CS 441 Discrete Mathematics for CS
Lecture 2

Propositional logic

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Course administration

Homework 1

- First homework assignment is out today will be posted on the course web page
- Due next week on Thurday

Recitations:

- **today at 4:00pm SENSQ 5313**
- **tomorrow at 11:00 SENSQ 5313**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Propositional logic: review

- **Propositional logic:** a formal language for representing knowledge and for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Compound propositions

- Let p: 2 is a prime ….. **T**
  q: 6 is a prime ….. **F**
- Determine **the truth value** of the following statements:

  $\neg p$: **F**

  $p \wedge q$: **F**

  $p \wedge \neg q$: **T**

  $p \vee q$: **T**

  $p \oplus q$: **T**

  $p \rightarrow q$: **F**

  $q \rightarrow p$: **T**

Constructing the truth table

- **Example:** Construct the truth table for
  (p → q) ∧ (¬p ↔ q)

Constructing the truth table

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Rows: all possible combinations of values for elementary propositions: 2^n values

p	q	¬p			(p→q)∧ (¬p↔q)
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

- Example: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Constructing the truth table

- Examples: Construct a truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Computer representation of True and False

We need to encode two values True and False:

- **Computers represents data and programs using 0s and 1s**
- Logical truth values – True and False
- A bit is sufficient to represent two possible values:
  - 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a **Boolean variable**.

- **Definition**: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Bitwise operations

- T and F replaced with 1 and 0

p	q	$p \vee q$	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

p	$\neg p$
1	0
0	1

Bitwise operations

- **Examples:**

1011 0011	1011 0011	1011 0011
$\vee$ <u>0110 1010</u>	$\wedge$ <u>0110 1010</u>	$\oplus$ <u>0110 1010</u>
1111 1011	0010 0010	1101 1001

Applications of propositional logic

- **Translation of English sentences**
- **Inference and reasoning:**
  - new true propositions are inferred from existing ones
  - Used in Artificial Intelligence:
    - Rule based (expert) systems
    - Automatic theorem provers
- **Design of logic circuit**

Translation

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (**you are older than 13 **or** you are with your parents **) then (**you can attend a PG-13 movie**)**

Atomic (elementary) propositions:

- A= you are older than 13
- B= you are with your parents
- C=you can attend a PG-13 movie

- **Translation:** $\mathbf{A \vee B \rightarrow C}$

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

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- **General rule for translation.**
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You can have free coffee if you are senior citizen and it is a Tuesday

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

a **b** **c**

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation .**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

a **b** **c**

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b \wedge c \rightarrow a}$$

Translation

- Assume two elementary statements:
  - **p: you drive over 65 mph** ; **q: you get a speeding ticket**
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \wedge \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $(q \wedge \neg p)$

Application: inference

Assume we know the following sentences are true:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie. You are older than 13.

Translation:

- **If (**you are older than 13 **or** you are with your parents **) then (**you can attend a PG-13 movie**)** . (You are older than 13).
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- $(A \vee B \rightarrow C)$, **A**
- $(A \vee B \rightarrow C) \wedge A$ **is true**
- **With the help of the logic we can infer the following statement (proposition):**
  - You can attend a PG-13 movie or C **is True**

Application: inference

The field of Artificial Intelligence:

- Builds programs that act intelligently
- Programs often rely on symbolic manipulations

Expert systems:

- Encode knowledge about the world in logic
- Support inferences where new facts are inferred from existing facts following the semantics of logic

Theorem provers:

- Encode existing knowledge (e.g. about math) using logic
- Show that some hypothesis is true

Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **It** represents
  - Facts about a specific patient case
  - Rules describing relations between entities in the bacterial infection domain

If	1. The stain of the organism is gram-positive, and 2. The morphology of the organism is coccus, and 3. The growth conformation of the organism is chains
Then	the identity of the organism is streptococcus

- **Inferences:**
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \vee \neg p$ is a **tautology.**

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

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Example: $p \wedge \neg p$ is a **contradiction.**

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Equivalence

- We have seen that some of the propositions are equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \to q}$ is equivalent to $\mathbf{\neg q \to \neg p}$ **(contrapositive)**

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- Equivalent statements are important for **logical reasoning** since they can be substituted and can help us to:

  (1) make a logical argument and (2) infer new propositions

Logical equivalence

<u>Definition</u>: The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q)$ <=> $\neg p \wedge \neg q$
- 2) $\neg(p \wedge q)$ <=> $\neg p \vee \neg q$

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: ?

Equivalence

- **Definition**: The propositions p and q are called **logically equivalent** if p ↔ q is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Example of important equivalences

- **DeMorgan's Laws:**
- 1) ¬(p ∨ q) <=> ¬p ∧ ¬q
- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: "The summer in Mexico is not cold or not sunny."

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) ¬(p ∨ q) <=> ¬p ∧ ¬q
- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
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To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	
T	F	F	T	**F**	
F	T	T	F	**F**	
F	F	T	T	**T**	

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
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To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) ¬(p ∨ q) <=> ¬p ∧ ¬q
- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Important logical equivalences

- **Identity**
  - p ∧ T <=> p
  - p ∨ F <=> p
- **Domination**
  - p ∨ T <=> T
  - p ∧ F <=> F
- **Idempotent**
  - p ∨ p <=> p
  - p ∧ p <=> p

Important logical equivalences

- **Double negation**
  - $\neg(\neg p) <=> p$
- **Commutative**
  - $p \vee q <=> q \vee p$
  - $p \wedge q <=> q \wedge p$
- **Associative**
  - $(p \vee q) \vee r <=> p \vee (q \vee r)$
  - $(p \wedge q) \wedge r <=> p \wedge (q \wedge r)$

Important logical equivalences

- **Distributive**
  - $p \vee (q \wedge r) <=> (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) <=> (p \wedge q) \vee (p \wedge r)$
- **De Morgan**
  - $\neg(p \vee q) <=> \neg p \wedge \neg q$
  - $(p \wedge q) <=> \neg p \vee \neg q$
- **Other useful equivalences**
  - $p \vee \neg p <=> T$
  - $p \wedge \neg p <=> F$
  - **$p \rightarrow q <=> (\neg p \vee q)$**

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example:** **Show $(p \wedge q) \rightarrow p$ is a tautology.**
- **Proof:** (we must show $(p \wedge q) \rightarrow p \;<=>\; T$)

  $(p \wedge q) \rightarrow p \;<=>\; \neg(p \wedge q) \vee p$ — Useful
- $<=>\; [\neg p \vee \neg q] \vee p$ — DeMorgan
- $<=>\; [\neg q \vee \neg p] \vee p$ — Commutative
- $<=>\; \neg q \vee [\, \neg p \vee p \,]$ — Associative
- $<=>\; \neg q \vee [\, T \,]$ — Useful
- $<=>\; T$ — Domination

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example:** Show $(p \wedge q) \rightarrow p$ is a tautology.
- Alternate proof:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** **Show** **$(p \rightarrow q)$ <=> $(\neg q \rightarrow \neg p)$**
  Proof:
- $(p \rightarrow q)$ <=> $(\neg q \rightarrow \neg p)$
- <=> ?

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q)$ <=> $(\neg q \rightarrow \neg p)$
  Proof:
- $(p \rightarrow q)$ <=> $(\neg q \rightarrow \neg p)$
- <=> $\neg(\neg q) \vee (\neg p)$ Useful
- <=> ?

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \to q) \iff (\neg q \to \neg p)$

  Proof:

- $(p \to q) \iff (\neg q \to \neg p)$
- $\iff \neg(\neg q) \vee (\neg p)$ Useful
- $\iff q \vee (\neg p)$ Double negation
- $\iff$?

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \to q) \iff (\neg q \to \neg p)$

  Proof:

- $(p \to q) \iff (\neg q \to \neg p)$
- $\iff \neg(\neg q) \vee (\neg p)$ Useful
- $\iff q \vee (\neg p)$ Double negation
- $\iff \neg p \vee q$ Commutative
- $\iff$?

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$

  Proof:

- $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
- $\Leftrightarrow \neg(\neg q) \vee (\neg p)$ — Useful
- $\Leftrightarrow q \vee (\neg p)$ — Double negation
- $\Leftrightarrow \neg p \vee q$ — Commutative
- $\Leftrightarrow p \rightarrow q$ — Useful

  End of proof