

Taylor Series

1. Find the degree 6 Taylor polynomial approximation for $f(x) = \sin x$ centered at 0.

Solution. We are looking for a polynomial $P_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ such that the k -th derivative $P_6^{(k)}(0)$ is equal to the k -th derivative $f^{(k)}(0)$. The k -th derivative $P_6^{(k)}(0)$ is just equal to $k!a_k$, so we want $a_k = \frac{f^{(k)}(0)}{k!}$. The derivatives of $f(x) = \sin x$ are:

$$\begin{aligned} f(x) &= \sin x &\Rightarrow f(0) &= 0 \\ f'(x) &= \cos x &\Rightarrow f'(0) &= 1 \\ f''(x) &= -\sin x &\Rightarrow f''(0) &= 0 \\ f'''(x) &= -\cos x &\Rightarrow f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x &\Rightarrow f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x &\Rightarrow f^{(5)}(0) &= 1 \\ f^{(6)}(x) &= -\sin x &\Rightarrow f^{(6)}(0) &= 0 \end{aligned}$$

Therefore, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{3!}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$, and $a_6 = 0$. So, $P_6(x) =$

$$\boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5}.$$

2. (a) If you want to find a Taylor polynomial approximation $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ (centered at 0) to $f(x)$, write a formula for the coefficient a_k .

Solution. $\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$. (You might wonder what happens when $k = 0$: $0!$ is defined to be 1, so the formula still works.)

- (b) If you want to find a Taylor polynomial approximation $a_0 + a_1(x-3) + a_2(x-3)^2 + \dots + a_n(x-3)^n$ (centered at 3) to $f(x)$, write a formula for the coefficient a_k .

Solution. $\boxed{a_k = \frac{f^{(k)}(3)}{k!}}$.

3. How do you think you would represent $\sin x$ as an infinite polynomial centered at 0? This is called the Taylor series (rather than Taylor polynomial) generated by $\sin x$ about 0.

Solution. Based on the pattern we started to see in #1, it seems like we should get

$$\boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots}.$$

4. What is the Taylor series generated by $\cos x$ about 0?

Solution. Using $f(x) = \cos x$, we have

$$\begin{aligned} f(x) &= \cos x &\Rightarrow f(0) &= 1 \\ f'(x) &= -\sin x &\Rightarrow f'(0) &= 0 \\ f''(x) &= -\cos x &\Rightarrow f''(0) &= -1 \\ f'''(x) &= \sin x &\Rightarrow f'''(0) &= 0 \end{aligned}$$

After this, the derivatives repeat, and we continue to get $1, 0, -1, 0, 1, 0, -1, 0, \dots$. So, our coefficients are $a_0 = 1, a_1 = 0, a_2 = -\frac{1}{2!}, a_3 = 0, a_4 = \frac{1}{4!}, a_5 = 0$, and so on. Thus, the Taylor series should be

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

5. What is the Taylor series generated by e^x about 0?

Solution. If $f(x) = e^x$, then the k -th derivative $f^{(k)}(x)$ is always e^x , so $f^{(k)}(0) = 1$. Therefore, the k -th coefficient a_k in the Taylor series is $\frac{1}{k!}$. Thus, the Taylor series is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

If we wanted to write this in summation notation, we would write $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

6. We hope that, by using “polynomials of infinite degree,” we end up with something that is not just an approximation for our function but is actually equal to the function. We don’t really know if this is true yet. Taking on faith that e^x is actually equal to its Taylor expansion about 0, can you write a power series expansion (or “infinite polynomial representation”) of:

(a) e^{-x^2} ?

Solution. Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, we can get e^{-x^2} just by replacing all of the x ’s in the series for e^x with $-x^2$: $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots$. In summation notation,

$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$. We often simplify $(-x^2)^k$ as $[(-1)(x^2)]^k = (-1)^k x^{2k}$, so you might also see

this as $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$.

(b) $\int e^{-x^2} dx$?

Solution. If we believe that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots$, then it seems plausible that we can integrate this using the reverse of the Power Rule to get $\int e^{-x^2} dx = C + x - \frac{1}{3}x^3 + \frac{1}{5} \cdot \frac{x^5}{2!} - \frac{1}{7} \cdot \frac{x^7}{3!} + \frac{1}{9} \cdot \frac{x^9}{4!} - \dots$, where C is any constant.

7. (a) Write a general formula for the Taylor series of $f(x)$ centered at 0.

Solution. We know it should look like $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ where $a_k = \frac{f^{(k)}(0)}{k!}$. So, it is

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$. In summation notation, it is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$.

(b) What if you wanted to center at 5?

Solution. Then, we would get $a_0 + a_1(x-5) + a_2(x-5)^2 + a_3(x-5)^3 + \dots$ where $a_k = \frac{f^{(k)}(5)}{k!}$.

In other words, we would get $f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \dots$.