \\
\{t = u\} \vdash u = t & (= Sub \text{ on 1 and 2 with } \mathsf{x} = t) & (3)
\end{array}$$

□

Proposition 5.2.6. *For any $t, u, w \in Term_{\mathcal{L}}$, we have $\{t = u, u = w\} \vdash t = w$.*

Proof. Fix $t, u, w \in Term_{\mathcal{L}}$.

$$\begin{array}{lr r}
\{t = u, u = w\} \vdash t = u & (Assume_{\mathcal{L}}) & (1) \\
\{t = u, u = w\} \vdash u = w & (Assume_{\mathcal{L}}) & (2) \\
\{t = u, u = w\} \vdash t = w & (= Sub \text{ on 1 and 2 with } t = \mathsf{x}) & (3)
\end{array}$$

□

Proposition 5.2.7. *For any $\mathsf{R} \in \mathcal{R}_k$ and any $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$, we have*

$$\{\mathsf{R}t_1t_2\cdots t_k, t_1 = u_1, t_2 = u_2, \dots, t_k = u_k\} \vdash \mathsf{R}u_1u_2\cdots u_k$$

Proof. Fix $\mathsf{R} \in \mathcal{R}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. Fix $\mathsf{x} \notin \bigcup_{i=1}^{k}(OccurVar(t_i) \cup OccurVar(u_i))$. Let $\Gamma = \{\mathsf{R}t_1t_2\cdots t_k, t_1 = u_1, t_2 = u_2, \dots, t_k = u_k\}$. We have

$$\begin{array}{lr r}
\Gamma \vdash \mathsf{R}t_1t_2\cdots t_k & (Assume_{\mathcal{L}}) & (1) \\
\Gamma \vdash t_1 = u_1 & (Assume_{\mathcal{L}}) & (2) \\
\Gamma \vdash \mathsf{R}u_1t_2t_3\cdots t_k & (= Sub \text{ on 1 and 2 with } \mathsf{R}\mathsf{x}t_2t_3\cdots t_k) & (3) \\
\Gamma \vdash t_2 = u_2 & (Assume_{\mathcal{L}}) & (4) \\
\Gamma \vdash \mathsf{R}u_1u_2t_3\cdots t_k & (= Sub \text{ on 3 and 4 with } \mathsf{R}u_1\mathsf{x}t_3\cdots t_k) & (5) \\
\vdots & & \\
\Gamma \vdash t_k = u_k & (Assume_{\mathcal{L}}) & (2k) \\
\Gamma \vdash \mathsf{R}u_1u_2\cdots u_k & (= Sub \text{ on } 2k-1 \text{ and } 2k \text{ with } \mathsf{R}u_1u_2\cdots\mathsf{x}) & (2k+1)
\end{array}$$

□

Proposition 5.2.8. *For any $\mathsf{f} \in \mathcal{F}_k$ and any $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$, we have*

$$\{t_1 = u_2, t_2 = u_2, \dots, t_k = u_k\} \vdash \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1u_2\cdots u_k$$

Proof. Fix $\mathsf{f} \in \mathcal{F}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. Fix $\mathsf{x} \notin \bigcup_{i=1}^{k}(OccurVar(t_i) \cup OccurVar(u_i))$. Let $\Gamma = \{t_1 =$

$u_1, t_2 = u_2, \dots, t_k = u_k\}$. We have

$$
\begin{array}{llr}
\Gamma \vdash \mathsf{f}t_1t_2\cdots t_k = ft_1t_2\cdots t_k & (EqRefl) & (1) \\
\Gamma \vdash t_1 = u_1 & (Assume_{\mathcal{L}}) & (2) \\
\Gamma \vdash \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1t_2\cdots t_k & (= Sub \text{ on 1 and 2 with } \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}\mathsf{x}t_2\cdots t_k) & (3) \\
\Gamma \vdash t_2 = u_2 & (Assume_{\mathcal{L}}) & (4) \\
\Gamma \vdash \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1u_2\cdots t_k & (= Sub \text{ on 1 and 2 with } \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1\mathsf{x}\cdots t_k) & (3) \\
\vdots & & \\
\Gamma \vdash t_k = u_k & (Assume_{\mathcal{L}}) & (2k) \\
\Gamma \vdash \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1u_2\cdots u_k & (= Sub \text{ on } 2k-1 \text{ and } 2k \text{ with } \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}u_1u_2\cdots\mathsf{x}) & (2k+1)
\end{array}
$$

Similar to the previous proposition, but start with the line $\emptyset \vdash \mathsf{f}t_1t_2\cdots t_k = \mathsf{f}t_1t_2\cdots t_k$ using the *EqRefl* rule. □

Proposition 5.2.9. $\exists\mathsf{x}\varphi \vdash \neg\forall\mathsf{x}\neg\varphi$.

Proof. Fix $\mathsf{y} \neq \mathsf{x}$ with $\mathsf{y} \notin OccurVar(\varphi)$.

$$
\begin{array}{rlr}
\{\varphi^{\mathsf{y}}_{\mathsf{x}}, \neg\neg\forall\mathsf{x}\neg\varphi, \neg\forall\mathsf{x}\neg\varphi\} \vdash \neg\forall\mathsf{x}\neg\varphi & (Assume_{\mathcal{L}}) & (1) \\
\{\varphi^{\mathsf{y}}_{\mathsf{x}}, \neg\neg\forall\mathsf{x}\neg\varphi, \neg\forall\mathsf{x}\neg\varphi\} \vdash \neg\neg\forall\mathsf{x}\neg\varphi & (Assume_{\mathcal{L}}) & (2) \\
\{\varphi^{\mathsf{y}}_{\mathsf{x}}, \neg\neg\forall\mathsf{x}\neg\varphi\} \vdash \forall\mathsf{x}\neg\varphi & (Contr \text{ on 1 and 2}) & (3) \\
\{\varphi^{\mathsf{y}}_{\mathsf{x}}, \neg\neg\forall\mathsf{x}\neg\varphi\} \vdash \neg(\varphi^{\mathsf{y}}_{\mathsf{x}}) & (\forall E \text{ on } 3) & (4) \\
\{\varphi^{\mathsf{y}}_{\mathsf{x}}, \neg\neg\forall\mathsf{x}\neg\varphi\} \vdash \varphi^{\mathsf{y}}_{\mathsf{x}} & (Assume_{\mathcal{L}}) & (5) \\
\{\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vdash \neg\forall\mathsf{x}\neg\varphi & (Contr \text{ on 4 and 5}) & (6) \\
\{\exists\mathsf{x}\varphi\} \vdash \neg\forall\mathsf{x}\neg\varphi & (\exists P \text{ on } 6) & (7)
\end{array}
$$

□

Proposition 5.2.10. $\neg\exists\mathsf{x}\neg\varphi \vdash \forall\mathsf{x}\varphi$.

Proof. Fix $\mathsf{y} \neq \mathsf{x}$ with $\mathsf{y} \notin OccurVar(\varphi)$.

$$
\begin{array}{rlr}
\{\neg\exists\mathsf{x}\neg\varphi, \neg\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vdash \neg\exists\mathsf{x}\neg\varphi & (Assume_{\mathcal{L}}) & (1) \\
\{\neg\exists\mathsf{x}\neg\varphi, \neg\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vdash (\neg\varphi)^{\mathsf{y}}_{\mathsf{x}} & (Assume_{\mathcal{L}}) & (2) \\
\{\neg\exists\mathsf{x}\neg\varphi, \neg\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vdash \exists\mathsf{x}\neg\varphi & (\exists I \text{ on } 2) & (3) \\
\{\neg\exists\mathsf{x}\neg\varphi\} \vdash \varphi^{\mathsf{y}}_{\mathsf{x}} & (Contr \text{ on 1 and 3}) & (4) \\
\{\neg\exists\mathsf{x}\neg\varphi\} \vdash \forall\mathsf{x}\varphi & (\forall I \text{ on } 4) & (5)
\end{array}
$$

□

5.2.3 Theorems About $\vdash$

Proposition 5.2.11. *If* Γ *is inconsistent, then* $\Gamma \vdash \varphi$ *for all* $\varphi \in Form_P$.

Proof. Fix θ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$, and fix $\varphi \in Form_{\mathcal{L}}$. We have that $\Gamma \cup \{\neg\varphi\} \vdash \theta$ and $\Gamma \cup \{\neg\varphi\} \vdash \neg\theta$ by the *Super* rule. Therefore, $\Gamma \vdash \varphi$ by using the *Contr* rule. □

Proposition 5.2.12.

1. *If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$.*

2. *If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \vdash \varphi$.*

Proof.

1. Since $\Gamma \cup \{\varphi\}$ is inconsistent, we know that $\Gamma \cup \{\varphi\} \vdash \neg\varphi$ by Proposition 5.2.11. Since we also have that $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by *Assume*, it follows that $\Gamma \vdash \neg\varphi$ by the $\neg PC$ rule.

2. Since $\Gamma \cup \{\neg\varphi\}$ is inconsistent, we know that $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition 5.2.11. Since we also have that $\Gamma \cup \{\varphi\} \vdash \varphi$ by *Assume*, it follows that $\Gamma \vdash \varphi$ by the $\neg PC$ rule.

□

Corollary 5.2.13. *If $\Gamma \subseteq Form_{\mathcal{L}}$ is consistent and $\varphi \in Form_{\mathcal{L}}$, then either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{\neg\varphi\}$ is consistent.*

Proof. If both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent, then both $\Gamma \vdash \neg\varphi$ and $\Gamma \vdash \varphi$ by Proposition 5.2.12, so Γ is inconsistent. □

Proposition 5.2.14.

1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \psi$.*

2. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.*

Proof.

1. Since $\Gamma \vdash \varphi$, it follows from the *Super* rule that $\Gamma \cup \{\neg\varphi\} \vdash \varphi$. Since we also have $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by *Assume*, we may conclude that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. Therefore, by Proposition 5.2.11, we have that $\Gamma \cup \{\neg\varphi\} \vdash \psi$. Now we also have $\Gamma \cup \{\varphi\} \vdash \psi$ by assumption, so the $\neg PC$ rule gives that $\Gamma \vdash \psi$.

2. Since $\Gamma \vdash \varphi \to \psi$, we can conclude that $\Gamma \cup \{\varphi\} \vdash \psi$ by rule $\to E$. The result follows from part 1.

□

Proposition 5.2.15. *Let $G_{fin} = G(\mathcal{P}_{fin}(Form_{\mathcal{L}}) \times Form_{\mathcal{L}}, Assume_{\mathcal{L}} \cup EqRefl, \mathcal{H})$, i.e. we insist that the set Γ is finite but otherwise have exactly the same proof rules. Let $\Gamma \vdash_{fin} \varphi$ denote that $(\Gamma, \varphi) \in G_{fin}$*

1. *If $\Gamma \vdash_{fin} \varphi$, then $\Gamma \vdash \varphi$.*

2. *If $\Gamma \vdash \varphi$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{fin} \varphi$*

In particular, if $\Gamma \vdash \varphi$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

Proof. 1 is a completely straightforward induction because the starting points are the same and we have the exact same rules. The proof of 2 goes in much the same way as the corresponding result for propositional logic. □

Corollary 5.2.16. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. Suppose that Γ is inconsistent, and fix $\theta \in Form_{\mathcal{L}}$ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. By Proposition 5.2.15, there exists finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_0 \vdash \theta$ and $\Gamma_1 \vdash \neg\theta$. Using the *Super* rule, it follows that $\Gamma_0 \cup \Gamma_1 \vdash \theta$ and $\Gamma_0 \cup \Gamma_1 \vdash \neg\theta$, so $\Gamma_0 \cup \Gamma_1$ is a finite inconsistent subset of Γ. □

Chapter 6

Soundness, Completeness, and Compactness

6.1 Soundness

Theorem 6.1.1 (Soundness Theorem)**.**

1. *If* $\Gamma \vdash \varphi$*, then* $\Gamma \vDash \varphi$*.*
2. *Every satisfiable set of formulas is consistent.*

Proof.

1. The proof is by induction. We let $X = \{(\Gamma, \varphi) \in G : \Gamma \vDash \varphi\}$ and we show by induction on G that $X = G$. We begin by noting that if $\varphi \in \Gamma$, then $\Gamma \vDash \varphi$ because if $(\mathcal{M}, s)$ is a model of Γ, then $(\mathcal{M}, s)$ is a model of γ simply because $\varphi \in \Gamma$. Therefore, $(\Gamma, \varphi) \in X$ for all $(\Gamma, \varphi) \in Assume_{\mathcal{L}}$. Also, for any $\Gamma \subseteq Form_{\mathcal{L}}$ and any $t \in Term_{\mathcal{L}}$, we have $\Gamma \vDash t = t$ because for any any model $(\mathcal{M}, s)$ of Γ we have $\overline{s}(t) = \overline{s}(t)$, hence $(\mathcal{M}, s) \vDash t = t$.

   We now handle the inductive steps. All of the old rules go through in a similar manner as before, and the *Super* rule is trivial.

   - We first handle the $= Sub$ rule. Suppose that $\Gamma \vDash \varphi^t_{\mathsf{x}}$, that $\Gamma \vDash t = u$, and that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1 = ValidSubst^u_{\mathsf{x}}(\varphi)$. We need to show that $\Gamma \vDash \varphi^u_{\mathsf{x}}$. Fix a model $(\mathcal{M}, s)$ of Γ. Since $\Gamma \vDash \varphi^t_{\mathsf{x}}$, we have that $(\mathcal{M}, s) \vDash \varphi^t_{\mathsf{x}}$. Also, since $\Gamma \vDash t = u$, we htave that $(\mathcal{M}, s) \vDash t = u$, and hence $\overline{s}(t) = \overline{s}(u)$. Now using the fact that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1 = ValidSubst^u_{\mathsf{x}}(\varphi) = 1$, we have

   $$\begin{aligned}(\mathcal{M}, s) \vDash \varphi^t_{\mathsf{x}} &\Rightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi && \text{(by the Substitution Theorem)}\\ &\Rightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(u)]) \vDash \varphi\\ &\Rightarrow (\mathcal{M}, s) \vDash \varphi^u_{\mathsf{x}} && \text{(by the Substitution Theorem)}\end{aligned}$$

   - We now handle the $\exists I$ rule. Suppose that $\Gamma \vDash \varphi^t_{\mathsf{x}}$ where $ValidSubst_{t,\mathsf{x}}(\varphi) = 1$. We need to show that $\Gamma \vDash \exists \mathsf{x} \varphi$. Fix a model $(\mathcal{M}, s)$ of Γ. Since $\Gamma \vDash \varphi^t_{\mathsf{x}}$, it follows that $(\mathcal{M}, s) \vDash \varphi^t_{\mathsf{x}}$. Since $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$, we have

   $$\begin{aligned}(\mathcal{M}, s) \vDash \varphi^t_{\mathsf{x}} &\Rightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi && \text{(by the Substitution Theorem)}\\ &\Rightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi\\ &\Rightarrow (\mathcal{M}, s) \vDash \exists \mathsf{x} \varphi\end{aligned}$$

- Let's next attack the $\exists P$ rule. Suppose that $\Gamma \cup \{\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vDash \psi$, that $\mathsf{y} \notin FreeVar(\Gamma \cup \{\exists \mathsf{x}\varphi, \psi\})$, and that $ValidSubst^{\mathsf{y}}_{\mathsf{x}}(\varphi) = 1$. We need to show that $\Gamma \cup \{\exists \mathsf{x}\varphi\} \vDash \psi$. Fix a model $(\mathcal{M}, s)$ of $\Gamma \cup \{\exists \mathsf{x}\varphi\}$. Since $(\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi$, we may fix $a \in M$ such that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$. We first divide into two cases to show that $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$.

  Case 1: Suppose that $\mathsf{y} = \mathsf{x}$. We then have $\varphi^{\mathsf{y}}_{\mathsf{x}} = \varphi^{\mathsf{x}}_{\mathsf{x}} = \varphi$ and $s[\mathsf{x} \Rightarrow a] = s[\mathsf{y} \Rightarrow a]$, hence $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$ because $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$.

  Case 2: Suppose that $\mathsf{y} \neq \mathsf{x}$. We then have

  $$\begin{aligned} (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi &\Rightarrow (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow a]) \vDash \varphi && \text{(since } \mathsf{y} \notin FreeVar(\varphi) \text{ and } \mathsf{y} \neq \mathsf{x}) \\ &\Rightarrow (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s[\mathsf{y} \Rightarrow a]}(\mathsf{y})]) \vDash \varphi \\ &\Rightarrow (\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}} && \text{(by the Substitution Theorem)} \end{aligned}$$

  Thus, $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$ in either case. Now since $(\mathcal{M}, s) \vDash \gamma$ for all $\gamma \in \Gamma$ and $\mathsf{y} \notin FreeVar(\Gamma)$, we have $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \gamma$ for all $\gamma \in \Gamma$. Thus, $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \psi$ because $\Gamma \cup \{\varphi^{\mathsf{y}}_{\mathsf{x}}\} \vDash \psi$. Finally, since $\mathsf{y} \notin FreeVar(\psi)$, it follows that $(\mathcal{M}, s) \vDash \psi$.

- We next do the $\forall E$ rule. Suppose that $\Gamma \vDash \forall \mathsf{x}\varphi$ and that $t \in Term_{\mathcal{L}}$ is such that $ValidSubst^{t}_{\mathsf{x}}(\varphi) = 1$. We need to show that $\Gamma \vDash \varphi^{t}_{\mathsf{x}}$. Fix a model $(\mathcal{M}, s)$ of Γ. Since $\Gamma \vDash \forall x\varphi$, it follows that that $(\mathcal{M}, s) \vDash \forall x\varphi$. Since $ValidSubst^{t}_{\mathsf{x}}(\varphi) = 1$, we have

  $$\begin{aligned} (\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi &\Rightarrow \text{For all } a \in M, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Rightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(t)]) \vDash \varphi \\ &\Rightarrow (\mathcal{M}, s) \vDash \varphi^{t}_{\mathsf{x}} && \text{(by the Substitution Theorem)} \end{aligned}$$

- We finally end with the $\forall I$ rule. Suppose that $\Gamma \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$, that $\mathsf{y} \notin FreeVar(\Gamma \cup \{\forall \mathsf{x}\varphi\})$, and that $ValidSubst^{\mathsf{y}}_{\mathsf{x}}(\varphi) = 1$. We need to show that $\Gamma \vDash \forall \mathsf{x}\varphi$. Fix a model $(\mathcal{M}, s)$ of Γ.

  Case 1: Suppose that $\mathsf{y} = \mathsf{x}$. Since $\mathsf{y} = \mathsf{x}$, we have $\varphi^{\mathsf{y}}_{\mathsf{x}} = \varphi^{\mathsf{x}}_{\mathsf{x}} = \varphi$. Fix $a \in M$. Since $(\mathcal{M}, s) \vDash \gamma$ for all $\gamma \in \Gamma$ and $\mathsf{x} = \mathsf{y} \notin FreeVar(\Gamma)$, we may conclude that $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \gamma$ for all $\gamma \in \Gamma$. Therefore, $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$, i.e. $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$, because $\Gamma \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$. Now $a \in \mathcal{M}$ was arbitrary, so $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$ for every $a \in M$, hence $(\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi$.

  Case 2: Suppose that $\mathsf{y} \neq \mathsf{x}$. Fix M. Since $(\mathcal{M}, s) \vDash \gamma$ for all $\gamma \in \Gamma$ and $\mathsf{y} \notin FreeVar(\Gamma)$, we may conclude that $(\mathcal{M}, s[\mathsf{y} \Rightarrow a] \vDash \gamma$ for all $\gamma \in \Gamma$. Therefore, $(\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$ because $\Gamma \vDash \varphi^{\mathsf{y}}_{\mathsf{x}}$. Since $ValidSubst^{\mathsf{y}}_{\mathsf{x}}(\varphi) = 1$, we have

  $$\begin{aligned} (\mathcal{M}, s[\mathsf{y} \Rightarrow a]) \vDash \varphi^{\mathsf{y}}_{\mathsf{x}} &\Rightarrow (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow \overline{s[\mathsf{y} \Rightarrow a]}(y)]) \vDash \varphi && \text{(by the Substitution Theorem)} \\ &\Rightarrow (\mathcal{M}, (s[\mathsf{y} \Rightarrow a])[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Rightarrow (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi && \text{(since } \mathsf{y} \notin FreeVar(\varphi) \text{ and } \mathsf{y} \neq \mathsf{x}) \end{aligned}$$

  Now $a \in M$ was arbitrary, so $(\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi$ for every $a \in M$, hence $(\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi$.

The result follows by induction.

2. Let Γ be a satisfiable set of formulas. Fix a model $(\mathcal{M}, s)$ of Γ. Suppose that Γ is inconsistent, and fix $\theta \in Form_{\mathcal{L}}$ such that $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. We then have $\Gamma \vDash \theta$ and $\Gamma \vDash \neg\theta$ by part 1, hence $(\mathcal{M}, s) \vDash \theta$ and $(\mathcal{M}, s) \vDash (\neg\theta)$, a contradiction. It follows that Γ is consistent.

□

6.2 Prime Formulas

Definition 6.2.1. *A formula $\varphi \in Form_{\mathcal{L}}$ is* prime *if one the following holds.*

1. $\varphi \in AtomicForm_{\mathcal{L}}$.
2. $\varphi = \exists \mathsf{x} \psi$ *for some* $\mathsf{x} \in Var$ *and some* $\psi \in Form_{\mathcal{L}}$.
3. $\varphi = \forall \mathsf{x} \psi$ *for some* $\mathsf{x} \in Var$ *and some* $\psi \in Form_{\mathcal{L}}$.

We denote the set of prime formulas by $PrimeForm_{\mathcal{L}}$.

Definition 6.2.2. *Let $\mathcal{L}$ be a language. Let $P(\mathcal{L}) = \{A_\varphi : \varphi \in PrimeForm_{\mathcal{L}}\}$.*

- *We define a function $h\colon Form_{\mathcal{L}} \to Form_{P(\mathcal{L})}$ recursively as follows.*
  1. $h(\varphi) = \mathsf{A}_\varphi$ *for all* $\varphi \in AtomicForm_{\mathcal{L}}$.
  2. $h(\neg\varphi) = \neg h(\varphi)$ *for all* $\varphi \in Form_{\mathcal{L}}$.
  3. $h(\Diamond\varphi\psi) = \Diamond h(\varphi)h(\psi)$ *for all* $\varphi, \psi \in Form_{\mathcal{L}}$ *and all* $\Diamond \in \{\wedge, \vee, \rightarrow\}$.
  4. $h(\mathsf{Qx}\varphi) = \mathsf{A}_{\mathsf{Qx}\varphi}$ *for all* $\varphi \in Form_{\mathcal{L}}$, $\mathsf{x} \in Var$, *and* $\mathsf{Q} \in \{\forall, \exists\}$.

  For $\varphi \in Form_{\mathcal{L}}$, we write $\varphi^\#$ for $h(\varphi)$. For $\Gamma \subseteq Form_{\mathcal{L}}$, we write $\Gamma^\#$ for $\{\gamma^\# : \gamma \in \Gamma\}$.

- *We also define a function $g\colon Form_{P(\mathcal{L})} \to Form_{\mathcal{L}}$ recursively as follows.*
  1. $g(\mathsf{A}_\varphi) = \varphi$ *for all* $\varphi \in PrimeForm_{\mathcal{L}}$.
  2. $g(\neg\alpha) = \neg g(\alpha)$ *for all* $\alpha \in Form_{P_{\mathcal{L}}}$.
  3. $g(\Diamond\alpha\beta) = \Diamond g(\alpha)g(\beta)$ *for all* $\alpha, \beta \in Form_{P_{\mathcal{L}}}$ *and all* $\Diamond \in \{\wedge, \vee, \rightarrow\}$.

  For $\alpha \in Form_{P(\mathcal{L})}$, we write $\alpha^\star$ for $g(\alpha)$. For $\Gamma \subseteq Form_{P(\mathcal{L})}$, we write $\Gamma^\star$ for $\{\gamma^\star : \gamma \in \Gamma\}$.

Proposition 6.2.3.

1. *For all $\varphi \in Form_{\mathcal{L}}$, we have $(\varphi^\#)^\star = \varphi$.*
2. *For all $\alpha \in Form_{P(\mathcal{L})}$, we have $(\alpha^\star)^\# = \alpha$.*

Proof. A trivial induction. □

Proposition 6.2.4. *Let $\mathcal{L}$ be a language, let $\Gamma \subseteq Form_{P(\mathcal{L})}$, and let $\varphi \in Form_{P(\mathcal{L})}$.*

1. *If $\Gamma \vdash_{P(\mathcal{L})} \varphi$ (in the propositional language $P(\mathcal{L})$), then $\Gamma^\star \vdash_{\mathcal{L}} \varphi^\star$ (in the first-order language $\mathcal{L}$).*
2. *If $\Gamma \vDash_{P(\mathcal{L})} \varphi$ (in the propositional language $P(\mathcal{L})$), then $\Gamma^\star \vdash_{\mathcal{L}} \varphi^\star$ (in the first-order language $\mathcal{L}$).*

Proof.

1. This follows by induction because all propositional rules are included as first-order logic rules.
2. If $\Gamma \vDash_{P(\mathcal{L})} \varphi$, then $\Gamma \vdash_{P(\mathcal{L})} \varphi$ by the Completeness Theorem for propositional logic, hence $\Gamma^\star \vdash_{\mathcal{L}} \varphi^\star$ by part 1.

□

Corollary 6.2.5. *Let $\mathcal{L}$ be a language, let $\Gamma \subseteq Form_{\mathcal{L}}$, and let $\varphi \in Form_{\mathcal{L}}$. If $\Gamma^\# \vDash_{P(\mathcal{L})} \varphi^\#$ (in the propositional language $P(\mathcal{L})$), then $\Gamma \vdash_{\mathcal{L}} \varphi$ (in the first-order language $\mathcal{L}$).*

Proof. Suppose that $\Gamma^\# \vDash_{P(\mathcal{L})} \varphi^\#$. By Proposition 6.2.4, it follows that $(\Gamma^\#)^\star \vDash_{P(\mathcal{L})} (\varphi^\#)^\star$, hence $\Gamma \vdash_{\mathcal{L}} \varphi$ by Proposition 6.2.3 □

Example 6.2.6. *Let $\mathcal{L}$ be a languague and let $\varphi, \psi \in Form_{\mathcal{L}}$ We have $\neg(\varphi \to \psi) \vdash_{\mathcal{L}} \varphi \wedge (\neg\psi)$*

Proof. We show that $(\neg(\varphi \to \psi))^\# \vDash_{P(\mathcal{L})} (\varphi \wedge (\neg\psi))^\#$. Notice that

1. $(\neg(\varphi \to \psi))^\# = \neg(\varphi^\# \to \psi^\#)$
2. $(\varphi \wedge (\neg\psi))^\# = \varphi^\# \wedge \neg(\psi^\#)$.

Suppose that $v\colon P(\mathcal{L}) \to \{0,1\}$ is a truth assignment such that $\overline{v}(\neg(\varphi^\# \to \psi^\#)) = 1$. We then have $\overline{v}(\varphi^\# \to \psi^\#) = 0$, hence $\overline{v}(\varphi^\#) = 1$ and $\overline{v}(\psi^\#) = 0$. We therefore have $\overline{v}(\neg(\psi^\#)) = 1$ and hence $\overline{v}(\varphi^\# \wedge \neg(\psi^\#)) = 1$. It follows that $(\neg(\varphi \to \psi))^\# \vDash_{P(\mathcal{L})} (\varphi \wedge (\neg\psi))^\#$. □

Corollary 6.2.7. *Let $\mathcal{L}$ be a language, let $\Gamma \subseteq Form_{\mathcal{L}}$, and let $\varphi, \psi \in Form_{\mathcal{L}}$. If $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\varphi^\# \vDash_{P(\mathcal{L})} \psi^\#$, then $\Gamma \vdash_{\mathcal{L}} \psi$.*

Proof. Since $\varphi^\# \vDash_{P(\mathcal{L})} \psi^\#$, we have that $\varphi \vdash_{\mathcal{L}} \psi$ by Corollary 6.2.5. It follows from the *Super* rule that $\Gamma \cup \{\varphi\} \vdash_{\mathcal{L}} \psi$. Using Proposition 5.2.14 (since $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathcal{L}} \psi$), we may conclude that $\Gamma \vdash_{\mathcal{L}} \psi$. □

6.3 Completeness

6.3.1 Motivating the Proof

We first give an overview of the key ideas in our proof of completeness. Let $\mathcal{L}$ be a language, and suppose that $\Gamma \subseteq \mathcal{L}$ is consistent.

Definition 6.3.1. *Suppose that $\mathcal{L}$ is a language and that $\Delta \subseteq Form_{\mathcal{L}}$. We say that Δ is* complete *if for all $\varphi \in Form_{\mathcal{L}}$, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.*

As we saw in propositional logic, it will aid use greatly to extend Γ to a set $\Delta \supseteq \Gamma$ which is both consistent and complete, so let's assume that we can do that (we will prove it exactly the same way below). We need to construct an $\mathcal{L}$-structure $\mathcal{M}$ and a variable assignment $s\colon Var \to M$ such that $(\mathcal{M}, s) \vDash \delta$ for all $\delta \in \Delta$. Now all that we have is the syntactic information that Δ provides, so it seems that the only way to proceed is to define our $\mathcal{M}$ from these syntactic objects. Since terms intuitively name elements, it is natural to try to define the universe M to simply be $Term_{\mathcal{L}}$. We would then define the structure as follows

1. $\mathsf{c}^{\mathcal{M}} = \mathsf{c}$ for all $\mathsf{c} \in \mathcal{C}$.
2. $\mathsf{R}^{\mathcal{M}} = \{(t_1, t_2, \dots, t_k) \in M^k : \mathsf{R}t_1t_2\dots t_k \in \Delta\}$ for all $\mathsf{R} \in \mathcal{R}_k$.
3. $\mathsf{f}^{\mathcal{M}}(t_1, t_2, \dots, t_k) = \mathsf{f}t_1t_2\cdots t_k$ for all $\mathsf{f} \in \mathcal{F}_k$ and all $t_1, t_2, \dots, t_k \in M$.

and let $s\colon Var \to M$ be the variable assignment defined by $s(\mathsf{x}) = \mathsf{x}$ for all $\mathsf{x} \in Var$.

However, there are two problems with this approach, one of which is minor and the other is quite serious. First, let's think about the minor problem. Suppose that $\mathcal{L} = \{\mathsf{f}, \mathsf{e}\}$ where f is a binary function symbol and e is a constant symbol, and that Γ is the set of group axioms. Suppose that $\Delta \supseteq \Gamma$ is consistent and complete. We then have $\mathsf{fee} = \mathsf{e} \in \Delta$ because $\Gamma \vdash \mathsf{fee} = \mathsf{e}$. However, the two terms fee and e are syntactically different objects, so if we were to let M be $Term_{\mathcal{L}}$ this would cause a problem because fee and e are distinct despite the fact that Δ says they must be equal. Of course, when you have distinct objects which you want to consider equivalent, you should define an equivalence relation. Thus, we should define $\sim$ on $Term_{\mathcal{L}}$ by letting $t \sim u$ if $t = u \in \Delta$. We would then need to check that $\sim$ is an equivalence relation and that the definition of the structure above is independent of our choice of representatives for the classes. This is all fairly straightfoward, and will be carried out below.

On to the more serious obstactle. Suppose that $\mathcal{L} = \{\mathsf{P}\}$ where P is a unary relation symbol. Let $\Gamma = \{\neg\mathsf{Px} : \mathsf{x} \in Var\} \cup \{\neg(\mathsf{x} = \mathsf{y}) : \mathsf{x}, \mathsf{y} \in Var \text{ with } \mathsf{x} \neq \mathsf{y}\} \cup \{\exists\mathsf{xPx}\}$ and notice that Γ is consistent because it is satisfiable. Suppose that $\Delta \supseteq \Gamma$ is consistent and complete. In the model $\mathcal{M}$ constructed, we have $M = Term_{\mathcal{L}} = Var$ (notice that the equivalence relation defined above will be trivial in this case). Thus, since $(\mathcal{M}, s) \vDash \neg\mathsf{Px}$ for all $\mathsf{x} \in Var$, it follows that $(\mathcal{M}, s) \not\vDash \exists\mathsf{xPx}$. Hence, $\mathcal{M}$ is not a model of Δ.

The problem in the above example is that there was an existential statement in Δ, but whenever you plugged a term in for the quantied variable, the resulting formula was not in Δ. Since we are building our model from the terms, this is a serious problem. However, if Δ had the following, then this problem would not arise.

Definition 6.3.2. *Let $\mathcal{L}$ be a language and let $\Gamma \subseteq Form_{\mathcal{L}}$. We say that Γ* contains witnesses *if for all $\varphi \in Form_{\mathcal{L}}$ and all $\mathsf{x} \in Var$, there exists $\mathsf{c} \in \mathcal{C}$ such that $(\exists\mathsf{x}\varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}} \in \Gamma$.*

Our goal then is to show that if Γ is consistent, then there exists a $\Delta \supseteq \Gamma$ which is consistent, complete, and contains witnesses. On the face of it, this is not true, as the above example shows (because there are no constant symbols). However, if we allow ourselves to expand our language with new constant symbols, we can repeatedly add witnessing statements by using these fresh constant symbols as our witnesses.

6.3.2 The Proof

We can also define substitution of variables for constants in the obvious recursive fashion. Ignore the following lemma until you see why we need it later.

Lemma 6.3.3. *Let $\varphi \in Form_{\mathcal{L}}$, let $t \in Term_{\mathcal{L}}$, let $\mathsf{c} \in \mathcal{C}$, and let $\mathsf{x}, \mathsf{z} \in Var$. Suppose that $\mathsf{z} \notin OccurVar(\varphi)$.*

- *$(\varphi^t_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ equals $(\varphi^{\mathsf{z}}_{\mathsf{c}})^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$.*
- *If $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$, then $ValidSubst^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}(\varphi^{\mathsf{z}}_{\mathsf{c}}) = 1$.*

Lemma 6.3.4. *Let $\mathcal{L}$ be a language, and let $\mathcal{L}'$ be $\mathcal{L}$ together with a new constant symbol c. Suppose that*

$$\begin{gathered}\Gamma_0 \vdash_{\mathcal{L}'} \varphi_0 \\ \Gamma_1 \vdash_{\mathcal{L}'} \varphi_1 \\ \Gamma_2 \vdash_{\mathcal{L}'} \varphi_2 \\ \vdots \\ \Gamma_n \vdash_{\mathcal{L}'} \varphi_n\end{gathered}$$

is an $\mathcal{L}'$-deduction. For any $\mathsf{z} \in Var$ with $z \notin OccurVar(\bigcup_{i=0}^{n}(\Gamma_i \cup \{\varphi_i\}))$, we have that

$$\begin{gathered}(\Gamma_0)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_0)^{\mathsf{z}}_{\mathsf{c}} \\ (\Gamma_1)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_1)^{\mathsf{z}}_{\mathsf{c}} \\ (\Gamma_2)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_2)^{\mathsf{z}}_{\mathsf{c}} \\ \vdots \\ (\Gamma_n)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_n)^{\mathsf{z}}_{\mathsf{c}}\end{gathered}$$

is an $\mathcal{L}$-deduction.

Proof. We prove by induction on i that

$$\begin{gathered}(\Gamma_0)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_0)^{\mathsf{z}}_{\mathsf{c}} \\ (\Gamma_1)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_1)^{\mathsf{z}}_{\mathsf{c}} \\ (\Gamma_2)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_2)^{\mathsf{z}}_{\mathsf{c}} \\ \vdots \\ (\Gamma_i)^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_i)^{\mathsf{z}}_{\mathsf{c}}\end{gathered}$$

is an $\mathcal{L}$-deduction.

If $\varphi \in \Gamma$, then $\varphi^{\mathsf{z}}_{\mathsf{c}} \in \Gamma^{\mathsf{z}}_{\mathsf{c}}$.

Suppose that line i is $\Gamma \vdash t = t$ where $t \in Term_{\mathcal{L}'}$. Since $(t = t)^{\mathsf{z}}_{\mathsf{c}}$ equals $t^{\mathsf{z}}_{\mathsf{c}} = t^{\mathsf{z}}_{\mathsf{c}}$, we can place $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash t^{\mathsf{z}}_{\mathsf{c}} = t^{\mathsf{z}}_{\mathsf{c}}$ on line i by the $EqRefl$ rule.

Suppose that $\Gamma \vdash_{\mathcal{L}'} \varphi \wedge \psi$ was a previous line and we inferred $\Gamma \vdash_{\mathcal{L}'} \varphi$. Inductively, we have $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi \wedge \psi)^{\mathsf{z}}_{\mathsf{c}}$ on the corresponding line. Since $(\varphi \wedge \psi)^{\mathsf{z}}_{\mathsf{c}} = \varphi^{\mathsf{z}}_{\mathsf{c}} \wedge \psi^{\mathsf{z}}_{\mathsf{c}}$, we may use the $\wedge EL$ rule to put $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} \varphi^{\mathsf{z}}_{\mathsf{c}}$ on the corresponding line. The other propositional rules are similarly uninteresting.

Suppose that $\Gamma \vdash_{\mathcal{L}'} \varphi^t_{\mathsf{x}}$ and $\Gamma \vdash_{\mathcal{L}'} t = u$ were previous lines, that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1 = ValidSubst^u_{\mathsf{x}}(\varphi)$, and we inferred $\Gamma \vdash_{\mathcal{L}'} \varphi^u_{\mathsf{x}}$. Inductively, we have $\Gamma^{\mathsf{z}}_{c} \vdash_{\mathcal{L}} (\varphi^t_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ and $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (t = u)^{\mathsf{z}}_{\mathsf{c}}$ on the corresponding lines. Now $(\varphi^t_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ equals $(\varphi^{\mathsf{z}}_{\mathsf{c}})^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$ by the previous lemma, and $(t = u)^{\mathsf{z}}_{\mathsf{c}}$ equals $t^{\mathsf{z}}_{\mathsf{c}} = u^{\mathsf{z}}_{\mathsf{c}}$. Thus, we have $\Gamma^{\mathsf{z}}_{c} \vdash_{\mathcal{L}} (\varphi^{\mathsf{z}}_{\mathsf{c}})^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$ and $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} t^{\mathsf{z}}_{\mathsf{c}} = u^{\mathsf{z}}_{\mathsf{c}}$ on the corresponding lines. Using the fact that $ValidSubst^t_{\mathsf{x}}(\varphi) = 1 = ValidSubst^u_{\mathsf{x}}(\varphi)$, we can use the previous lemma to conclude that that $ValidSubst^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}(\varphi^{\mathsf{z}}_{\mathsf{c}}) = 1 = ValidSubst^{u^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}(\varphi^{\mathsf{z}}_{\mathsf{c}})$. Hence, we may use that $= Sub$ rule to put $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi^{\mathsf{z}}_{\mathsf{c}})^{u^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$ on the corresponding line. We now need only note that $(\varphi^{\mathsf{z}}_{\mathsf{c}})^{u^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$ equals $(\varphi^u_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ by the previous lemma.

Suppose that $\Gamma \vdash_{\mathcal{L}'} \varphi^t_{\mathsf{x}}$ where $ValidSubst^t_{\mathsf{x}}(\varphi) = 1$ was a previous line and we inferred $\Gamma \vdash_{\mathcal{L}'} \exists \mathsf{x} \varphi$. Inductively, we have $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi^t_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ on the corresponding line. Now $(\varphi^t_{\mathsf{x}})^{\mathsf{z}}_{\mathsf{c}}$ equals $(\varphi^{\mathsf{z}}_{\mathsf{c}})^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}$ and $ValidSubst^{t^{\mathsf{z}}_{\mathsf{c}}}_{\mathsf{x}}(\varphi^{\mathsf{z}}_{\mathsf{c}}) = 1$ by the previous lemma. Hence, we may use the $\exists I$ rule to put $\Gamma^{\mathsf{z}}_{\mathsf{c}} \vdash_{\mathcal{L}} \exists \mathsf{x}(\varphi^{\mathsf{z}}_{\mathsf{c}})$ on the corresponding line. We now need only note that $\exists \mathsf{x}(\varphi^{\mathsf{z}}_{\mathsf{c}})$ equals $(\exists \mathsf{x} \varphi)^{\mathsf{z}}_{\mathsf{c}}$.

The other rules are similarly awful. □

Corollary 6.3.5 (Generalization on Constants)**.** *Let $\mathcal{L}$ be a language, and let $\mathcal{L}'$ be $\mathcal{L}$ together with a new constant symbol* c*. Suppose that $\Gamma \subseteq Form_{\mathcal{L}}$ and $\varphi \in Form_{\mathcal{L}}$. If $\Gamma \vdash_{\mathcal{L}'} \varphi^{\mathsf{c}}_{\mathsf{x}}$, then $\Gamma \vdash_{\mathcal{L}} \forall \mathsf{x} \varphi$.*

Proof. Since $\Gamma \vdash_{\mathcal{L}'} \varphi$, we may use Proposition 5.2.15 to fix an $\mathcal{L}'$-deduction

$$\begin{gathered}\Gamma_0 \vdash_{\mathcal{L}'} \varphi_0 \\ \Gamma_1 \vdash_{\mathcal{L}'} \varphi_1 \\ \Gamma_2 \vdash_{\mathcal{L}'} \varphi_2 \\ \vdots \\ \Gamma_n \vdash_{\mathcal{L}'} \varphi_n\end{gathered}$$

such that each $\Gamma_i \subseteq Form_{\mathcal{L}'}$ is finite and $\Gamma_n \subseteq \Gamma$. Fix $\mathsf{y} \in Var$ such that

$$\mathsf{y} \notin OccurVar(\bigcup_{i=0}^{n}(\Gamma_i \cup \{\varphi_i\}))$$

From Lemma 6.3.4, we have that

$$\begin{gathered}
(\Gamma_0)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_0)^{\mathsf{y}}_{\mathsf{c}} \\
(\Gamma_1)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_1)^{\mathsf{y}}_{\mathsf{c}} \\
(\Gamma_2)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_2)^{\mathsf{y}}_{\mathsf{c}} \\
\vdots \\
(\Gamma_n)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_n)^{\mathsf{y}}_{\mathsf{c}}
\end{gathered}$$

is an $\mathcal{L}$-deduction. Since $\Gamma_n \subseteq \Gamma \subseteq Form_{\mathcal{L}}$, we have $(\Gamma_n)^{\mathsf{y}}_{\mathsf{c}} = \Gamma_n$. Now $(\varphi_n)^{\mathsf{y}}_{\mathsf{c}} = (\varphi^{\mathsf{c}}_{\mathsf{x}})^{\mathsf{y}}_{\mathsf{c}} = \varphi^{\mathsf{y}}_{\mathsf{x}}$. We therefore have $\Gamma_n \vdash_{\mathcal{L}} \varphi^{\mathsf{y}}_{\mathsf{x}}$. We may then use the $\forall I$ rule to conclude that $\Gamma_n \vdash_{\mathcal{L}} \forall \mathsf{x} \varphi$. Finally, $\Gamma \vdash_{\mathcal{L}} \forall \mathsf{x} \varphi$ by the *Super* rule. □

Corollary 6.3.6. *Let $\mathcal{L}$ be a language, let $\Gamma \subseteq Form_{\mathcal{L}}$, and let $\varphi \in Form_{\mathcal{L}}$.*

1. *Let $\mathcal{L}'$ be $\mathcal{L}$ together with a new constant symbol. If $\Gamma \vdash_{\mathcal{L}'} \varphi$, then $\Gamma \vdash_{\mathcal{L}} \varphi$.*
2. *Let $\mathcal{L}'$ be $\mathcal{L}$ together with finitely many new constant symbols. If $\Gamma \vdash_{\mathcal{L}'} \varphi$, then $\Gamma \vdash_{\mathcal{L}} \varphi$.*
3. *Let $\mathcal{L}'$ be $\mathcal{L}$ together with (perhaps infinitely many) new constant symbols. If $\Gamma \vdash_{\mathcal{L}'} \varphi$, then $\Gamma \vdash_{\mathcal{L}} \varphi$.*

Proof.

1. Since $\Gamma \vdash_{\mathcal{L}'} \varphi$, we may use Proposition 5.2.15 to fix an $\mathcal{L}'$-deduction

$$\begin{gathered}
\Gamma_0 \vdash_{\mathcal{L}'} \varphi_0 \\
\Gamma_1 \vdash_{\mathcal{L}'} \varphi_1 \\
\Gamma_2 \vdash_{\mathcal{L}'} \varphi_2 \\
\vdots \\
\Gamma_n \vdash_{\mathcal{L}'} \varphi_n
\end{gathered}$$

   such that each $\Gamma_i \subseteq Form_{\mathcal{L}'}$ is finite, $\Gamma_n \subseteq \Gamma$, and $\varphi_n = \varphi$. Fix $\mathsf{y} \in Var$ such that

$$\mathsf{y} \notin OccurVar(\bigcup_{i=0}^{n}(\Gamma_i) \cup \{\varphi_i\}))$$

   From Lemma 6.3.4, we have that

$$\begin{gathered}
(\Gamma_0)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_0)^{\mathsf{y}}_{\mathsf{c}} \\
(\Gamma_1)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_1)^{\mathsf{y}}_{\mathsf{c}} \\
(\Gamma_2)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_2)^{\mathsf{y}}_{\mathsf{c}} \\
\vdots \\
(\Gamma_n)^{\mathsf{y}}_{\mathsf{c}} \vdash_{\mathcal{L}} (\varphi_n)^{\mathsf{y}}_{\mathsf{c}}
\end{gathered}$$

   is an $\mathcal{L}$-deduction. Since $\Gamma_n \subseteq \Gamma \subseteq Form_{\mathcal{L}}$ and $\varphi \in Form_{\mathcal{L}}$, it follows that $(\Gamma_n)^{\mathsf{y}}_{\mathsf{c}} = \Gamma_n$ and $(\varphi_n)^{\mathsf{y}}_{\mathsf{c}} = \varphi$. Thus, $\Gamma_n \vdash_{\mathcal{L}} \varphi$ and so $\Gamma \vdash_{\mathcal{L}} \varphi$ by the *Super* rule.

2. This is proved by induction on the number of new constant symbols, using part 1 for the base case and the inductive step.

3. Suppose that $\Gamma \vdash_{\mathcal{L}'} \varphi$, and use Proposition 5.2.15 to fix an $\mathcal{L}'$-deduction

$$\begin{gathered} \Gamma_0 \vdash_{\mathcal{L}'} \varphi_0 \\ \Gamma_1 \vdash_{\mathcal{L}'} \varphi_1 \\ \Gamma_2 \vdash_{\mathcal{L}'} \varphi_2 \\ \vdots \\ \Gamma_n \vdash_{\mathcal{L}'} \varphi_n \end{gathered}$$

such that each $\Gamma_i \subseteq Form_{\mathcal{L}'}$ is finite, $\Gamma_n \subseteq \Gamma$, and $\varphi_n = \varphi$. Let $\{\mathsf{c}_0, \mathsf{c}_1, \dots, \mathsf{c}_m\}$ be all constants appearing in $\bigcup_{i=0}^{n}(\Gamma_i \cup \{\varphi_i\})$ and let $\mathcal{L}_0 = \mathcal{L} \cup \{\mathsf{c}_0, \mathsf{c}_1, \dots, \mathsf{c}_m\}$. We then have that

$$\begin{gathered} \Gamma_0 \vdash_{\mathcal{L}_0} \varphi_0 \\ \Gamma_1 \vdash_{\mathcal{L}_0} \varphi_1 \\ \Gamma_2 \vdash_{\mathcal{L}_0} \varphi_2 \\ \vdots \\ \Gamma_n \vdash_{\mathcal{L}_0} \varphi_n \end{gathered}$$

is an $\mathcal{L}_0$-deduction, so $\Gamma_n \vdash_{\mathcal{L}_0} \varphi$. Using the *Super* rule, we conclude that $\Gamma \vdash_{\mathcal{L}_0} \varphi$. Therefore, $\Gamma \vdash_{\mathcal{L}} \varphi$ by part 2.

□

Corollary 6.3.7. *Let $\mathcal{L}$ be a language and let $\mathcal{L}'$ be $\mathcal{L}$ together with (perhaps infinitely many) new constant symbols. Let $\Gamma \subseteq Form_{\mathcal{L}}$. Γ is $\mathcal{L}$-consistent if and only if Γ is $\mathcal{L}'$-consistent.*

Proof. Since any $\mathcal{L}$-deduction is also a $\mathcal{L}'$-deduction, if Γ is $\mathcal{L}$-inconsistent then it is $\mathcal{L}'$-inconsistent . Suppose that Γ is $\mathcal{L}'$-inconsistent. We then have that $\Gamma \vdash_{\mathcal{L}'} \varphi$ for all $\varphi \in Form_{\mathcal{L}}$ by Proposition 5.2.11, hence $\Gamma \vdash_{\mathcal{L}} \varphi$ for all $\varphi \in Form_{\mathcal{L}}$ by Corollary 6.3.6. Therefore, Γ is $\mathcal{L}$-inconsistent. □

Lemma 6.3.8. *Let $\mathcal{L}$ be a language, and let $\mathcal{L}'$ be $\mathcal{L}$ together with a new constant symbol c. Suppose that $\Gamma \subseteq Form_{\mathcal{L}}$ is $\mathcal{L}$-consistent and that $\varphi \in Form_{\mathcal{L}}$. We then have that $\Gamma \cup \{(\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}}\}$ is $\mathcal{L}'$-consistent.*

Proof. Suppose that $\Gamma \cup \{(\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}}\}$ is $\mathcal{L}'$-inconsistent. We then have that $\Gamma \vdash_{\mathcal{L}'} \neg((\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}})$, hence $\Gamma \vdash_{\mathcal{L}'} (\exists \mathsf{x} \varphi) \wedge \neg(\varphi^{\mathsf{c}}_{\mathsf{x}})$ by Corollary 6.2.7 (because $(\neg((\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}}))^{\#} \vDash_{P(\mathcal{L}')} ((\exists \mathsf{x} \varphi) \wedge \neg(\varphi^{\mathsf{c}}_{\mathsf{x}}))^{\#}$). Thus, $\Gamma \vdash_{\mathcal{L}'} \exists \mathsf{x} \varphi$ by the $\wedge EL$ rule, so $\Gamma \vdash_{\mathcal{L}'} \neg \forall \mathsf{x} \neg \varphi$ (by Proposition 5.2.9 and Proposition 5.2.14), and hence $\Gamma \vdash_{\mathcal{L}} \neg \forall \mathsf{x} \neg \varphi$ by Corollary 6.3.6. We also have $\Gamma \vdash_{\mathcal{L}'} (\neg \varphi)^{\mathsf{c}}_{\mathsf{x}}$ by the $\wedge ER$ rule, so $\Gamma \vdash_{\mathcal{L}} \forall \mathsf{x} \neg \varphi$ by Generalization on Constants. This contradicts the fact that Γ is $\mathcal{L}$-consistent. □

Lemma 6.3.9. *Let $\mathcal{L}$ be a language and let $\Gamma \subseteq Form_{\mathcal{L}}$ be $\mathcal{L}$-consistent. There exists a language $\mathcal{L}' \supseteq \mathcal{L}$ and $\Gamma' \subseteq Form_{\mathcal{L}'}$ such that*

1. *$\Gamma \subseteq \Gamma'$.*

2. *Γ' is $\mathcal{L}'$-consistent.*

3. *For all $\varphi \in Form_{\mathcal{L}}$ and all $\mathsf{x} \in Var$, there exists $\mathsf{c} \in \mathcal{C}$ such that $(\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}} \in \Gamma'$.*

Proof. For each $\varphi \in Form_{\mathcal{L}}$ and each $\mathsf{x} \in Var$, let $\mathsf{c}_{\varphi,\mathsf{x}}$ be a new constant symbol (distinct from all symbols in $\mathcal{L}$). Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}_{\varphi,\mathsf{x}} : \varphi \in Form_{\mathcal{L}} \text{ and } \mathsf{x} \in Var\}$. Let

$$\Gamma' = \Gamma \cup \{(\exists \mathsf{x} \varphi) \to \varphi^{\mathsf{c}_{\varphi,\mathsf{x}}}_{\mathsf{x}} : \varphi \in Form_{\mathcal{L}} \text{ and } \mathsf{x} \in Var\}\}$$

Conditions 1 and 3 are clear, so we need only check that Γ' is $\mathcal{L}'$-consistent. By Corollary 5.2.16, it suffices to check that all finite subsets of Γ' are $\mathcal{L}'$-consistent, and for this it suffices to show that

$$\Gamma \cup \{(\exists \mathsf{x}_1 \varphi_1) \to (\varphi_1)^{\mathsf{c}_{\varphi_1,\mathsf{x}_1}}_{\mathsf{x}_1}, (\exists \mathsf{x}_2 \varphi_2) \to (\varphi_2)^{\mathsf{c}_{\varphi_2,\mathsf{x}_2}}_{\mathsf{x}_2}, \dots, (\exists \mathsf{x}_\mathsf{n} \varphi_\mathsf{n}) \to (\varphi_\mathsf{n})^{\mathsf{c}_{\varphi_\mathsf{n},\mathsf{x}_\mathsf{n}}}_{\mathsf{x}_\mathsf{n}}\}$$

is $\mathcal{L}'$-consistent whenever $\varphi_1, \varphi_2, \dots, \varphi_n \in Form_{\mathcal{L}}$ and $\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_n \in Var$. Formally, one can prove this by induction on n. A slightly informal argument is as as follows. Fix $\varphi_1, \varphi_2, \dots, \varphi_n \in Form_{\mathcal{L}}$ and $\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_\mathsf{n} \in Var$. We have

- Γ is $\mathcal{L}$-consistent, so
- $\Gamma \cup \{(\exists \mathsf{x}_1 \varphi_1) \to (\varphi_1)^{\mathsf{c}_{\varphi_1,\mathsf{x}_1}}_{\mathsf{x}_1}\}$ is $(\mathcal{L} \cup \{\mathsf{c}_{\varphi_1,\mathsf{x}_1}\})$-consistent by Lemma 6.3.8, so
- $\Gamma \cup \{(\exists \mathsf{x}_1 \varphi_1) \to (\varphi_1)^{\mathsf{c}_{\varphi_1,\mathsf{x}_1}}_{\mathsf{x}_1}, (\exists \mathsf{x}_2 \varphi_2) \to (\varphi_2)^{\mathsf{c}_{\varphi_2,\mathsf{x}_2}}_{\mathsf{x}_2}\}$ is $(\mathcal{L} \cup \{\mathsf{c}_{\varphi_1,\mathsf{x}_1}, \mathsf{c}_{\varphi_2,\mathsf{x}_2}\})$-consistent by Lemma 6.3.8, so
- ...
- $\Gamma \cup \{(\exists \mathsf{x}_1 \varphi_1) \to (\varphi_1)^{\mathsf{c}_{\varphi_1,\mathsf{x}_1}}_{\mathsf{x}_1}, (\exists \mathsf{x}_2 \varphi_2) \to (\varphi_2)^{\mathsf{c}_{\varphi_2,\mathsf{x}_2}}_{\mathsf{x}_2}, \dots, (\exists \mathsf{x}_\mathsf{n} \varphi_\mathsf{n}) \to (\varphi_\mathsf{n})^{\mathsf{c}_{\varphi_\mathsf{n},\mathsf{x}_\mathsf{n}}}_{\mathsf{x}_\mathsf{n}}\}$ is $(\mathcal{L} \cup \{\mathsf{c}_{\varphi_1,\mathsf{x}_1}, \mathsf{c}_{\varphi_2,\mathsf{x}_2}, \dots, \mathsf{c}_{\varphi_\mathsf{n},\mathsf{x}_\mathsf{n}}\})$-consistent

Therefore,

$$\Gamma \cup \{(\exists \mathsf{x}_1 \varphi_1) \to (\varphi_1)^{\mathsf{c}_{\varphi_1,\mathsf{x}_1}}_{\mathsf{x}_1}, (\exists \mathsf{x}_2 \varphi_2) \to (\varphi_2)^{\mathsf{c}_{\varphi_2,\mathsf{x}_2}}_{\mathsf{x}_2}, \dots, (\exists \mathsf{x}_\mathsf{n} \varphi_\mathsf{n}) \to (\varphi_\mathsf{n})^{\mathsf{c}_{\varphi_\mathsf{n},\mathsf{x}_\mathsf{n}}}_{\mathsf{x}_\mathsf{n}}\}$$

is $\mathcal{L}'$-consistent by Corollary 6.3.7. □

Proposition 6.3.10. *Let $\mathcal{L}$ be a language and let $\Gamma \subseteq Form_{\mathcal{L}}$ be consistent. There exists a language $\mathcal{L}' \supseteq \mathcal{L}$ and $\Gamma' \subseteq Form_{\mathcal{L}'}$ such that*

1. $\Gamma \subseteq \Gamma'$.
2. Γ' *is* $\mathcal{L}'$*-consistent.*
3. Γ' *contains witnesses.*

Proof. Let $\mathcal{L}_0 = \mathcal{L}$ and $\Gamma_0 = \Gamma$. For each $n \in \mathbb{N}$, use the previous lemma to get $\mathcal{L}_{n+1}$ and Γ_{n+1} from $\mathcal{L}_n$ and Γ_n. Set $\mathcal{L}' = \bigcup_{n \in \mathbb{N}} \mathcal{L}$ and set $\Gamma' = \bigcup_{n \in \mathbb{N}} \Gamma_n$. □

Proposition 6.3.11. *(Suppose that $\mathcal{L}$ is countable.) If Γ is consistent, then there exists a set $\Delta \supseteq \Gamma$ which is consistent and complete.*

Proof. Exactly the same proof as the propositional logic case, using Zorn's Lemma in the uncountable case. □

Proposition 6.3.12. *Let $\mathcal{L}$ be a language. If $\Gamma \subseteq \mathcal{L}$ is consistent, then there a language $\mathcal{L}' \supseteq \mathcal{L}$ (which is $\mathcal{L}$ together with new constant symbols) and $\Delta \subseteq Form_{\mathcal{L}'}$ such that*

- $\Gamma \subseteq \Delta$.
- Δ *is consistent.*
- Δ *is complete.*
- Δ *contains witnesses.*

Proof. □

Lemma 6.3.13. *Suppose that Δ is consistent and complete. If $\Delta \vdash \varphi$, then $\varphi \in \Delta$.*

Proof. Suppose that $\Delta \vdash \varphi$. Since Δ is complete, we have that either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$. Now if $\neg\varphi \in \Delta$, then $\Delta \vdash \neg\varphi$, hence Δ is inconsistent contradicting our assumption. It follows that $\varphi \in \Delta$. □

Lemma 6.3.14. *Suppose that Δ is consistent, complete, and contains witnesses. For every $t \in Term_{\mathcal{L}}$, there exists $\mathsf{c} \in \mathcal{C}$ such that $t = \mathsf{c} \in \Delta$.*

Proof. Let $t \in Term_{\mathcal{L}}$. Fix $\mathsf{x} \in Var$ such that $\mathsf{x} \notin OccurVar(t)$. Since Δ contains witnesses, we may fix $\mathsf{c} \in \mathcal{C}$ such that $(\exists\mathsf{x}(t = \mathsf{x})) \rightarrow (t = \mathsf{c}) \in \Delta$ (using the formula $t = \mathsf{x}$). Now $\emptyset \vdash (t = \mathsf{x})^t_{\mathsf{x}}$, so we may use the $\exists I$ rule (because $ValidSubst^t_{\mathsf{x}}(t = \mathsf{x}) = 1$) to conclude that $\emptyset \vdash \exists\mathsf{x}(t = \mathsf{x})$. From here we can use the *Super* rule to conclude that $\Delta \vdash \exists\mathsf{x}(t = \mathsf{x})$. We therefore have $\Delta \vdash \exists\mathsf{x}(t = \mathsf{x})$ and $\Delta \vdash (\exists\mathsf{x}(t = \mathsf{x})) \rightarrow (t = \mathsf{c})$, hence $\Delta \vdash t = \mathsf{c}$ by Proposition 5.2.14. Using Lemma 6.3.13, we conclude that $t = \mathsf{c} \in \Delta$. □

Lemma 6.3.15. *Suppose that Δ is consistent, complete, and contains witnesses. We have*

1. *$\neg\varphi \in \Delta$ if and only if $\varphi \notin \Delta$.*
2. *$\varphi \wedge \psi \in \Delta$ if and only if $\varphi \in \Delta$ and $\psi \in \Delta$.*
3. *$\varphi \vee \psi \in \Delta$ if and only if $\varphi \in \Delta$ or $\psi \in \Delta$.*
4. *$\varphi \rightarrow \psi \in \Delta$ if and only if $\varphi \notin \Delta$ or $\psi \in \Delta$.*
5. *$\exists\mathsf{x}\varphi \in \Delta$ if and only if there exists $\mathsf{c} \in \mathcal{C}$ such that $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$.*
6. *$\forall\mathsf{x}\varphi \in \Delta$ if and only if $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$ for all $\mathsf{c} \in \mathcal{C}$.*

Proof.

1. If $\neg\varphi \in \Delta$, then $\varphi \notin \Delta$ because otherwise $\Delta \vdash \varphi$ and so Δ would be inconsistent.

   Conversely, if $\varphi \notin \Delta$, then $\neg\varphi \in \Delta$ because Δ is complete.

2. Suppose first that $\varphi \wedge \psi \in \Delta$. We then have that $\Delta \vdash \varphi \wedge \psi$, hence $\Delta \vdash \varphi$ by the $\wedge EL$ rule and $\Delta \vdash \psi$ by the $\wedge ER$ rule. Therefore, $\varphi \in \Delta$ and $\psi \in \Delta$ by Lemma 6.3.13.

   Conversely, suppose that $\varphi \in \Delta$ and $\psi \in \Delta$. We then have $\Delta \vdash \varphi$ and $\Delta \vdash \psi$, hence $\Delta \vdash \varphi \wedge \psi$ by the $\wedge I$ rule. Therefore, $\varphi \wedge \psi \in \Delta$ by Lemma 6.3.13.

3. Suppose first that $\varphi \vee \psi \in \Delta$. Suppose that $\varphi \notin \Delta$. Since Δ is complete, we have that $\neg\varphi \in \Delta$. From Proposition 3.4.10, we know that $\{\neg\varphi, \varphi \vee \psi\} \vdash \psi$, hence $\Delta \vdash \psi$ by the *Super* rule. Therefore, $\psi \in \Delta$ by Lemma 6.3.13. It follows that either $\varphi \in \Delta$ or $\psi \in \Delta$.

   Conversely, suppose that either $\varphi \in \Delta$ or $\psi \in \Delta$.

   Case 1: Suppose that $\varphi \in \Delta$. We have $\Delta \vdash \varphi$, hence $\Delta \vdash \varphi \vee \psi$ by the $\vee IL$ rule. Therefore, $\varphi \vee \psi \in \Delta$ by Lemma 6.3.13.

   Case 2: Suppose that $\psi \in \Delta$. We have $\Delta \vdash \psi$, hence $\Delta \vdash \varphi \vee \psi$ by the $\vee IR$ rule. Therefore, $\varphi \vee \psi \in \Delta$ by Lemma 6.3.13.

4. Suppose first that $\varphi \rightarrow \psi \in \Delta$. Suppose that $\varphi \in \Delta$. We then have that $\Delta \vdash \varphi$ and $\Delta \vdash \varphi \rightarrow \psi$, hence $\Delta \vdash \psi$ by Proposition 5.2.14. Therefore, $\psi \in \Delta$ by Lemma 6.3.13. It follows that either $\varphi \notin \Delta$ or $\psi \in \Delta$.

   Conversely, suppose that either $\varphi \notin \Delta$ or $\psi \in \Delta$.

   Case 1: Suppose that $\varphi \notin \Delta$. We have $\neg\varphi \in \Delta$ because Δ is complete, hence $\Delta \cup \{\varphi\}$ is inconsistent (as $\Delta \cup \{\varphi\} \vdash \varphi$ and $\Delta \cup \{\neg\varphi\} \vdash \neg\varphi$). It follows that $\Delta \cup \{\varphi\} \vdash \psi$ by Proposition 5.2.11, hence $\Delta \vdash \varphi \rightarrow \psi$ by the $\rightarrow I$ rule. Therefore, $\varphi \rightarrow \psi \in \Delta$ by Lemma 6.3.13.

   Case 2: Suppose that $\psi \in \Delta$. We have $\psi \in \Delta \cup \{\varphi\}$, hence $\Delta \cup \{\varphi\} \vdash \psi$, and so $\Delta \vdash \varphi \rightarrow \psi$ by the $\rightarrow I$ rule. Therefore, $\varphi \rightarrow \psi \in \Delta$ by Lemma 6.3.13.

5. Suppose first that $\exists \mathsf{x}\varphi \in \Delta$. Since Δ contains witnesses, we may fix $\mathsf{c} \in \mathcal{C}$ such that $(\exists \mathsf{x}\varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$. We therefore have $\Delta \vdash \exists \mathsf{x}\varphi$ and $\Delta \vdash (\exists \mathsf{x}\varphi) \to \varphi^{\mathsf{c}}_{\mathsf{x}}$, hence $\Delta \vdash \varphi^{\mathsf{c}}_{\mathsf{x}}$ by Proposition 5.2.14. Using Lemma 6.3.13, we conclude that $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$.

   Conversely, suppose that there exists $\mathsf{c} \in \mathcal{C}$ such that $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$. We then have $\Delta \vdash \varphi^{\mathsf{c}}_{\mathsf{x}}$, hence $\Delta \vdash \exists \mathsf{x}\varphi$ using the $\exists I$ rule (notice that $ValidSubst^{\mathsf{c}}_{\mathsf{x}}(\varphi) = 1$). Using Lemma 6.3.13, we conclude that $\exists \mathsf{x}\varphi \in \Delta$.

6. Suppose first that $\forall \mathsf{x}\varphi \in \Delta$. We then have $\Delta \vdash \forall \mathsf{x}\varphi$, hence $\Delta \vdash \varphi^{\mathsf{c}}_{\mathsf{x}}$ for all $\mathsf{c} \in \mathcal{C}$ using the $\forall E$ rule (notice that $ValidSubst^{\mathsf{c}}_{\mathsf{x}}(\varphi) = 1$ for all $\mathsf{c} \in \mathcal{C}$). Using Lemma 6.3.13, we conclude that $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$ for all $c \in \mathcal{C}$.

   Conversely, suppose that $\varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta$ for all $\mathsf{c} \in \mathcal{C}$. Since Δ is consistent, this implies that there does not exist $\mathsf{c} \in \mathcal{C}$ with $\neg(\varphi^{\mathsf{c}}_{\mathsf{x}}) = (\neg\varphi)^{\mathsf{c}}_{\mathsf{x}} \in \Delta$. Therefore, $\exists \mathsf{x}\neg\varphi \notin \Delta$ by part 5, so $\neg\exists \mathsf{x}\neg\varphi \in \Delta$ by part 1. It follows from Proposition 5.2.10 that $\Delta \vdash \forall \mathsf{x}\varphi$. Using Lemma 6.3.13, we conclude that $\forall \mathsf{x}\varphi \in \Delta$.

□

Proposition 6.3.16. *If Δ is consistent, complete, and contains witnesses, then Δ is satisfiable.*

Proof. Suppose that Δ is consistent, complete, and contains witnesses.

Define a relation $\sim$ on $Term_{\mathcal{L}}$ by letting $t \sim u$ if $t = u \in \Delta$. We first check that $\sim$ is an equivalence relation. Reflexivity follows from the $EqRefl$ rule and Lemma 6.3.13. Symmetry and transitivity follow from Proposition 5.2.5 and Proposition 5.2.6, together with the $Super$ rule and Lemma 6.3.13).

We now define our $\mathcal{L}$-structure $\mathcal{M}$. We first let $M = Term_{\mathcal{L}}/\sim$. For each $t \in Term_{\mathcal{L}}$, we let $[t]$ denote the equivalence class of t. Notice that $M = \{[\mathsf{c}] : \mathsf{c} \in \mathcal{C}\}$ be Lemma 6.3.14. We now finish our description of the $\mathcal{L}$-structure $\mathcal{M}$ by saying how to interpret the constant, relation, and function symbols. We let

1. $\mathsf{c}^{\mathcal{M}} = [\mathsf{c}]$ for all $\mathsf{c} \in \mathcal{C}$.

2. $\mathsf{R}^{\mathcal{M}} = \{([t_1], [t_2], \dots, [t_k]) \in M^k : \mathsf{R}t_1t_2\cdots t_k \in \Delta\}$ for all $\mathsf{R} \in \mathcal{R}_k$.

3. $\mathsf{f}^{\mathcal{M}}([t_1], [t_2], \dots, [t_k]) = [\mathsf{f}t_1t_2\cdots t_k]$ for all $\mathsf{f} \in \mathcal{F}_k$.

Notice that our definitions of $\mathsf{R}^{\mathcal{M}}$ do not depend on our choice of representatives for the equivalence classes by Proposition 5.2.7. Similarly, our definitions of $\mathsf{f}^{\mathcal{M}}$ do not depend on our choice of representatives for the equivalences classes by Proposition 5.2.8. Finally, define $s \colon Var \to M$ by letting $s(\mathsf{x}) = [\mathsf{x}]$ for all $\mathsf{x} \in Var$.

We first show that $\overline{s}(t) = [t]$ for all $t \in Term_{\mathcal{L}}$ by induction. We have $\overline{s}(\mathsf{c}) = \mathsf{c}^{\mathcal{M}} = [\mathsf{c}]$ for all $\mathsf{c} \in \mathcal{C}$ and $\overline{s}(\mathsf{x}) = s(\mathsf{x}) = [\mathsf{x}]$ for all $\mathsf{x} \in Var$. Suppose that $\mathsf{f} \in \mathcal{F}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$ are such that $\overline{s}(t_i) = [t_i]$ for all i. We then have

$$\begin{aligned} \overline{s}(\mathsf{f}t_1t_2\cdots t_k) &= \mathsf{f}^{\mathcal{M}}(\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k)) \\ &= f^{\mathcal{M}}([t_1], [t_2], \dots, [t_k]) && \text{(by induction)} \\ &= [\mathsf{f}t_1t_2\cdots t_k] \end{aligned}$$

Therefore, $\overline{s}(t) = [t]$ for all $t \in Term_{\mathcal{L}}$.

We now show that $\varphi \in \Delta$ if and only if $(\mathcal{M}, s) \vDash \varphi$ for all $\varphi \in Form_{\mathcal{L}}$ by induction. We first prove the result for $\varphi \in AtomicForm_{\mathcal{L}}$. Suppose that $\mathsf{R} \in \mathcal{R}_k$ and $t_1, t_2, \dots, t_k \in Term_{\mathcal{L}}$. We have

$$\begin{aligned} \mathsf{R}t_1t_2\cdots t_k \in \Delta &\Leftrightarrow ([t_1], [t_2], \dots, [t_k]) \in \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow (\overline{s}(t_1), \overline{s}(t_2), \dots, \overline{s}(t_k)) \in \mathsf{R}^{\mathcal{M}} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \mathsf{R}t_1t_2\cdots t_k \end{aligned}$$

Suppose now that $t_1, t_2 \in Term_{\mathcal{L}}$. We have

$$\begin{aligned} t_1 = t_2 \in \Delta &\Leftrightarrow [t_1] = [t_2] \\ &\Leftrightarrow \overline{s}(t_1) = \overline{s}(t_2) \\ &\Leftrightarrow (\mathcal{M}, s) \vDash t_1 = t_2 \end{aligned}$$

Suppose that the result holds for φ. We have

$$\begin{aligned} \neg\varphi \in \Delta &\Leftrightarrow \varphi \notin \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow (\mathcal{M}, s) \nvDash \varphi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \end{aligned}$$

Suppose that the result holds for φ and ψ. We have

$$\begin{aligned} \varphi \wedge \psi \in \Delta &\Leftrightarrow \varphi \in \Delta \text{ and } \psi \in \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \text{ and } (\mathcal{M}, s) \vDash \psi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \wedge \psi \end{aligned}$$

and

$$\begin{aligned} \varphi \vee \psi \in \Delta &\Leftrightarrow \varphi \in \Delta \text{ or } \psi \in \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \text{ or } (\mathcal{M}, s) \vDash \psi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \vee \psi \end{aligned}$$

and finally

$$\begin{aligned} \varphi \rightarrow \psi \in \Delta &\Leftrightarrow \varphi \notin \Delta \text{ or } \psi \in \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow (\mathcal{M}, s) \nvDash \varphi \text{ or } (\mathcal{M}, s) \vDash \psi && \text{(by induction)} \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \varphi \rightarrow \psi \end{aligned}$$

Suppose that the result holds for φ and that $\mathsf{x} \in Var$. We have

$$\begin{aligned} \exists \mathsf{x}\varphi \in \Delta &\Leftrightarrow \text{There exists } \mathsf{c} \in \mathcal{C} \text{ such that } \varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow \text{There exists } \mathsf{c} \in \mathcal{C} \text{ such that } (\mathcal{M}, s) \vDash \varphi^{\mathsf{c}}_{\mathsf{x}} && \text{(by induction)} \\ &\Leftrightarrow \text{There exists } \mathsf{c} \in \mathcal{C} \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(\mathsf{c})]) \vDash \varphi && \text{(by the Substitution Theorem)} \\ &\Leftrightarrow \text{There exists } \mathsf{c} \in \mathcal{C} \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow [\mathsf{c}]]) \vDash \varphi \\ &\Leftrightarrow \text{There exists } a \in M \text{ such that } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \exists \mathsf{x}\varphi \end{aligned}$$

and also

$$\begin{aligned} \forall x\varphi \in \Delta &\Leftrightarrow \text{For all } \mathsf{c} \in \mathcal{C}, \text{ we have } \varphi^{\mathsf{c}}_{\mathsf{x}} \in \Delta && \text{(by Lemma 6.3.15)} \\ &\Leftrightarrow \text{For all } \mathsf{c} \in \mathcal{C}, \text{ we have } (\mathcal{M}, s) \vDash \varphi^{\mathsf{c}}_{\mathsf{x}} && \text{(by induction)} \\ &\Leftrightarrow \text{For all } \mathsf{c} \in \mathcal{C}, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow \overline{s}(\mathsf{c})]) \vDash \varphi && \text{(by the Substitution Theorem)} \\ &\Leftrightarrow \text{For all } \mathsf{c} \in \mathcal{C}, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow [\mathsf{c}]]) \\ &\Leftrightarrow \text{For all } a \in M, \text{ we have } (\mathcal{M}, s[\mathsf{x} \Rightarrow a]) \vDash \varphi \\ &\Leftrightarrow (\mathcal{M}, s) \vDash \forall \mathsf{x}\varphi \end{aligned}$$

Therefore, by induction, we have $\varphi \in \Delta$ if and only if $(\mathcal{M}, s) \vDash \varphi$. In particular, we have $(\mathcal{M}, s) \vDash \varphi$ for all $\varphi \in \Delta$, hence Δ is satisfiable. □

Theorem 6.3.17 (Completeness Theorem)**.** *(Suppose that $\mathcal{L}$ is countable.)*

1. *Every consistent set of formulas is satisfiable.*

2. *If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.*

Proof.

1. Suppose that Γ is consistent. By Proposition 6.3.12, we may fix a language $\mathcal{L}' \supseteq \mathcal{L}$ and $\Delta \subseteq Form_{\mathcal{L}'}$ such that $\Delta \supseteq \Gamma$ is consistent, complete, and contains witnesses. Now Δ is satisfiable by Proposition 6.3.16, so we may fix an $\mathcal{L}'$-structure $\mathcal{M}'$ together with $s\colon Var \to M'$ such that $(\mathcal{M}', s) \vDash \varphi$ for all $\varphi \in \Delta$. We then have $(\mathcal{M}', s) \vDash \gamma$ for all $\gamma \in \Gamma$. Letting $\mathcal{M}$ be the restiction of $\mathcal{M}'$ to $\mathcal{L}$, we then have $(\mathcal{M}, s) \vDash \gamma$ for all $\gamma \in \Gamma$. Therefore, Γ is satisfiable.

2. Suppose that $\Gamma \vDash \varphi$. We then have that $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable, hence $\Gamma \cup \{\neg\varphi\}$ is inconsistent by part 1. It follows from Proposition 5.2.12 that $\Gamma \vdash \varphi$.

□

We now give another proof of the Countable Lowenheim-Skolem Theorem which does not go through the concept of elementary substructures.

Corollary 6.3.18 (Countable Lowenheim-Skolem Theorem)**.** *Suppose that $\mathcal{L}$ is countable and $\Gamma \subseteq Form_{\mathcal{L}}$ is consistent. There exists a countable model of Γ.*

Proof. Notice that if $\mathcal{L}$ is consistent, then the $\mathcal{L}'$ formed in Lemma 6.3.9 is countable because $Form_{\mathcal{L}} \times Var$ is countable. Thus, each $\mathcal{L}_n$ in the proof of Proposition 6.3.10 is countable, so the $\mathcal{L}'$ formed in Proposition 6.3.10 is countable. It follows that $Term_{\mathcal{L}'}$ is countable, and since the $\mathcal{L}'$-structure $\mathcal{M}$ we construct in the proof of Proposition 6.3.16 is formed by taking the quotient from an equivalence relation on the countable $Term_{\mathcal{L}'}$, we can conclude that $\mathcal{M}$ is countable. Therefore, the $\mathcal{L}$-structure which is the restriction of $\mathcal{M}$ to $\mathcal{L}$ from the proof of the Completeness Theorem is countable. □

6.4 Compactness

Corollary 6.4.1 (Compactness Theorem)**.**

1. *If $\Gamma \vDash \varphi$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi$.*

2. *If every finite subset of Γ is satisfiable, then Γ is satisfiable.*

Proof.

1. Suppose that $\Gamma \vDash \varphi$. By the Completeness Theorem, we have $\Gamma \vdash \varphi$. Using Proposition 5.2.15, we may fix a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$. By the Soundness Theorem, we have $\Gamma_0 \vDash \varphi$.

2. If every finite subset of Γ is satisfiable, then every finite subset of Γ is consistent by the Soundness Theorem, hence Γ is consistent by Corollary 5.2.16, and so Γ is satisfiable by the Soundness Theorem.

□

6.5 Applications of Compactness

The next proposition is another result which expresses that first-order logic is not powerful enough to distinguish certain aspects of cardinality. Here the distinction is between large finite numbers and the infinite.

Proposition 6.5.1. *Let $\mathcal{L}$ be a language. Suppose that $\Gamma \subseteq Form_{\mathcal{L}}$ is such that for all $n \in \mathbb{N}$, there exists a model $(\mathcal{M}, s)$ of Γ such that $|M| > n$. We then have that there exists a model $(\mathcal{M}, s)$ of Γ such that M is infinite.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}_k : k \in \mathbb{N}\}$ where the c_k are new distinct constant symbols. Let

$$\Gamma' = \Gamma \cup \{\mathsf{c}_k \neq \mathsf{c}_\ell : k, \ell \in \mathbb{N} \text{ and } k \neq \ell\}$$

We claim that every finite subset of Γ' is satisfiable. Fix a finite $\Gamma'_0 \subseteq \Gamma'$. Fix $N \in \mathbb{N}$ such that

$$\Gamma'_0 \subseteq \Gamma \cup \{\mathsf{c}_k \neq \mathsf{c}_\ell : k, \ell \leq N \text{ and } k \neq \ell\}$$

By assumption, we may fix a model $(\mathcal{M}, s)$ of Γ such that $|M| > N$. Let $\mathcal{M}'$ be the $\mathcal{L}'$ structure $\mathcal{M}$ together with interpreting the constants $\mathsf{c}_0, \mathsf{c}_1, \dots, \mathsf{c}_N$ as distinct elements of M and interpreting each c_i for $i > N$ arbitrarily. We then have $(\mathcal{M}', s)$ is a model of Γ'. Hence, every finite subset of Γ' is satisfiable.

By the Compactness Theorem we may conclude that Γ' is satisfiable. Fix a model $(\mathcal{M}', s)$ of Γ'. If we let $\mathcal{M}$ be the restriction of $\mathcal{M}'$ to $\mathcal{L}$, then $(\mathcal{M}, s)$ is a model of Γ which is infinite. □

Corollary 6.5.2. *The class of all finite groups is not a weak elementary class in the language $\mathcal{L} = \{F, e\}$.*

Proof. If $\Sigma \subseteq Sent_{\mathcal{L}}$ is such that $Mod(\Sigma)$ includes all finite groups, then we may use the trivial fact that there are arbitrarily large finite groups and Proposition 6.5.1 to conclude that it contains an infinite structure. □

Proposition 6.5.3. *Let $\mathcal{L}$ be a language. Suppose that $\Gamma \subseteq Form_{\mathcal{L}}$ is such there exists a model $(\mathcal{M}, s)$ of Γ with M infinite. We then have that there exists a model $(\mathcal{M}, s)$ of Γ such that M is uncountable.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}_r : r \in \mathbb{R}\}$ where the c_r are new distinct constant symbols. Let

$$\Gamma' = \Gamma \cup \{\mathsf{c}_r \neq \mathsf{c}_t : r, t \in \mathbb{R} \text{ and } r \neq t\}$$

We claim that every finite subset of Γ' is satisfiable. Fix a finite $\Gamma'_0 \subseteq \Gamma'$. Fix a finite $Z \subseteq \mathbb{R}$ such that

$$\Gamma'_0 \subseteq \Gamma \cup \{\mathsf{c}_r \neq \mathsf{c}_t : r, t \in Z\}$$

By assumption, we may fix a model $(\mathcal{M}, s)$ of Γ such that M is infinite. Let $\mathcal{M}'$ be the $\mathcal{L}'$ structure $\mathcal{M}$ together with interpreting the constants c_r for $r \in Z$ as distinct elements of M and interpreting each c_t for $t \notin Z$ arbitrarily. We then have $(\mathcal{M}', s)$ is a model of Γ'. Hence, every finite subset of Γ' is satisfiable.

By the Compactness Theorem we may conclude that Γ' is satisfiable. Fix a model $(\mathcal{M}', s)$ of Γ'. If we let $\mathcal{M}$ be the restriction of $\mathcal{M}'$ to $\mathcal{L}$, then $(\mathcal{M}, s)$ is a model of Γ which is uncountable. □

Proposition 6.5.4. *The class $\mathcal{K}$ of all torsion groups is not a weak elementary class in the language $\mathcal{L} = \{\mathsf{f}, \mathsf{e}\}$.*

Proof. Suppose that $\Sigma \subseteq Sent_{\mathcal{L}}$ is such that $\mathcal{K} \subseteq Mod(\Sigma)$. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}\}$ where c is new constant symbol. For each $n \in \mathbb{N}^+$, let $\tau_n \in Sent_{\mathcal{L}'}$ be $\mathsf{c}^n \neq e$ (more formally, $\mathsf{fcfc} \cdots \mathsf{fcc}$ where there are $n-1$ f's). Let

$$\Sigma' = \Sigma \cup \{\tau_n : n \in \mathbb{N}\}$$

We claim that every finite subset of Σ' has a model. Suppose that $\Sigma_0 \subseteq \Sigma'$ is finite. Fix $N \in \mathbb{N}$ such that

$$\Sigma_0 \subseteq \Sigma \cup \{\tau_n : n < N\}$$

Notice that if we let $\mathcal{M}'$ be the group $\mathbb{Z}/N\mathbb{Z}$ and let $\mathsf{c}^{\mathcal{M}'} = \overline{1}$, then $\mathcal{M}'$ is a model of Σ_0. Thus, every finite subset of Σ' has a model, so Σ' has a model by Compactness. If we restrict this model to $\mathcal{L}$, we get an element of $Mod(\Sigma)$ which is not int $\mathcal{K}$ because it has an element of infinite order. ☐

Proposition 6.5.5. *The class $\mathcal{K}$ of all equivalence relations in which all equivalence classes are finite is not a weak elementary class in the language $\mathcal{L} = \{\mathsf{R}\}$.*

Proof. Suppose that $\Sigma \subseteq Sent_{\mathcal{L}}$ is such that $\mathcal{K} \subseteq Mod(\Sigma)$. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}\}$ where c is new constant symbol. For each $n \in \mathbb{N}^+$, let $\tau_n \in Sent_{\mathcal{L}'}$ be

$$\exists \mathsf{x}_1 \exists \mathsf{x}_2 \cdots \exists \mathsf{x}_n (\bigwedge_{1 \le i < j \le n} (\mathsf{x}_i \neq \mathsf{x}_j) \wedge \bigwedge_{i=1}^{n} \mathsf{R}\mathsf{c}\mathsf{x}_i)$$

and let

$$\Sigma' = \Sigma \cup \{\tau_n : n \in \mathbb{N}\}$$

We claim that every finite subset of Σ' has a model. Suppose that $\Sigma_0 \subseteq \Sigma'$ is finite. Fix $N \in \mathbb{N}$ such that

$$\Sigma_0 \subseteq \Sigma \cup \{\tau_n : n \le N\}$$

Notice that if we let $M' = \{0, 1, 2, \dots, N\}$, $\mathsf{R}^{\mathcal{M}'} = (M')^2$, and $\mathsf{c}^{\mathcal{M}'} = 0$, then $\mathcal{M}'$ is a model of Σ_0. Thus, every finite subset of Σ' has a model, so Σ' has a model by Compactness. If we restrict this model to $\mathcal{L}$, we get an element of $Mod(\Sigma)$ which is not int $\mathcal{K}$ because it has an infinite equivalence class. ☐

Proposition 6.5.6. *Suppose that $\mathcal{K}$ is an elementary class, that $\Sigma \subseteq Sent_{\mathcal{L}}$, and that $\mathcal{K} = Mod(\Sigma)$. There exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\mathcal{K} = Mod(\Sigma_0)$.*

Proof. Since $\mathcal{K}$ is an elementary class, we may fix $\tau \in Sent_{\mathcal{L}}$ with $\mathcal{K} = Mod(\tau)$. We then have $\Sigma \vDash \tau$, so by the Compactness Theorem we may fix a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$. Notice that $\mathcal{K} = Mod(\Sigma_0)$. ☐

Corollary 6.5.7. *The class $\mathcal{K}$ of all fields of characteristic 0 is a weak elementary class, but not an elementary class, in the language $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +, \cdot\}$.*

Proof. We already know that $\mathcal{K}$ is a weak elementary class because if we let σ be the conjunction of the fields axioms and let τ_n be $\mathsf{1} + \mathsf{1} + \cdots + \mathsf{1} \neq \mathsf{0}$ (where there are n $\mathsf{1}$'s) for each $n \in \mathbb{N}^+$, then $\mathcal{K} = Mod(\Sigma)$ where

$$\Sigma = \{\sigma\} \cup \{\tau_n : n \in \mathbb{N}^+\}$$

Suppose that $\mathcal{K}$ is an elementary class. By the previous proposition, we may fix a finite $\Sigma_0 \subseteq \Sigma$ such that $\mathcal{K} = Mod(\Sigma_0)$. Fix $N \in \mathbb{N}$ such that

$$\Sigma_0 \subseteq \{\sigma\} \cup \{\tau_n : n \le N\}$$

Now if fix a prime $p > N$ we see that $(\mathbb{F}_p, 0, 1, +, \cdot)$ (the field with p elements) is a model of Σ_0 which is not an element of $\mathcal{K}$. This is a contradiction, so $\mathcal{K}$ is not an elementary class. ☐

6.6 An Example: The Random Graph

Throughout this section, we work in the language $\mathcal{L} = \{\mathsf{R}\}$ where R is binary relation symbol. We think of graphs as $\mathcal{L}$-structures which are model of $\{\forall \mathsf{x} \neg \mathsf{Rxx}, \forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \to \mathsf{Ryx})\}$.

Definition 6.6.1. *For each $n \in \mathbb{N}^+$, let $\mathcal{G}_n$ be the set of of all models of $\{\forall \mathsf{x} \neg \mathsf{Rxx}, \forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \to \mathsf{Ryx})\}$ with universe $[n]$.*

Definition 6.6.2. *For each $\mathcal{A} \subseteq \mathcal{G}_n$, we let*

$$Pr_n(\mathcal{A}) = \frac{|\mathcal{A}|}{|\mathcal{G}_n|}$$

For each $\sigma \in Sent_{\mathcal{L}}$, we let

$$Pr_n(\sigma) = \frac{|\{\mathcal{M} \in \mathcal{G}_n : \mathcal{M} \vDash \sigma\}|}{|\mathcal{G}_n|}$$

We use the suggestive Pr because we think of constucting a graph randomly by flipping a fair coin for each pair i, j to determine whether or not there is an edge linking them. In this context, $Pr_n(A)$ is the probability the graph so constucted on $\{1, 2, \dots, n\}$ is in A. Notice that given i, j and i', j' distinct two element subsets of $\{1, 2, \dots, n\}$, the question of whether there is an edge linking i, j and the question of whether there is an edge linking i', j' are independent. We aim to prove the following.

Theorem 6.6.3. *For all $\sigma \in Sent_{\mathcal{L}}$, either $\lim_{n\to\infty} Pr_n(\sigma) = 1$ or $\lim_{n\to\infty} Pr_n(\sigma) = 0$.*

Definition 6.6.4. *For each $r, s \in \mathbb{N}$ with $\max\{r, s\} > 0$, let $\sigma_{r,s}$ be the sentence*

$$\forall \mathsf{x}_1 \forall \mathsf{x}_2 \cdots \forall \mathsf{x}_r \forall \mathsf{y}_1 \forall \mathsf{y}_2 \cdots \forall \mathsf{y}_s \Big(\bigwedge_{1 \le i < j \le r} (\mathsf{x}_i \neq \mathsf{x}_j) \wedge \bigwedge_{1 \le i < j \le s} (\mathsf{y}_i \neq \mathsf{y}_j) \wedge \bigwedge_{i=1}^{r} \bigwedge_{j=1}^{s} (\mathsf{x}_i \neq \mathsf{y}_j)$$
$$\to \exists \mathsf{z} \Big(\bigwedge_{i=1}^{r} (\mathsf{z} \neq \mathsf{x}_i) \wedge \bigwedge_{j=1}^{s} (\mathsf{z} \neq \mathsf{y}_j) \wedge \bigwedge_{i=1}^{r} \mathsf{Rx}_i\mathsf{z} \wedge \bigwedge_{j=1}^{s} \neg \mathsf{Ry}_j\mathsf{z} \Big) \Big)$$

Proposition 6.6.5. *For all $r, s \in \mathbb{N}$ with $\max\{r, s\} > 0$, we have $\lim_{n\to\infty} Pr_n(\sigma_{r,s}) = 1$.*

Proof. Fix $r, s \in \mathbb{N}$. Suppose that $n \in \mathbb{N}$ with $n > r, s$. Fix distinct $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s \in \{1, 2, \dots, n\}$. For each c distinct from the a_i and b_j, let

$$\mathcal{A}_c = \{\mathcal{M} \in \mathcal{G}_n : c \text{ is linked to each } a_i \text{ and to no } b_j\}$$

For each such c, we have $Pr_n(\mathcal{A}_c) = \frac{1}{2^{r+s}}$, so $Pr_n(\overline{\mathcal{A}_c}) = 1 - \frac{1}{2^{r+s}}$ and hence the probability that no c works is

$$(1 - \frac{1}{2^{r+s}})^{n-r-s}$$

Therefore,

$$\begin{aligned} Pr_n(\neg \sigma_{r,s}) &\le \binom{n}{r}\binom{n-r}{s}(1 - \frac{1}{2^{r+s}})^{n-r-s} \\ &\le n^{r+s}(1 - \frac{1}{2^{r+s}})^{n-r-s} \\ &= (1 - \frac{1}{2^{r+s}})^{-r-s} \cdot n^{r+s}(1 - \frac{1}{2^{r+s}})^{n} \\ &= (1 - \frac{1}{2^{r+s}})^{-r-s} \cdot n^{r+s}(\frac{2^{r+s} - 1}{2^{r+s}})^{n} \\ &= (1 - \frac{1}{2^{r+s}})^{-r-s} \cdot \frac{n^{r+s}}{(\frac{2^{r+s}}{2^{r+s}-1})^{n}} \end{aligned}$$

Thus, $\lim_{n\to\infty} Pr_n(\neg\sigma_{r,s}) = 0$, and hence $\lim_{n\to\infty} Pr_n(\sigma_{r,s}) = 1$. $\square$

Proposition 6.6.6. *Let* $\Sigma = \{\forall \mathsf{x}\neg\mathsf{Rxx}, \forall\mathsf{x}\forall\mathsf{y}(\mathsf{Rxy} \to \mathsf{Ryx})\} \cup \{\sigma_{r,s} : r, s \in \mathbb{N}^+$ *and* $\max\{r, s\} > 0\}$ *and let* $RG = Cn(\Sigma)$.

Proposition 6.6.7. *RG is satisfiable.*

Proof. We build a countable model $\mathcal{M}$ of RG with $M = \mathbb{N}$. Notice first that since $\mathcal{P}_{fin}(\mathbb{N})$ (the set of all finite subsets of $\mathbb{N}$) is countable, so is the set $\mathcal{P}_{fin}(\mathbb{N})^2$, Hence the set

$$\{(A, B) \in \mathcal{P}_{fin}(\mathbb{N})^2 : A \cap B = \emptyset \text{ and } A \cup B \neq \emptyset\}$$

is countable. Therefore, we may list it as

$$(A_1, B_1), (A_2, B_2), (A_3, B_3), \ldots$$

and furthermore we may assume that $\max(A_n \cup B_n) < n$ for all $n \in \mathbb{N}$. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{N}$ and $\mathsf{R}^{\mathcal{M}} = \{(k, n) : k \in A_n\} \cup \{(n, k) : k \in A_n\}$. Suppose now that $A, B \subseteq \mathbb{N}$ are finite with $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$. Fix $n \in \mathbb{N}$ with $A = A_n$. We then have that $(k, n) \in \mathsf{R}^{\mathcal{M}}$ for all $k \in A$ and $(\ell, n) \notin \mathsf{R}^{\mathcal{M}}$ for all $\ell \in B$. Fix $n \in \mathbb{N}$ with $A = A_n$ and $B = B_n$. We then have that $(k, n) \in \mathsf{R}^{\mathcal{M}}$ for all $k \in A$ (because $k \in A_n$) and $(\ell, n) \notin \mathsf{R}^{\mathcal{M}}$ for all $\ell \in B$ (because $\ell \notin A_n$ and $n \notin A_\ell$ since $\ell < n$). Therefore, $\mathcal{M} \vDash \sigma_{r,s}$ for all $r, s \in \mathbb{N}$ with $\max r, s > 0$. Thus, $\mathcal{M}$ is a model of RG. $\square$

Theorem 6.6.8. *All models of RG are infinite, and any two countable models of RG are isomorphic.*

Proof. Suppose that $\mathcal{M}$ is model of RG which is finite. Let $n = |M|$. Since $\mathcal{M} \vDash \sigma_{n,0}$, there exists $b \in M$ such that $(b, a) \in \mathsf{R}^{\mathcal{M}}$ for all $a \in M$. However, this is a contradiction because $(a, a) \notin \mathsf{R}^{\mathcal{M}}$ for all $a \in M$. It follows that all models of RG are infinite.

Suppose now that $\mathcal{M}$ and $\mathcal{N}$ are two countable models of RG. From above, we know that M and N are both countably infinite. List M as $m_0, m_1, m_2, \ldots$ and list N as $n_0, n_1, n_2, \ldots$. We build an isomorphism via a back-and-forth construction as in the proof of the corresponding result for DLO. That is, we define $\sigma_k \in \mathcal{P}_{fin}(M \times N)$ for $k \in \mathbb{N}$ recursively such that

1. $\sigma_k \subseteq \sigma_{k+1}$.
2. If $(m, n) \in \sigma_k$ and $(m', n) \in \sigma_k$, then $m = m'$.
3. If $(m, n) \in \sigma_k$ and $(m, n') \in \sigma_k$, then $n = n'$.
4. $m_i \in dom(\sigma_{2i})$.
5. $n_j \in ran(\sigma_{2j+1})$.
6. If $(m, n) \in \sigma$ and $(m', n') \in \sigma$, then $(m, m') \in \mathsf{R}^{\mathcal{M}}$ if and only if $(n, n') \in \mathsf{R}^{\mathcal{N}}$

Suppose that we are successful. Define $h\colon M \to N$ be letting $h(m)$ be the unique n such that $(m, n) \in \bigcup_{k\in\mathbb{N}} \sigma_k$, and notice that h is isomorphism.

We now define the σ_k. Let $\sigma_0 = (m_0, n_0)$. Suppose that $k \in \mathbb{N}$ and we've defined σ_k. Suppose first that k is odd, say $k = 2i + 1$. If $m_i \in dom(\sigma_k)$, let $\sigma_{k+1} = \sigma_k$. Suppose then that $m_i \notin dom(\sigma_k)$. Let $A = \{m \in dom(\sigma_k) : (m, m_i) \in \mathsf{R}^{\mathcal{M}}\}$ and let $B = \{m \in dom(\sigma_k) : (m, m_i) \notin \mathsf{R}^{\mathcal{M}}\}$. Since $\mathcal{N}$ is a model of RG and $A \cap B = \emptyset$, we may fix $n \in N \backslash ran(\sigma_k)$ such that $(\sigma_k(m), n) \in \mathsf{R}^{\mathcal{M}}$ for all $m \in A$ and $(\sigma_k(m), n) \notin \mathsf{R}^{\mathcal{M}}$ for all $m \in B$. Let $\sigma_{k+1} = \sigma_k \cup \{(m_i, n)\}$.

Suppose now that k is even, say $k = 2j$. If $n_j \in ran(\sigma_k)$, let $\sigma_{k+1} = \sigma_k$. Suppose then that $n_j \notin ran(\sigma_k)$. Let $A = \{n \in ran(\sigma_k) : (n, n_j) \in \mathsf{R}^{\mathcal{N}}\}$ and let $B = \{n \in ran(\sigma_k) : (n, n_j) \notin \mathsf{R}^{\mathcal{N}}\}$. Since $\mathcal{M}$ is a model of RG and $A \cap B = \emptyset$, we may fix $m \in M \backslash dom(\sigma_k)$ such that $(\sigma_k^{-1}(n), m) \in \mathsf{R}^{\mathcal{M}}$ for all $n \in A$ and $(\sigma_k^{-1}(n), m) \notin \mathsf{R}^{\mathcal{M}}$ for all $n \in B$. Let $\sigma_{k+1} = \sigma_k \cup \{(m, n_j)\}$. $\square$

Corollary 6.6.9. *RG is a complete theory.*

Proof. Apply the Countable Los-Vaught Test. □

Theorem 6.6.10. *Let $\tau \in Sent_{\mathcal{L}}$.*

1. *If $\tau \in RG$, then $\lim_{n\to\infty} Pr_n(\tau) = 1$.*
2. *If $\tau \notin RG$, then $\lim_{n\to\infty} Pr_n(\tau) = 0$.*

Proof.

1. Suppose that $\tau \in RG$. We then have $\Sigma \vDash \tau$, so by Compactness we may fix $N \in \mathbb{N}$ such that

   $$\{\forall \mathsf{x} \neg \mathsf{Rxx}, \forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \rightarrow \mathsf{Ryx})\} \cup \{\sigma_{r,s} : r, s \leq N\} \vDash \tau$$

   We then have that if $\mathcal{M} \in \mathcal{G}_n$ is such that $\mathcal{M} \vDash \neg\tau$, then

   $$\mathcal{M} \vDash \bigvee_{0 \leq r,s \leq N, \max\{r,s\} > 0} \neg\sigma_{r,s}$$

   Hence for every $n \in \mathbb{N}$ we have

   $$Pr_n(\neg\tau) \leq \sum_{0 \leq r,s \leq N, \max\{r,s\} > 0} Pr_n(\neg\sigma_{r,s})$$

   Therefore, $\lim_{n\to\infty} Pr_n(\neg\tau) = 0$, and hence $\lim_{n\to\infty} Pr_n(\tau) = 1$.

2. Suppose that $\tau \notin RG$. Since RG is complete, it follows that $\neg\tau \in RG$. Thus, $\lim_{n\to\infty} Pr_n(\neg\tau) = 1$ by part 1, and hence $\lim_{n\to\infty} Pr_n(\tau) = 0$.

□

Chapter 7

Quantifier Elimination

7.1 Motivation and Definition

Quantifiers make life hard, so it's always nice when we can find a way to express a statement involving quantifiers using an equivalent statement without quantifiers.

Examples.

1. Let $\mathcal{L} = \{0, 1, +, \cdot\}$ and let $\varphi(\mathsf{a}, \mathsf{b}, \mathsf{c})$ (where $\mathsf{a}, \mathsf{b}, \mathsf{c} \in Var$) be the formula
$$\exists \mathsf{x}(\mathsf{a}\mathsf{x}^2 + \mathsf{b}\mathsf{x} + \mathsf{c} = 0)$$
or more formally
$$\exists \mathsf{x}(\mathsf{a} \cdot \mathsf{x} \cdot \mathsf{x} + \mathsf{b} \cdot \mathsf{x} + \mathsf{c} = 0)$$
Let $\mathcal{M}$ be the $\mathcal{L}$-structure $(\mathbb{C}, 0, 1, +, \cdot)$. Since $\mathbb{C}$ is algebraically closed, we have that
$$(\mathcal{M}, s) \vDash \varphi \leftrightarrow (\mathsf{a} \neq 0 \vee \mathsf{b} \neq 0 \vee \mathsf{c} = 0)$$
for all $s \colon Var \to \mathbb{C}$.

2. Let $\mathcal{L} = \{0, 1, +, \cdot, <\}$ and let $\varphi(\mathsf{a}, \mathsf{b}, \mathsf{c})$ (where $\mathsf{a}, \mathsf{b}, \mathsf{c} \in Var$) be the formula
$$\exists \mathsf{x}(\mathsf{a}\mathsf{x}^2 + \mathsf{b}\mathsf{x} + \mathsf{c} = 0)$$
Let $\mathcal{M}$ be the $\mathcal{L}$-structure $(\mathbb{R}, 0, 1, +, \cdot)$. Using the quadratic formula, we have
$$(\mathcal{M}, s) \vDash \varphi \leftrightarrow ((\mathsf{a} \neq 0 \wedge \mathsf{b}^2 - 4\mathsf{a}\mathsf{c} \geq 0) \vee (\mathsf{a} = 0 \wedge \mathsf{b} \neq 0) \vee (\mathsf{a} = 0 \wedge \mathsf{b} = 0 \wedge \mathsf{c} = 0))$$
for all $s \colon Var \to \mathbb{R}$.

□

The above examples focused on one structure rather than a theory (which could have many models). The next example uses a theory.

Example. Let $\mathcal{L} = \{0, 1, +, \cdot\}$ and let T be the theory of fields, i.e. $T = Cn(\Sigma)$ where Σ is the set of field axioms. Let $\varphi(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d})$ (where $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \in Var$) be the formula
$$\exists \mathsf{w} \exists \mathsf{x} \exists \mathsf{y} \exists \mathsf{z}(\mathsf{wa} + \mathsf{xc} = 1 \wedge \mathsf{wb} + \mathsf{xd} = 0 \wedge \mathsf{ya} + \mathsf{zc} = 0 \wedge \mathsf{yb} + \mathsf{zd} = 1)$$
Using determinants, we have
$$T \vDash \varphi \leftrightarrow \mathsf{ad} \neq \mathsf{bc}$$

□

Definition 7.1.1. *Let T be a theory. We say that T has* quantifier elimination*, or has QE, if for every $k \geq 1$ and every $\varphi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k) \in Form_{\mathcal{L}}$, there exists a quantifier-free $\psi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k)$ such that*

$$T \vDash \varphi \leftrightarrow \psi$$

7.2 What Quantifier Elimination Provides

The first application of a using QE is to show that certain theories are complete. QE itself is not sufficient, but a very mild additional assumption gives us what we want.

Proposition 7.2.1. *Let T be a theory that has QE. If there exists an $\mathcal{L}$-structure $\mathcal{N}$ such that for every model $\mathcal{M}$ of T there is an embedding $h\colon N \to M$ from $\mathcal{N}$ to $\mathcal{M}$, then T is complete. (Notice, there is no assumption that $\mathcal{N}$ is a model of T.)*

Proof. Fix an $\mathcal{L}$-structure $\mathcal{N}$ such that for every model $\mathcal{M}$ of T there is an embedding $h\colon N \to M$ from $\mathcal{N}$ to $\mathcal{M}$, and fix $n \in N$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models of T. For each $i \in \{1, 2\}$, fix an embedding $h_i\colon N \to M_i$ from $\mathcal{N}$ to $\mathcal{M}_i$. For each i, let $A_i = \mathrm{ran}(h_i)$, and notice that A_i is the universe of a substructure $\mathcal{A}_i$ of $\mathcal{M}_i$. Furthermore, notice that h_i is an isomorphism from $\mathcal{N}$ to $\mathcal{A}_i$.

Let $\sigma \in Sent_{\mathcal{L}}$ and let $\varphi(\mathsf{x}) \in Form_{\mathcal{L}}$ be the formula $\sigma \wedge (\mathsf{x} = \mathsf{x})$. Since T has QE, we may fix a quantifier-free $\psi(\mathsf{x}) \in Form_{\mathcal{L}}$ such that $T \vDash \varphi \leftrightarrow \psi$. We then have

$$\begin{aligned}
\mathcal{M}_1 \vDash \sigma &\Leftrightarrow (\mathcal{M}_1, h_1(n)) \vDash \varphi \\
&\Leftrightarrow (\mathcal{M}_1, h_1(n)) \vDash \psi \\
&\Leftrightarrow (\mathcal{A}_1, h_1(n)) \vDash \psi && \text{(since } \psi \text{ is quantifier-free)} \\
&\Leftrightarrow (\mathcal{N}, n) \vDash \psi && \text{(since } h_1 \text{ is an isomorphism from } \mathcal{N} \text{ to } \mathcal{A}_1) \\
&\Leftrightarrow (\mathcal{A}_2, h_2(n)) \vDash \psi && \text{(since } h_2 \text{ is an isomorphism from } \mathcal{N} \text{ to } \mathcal{A}_2) \\
&\Leftrightarrow (\mathcal{M}_2, h_2(n)) \vDash \psi && \text{(since } \psi \text{ is quantifier-free)} \\
&\Leftrightarrow (\mathcal{M}_2, h_2(n)) \vDash \varphi \\
&\Leftrightarrow \mathcal{M}_2 \vDash \sigma
\end{aligned}$$

□

Proposition 7.2.2. *Let T be a theory that has QE. Suppose that $\mathcal{A}$ and $\mathcal{M}$ are models of T and that $\mathcal{A} \subseteq \mathcal{M}$. We then have that $\mathcal{A} \preceq \mathcal{M}$.*

Proof. Let $\varphi \in Form_{\mathcal{L}}$ and let $s\colon Var \to A$ be a variable assignment. Suppose first that $\varphi \notin Sent_{\mathcal{L}}$. Since T has QE, we may fix a quantifier-free $\psi(\mathsf{x}) \in Form_{\mathcal{L}}$ such that $T \vDash \varphi \leftrightarrow \psi$. We then have

$$\begin{aligned}
(\mathcal{M}, s) \vDash \varphi &\Leftrightarrow (\mathcal{M}, s) \vDash \psi \\
&\Leftrightarrow (\mathcal{A}, s) \vDash \psi && \text{(since } \psi \text{ is quantifier-free)} \\
&\Leftrightarrow (\mathcal{A}, s) \vDash \varphi
\end{aligned}$$

If φ is a sentence, we may tack on a dummy $\mathsf{x} = \mathsf{x}$ as in the previous proof. □

Proposition 7.2.3. *Let T be a theory that has QE, let $\mathcal{M}$ be a model of T, and let $k \in \mathbb{N}^+$. Let $\mathcal{Z}$ be the set of all subsets of M^k which are definable by atomic formuals. The set of definable subsets of M^k equals $G(\mathcal{P}(M^k), \mathcal{Z}, \{h_1, h_2\})$ where $h_1\colon \mathcal{P}(M^k) \to \mathcal{P}(M^k)$ is the complement function and $h_2\colon \mathcal{P}(M^k)^2 \to \mathcal{P}(M^k)$ is the union function.*

Proof. □

7.3 Quantifier Manipulation Rules

Definition 7.3.1. *Let $\mathcal{L}$ be a language, and let $\varphi, \psi \in Form_{\mathcal{L}}$. We say that φ and ψ are semantically equivalent if $\varphi \vDash \psi$ and $\psi \vDash \varphi$.*

We now list a bunch of simple rules for manipulating formulas while maintaing.

1. $\neg(\exists \mathsf{x}\varphi)$ and $\forall \mathsf{x}(\neg\varphi)$ are s.e.
2. $\neg(\forall \mathsf{x}\varphi)$ and $\exists \mathsf{x}(\neg\varphi)$ are s.e.
3. $(\exists \mathsf{x}\varphi) \wedge \psi$ and $\exists \mathsf{x}(\varphi \wedge \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.
4. $(\forall \mathsf{x}\varphi) \wedge \psi$ and $\forall \mathsf{x}(\varphi \wedge \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.
5. $(\exists \mathsf{x}\varphi) \vee \psi$ and $\exists \mathsf{x}(\varphi \vee \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.
6. $(\forall \mathsf{x}\varphi) \vee \psi$ and $\forall \mathsf{x}(\varphi \vee \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.
7. $(\exists \mathsf{x}\varphi) \to \psi$ and $\forall \mathsf{x}(\varphi \to \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.
8. $(\forall \mathsf{x}\varphi) \to \psi$ and $\exists \mathsf{x}(\varphi \to \psi)$ are s.e. if $\mathsf{x} \notin FreeVar(\psi)$.

We'll need the following to change annoying variables.

1. $\exists \mathsf{x}\varphi$ and $\exists \mathsf{y}(\varphi^{\mathsf{y}}_{\mathsf{x}})$ are s.e. if $\mathsf{y} \notin OccurVar(\varphi)$.
2. $\forall \mathsf{x}\varphi$ and $\forall \mathsf{y}(\varphi^{\mathsf{y}}_{\mathsf{x}})$ are s.e. if $\mathsf{y} \notin OccurVar(\varphi)$.

We'll also need to know that if φ and ψ are s.e., then

1. $\neg\varphi$ and $\neg\psi$ are s.e.
2. $\exists \mathsf{x}\varphi$ and $\exists \mathsf{x}\psi$ are s.e.
3. $\forall \mathsf{x}\varphi$ and $\forall \mathsf{x}\psi$ are s.e.

and also that if φ_1 are φ_2 s.e., and ψ_1 and ψ_2 are s.e., then

1. $\varphi_1 \wedge \psi_1$ and $\varphi_2 \wedge \psi_2$ are s.e.
2. $\varphi_1 \vee \psi_1$ and $\varphi_2 \vee \psi_2$ are s.e.
3. $\varphi_1 \to \psi_1$ and $\varphi_2 \to \psi_2$ are s.e.

Definition 7.3.2. *A quantifier-free formula φ is in* disjunctive normal form *if there exists ψ_i for $1 \le i \le n$ such that*

$$\varphi = \psi_1 \vee \psi_2 \vee \cdots \vee \psi_n$$

where for each i, there exists $\alpha_{i,j}$ which is either an atomic formula or the negation of an atomic formula for $1 \le j \le m_i$ such that

$$\psi_i = \alpha_{i,1} \wedge \alpha_{i,2} \wedge \cdots \wedge \alpha_{i,m_i}$$

Proposition 7.3.3. *Suppose that $\varphi(x_1, x_2, \ldots, x_k) \in Form_{\mathcal{L}}$ is quantifier-free. There exists a quantifier-free formula $\theta(x_1, x_2, \ldots, x_k)$ in disjunctive normal form such that φ and θ are s.e.*

Proof. □

Definition 7.3.4. *A formula φ is called a* prenex formula *if it is an element of*

$$G(Sym^*_{\mathcal{L}}, QuantFreeForm_{\mathcal{L}}, \{h_{\forall,\mathsf{x}}, h_{\exists,\mathsf{x}} : \mathsf{x} \in Var\})$$

Proposition 7.3.5. *For every $\varphi \in Form_{\mathcal{L}}$, there exists a prenex formula ψ such that φ and ψ are semantically equivalent.*

Proposition 7.3.6. *Let T be a theory. The following are equivalent*

1. *T has QE.*
2. *For all $k \geq 1$ and all $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Form_{\mathcal{L}}$ with*
   (a) *$FreeVar(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \subseteq \{\mathsf{y}, \mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k\}$.*
   (b) *$\mathsf{y} \in FreeVar(\alpha_i)$ for all i and $\mathsf{y} \in FreeVar(\beta_j)$ for all j.*
   (c) *Each α_i and β_j is an atomic formula.*

   there exists a quantifier-free $\psi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k) \in Form_{\mathcal{L}}$ such that

$$T \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg \beta_j) \leftrightarrow \psi$$

Proof. □

7.4 Examples of Theories With QE

Theorem 7.4.1. *Let $\mathcal{L} = \emptyset$. For each $n \in \mathbb{N}^+$, let σ_n be the sentence*

$$\exists \mathsf{x}_1 \exists \mathsf{x}_2 \cdots \exists \mathsf{x}_n \bigwedge_{1 \leq i < j \leq n} \neg(\mathsf{x}_i = \mathsf{x}_j)$$

Let $T = Cn(\{\sigma_n : n \in \mathbb{N}^+\})$. T has QE and is complete.

Proof. Suppose that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Form_{\mathcal{L}}$ with

1. $FreeVar(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \subseteq \{\mathsf{y}, \mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k\}$.
2. $\mathsf{y} \in FreeVar(\alpha_i)$ for all i and $\mathsf{y} \in FreeVar(\beta_j)$ for all j.
3. Each α_i and β_j is an atomic formula.

We need to show that there exists a quantifier-free $\psi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k) \in Form_{\mathcal{L}}$ such that

$$T \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg \beta_j) \leftrightarrow \psi$$

Now each α_i and β_j is s.e. with, and hence we may assume is, one of the following:

1. $\mathsf{x}_\ell = \mathsf{y}$
2. $\mathsf{y} = \mathsf{y}$

If some α_i is $\mathsf{x}_\ell = \mathsf{y}$, then

$$T \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow (\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j)^{\mathsf{x}_\ell}_{\mathsf{y}}$$

If some β_j is $\mathsf{y} = \mathsf{y}$, then

$$T \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow \neg(\mathsf{x}_1 = \mathsf{x}_1)$$

Suppose then that each α_i is $\mathsf{y} = \mathsf{y}$ and each β_j is $\mathsf{x}_\ell = \mathsf{y}$. We then have

$$T \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow \mathsf{x}_1 = \mathsf{x}_1$$

because all models of T are infinite. Therefore, T has QE.

Notice that T is complete because the structure $\mathcal{M}$ given by $M = \{0\}$ trivially embeds into all models of T. □

Theorem 7.4.2. *RG has QE and is complete.*

Proof. Suppose that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Form_{\mathcal{L}}$ with

1. $FreeVar(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \subseteq \{\mathsf{y}, \mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k\}$.
2. $\mathsf{y} \in FreeVar(\alpha_i)$ for all i and $\mathsf{y} \in FreeVar(\beta_j)$ for all j.
3. Each α_i and β_j is an atomic formula.

We need to show that there exists a quantifier-free $\psi(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k) \in Form_{\mathcal{L}}$ such that

$$RG \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow \psi$$

Now each α_i and β_j is RG-equivalent with, and hence we may assume is, one of the following:

1. $\mathsf{x}_\ell = \mathsf{y}$
2. $\mathsf{R}\mathsf{x}_\ell\mathsf{y}$
3. $\mathsf{y} = \mathsf{y}$
4. Ryy

If some α_i is $\mathsf{x}_\ell = \mathsf{y}$, then

$$RG \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow (\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j)^{\mathsf{x}_\ell}_{\mathsf{y}}$$

If some α_i is Ryy or some β_j is $\mathsf{y} = \mathsf{y}$, then

$$RG \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow \neg(\mathsf{x}_1 = \mathsf{x}_1)$$

Suppose then that no α_i is $\mathsf{x}_\ell = \mathsf{y}$, no α_i is Ryy, and no β_j is $\mathsf{y} = \mathsf{y}$. Let

$$A = \{\ell \in \{1, 2, \dots, k\} : \text{ there exists } i \text{ such that } \alpha_i \text{ is } \mathsf{Rx}_\ell\mathsf{y}\}$$

and let

$$B = \{\ell \in \{1, 2, \dots, k\} : \text{ there exists } j \text{ such that } \beta_j \text{ is } \mathsf{Rx}_\ell\mathsf{y}\}$$

We then have that

$$RG \vDash \exists \mathsf{y}(\bigwedge_{i=1}^{m} \alpha_i \wedge \bigwedge_{j=1}^{n} \neg\beta_j) \leftrightarrow \bigwedge_{a\in A}\bigwedge_{b\in B} \neg(\mathsf{x}_a = \mathsf{x}_b)$$

because in models of RG, given disjoint finite sets A and B of vertices, there are infinitely many vertices linked to everything in A and not linked to everything in B. Therefore, RG has QE.

Notice that RG is complete because the structure $\mathcal{M}$ given by $M = \{0\}$ and $\mathsf{R}^{\mathcal{M}} = \emptyset$ trivially embeds into all models of RG. □

7.5 Algebraically Closed Fields

Definition 7.5.1. *Let $\mathcal{L} = \{0, 1, +, \cdot\}$. Let $\Sigma \subseteq Sent_{\mathcal{L}}$ be the field axioms together with the sentences*

$$\forall \mathsf{a_0} \forall \mathsf{a_1} \cdots \forall \mathsf{a_n}(\mathsf{a_n} \neq 0 \rightarrow \exists \mathsf{x}(\mathsf{a_n x^n} + \cdots + \mathsf{a_1 x} + \mathsf{a_0} = 0))$$

for each $n \in \mathbb{N}^+$. Let $ACF = Cn(\Sigma)$.

Theorem 7.5.2. *ACF has QE.*

Proof Sketch. The fundamental observation is that we can think of atomic formulas with free variables in $\{\mathsf{y}, \mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_k\}$ as equations $p(\vec{\mathsf{x}}, \mathsf{y}) = 0$ where $p(\vec{\mathsf{x}}, \mathsf{y}) \in \mathbb{Z}[\vec{\mathsf{x}}, \mathsf{y}]$ is a polynomial.

Thus, we have to find quantifier-free equivalents to formulas of the form

$$\exists \mathsf{y}[\bigwedge_{i=1}^{m} (p_i(\vec{\mathsf{x}}, \mathsf{y}) = 0) \wedge \bigwedge_{j=1}^{n} (q_j(\vec{\mathsf{x}}, \mathsf{y}) \neq 0)]$$

which, letting $q(\vec{\mathsf{x}}, \mathsf{y}) = \prod_{j=1}^{n} q_j(\vec{\mathsf{x}}, \mathsf{y})$, is equivalent in ACF to

$$\exists \mathsf{y}[\bigwedge_{i=1}^{m} (p_i(\vec{\mathsf{x}}, \mathsf{y}) = 0) \wedge q(\vec{\mathsf{x}}, \mathsf{y}) \neq 0]$$

Suppose now that R is a ring and $p_1, p_2, \dots, p_m, q \in R[y]$ listed in decreasing order of degrees. Let the leading term of p_1 be ay^n and let the leading term of p_m be by^k. We then have that there is a simlutaneous root of polynomials $p_1, p_2, \dots, p_m$ which is not a root of q if and only if one of the following happens:

1. $b = 0$ and there is simlutaneous root of polynomials $p_1, p_2, \dots, p_{m-1}, p_m - by^k$ which is not a root of q.
2. $b \neq 0$ and there is a simultaneous root of the polynomials $bp_1 - ay^{n-k}p_m, p_2, \dots, p_m$ which is not a root of q.

Repeating this, we may assume that we have a forumla of the form

$$\exists \mathsf{y}[p(\vec{\mathsf{x}}, \mathsf{y}) = 0 \wedge q(\vec{\mathsf{x}}, \mathsf{y}) \neq 0]$$

If q is not present, then we may use the fact that in an algebraically closed field, the polynomial $a_n y^n + \cdots + a_1 y + a_0$ has a root if and only if some $a_i \neq 0$ for $i > 0$, or $a_0 = 0$. If p is not present, then we use the fact that every algebraically closed field is infinite and also that every nonzero has finitely many roots to conclude that the there is an element which is not a root of the polynomial $a_n y^n + \cdots a_1 y + a_0$ if and only if some $a_i \neq 0$.

Suppose then that both p and q are present. The key fact is to use here is that if p and q are polynomials over an algebraically closed field and the degree of p is at most n, then every root of p is a root of q if and only if $p \mid q^n$.

Thus, we have two polynomials p and q, and we want to find a quantifier formula equivlent to $p \mid q$. We may now use the Euclidean algorithm to repeatedly reduce the problem. □

Corollary 7.5.3. *ACF_0 is complete and ACF_p is complete for all primes p.*

Corollary 7.5.4. *If F and K are algebraically closed fields such that F is a subfield of K, then $(F, 0, 1, +, \cdot) \preceq (K, 0, 1, +, \cdot)$.*

Corollary 7.5.5. $(\overline{\mathbb{Q}}, 0, 1, +, \cdot) \preceq (\mathbb{C}, 0, 1, +, \cdot)$.

Corollary 7.5.6. *Suppose that F is an algebraically closed field. A subset $X \subseteq F$ is definable in $(F, 0, 1, +, \cdot)$ if and only if X is either finite or cofinite.*

Proposition 7.5.7. *Let $\sigma \in Sent_{\mathcal{L}}$. The following are equivalent.*

1. *$ACF_0 \vDash \sigma$.*
2. *There exists m such that $ACF_p \vDash \sigma$ for all primes $p > m$.*
3. *$ACF_p \vDash \sigma$ for infinitely many primes p.*

Proof. 1 imples 2 is Compactness. 2 implies 3 is trivial. 3 implies 1 using completeness of ACF_0 and the ACF_p for each prime p, together with 1 implies 2. □

Proposition 7.5.8. *Let p be prime. Every finitely generated subfield of $\overline{\mathbb{F}}_p$ is finite.*

Proof. Let p be prime. For every n, let K_n be the set of roots of $x^{p^n} - x$ in $\overline{\mathbb{F}}_p$. By standard results in algebra, we have that K_n is a field of order p^n, and furthermore is the unique subfield of $\overline{\mathbb{F}}_p$ of order p^n. If $d \mid n$, we then have that $K_d \subseteq K_n$ because if $a^{p^d} = a$, then $a^{p^{2 \cdot d}} = (a^{p^d})^{p^d} = a^{p^d} = a$, so $a^{p^{3 \cdot d}} = (a^{p^{2 \cdot d}})^{p^d} = a^{p^d} = a$, etc. Let $K = \bigcup_{n \in \mathbb{N}} K_n$. Notice that K is a subfield of $\overline{\mathbb{F}}_p$ because if $a \in K_n$ and $b \in K_m$, then $a + b, a \cdot b \in K_{m \cdot n}$. Furthermore, notice that K is algebraically closed because a finite extension of a finite field is finite. Therefore, $K = \overline{\mathbb{F}}_p$. □

Theorem 7.5.9. *Every injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective.*

Proof. Let $\sigma_{n,d} \in Sent_{\mathcal{L}}$ be the sentence expressing that every injective polynomial map from F^n to F^n where each polynomial has degree at most d is surjective. We want to show that $\mathbb{C} \vDash \sigma_{n,d}$ for all n, d. To do this, it suffices to show that $\overline{\mathbb{F}}_p \vDash \sigma_{n,d}$ for all primes p and all $n, d \in \mathbb{N}$. Thus, it suffices to show that for all primes p, every injective polynomial map from $\overline{\mathbb{F}}_p^n$ to $\overline{\mathbb{F}}_p^n$ is surjective.

Fix a prime p and an $n \in \mathbb{N}$. Suppose that $f \colon \overline{\mathbb{F}}_p^n \to \overline{\mathbb{F}}_p^n$ is an injective polynomial map. Let $(b_1, b_2, \dots, b_n) \in \overline{\mathbb{F}}_p^n$. We need to show that there exists $(a_1, a_2, \dots, a_n) \in \overline{\mathbb{F}}_p^n$ with $f(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$. Let $f_1, f_2, \dots, f_n \in \overline{\mathbb{F}}_p[x_1, x_2, \dots, x_n]$ be such that $f = (f_1, f_2, \dots, f_n)$, and let C be the finite set of coefficients appearing in $f_1, f_2, \dots, f_n$. Let K be the subfield of $\overline{\mathbb{F}}_p$ generated by $C \cup \{b_1, b_2, \dots, b_n\}$ and notice that K is a finite field. Now $f \upharpoonright K^n$ maps K^n into K^n and is injective, so it's surjective because K^n is finite. Thus, there exists $(a_1, a_2, \dots, a_n) \in K^n \subseteq \overline{\mathbb{F}}_p^n$ such that $f(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$. □

Chapter 8

Nonstandard Models of Arithmetic and Analysis

8.1 Nonstandard Models of Arithmetic

Throughout this section, we work in the language $\mathcal{L} = \{0, 1, <, +, \cdot\}$ where $0, 1$ are constant symbols, $<$ is a binary relation symbol, and $+, \cdot$ are binary function symbols. We also let $\mathfrak{N} = (\mathbb{N}, 0, 1, <, +, \cdot)$ where the symbol 0 is interpreted as the "real" 0, the symbol $+$ is interpreted as "real" addition, etc. Make sure that you understand when $+$ means the symbol in the language $\mathcal{L}$ and when it mean the addition function on $\mathbb{N}$.

A basic question is whether $Th(\mathfrak{N})$ compeletely determines the model $\mathfrak{N}$. More precisely, we have the following question.

Question 8.1.1. *Are all models of $Th(\mathfrak{N})$ isomorphic to $\mathfrak{N}$?*

Using Proposition 6.5.3, we can immediately give a negative answer to this question because there is an uncountable model of $Th(\mathfrak{N})$, and an uncountable model can't be isomorphic to $\mathfrak{N}$. What would such a model look like? In order to answer this, let's think a little about the kinds of sentences that are in $Th(\mathfrak{N})$.

Definition 8.1.2. *For each $n \in \mathbb{N}$, we define a term $\underline{n} \in Term_{\mathcal{L}}$ as follows. Let $\underline{0} = 0$ and let $\underline{1} = 1$. Now define the $\underline{n}$ recursively by letting $\underline{n+1} = \underline{n} + 1$ for each $n \geq 1$. Notice here that the 1 and the + in $\underline{n+1}$ mean the actual number 1 and the actual addition function, whereas the 1 and + in $\underline{n} + 1$ mean the symbols 1 and + in our language $\mathcal{L}$. Thus, for example, $\underline{2}$ is the term $1 + 1$ and $\underline{3}$ is the term $(1 + 1) + 1$.*

Definition 8.1.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We know that given any $t \in Term_{\mathcal{L}}$ containing no variables, t corresponds to an element of M given by $\overline{s}(t)$ for some (any) variable assignment $s\colon Var \to M$. We denote this value by $t^{\mathcal{M}}$.*

Notice that $\underline{n}^{\mathfrak{N}} = n$ for all $n \in \mathbb{N}$ be a simple induction. Here are some important examples of the kinds of things in $Th(\mathfrak{N})$.

Examples of Sentences in $Th(\mathfrak{N})$.

1. $\underline{2} + \underline{2} = \underline{4}$ and in general $\underline{m} + \underline{n} = \underline{m+n}$ and $\underline{m} \cdot \underline{n} = \underline{m \cdot n}$.

2. $\forall x \forall y (x + y = y + x)$

3. $\forall x (x \neq 0 \to \exists y (y + 1 = x))$

4. For each $\varphi(\mathsf{x}) \in Form_{\mathcal{L}}$, the sentence

$$(\varphi^{\mathsf{0}}_{\mathsf{x}} \wedge \forall \mathsf{x}(\varphi \to \varphi^{\mathsf{x}+\mathsf{1}}_{\mathsf{x}})) \to \forall \mathsf{x}\varphi$$

□

Now any model $\mathcal{M}$ of $Th(\mathfrak{N})$ must satisfy all of these sentences. The basic sentences in 1 above roughly tell us that $\mathcal{M}$ has a piece which looks just like $\mathfrak{N}$. We make this precise as follows.

Proposition 8.1.4. *For any model $\mathcal{M}$ of $Th(\mathfrak{N})$, the function $h\colon \mathbb{N} \to M$ given by $h(n) = \underline{n}^{\mathcal{M}}$ is an embedding of $\mathfrak{N}$ into $\mathcal{M}$.*

Proof. Notice that

$$h(\mathsf{0}^{\mathfrak{N}}) = h(0) = \underline{0}^{\mathcal{M}} = \mathsf{0}^{\mathcal{M}}$$

and

$$h(\mathsf{1}^{\mathfrak{N}}) = h(1) = \underline{1}^{\mathcal{M}} = \mathsf{1}^{\mathcal{M}}$$

Now let $m, n \in \mathbb{N}$. We have

$$\begin{aligned} m < n &\Leftrightarrow \mathfrak{N} \vDash \underline{m} < \underline{n} \\ &\Leftrightarrow \underline{m} < \underline{n} \in Th(\mathfrak{N}) \\ &\Leftrightarrow \mathcal{M} \vDash \underline{m} < \underline{n} \\ &\Leftrightarrow \underline{m}^{\mathcal{M}} <^{\mathcal{M}} \underline{n}^{\mathcal{M}} \\ &\Leftrightarrow h(m) <^{\mathcal{M}} h(n) \end{aligned}$$

Also, since $\underline{m} + \underline{n} = \underline{m+n} \in Th(\mathfrak{N})$ we have

$$\begin{aligned} h(m+n) &= (\underline{m+n})^{\mathcal{M}} \\ &= \underline{m}^{\mathcal{M}} +^{\mathcal{M}} \underline{n}^{\mathcal{M}} \\ &= h(m) +^{\mathcal{M}} h(n) \end{aligned}$$

and since $\underline{m} \cdot \underline{n} = \underline{m \cdot n} \in Th(\mathfrak{N})$ we have

$$\begin{aligned} h(m \cdot n) &= (\underline{m \cdot n})^{\mathcal{M}} \\ &= \underline{m}^{\mathcal{M}} \cdot^{\mathcal{M}} \underline{n}^{\mathcal{M}} \\ &= h(m) \cdot^{\mathcal{M}} h(n) \end{aligned}$$

Finally, for any $m, n \in \mathbb{N}$ with $m \neq n$, we have $\underline{m} \neq \underline{n} \in Th(\mathfrak{N})$, so $\mathcal{M} \vDash \underline{m} \neq \underline{n}$, and hence $h(m) \neq h(n)$. □

Proposition 8.1.5. *Let $\mathcal{M}$ be a model of $Th(\mathfrak{N})$. The following are equivalent.*

1. *$\mathcal{M} \cong \mathfrak{N}$.*
2. *$M = \{\underline{n}^{\mathcal{M}} : n \in \mathbb{N}\}$.*

Proof. If 2 holds, then the h of the Proposition 8.1.4 is surjective and hence an isomorphism. Suppose then that 1 holds and fix an isomorphism $h\colon \mathbb{N} \to \mathcal{M}$ from $\mathfrak{N}$ to $\mathcal{M}$. We show that $h(n) = \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$ by induction. We have

$$h(0) = h(\mathsf{0}^{\mathfrak{N}}) = \mathsf{0}^{\mathcal{M}}$$

and

$$h(1) = h(\mathsf{1}^{\mathfrak{N}}) = \mathsf{1}^{\mathcal{M}}$$

Suppose that $n \in \mathbb{N}$ and $h(n) = \underline{n}^{\mathcal{M}}$. We then have

$$\begin{aligned} h(n+1) &= h(n) +^{\mathcal{M}} h(1) \\ &= \underline{n}^{\mathcal{M}} +^{\mathcal{M}} \mathsf{1}^{\mathcal{M}} \\ &= (\underline{n+1})^{\mathcal{M}} \end{aligned}$$

Therefore, $h(n) = \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$, so $M = \{\underline{n}^{\mathcal{M}} : n \in \mathbb{N}\}$ because h is surjective. □

Definition 8.1.6. *A* nonstandard model of arithmetic *is a model $\mathcal{M}$ of $Th(\mathfrak{N})$ such that $\mathcal{M} \not\cong \mathfrak{N}$.*

We've already seen that there are nonstandard models of arithmetic by cardinality considerations, but we can also build countable nonstandard models of arithmetic using the Compactness Theorem and the Countable Lowenheim-Skolem Theorem.

Theorem 8.1.7. *There exists a countable nonstandard model of arithmetic.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}\}$ where c is a new constant symbol. Consider the following set of $\mathcal{L}'$-sentences.

$$\Sigma' = Th(\mathfrak{N}) \cup \{\mathsf{c} \neq \underline{n} : n \in \mathbb{N}\}$$

Notice that every finite subset of Σ' has a model (by taking $\mathfrak{N}$ and interpreting c large enough), so Σ' has a countable model $\mathcal{M}$ (notice that $\mathcal{L}'$ is countable) by the Compactness Theorem and the Countable Lowenheim-Skolem Theorem. Restricting this model to the original language $\mathcal{L}$, we may use the Proposition 8.1.5 to conclude that $\mathcal{M}$ is a countable nonstandard model of arithmetic. □

8.2 The Structure of Nonstandard Models of Arithmetic

Throughout this section, let $\mathcal{M}$ be a nonstandard model of arithmetic. Anything we can express in the first-order language of $\mathcal{L}$ which is true of $\mathfrak{N}$ is in $Th(\mathfrak{N})$, and hence is true in $\mathcal{M}$. For example, we have the following.

Proposition 8.2.1.

- $+^{\mathcal{M}}$ *is associative on M.*
- $+^{\mathcal{M}}$ *is commutative on M.*
- $<^{\mathcal{M}}$ *is a linear ordering on M.*
- *For all $a \in M$ with $a \neq \underline{0}^{\mathcal{M}}$, there exists $b \in M$ with $a + 1 = b$.*

Proof. The sentences

- $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{x} + (\mathsf{y} + \mathsf{z}) = (\mathsf{x} + \mathsf{y}) + \mathsf{x})$
- $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} + \mathsf{y} = \mathsf{y} + \mathsf{x})$
- $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} < \mathsf{y} \vee \mathsf{y} < \mathsf{x} \vee \mathsf{x} = \mathsf{y})$
- $\forall \mathsf{x}(\mathsf{x} \neq \mathsf{0} \rightarrow \exists \mathsf{y}(\mathsf{y} + \mathsf{1} = \mathsf{x}))$

are in $Th(\mathfrak{N})$. □

Since we already know that $\mathfrak{N}$ is naturally embedded in $\mathcal{M}$, and it gets tiresome to write $+^{\mathcal{M}}$, $\cdot^{\mathcal{M}}$, and $<^{\mathcal{M}}$, we'll abuse notation by using just $+$, $\cdot$, and $<$ to denote these. Thus, these symbols now have three different meanings. They are used as formal symbols in our language, as the normal functions and relations in $\mathfrak{N}$, and as their interpretations in $\mathcal{M}$. Make sure you know how each appearance of these symbols is being used.

Definition 8.2.2. *We let $M_{fin} = \{\underline{n}^{\mathcal{M}} : n \in \mathbb{N}\}$ and we call M_{fin} the set of* finite *elements of $\mathcal{M}$. We also let $M_{inf} = M \backslash M_{fin}$ and we call M_{inf} the set of* infinite *elements of $\mathcal{M}$.*

The following definition justifies our choice of name.

Proposition 8.2.3. *Let $a \in M_{inf}$. For any $n \in \mathbb{N}$, we have $\underline{n}^{\mathcal{M}} < a$.*

Proof. For each $n \in \mathbb{N}$, the sentence

$$\forall \mathsf{x}(\mathsf{x} < \underline{n} \to \bigvee_{i=0}^{n-1} (\mathsf{x} = \underline{i}))$$

is in $Th(\mathfrak{N})$. Since $a \neq \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$, it follows that it's not the case that $a < \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$. Since $<$ is a linear ordering on M, we may conclude that $\underline{n}^{\mathcal{M}} < a$ for all $n \in \mathbb{N}$. □

Definition 8.2.4. *Define a relation $\sim$ on M by letting $a \sim b$ if either*

- *$a = b$.*
- *$a < b$ and there exists $n \in \mathbb{N}$ such that $a + \underline{n}^{\mathcal{M}} = b$.*
- *$b < a$ and there exists $n \in \mathbb{N}$ such that $b + \underline{n}^{\mathcal{M}} = a$.*

In other words, $a \sim b$ if a and b are "finitely" far apart.

Proposition 8.2.5. *$\sim$ is an equivalence relation on M.*

Proof. $\sim$ is clearly relexive and symmetric. Suppose that $a, b, c \in M$, that $a \sim b$, and that $b \sim c$. We handle one case. Suppose that $a < b$ and $b < c$. Fix $m, n \in \mathbb{N}$ with $a + \underline{m}^{\mathcal{M}} = b$ and $b + \underline{n}^{\mathcal{M}} = c$. We then have

$$\begin{aligned} a + (\underline{m+n})^{\mathcal{M}} &= a + (\underline{m}^{\mathcal{M}} + \underline{n}^{\mathcal{M}}) \\ &= (a + \underline{m}^{\mathcal{M}}) + \underline{n}^{\mathcal{M}} \\ &= b + \underline{n}^{\mathcal{M}} \\ &= c \end{aligned}$$

so $a \sim c$. The other cases are similar. □

Definition 8.2.6. *Let $a, b \in M$. We write $a \ll b$ to mean that $a < b$ and $a \not\sim b$.*

We'd like to know that that relation $\ll$ is well-defined on the equivalence classes of $\sim$. The following lemma is useful.

Lemma 8.2.7. *Let $a, b, c \in M$ be such that $a \leq b \leq c$ and suppose that $a \sim c$. We then have $a \sim b$ and $b \sim c$.*

Proof. If either $a = b$ or $b = c$, this is trivial, so assume that $a < b < c$. Since $a < c$ and $a \sim c$, there exists $n \in \mathbb{N}^+$ with $a + \underline{n}^{\mathcal{M}} = c$. Now the sentence

$$\forall \mathsf{x} \forall \mathsf{z} \forall \mathsf{w}(\mathsf{x} + \mathsf{w} = \mathsf{z} \to \forall \mathsf{y}((\mathsf{x} < \mathsf{y} \wedge \mathsf{y} < \mathsf{z}) \to \exists \mathsf{u}(\mathsf{u} < \mathsf{w} \wedge \mathsf{x} + \mathsf{u} = \mathsf{y})))$$

is in $Th(\mathfrak{N})$, so there exists $d \in M$ such that $d < \underline{n}^{\mathcal{M}}$ and $a + d = b$. Since $d < \underline{n}^{\mathcal{M}}$, there exists $i \in \mathbb{N}$ with $d = \underline{i}^{\mathcal{M}}$. We then have $a + \underline{i}^{\mathcal{M}} = b$, hence $a \sim b$. The proof that $b \sim c$ is similar. □

Proposition 8.2.8. *Suppose that $a_0, b_0 \in M$ are such that $a_0 \ll b_0$. For any $a, b \in M$ with $a \sim a_0$ and $b \sim b_0$, we have $a \ll b$.*

Proof. We first show that $a \not\sim b$. If $a \sim b$, then using $a_0 \sim a$ and $b_0 \sim b$, together with the fact that $\sim$ is an equivalence relation, we can conclude that $a_0 \sim b_0$, a contradiction. Therefore, $a \not\sim b$.

Thus, we need only show that $a < b$. Notice that $a_0 < b$ because otherwise $a_0 \sim b_0$ by Lemma 8.2.7. Similarly, $a < b_0$ because otherwise $a_0 \sim b_0$ by Lemma 8.2.7. Thus, if $b \leq a$, we have

$$a_0 < b \leq a < b_0.$$

so $b \sim a_0$ by Lemma 8.2.7, hence $a_0 \sim b_0$, a contradiction. It follows that $a < b$. □

This allows us to define an ordering on the equivalence classes.

Definition 8.2.9. *Given $a, b \in M$, we write $[a] \prec [b]$ to mean that $a \ll b$.*

The next proposition implies that there is no largest equivalence class under the ordering $\prec$.

Proposition 8.2.10. *For any $a \in M_{inf}$, we have $a \ll a + a$.*

Proof. Let $a \in M_{inf}$. For each $n \in \mathbb{N}$, the sentence

$$\forall \mathsf{x}(\underline{n} < \mathsf{x} \rightarrow \mathsf{x} + \underline{n} < \mathsf{x} + \mathsf{x})$$

is in $Th(\mathfrak{N})$. Using this when $n = 0$, we see that $a = a + \underline{0}^{\mathcal{M}} < a + a$. Since $a \in M_{inf}$, we have $\underline{n}^{\mathcal{M}} < a$ and hence $a + \underline{n}^{\mathcal{M}} < a + a$ for all $n \in \mathbb{N}$. Therefore, $a + \underline{n}^{\mathcal{M}} \neq a + a$ for all $n \in \mathbb{N}$, and so $a \not\sim a + a$. □

Lemma 8.2.11. *For all $a \in M$, one of the following holds*

1. *There exists $b \in M$ such that $a = \underline{2}^{\mathcal{M}} \cdot b$.*
2. *There exists $b \in M$ such that $a = \underline{2}^{\mathcal{M}} \cdot b + \underline{1}^{\mathcal{M}}$.*

Proof. The sentence

$$\forall \mathsf{x} \exists \mathsf{y}(\mathsf{x} = \underline{2} \cdot \mathsf{y} \vee \mathsf{x} = \underline{2} \cdot \mathsf{y} + \underline{1})$$

is in $Th(\mathfrak{N})$. □

Proposition 8.2.12. *For any $a \in M_{inf}$, there exists $b \in M_{inf}$ with $b \ll a$.*

Proof. Suppose first that we have a $b \in M$ such that $a = \underline{2}^{\mathcal{M}} \cdot b$. We then have $a = b + b$ (because $\forall \mathsf{x}(\underline{2} \cdot \mathsf{x} = \mathsf{x} + \mathsf{x})$ is in $Th(\mathfrak{N})$). Notice that $b \notin M_{fin}$ because otherwise we would have $a \in M_{fin}$. Therefore, $b \ll b + b = a$ using Proposition 8.2.10. Suppose instead that we have a $b \in M$ such that $a = \underline{2}^{\mathcal{M}} \cdot b + \underline{1}^{\mathcal{M}}$. We then have $a = (b + b) + \underline{1}^{\mathcal{M}}$ because $\forall \mathsf{x}(\underline{2} \cdot \mathsf{x} + \underline{1} = (\mathsf{x} + \mathsf{x}) + \underline{1})$ is in $Th(\mathfrak{N})$. Notice that $b \notin M_{fin}$ because otherwise we would have $a \in M_{fin}$. Therefore, $b \ll b + b$ using Proposition 8.2.10, so $b \ll (b + b) + 1 = a$ since $b + b \sim (b + b) + 1$. □

Proposition 8.2.13. *For any $a, b \in M_{inf}$ with $a \ll b$, there exists $c \in M_{inf}$ with $a \ll c \ll b$.*

Proof. Suppose first that we have a $c \in M$ such that $a + b = \underline{2}^{\mathcal{M}} \cdot c$. We then have $a + b = c + c$. Since

$$\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}((\mathsf{x} < \mathsf{y} \wedge \mathsf{x} + \mathsf{y} = \mathsf{z} + \mathsf{z}) \rightarrow (\mathsf{x} < \mathsf{z} \wedge \mathsf{z} < \mathsf{y}))$$

is in $Th(\mathfrak{N})$ it follows that $a < c < b$.

Suppose that $a \sim c$ and fix $n \in \mathbb{N}$ with $a + \underline{n}^{\mathcal{M}} = c$. We then have that $a + b = c + c = a + a + (\underline{2n})^{\mathcal{M}}$, so $b = a + (\underline{2n})^{\mathcal{M}}$ contradicting the fact that $a \ll b$. Therefore $a \not\sim c$.

Suppose that $c \sim b$ and fix $n \in \mathbb{N}$ with $c + \underline{n}^{\mathcal{M}} = b$. We then have that

$$\begin{aligned} a + (\underline{2n})^{\mathcal{M}} + b &= c + c + (\underline{2n})^{\mathcal{M}} \\ &= (c + \underline{n}^{\mathcal{M}}) + (c + \underline{n}^{\mathcal{M}}) \\ &= b + b \end{aligned}$$

so $a + (\underline{2n})^{\mathcal{M}} = b$ contradicting the fact that $a \not\sim b$. Therefore, $b \not\sim c$.

A similar argument handles the case when we have a $c \in M$ such that $a + b = \underline{2}^{\mathcal{M}} \cdot c + \underline{1}^{\mathcal{M}}$. □

This last proposition shows how nonstandard models can simplify quantifiers. It says that asking whether a first-order statement holds for infinitely many $n \in \mathbb{N}$ is equivalent to asking whether it holds for at least one infinite element of a nonstandard model.

Proposition 8.2.14. *Let $\varphi(\mathsf{x}) \in Form_{\mathcal{L}}$. The following are equivalent.*

1. *There are infinitely many $n \in \mathbb{N}$ such that $(\mathfrak{N}, n) \vDash \varphi$.*

2. *There exists $a \in M_{inf}$ such that $(\mathcal{M}, a) \vDash \varphi$.*

Proof. Suppose first that there are infinitely many $n \in \mathbb{N}$ such that $(\mathfrak{N}, n) \vDash \varphi$. In this case, the sentence

$$\forall \mathsf{y} \exists \mathsf{x} (\mathsf{y} < \mathsf{x} \wedge \varphi)$$

is in $Th(\mathfrak{N})$, so it holds in $\mathcal{M}$. Fixing any $b \in M_{inf}$, we may conclude that there exists $a \in M$ with $b < a$ such that $(\mathcal{M}, a) \vDash \varphi$. Since $b < a$ and $b \in M_{inf}$, we may conclude that $a \in M_{inf}$.

Conversely, suppose that there are only finitely many $n \in \mathbb{N}$ such that $(\mathfrak{N}, a) \vDash \varphi$. Fix $N \in \mathbb{N}$ such that $n < N$ for all n with $(\mathfrak{N}, n) \vDash \varphi$. We then have that the sentence

$$\forall \mathsf{x} (\varphi \to \mathsf{x} < \underline{N})$$

is in $Th(\mathfrak{N})$, so it holds in $\mathcal{M}$. Since there is no $a \in M_{inf}$ with $a < \underline{N}^{\mathcal{M}}$, it follows that there is no $a \in M_{inf}$ such that $\mathcal{M} \vDash \varphi(a)$. □

8.3 Nonstandard Models of Analysis

With a basic understanding of nonstandard models of arithmetic, let's think about nonstandard models of other theories. One of the more amazing and useful such theories is the theory of the real numbers. The idea is that we will have nonstandard models of the theory of the reals which contain in "infinite" and "infinitesimal" elements. We can then transfer first-order statements back-and-forth, and do "calculus" in this expanded stucture where the basic definitions (of say continuity) are simpler and more intuitive.

The first thing we need to decide on is what our language will be. Since we want to do calculus, we want to have analogs of all of our favorite functions (such as sin) in the nonstandard models. Once we through these in, it's hard to know where to draw the line. In fact, there is no reason to draw a line at all. Simply throw in relation symbols for every possible subset of $\mathbb{R}^k$, and throw in function symbols for every possible function $f \colon \mathbb{R}^k \to \mathbb{R}$. Thus, throughout this section, we work in the language $\mathcal{L} = \{\underline{r} : r \in \mathbb{R}\} \cup \{\underline{P} : P \subseteq \mathbb{R}^k\} \cup \{\underline{f} : f \colon \mathbb{R}^k \to \mathbb{R}\}$ where the $\underline{P}$ and $\underline{f}$ have the corresponding arities. We also let $\mathfrak{R}$ be the structure with universe $\mathbb{R}$ and where we interpret all symbols in the natural way.

Proposition 8.3.1. *For any model $\mathcal{M}$ of $Th(\mathfrak{R})$, the function $h \colon \mathbb{R} \to M$ given by $h(n) = \underline{r}^{\mathcal{M}}$ is an embedding of $\mathfrak{R}$ into $\mathcal{M}$.*

Proof. Notice that
$$h(\underline{r}^{\mathfrak{R}}) = h(r) = \underline{r}^{\mathcal{M}}$$
for every $r \in \mathbb{R}$. Now let $P \subseteq \mathbb{R}^k$ and let $r_1, r_2, \ldots, r_k \in \mathbb{R}$. We have

$$\begin{aligned}
(r_1, r_2, \ldots, r_n) \in \underline{P}^{\mathfrak{R}} &\Leftrightarrow \mathfrak{R} \vDash \underline{P}\ \underline{r_1}\ \underline{r_2} \cdots \underline{r_k} \\
&\Leftrightarrow \underline{P}\ \underline{r_1}\ \underline{r_2} \cdots \underline{r_k} \in Th(\mathfrak{R}) \\
&\Leftrightarrow \mathcal{M} \vDash \underline{P}\ \underline{r_1}\ \underline{r_2} \cdots \underline{r_k} \\
&\Leftrightarrow (\underline{r_1}^{\mathcal{M}}, \underline{r_2}^{\mathcal{M}}, \ldots, \underline{r_k}^{\mathcal{M}}) \in \underline{P}^{\mathcal{M}} \\
&\Leftrightarrow (h(r_1), h(r_2), \ldots, h(r_k)) \in \underline{P}^{\mathcal{M}}
\end{aligned}$$

Now let $f \colon \mathbb{R}^k \to \mathbb{R}$ and let $r_1, r_2, \ldots, r_k \in \mathbb{R}$. Since $\underline{f}\ \underline{r_1}\ \underline{r_2} \cdots \underline{r_k} = \underline{f(r_1, r_2, \ldots, r_k)} \in Th(\mathfrak{R})$ we have

$$\begin{aligned}
h(\underline{f}^{\mathfrak{R}}(r_1, r_2, \ldots, r_k)) &= h(f(r_1, r_2, \ldots, r_k)) \\
&= \underline{f(r_1, r_2, \ldots, r_k)}^{\mathcal{M}} \\
&= \underline{f}^{\mathcal{M}}(\underline{r_1}^{\mathcal{M}}, \underline{r_2}^{\mathcal{M}}, \ldots, \underline{r_k}^{\mathcal{M}}) \\
&= \underline{f}^{\mathcal{M}}(h(r_1), h(r_2), \ldots, h(r_k))
\end{aligned}$$

Finally, for any $r_1, r_2 \in \mathbb{R}$ with $r_1 \neq r_2$, we have $\underline{r_1} \neq \underline{r_2} \in Th(\mathfrak{R})$, so $\mathcal{M} \vDash \underline{r_1} \neq \underline{r_2}$, and hence $h(r_1) \neq h(r_2)$. □

Proposition 8.3.2. *Let $\mathcal{M}$ be a model of $Th(\mathfrak{R})$. The following are equivalent.*

1. *$\mathcal{M} \cong \mathfrak{R}$.*
2. *$M = \{\underline{r}^{\mathcal{M}} : r \in \mathbb{R}\}$.*

Proof. If 2 holds, then the h of the Proposition 8.3.1 is surjective and hence an isomorphism. Suppose then that 1 holds and fix an isomorphism $h \colon \mathbb{R} \to \mathcal{M}$ from $\mathfrak{R}$ to $\mathcal{M}$. For any $r \in \mathbb{R}$, we must have $h(r) = h(\underline{r}^{\mathfrak{R}}) = \underline{r}^{\mathcal{M}}$. Therefore, $M = \{\underline{r}^{\mathcal{M}} : r \in \mathbb{R}\}$ because h is surjective. □

Definition 8.3.3. *A* nonstandard model of analysis *is a model $\mathcal{M}$ of $Th(\mathfrak{R})$ such that $\mathcal{M} \ncong \mathfrak{R}$.*

Theorem 8.3.4. *There exists a nonstandard model of analysis.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}\}$ where c is a new constant symbol. Consider the following set of $\mathcal{L}'$-sentences.

$$\Sigma' = Th(\mathfrak{R}) \cup \{\mathsf{c} \neq \underline{r} : r \in \mathbb{R}\}$$

Notice that every finite subset of Σ' has a model (by taking $\mathfrak{R}$ and interpreting c distinct from each r such that $\underline{r}$ appears in Σ'), so Σ' has a model $\mathcal{M}$ by the Compactness Theorem. Restricting this model to the original language $\mathcal{L}$, we may use the Proposition 8.3.2 to conclude that $\mathcal{M}$ is a nonstandard model of analysis. □

Definition 8.3.5. *For the rest of this section, fix a nonstandard model of analysis and denote it by ${}^*\mathfrak{R}$. Instead of wrting $\underline{f}^{{}^*\mathfrak{R}}$ for each $f \colon \mathbb{R}^k \to \mathbb{R}$, we simply write ${}^*f$. We use similar notation for each $P \subseteq \mathbb{R}^k$. Also, since there is a natural embedding (the h above) from $\mathfrak{R}$ into ${}^*\mathfrak{R}$, we will identify $\mathbb{R}$ with it's image and hence think of $\mathbb{R}$ as a subset of ${}^*\mathbb{R}$. Finally, for operations like $+$ and $\cdot$, we will abuse notation and omit the $*$'s.*

Proposition 8.3.6. *There exists $z \in {}^*\mathbb{R}$ such that $z > 0$ and $z < \varepsilon$ for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.*

Proof. Fix $b \in {}^*\mathbb{R}$ such that $b \neq r$ for all $r \in \mathbb{R}$.

Case 1: Suppose that $b > r$ for all $r \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(r) = \begin{cases} \frac{1}{r} & \text{if } r \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $z = {}^*f(b)$. We then have have that $z > 0$ using the sentence

$$\forall \mathsf{x}(\underline{0} < \mathsf{x} \to \underline{0} < \underline{f}\mathsf{x})$$

Also, for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have that $b > \frac{1}{\varepsilon}$, hence $z < \varepsilon$ using the sentence

$$\forall \mathsf{x}(\underline{f}\ \underline{\varepsilon} < \mathsf{x} \to \underline{f}\mathsf{x} < \underline{\varepsilon})$$

Case 2: Suppose that $b < r$ for all $r \in \mathbb{R}$. We then have that $b < -r$ for all $r \in \mathbb{R}$ and hence $r < -b$ for all $r \in \mathbb{R}$. Thus, we may tak $z = {}^*f(-b)$ by the argument in Case 1.

Case 3: Suppose then that there exists $r \in \mathbb{R}$ with $r < b$ and there exists $r \in \mathbb{R}$ with $b < r$. Let

$$X = \{r \in \mathbb{R} : r < b\}$$

Notice that X is downward closed (if $r_1, r_2 \in \mathbb{R}$ with $r_2 \in X$ and $r_1 < r_2$, then $r_1 \in X$), nonempty, bounded above, Let $s = \sup X \in \mathbb{R}$. Now $b = s$ is impossible, so either $s < b$ or $b < s$.

Subcase 1: Suppose that $s < b$. We claim that we may take $z = b - s$. Since $s < b$, we have $z = b - s > 0$. Suppose that $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. We then have that $s + \varepsilon > s = \sup X$, so $s + \varepsilon \notin X$ and hence $s + \varepsilon \geq b$. Now $s + \varepsilon \neq b$ because $s + \varepsilon \in \mathbb{R}$, so $s + \varepsilon > b$. It follows that $z = b - s < \varepsilon$.

Subcase 2: Suppose that $b < s$. We claim that we may take $z = s - b$. Since $b < s$, we have $z = s - b > 0$. Suppose that $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. We then that $s - \varepsilon < s = \sup X$, so we may fix $r \in X$ with $s - \varepsilon < r$. Since X is downward closed, we have that $s - \varepsilon \in X$, so $s - \varepsilon < b$. It follows that $z = s - b < \varepsilon$. □

From now on, we'll use the more natural notation $\frac{1}{b}$ for ${}^*f(b)$ whenever $b \neq 0$.

Definition 8.3.7.

1. $\mathcal{Z} = \{a \in {}^*\mathbb{R} : |a| < \varepsilon$ *for all* $\varepsilon \in \mathbb{R}$ *with* $\varepsilon > 0\}$. *We call* $\mathcal{Z}$ *the set of* infinitesimals.

2. $\mathcal{F} = \{a \in {}^*\mathbb{R} : |a| < r$ *for some* $r \in \mathbb{R}$ *with* $r > 0\}$. *We call* $\mathcal{F}$ *the set of* finite *or* limited *elements.*

3. $\mathcal{I} = {}^*\mathbb{R} \backslash \mathcal{F}$. *We call* $\mathcal{I}$ *the set of* infinite *or* unlimited *elements.*

Proposition 8.3.8.

1. *$\mathcal{Z}$ is a subring of ${}^*\mathbb{R}$.*

2. *$\mathcal{F}$ is a subring of ${}^*\mathbb{R}$.*

3. *$\mathcal{Z}$ is a prime ideal of $\mathcal{F}$.*

Proof.

1. First notice that $\mathcal{Z} \neq \emptyset$ because $0 \in \mathcal{Z}$ (or we can use Proposition 8.3.6). Suppose that $a, b \in \mathcal{Z}$. Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.

We have that $\frac{\varepsilon}{2} \in \mathbb{R}$ and $\frac{\varepsilon}{2} > 0$, hence $|a| < \frac{\varepsilon}{2}$ and $|b| < \frac{\varepsilon}{2}$. It follows that

$$\begin{aligned} |a - b| &\leq |a + (-b)| \\ &\leq |a| + |-b| \\ &\leq |a| + |b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $a - b \in \mathcal{Z}$. We also have that $|a| < 1$ and $|b| < \varepsilon$, hence

$$\begin{aligned} |a \cdot b| &= |a| \cdot |b| \\ &< 1 \cdot \varepsilon \\ &= \varepsilon \end{aligned}$$

Therefore, $a \cdot b \in \mathcal{Z}$.

2. Clearly, $\mathcal{F} \neq \emptyset$. Suppose that $a, b \in \mathcal{F}$, and fix $r_1, r_2 \in \mathbb{R}$ with $r_1, r_2 > 0$ such that $|a| < r_1$ and $|b| < r_2$. We have

$$\begin{aligned} |a - b| &\leq |a + (-b)| \\ &\leq |a| + |-b| \\ &\leq |a| + |b| \\ &< r_1 + r_2 \end{aligned}$$

so $a - b \in \mathcal{F}$. We also have

$$\begin{aligned} |a \cdot b| &= |a| \cdot |b| \\ &< r_1 \cdot r_2 \end{aligned}$$

so $a \cdot b \in \mathcal{F}$.

3. We first show that $\mathcal{Z}$ is an ideal of $\mathcal{F}$. Suppose that $a \in \mathcal{F}$ and $b \in \mathcal{Z}$. Fix $r \in \mathbb{R}$ with $r > 0$ and $|a| < r$. Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. We then have that $\frac{\varepsilon}{r} \in \mathbb{R}$ and $\frac{\varepsilon}{r} > 0$, hence $|a| < \frac{\varepsilon}{r}$. It follows that

$$\begin{aligned} |a \cdot b| &= |a| \cdot |b| \\ &< \frac{\varepsilon}{r} \cdot r \\ &= \varepsilon \end{aligned}$$

Therefore, $a \cdot b \in \mathcal{Z}$.

We now show that $\mathcal{Z}$ is a prime ideal of $\mathcal{F}$. Suppose that $a, b \in \mathcal{F} \backslash \mathcal{Z}$. We have $a \cdot b \in \mathcal{F}$ by part 2. Fix $\varepsilon, \delta \in \mathbb{R}$ with $\varepsilon, \delta > 0$ such that $|a| > \varepsilon$ and $|b| > \delta$. We then have $|a \cdot b| = |a| \cdot |b| > \varepsilon \cdot \delta$, hence $a \cdot b \notin \mathcal{Z}$.

□

Definition 8.3.9. *Let $a, b \in {}^*\mathbb{R}$.*

1. *We write $a \approx b$ to mean that $a - b \in \mathcal{Z}$.*

2. *We write $a \sim b$ to mean that $a - b \in \mathcal{F}$.*

Proposition 8.3.10. *$\approx$ and $\sim$ are equivalence relations on ${}^*\mathbb{R}$.*

Definition 8.3.11. *Let $a \in {}^*\mathbb{R}$. The $\approx$-equivalence class of a is called the* halo *of a. The $\sim$-equivalence class of a is called the* galaxy *of a.*

Proposition 8.3.12. *Let $a_1, b_1, a_2, b_2 \in {}^*\mathbb{R}$ with $a_1 \approx b_1$ and $a_2 \approx b_2$.*

1. *$a_1 + a_2 \approx b_1 + b_2$.*
2. *$a_1 - a_2 \approx b_1 - b_2$.*
3. *If $a_1, b_1, a_2, b_2 \in \mathcal{F}$, then $a_1 \cdot a_2 \approx b_1 \cdot b_2$.*
4. *If $a_1, b_1, a_2, b_2 \in \mathcal{F} \backslash \mathcal{Z}$, then $\frac{a_1}{a_2} \approx \frac{b_1}{b_2}$.*

Proof.

1. We have $a_1 - b_1 \in \mathcal{Z}$ and $a_2 - b_2 \in \mathcal{Z}$, hence
$$(a_1 + a_2) - (b_1 + b_2) = (a_1 - b_1) + (a_2 - b_2)$$
is in $\mathcal{Z}$ by Proposition 8.3.8.
2. We have $a_1 - b_1 \in \mathcal{Z}$ and $a_2 - b_2 \in \mathcal{Z}$, hence
$$(a_1 - a_2) - (b_1 - b_2) = (a_1 - b_1) - (a_2 - b_2)$$
is in $\mathcal{Z}$ by Proposition 8.3.8.
3. We have $a_1 - b_1 \in \mathcal{Z}$ and $a_2 - b_2 \in \mathcal{Z}$. Now
$$a_1 \cdot a_2 - b_1 \cdot b_2 = a_1 \cdot a_2 - a_1 \cdot b_2 + a_1 \cdot b_2 - b_1 \cdot b_2 = a_1 \cdot (a_2 - b_2) + b_2 \cdot (a_1 - b_1)$$
so $a_1 \cdot a_2 - b_1 \cdot b_2 \in \mathcal{Z}$ by Proposition 8.3.8.
4. We have $a_1 - b_1 \in \mathcal{Z}$ and $a_2 - b_2 \in \mathcal{Z}$. Now
$$\frac{a_1}{a_2} - \frac{b_1}{b_2} = \frac{a_1 \cdot b_2 - a_2 \cdot b_1}{a_2 \cdot b_2} = \frac{1}{a_2 \cdot b_2} \cdot (a_1 \cdot b_2 - a_2 \cdot b_1)$$
and we know by part 3 that $a_1 \cdot b_2 - a_2 \cdot b_1 \in \mathcal{Z}$. Since $a_2, b_2 \in \mathcal{F} \backslash \mathcal{Z}$, it follows that $a_2 \cdot b_2 \in \mathcal{F} \backslash \mathcal{Z}$ by Proposition 8.3.8. Therefore, $\frac{1}{a_2 \cdot b_2} \in \mathcal{F}$ (if $\varepsilon > 0$ is such that $|a_2 \cdot b_2| > \varepsilon$, then $|\frac{1}{a_2 \cdot b_2}| < \frac{1}{\varepsilon}$), so $\frac{a_1}{a_2} - \frac{b_1}{b_2} \in \mathcal{Z}$ by Proposition 8.3.8.

□

Proposition 8.3.13. *For every $a \in \mathcal{F}$, there exists a unique $r \in \mathbb{R}$ such that $a \approx r$.*

Proof. Fix $a \in \mathcal{F}$. We first prove existence. Let
$$X = \{r \in \mathbb{R} : r < a\}$$
and notice that X is downward closed, nonempty, and bounded above because $a \in \mathcal{F}$. Now let $s = \sup X$ and argue as in Case 3 of Proposition 8.3.6 that $a \approx s$.

Suppose now that $r_1, r_2 \in \mathbb{R}$ are such that $a \approx r_1$ and $a \approx r_2$. We then have that $r_1 \approx r_2$ because $\approx$ is an equivalence relation. However, this is a contradiction because $|r_1 - r_2| > \frac{|r_1 - r_2|}{2} \in \mathbb{R}$. □

Definition 8.3.14. *We define a map $st\colon \mathcal{F} \to \mathbb{R}$ by letting $st(a)$ be the unique $r \in \mathbb{R}$ such that $a \approx r$. We call $st(a)$ the* standard part *or* shadow *of a.*

Corollary 8.3.15. *The function $st\colon \mathcal{F} \to \mathbb{R}$ is a ring homomorphism and $ker(st) = \mathcal{Z}$.*

Proposition 8.3.16. *Suppose that $A \subseteq \mathbb{R}$, that $f\colon A \to \mathbb{R}$, and that $r, \ell \in \mathbb{R}$. Suppose also that there exists $\delta > 0$ such that $(r-\delta, r+\delta)\backslash\{r\} \subseteq A$. The following are equivalent.*

1. *$\lim_{x\to r} f(x) = \ell$.*
2. *For all $a \approx r$ with $a \neq r$, we have ${}^*f(a) \approx \ell$.*

Proof. Suppose first that $\lim_{x\to r} f(x) = \ell$. Fix $a \in {}^*A\backslash\{r\}$ with $a \approx r$. Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. Since $\lim_{x\to r} f(x) = \ell$, we may fix $\delta \in \mathbb{R}$ with $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x \in A$ and $0 < |x - r| < \delta$. Now the sentence

$$\forall \mathsf{x}((\mathsf{x} \in \underline{A} \wedge 0 < |x - \underline{r}| < \underline{\delta}) \to |\underline{f}(\mathsf{x}) - \underline{\ell}| < \underline{\varepsilon})$$

is in $Th(\mathfrak{R}) = Th({}^*\mathfrak{R})$. Now we have $a \in {}^*A$ and $0 < |a-r| < \delta$, hence $|{}^*f(a) - \ell| < \varepsilon$. Since ε was arbitrary, it follows that ${}^*f(a) \approx \ell$.

Suppose now that for all $a \approx r$ with $a \neq r$, we have ${}^*f(a) \approx \ell$. Fix $z \in \mathcal{Z}$ with $z > 0$. Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. By assumption, whenever $a \in {}^*A$ and $0 < |a - r| < z$, we have that ${}^*f(a) \approx \ell$. Thus, the sentence

$$\exists \delta(\delta > 0 \wedge \forall \mathsf{x}((\mathsf{x} \in \underline{A} \wedge 0 < |\mathsf{x} - \underline{r}| < \delta) \to |\underline{f}(\mathsf{x}) - \underline{\ell}| < \underline{\varepsilon}))$$

is in $Th({}^*\mathfrak{R}) = Th(\mathfrak{R})$. By fixing a witnessing δ, we see that the limit condition holds for ε. □

Proposition 8.3.17. *Suppose that $A \subseteq \mathbb{R}$, that $f, g\colon A \to \mathbb{R}$, and that $r, \ell, m \in \mathbb{R}$. Suppose also that there exists $\delta > 0$ such that $(r - \delta, r + \delta)\backslash\{r\} \subseteq A$, that $\lim_{x\to r} f(x) = \ell$ and $\lim_{x\to r} g(x) = m$. We then have*

1. *$\lim_{x\to r}(f + g)(x) = \ell + m$.*
2. *$\lim_{x\to r}(f - g)(x) = \ell + m$.*
3. *$\lim_{x\to r}(f \cdot g)(x) = \ell \cdot m$.*
4. *If $m \neq 0$, then $\lim_{x\to r}(\frac{f}{g})(x) = \frac{\ell}{m}$.*

Proof. Fix $a \approx r$ with $a \neq r$. We then have ${}^*f(a) \approx \ell$ and ${}^*g(a) \approx m$. We have

1. ${}^*(f + g)(a) = {}^*f(a) + {}^*g(a) \approx \ell + m$.
2. ${}^*(f - g)(a) = {}^*f(a) - {}^*g(a) \approx \ell - m$.
3. ${}^*(f \cdot g)(a) = {}^*f(a) \cdot {}^*g(a) \approx \ell \cdot m$.
4. ${}^*(\frac{f}{g})(a) = \frac{{}^*f(a)}{{}^*g(a)} \approx \frac{\ell}{m}$ (notice ${}^*g(a) \notin \mathcal{Z}$ because $m \neq 0$).

□

Corollary 8.3.18. *Suppose that $A \subseteq \mathbb{R}$, that $f\colon A \to \mathbb{R}$, and that $r \in \mathbb{R}$. Suppose also that there exists $\delta > 0$ such that $(r - \delta, r + \delta) \subseteq A$. The following are equivalent.*

1. *f is continuous at r.*
2. *For all $a \approx r$, we have ${}^*f(a) \approx f(r)$.*

Taking the standard definition of the derivative, we immediately get the following.

Corollary 8.3.19. *Suppose that* $A \subseteq \mathbb{R}$*, that* $f\colon A \to \mathbb{R}$*, and that* $r, \ell \in \mathbb{R}$*. Suppose also that there exists* $\delta > 0$ *such that* $(r-\delta, r+\delta) \subseteq A$*. The following are equivalent.*

1. f *is differentiable at* r *with* $f'(r) = \ell$*.*
2. *For all* $a \approx r$ *with* $a \neq r$*, we have* $\frac{{}^*f(a)-f(r)}{a-r} \approx \ell$*.*

Proposition 8.3.20. *If* f *is differentiable at* r*, then* f *is continuous at* r*.*

Proof. Fix $a \approx r$ with $a \neq r$. Since f is differentiable at r, we have

$$\frac{{}^*f(a) - f(r)}{a - r} \approx f'(r)$$

Now $f'(r) \in \mathcal{F}$, so $\frac{{}^*f(a)-f(r)}{a-r} \in \mathcal{F}$, and hence ${}^*f(a) - f(r) \in \mathcal{Z}$ because $a - r \in \mathcal{Z}$. It follows that ${}^*f(a) \approx f(r)$. □

Proposition 8.3.21. *Suppose that* $f, g\colon \mathbb{R} \to \mathbb{R}$ *and* $r \in \mathbb{R}$*. Suppose also that* g *is differentiable at* r *and* f *is differentiable at* $g(r)$*. We then have that* $f \circ g$ *is differentiable at* r *and* $(f \circ g)'(r) = f'(g(r)) \cdot g'(r)$*.*

Proof. We know that for all $a \approx r$ with $a \neq r$, we have

$$\frac{{}^*g(a) - g(r)}{a - r} \approx g'(r)$$

Also, for all $b \approx g(r)$ with $b \neq g(r)$, we have

$$\frac{{}^*f(b) - f(g(r))}{b - g(r)} \approx f'(g(r))$$

Now fix $a \approx r$ with $a \neq r$. Since g is continuous at r, we have ${}^*g(a) \approx g(r)$. If ${}^*g(a) \neq g(r)$, then

$$\begin{aligned}
\frac{{}^*(f \circ g)(a) - (f \circ g)(r)}{a - r} &= \frac{{}^*f({}^*g(a)) - f(g(r))}{a - r} \\
&= \frac{{}^*f({}^*g(a)) - f(g(r))}{{}^*g(a) - g(r)} \cdot \frac{{}^*g(a) - g(r)}{a - r} \\
&\approx f'(g(r)) \cdot g'(r)
\end{aligned}$$

Suppose then that ${}^*g(a) = g(r)$. Since the first line above holds for every $a \approx r$ with $a \neq r$, we must have $g'(r) \approx 0$ and hence $g'(r) = 0$ because $g'(r) \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\frac{{}^*(f \circ g)(a) - (f \circ g)(r)}{a - r} &= \frac{{}^*f({}^*g(a)) - f(g(r))}{a - r} \\
&= 0 \\
&= f'(g(r)) \cdot g'(r)
\end{aligned}$$

□