English translation of
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THE MAGIC WAND THEOREM OF A. ESKIN AND M. MIRZAKHANI

ANTON ZORICH

Courtesy of Maryam Mirzakhani

On August 13, 2014 (the openning day of ICM at Seoul) Maryam Mirzakhani received the Fields Medal *"for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces"* becoming the first woman to win the Fields Prize. Citing the ICM laudation [ICM],

"Maryam Mirzakhani has made stunning advances in the theory of Riemann surfaces and their moduli spaces, and led the way to new frontiers in this area. Her insights have integrated methods from diverse fields, such as algebraic geometry, topology and probability theory.

In hyperbolic geometry, Mirzakhani established asymptotic formulas and statistics for the number of simple closed geodesics on a Riemann surface of genus g. She next used these results to give a new and completely unexpected proof of Witten's conjecture, a formula for characteristic classes for the moduli spaces of Riemann surfaces with marked points.

In dynamics, she found a remarkable new construction that bridges the holomorphic and symplectic aspects of moduli space, and used it to show that Thurston's earthquake flow is ergodic and mixing.

Most recently, in the complex realm, Mirzakhani and her coworkers produced the long sought-after proof of the conjecture that – while the closure of a real geodesic in moduli space can be a fractal cobweb, defying classification – the closure of a complex geodesic is always an algebraic subvariety.

Her work has revealed that the rigidity theory of homogeneous spaces (developed by Margulis, Ratner and others) has a definite resonance in the highly inhomogeneous, but equally fundamental realm of moduli spaces, where many developments are still unfolding."

We start this article with a short biographical note of Maryam Mirzakhani (for more details see excellent online article [Kl] of Erica Klarreich). Then we try to give an idea of her achievements focusing on one fundamental result as an example. To put this result into context we present in section 1 several natural problems in physics which lead to measured foliations on surfaces. In section 2 we describe the extremely elementary and convenient language of translation surfaces used to work with measured foliations. In section 3 we provide background material describing how dynamics on one individual translation surface leads to dynamics in a multidimensional space of translation surfaces. In section 4 we discuss why ergodic theory is usually absolutely powerless in describing specific trajectories. And, finally, in section 5 we present the Theorems of A. Eskin and M. Mirzakhani, and of A. Eskin, M. Mirzakhani, A. Mohammadi, which serve as a Magic Wand for numerous applications (the choice of applications in this exposition is mostly based on simplicity of presentation rather than on their significance).

This article is closely related to the part of the paper of P. Hubert and R. Krikorian in the current issue dedicated to results of Artur Avila on dynamics in Teichmüller space, see [HuKr]. The reader might find it interesting to read both companion papers: they are intended to complement each other.

For more ample presentation of mathematical works of Maryam Mirzakhani see the short paper [Mc2] written by C. McMullen for the ICM Proceedings.

Brief biographical note

Maryam Mirzakhani was born in 1977 in Teheran. After passing an entrance test she was accepted for the Farzanegan middle school and then high school for girls in Teheran administered by Irans National Organization for Development of Exceptional Talents. As Maryam told to Erica Klarreich, journalist of Quanta Magazine (see [Kl] for details) she, and her friend Roya Beheshti convinced the principal of their high school to organize classes of preparation for International Mathematical Olympiad (at this time Iranian team never contained girls). Maryam's determination gave excellent results: she won gold medals at International Mathematical Olympiads in 1994 and in 1995.

Maryam Mirzakhani completed undergraduate studies at Sharif University in Tehran in 1999 and moved for graduate studies to Harvard University choosing Curtis McMullen as scientific advisor. The Ph.D. Thesis defended in 2004 brought her international recognition. In her Thesis Maryam proved that the number $N(X, L)$ of simple closed geodesics (i.e. of those, which do not have self-intersections) on a Riemann surface X of length at most L grows asymptotically as

$$N(X, L) \sim const(X) \cdot L^{6g-6} \text{ as } L \to \infty\,.$$

(It is a classical result that the number of *all* closed geodesics grows much faster, as e^L/L: most of closed geodesics intersect themselves.) Another extraordinary result of the same Ph. D. Thesis was a new proof of Witten conjecture (the first proof was obtained by M. Kontsevich in 1992).

Having defended the Ph.D. Thesis Maryam Mirzakhani got a prestigious Clay Mathematics Institute Research Fellowship which provides a generous salary and research expenses leaving to a Fellow complete freedom of choice where to perform research[1].

Maryam Mirzakhani worked for several years at Princeton University; in 2008, at the age of 31, she became a Full Professor of Stanford University, where she has been working ever since. Maryam is a mother of a charming 3 years old daughter.

Personally Maryam is extremely nice and friendly, and not the least bit standoffish; meeting her at a conference you would take her at the first glance for a young postdoc rather than for a celebrated star. All her lectures which I have attended were full of

[1] Note that three out of four 2014 Fields Medalists are also former Clay Research Fellows.

sparkling and contagious enthusiasm, optimism, and appreciation of beauty of mathematics; they inspire you to attack fearlessly complicated problems and, following Maryam, not to give up when they resist.

1. From problems of solid state physics to surface foliations

In a sense, dynamical systems concern anything which moves; usually, when the motion has already achieved some kind of stable regime. "The thing that moves" might be the Solar system, or a system of particles in a chamber, or a billiard ball on a table (where the table is not necessarily a rectangular one), or currents in the ocean, or electrons in a metal, etc. One can observe certain common phenomena in large classes of dynamical systems; in particular, ideal billiards might be interpreted as toy models of a gas in a chamber. Such toy models allow to elaborate tools to study original dynamical systems of physical nature.

The language of measured foliations on surfaces (generalizing irrational winding lines on a torus) developed by Bill Thurston proved to be very useful in working with the class of dynamical systems including periodic billiards in the plane (like the *"windtree model"* introduced by Paule and Tatiana Ehrenfest a century ago; see the paper of P. Hubert and R. Krikorian in this issue for details) and dynamics of an electron on a Fermi-surface in the presence of a homogeneous magnetic field (*"the S. P. Novikov Problem"*; see the survey [MlNo] for details).

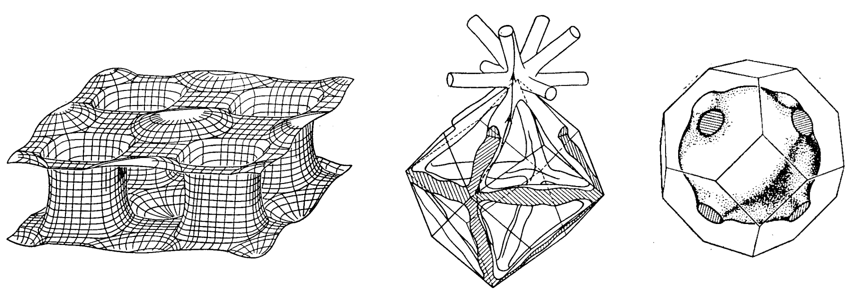

FIGURE 1. Fundamental domains of Fermi surfaces of tin, iron and gold reproduced from [ALK]. Electron trajectories in these metals in the presence of a homogeneous magnetic field correspond to plane sections of the corresponding periodic surfaces.

A flow following the leaves of such measured foliation on a surface decomposes the surface into two types of domains: periodic components filled with periodic trajectories and minimal components in which every trajectory is dense. After applying a natural surgery, the foliation on any minimal component can be in certain sense (described in the next section) "globally straightened": one can choose appropriate flat coordinates in which the foliation is represented by a family of vertical lines $x = const$. Versions of this "straightening theorem" were proved in different contexts and in different terms by E. Calabi [Ca], A. Katok [Ka], J. Hubbard and H. Masur [HdMa], and by other authors. These theorems imply that to study an important class of dynamical systems it is sufficient to confine the study to a simplified model and to study straight line foliations on "translation surfaces".

2. Translation surfaces

In this section we describe a very concrete construction of a vertical foliation on a surface endowed with a rather special flat metric with isolated singularities. The "straightening theorem" mentioned above asserts that all orientable measured foliations on surfaces can be reduced (in the way described above) to the ones as in the current section.

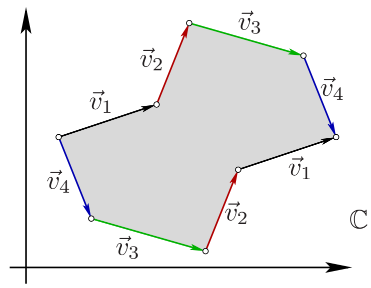

FIGURE 2. Identifying corresponding pairs of sides of this polygon by parallel translations we obtain a flat surface of genus two endowed with a flat metric having a single conical singularity.

Consider a collection of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^2$ and arrange these vectors into a broken line. Construct another broken line starting at the same point as the first one arranging the same vectors in the order $\vec{v}_{\pi(1)}, \dots, \vec{v}_{\pi(n)}$, where π is some permutation of n elements. By construction the two broken lines share the same endpoints; suppose that they bound a polygon as in Figure 2. Identifying the pairs of sides corresponding to the same vectors $\vec{v}_j$, $j = 1, \dots, n$, by parallel translations we obtain a closed topological surface.

By construction, the surface is endowed with a flat metric. When $n = 2$ and $\pi = (2, 1)$ we get a usual flat torus glued from a parallelogram. For larger number of elements we might get a surface of higher genus, where the genus is determined by the permutation π. It is convenient to impose from now on some simple restrictions on the permutation π which guarantee, in particular, non degeneracy of the surface; see [Ma1] or [Ve].

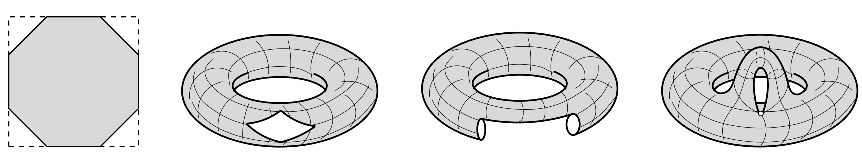

FIGURE 3. Cartoon movie of gluing a translation surface of genus two from a regular octagon.

For example, a regular octagon gives rise to a surface of genus two as in Figure 3. Indeed, identifying pairs of horizontal and vertical sides of a regular octagon we get a usual torus with a hole in the form of a square. We slightly cheat in the next frame, where we turn this hole by 45° and only then glue the next pair of sides. As a result we get a torus with two isolated holes as on the third frame. Identifying the remaining pair of sides (which represent the holes) we get a torus *with a handle*, or, in other words, a surface of genus two.

Similar to the torus case, the surface glued from the regular octagon or from an octagon as in Figure 2 also inherits from the polygon a flat metric, but now the resulting flat metric has a singularity at the point obtained from identified vertices of the octagon.

Note that the flat metric thus constructed is very special: since we identify the sides of the polygon only by translations, the parallel transport of any tangent vector along a closed cycle (avoiding conical singularities) on the resulting surface brings the vector back to itself. In other words, our flat metric has *trivial holonomy*. In particular, since a parallel transport along a small loop around any conical singularity brings the vector to itself, the cone angle at any singularity is an integer multiple of 2π. In the most general situation the flat surface of genus g would have several conical singularities with cone angles $2\pi(d_1+1), \dots 2\pi(d_m+1)$, where $d_1 + \dots + d_m = 2g-2$.

It is convenient to consider the vertical direction as part of the structure. A surface endowed with a flat metric with trivial holonomy and with a choice of a vertical direction is called a *translation surface*. Two polygons in the plane obtained one from another by a parallel translation give rise to the same translation surface, while polygons obtained one from another by a nontrivial rotation (usually) give rise to distinct translation surfaces.

We can assume that the polygon defining our translation surface is embedded into the complex plane $\mathbb{C} \simeq \mathbb{R}^2$ with coordinate z. The translation surface obtained by identifying the corresponding sides of the polygon inherits the complex structure. Moreover, since the gluing rule for the sides can be expressed in local coordinates as $z = \tilde{z} + const$, the closed 1-form dz is well-defined not only in the polygon, but on the surface. An exercise in complex analysis shows that the complex structure extends to the points coming from the vertices of the polygon, and that the 1-form $\omega = dz$ extends to the holomorphic 1-form on the resulting Riemann surface. This 1-form ω has zeroes of degrees $d_1, \dots, d_m$ exactly at the points where the flat metric has conical singularities of angles $2\pi(d_1+1), \dots, 2\pi(d_m+1)$.

Reciprocally, given a holomorphic 1-form ω on a Riemann surface one can always find a local coordinate z (in a simply-connected domain not containing zeroes of ω) such that $\omega = dz$. This coordinate is defined up to an additive constant. It defines the translation structure on the surface. Cutting up the surface along an appropriate collection of straight segments joining conical singularities we can unwrap the Riemann surface into a polygon as above.

This construction shows that the two structures are completely equivalent; the flat metric with trivial holonomy plus a choice of distinguished direction or a pair: Riemann surface and a holomorphic 1-form on it.

3. Families of translation surfaces and dynamics in the moduli space

The polygon in our construction depends continuously on the vectors $\vec{v}_i$. This means that the topology of the resulting translation surface (its genus g, the number and the types of the resulting conical singularities) does not change under small deformations of the vectors $\vec{v}_i$. For every collection of cone angles $2\pi(d_1+1), \dots, 2\pi(d_m+1)$ satisfying $d_1 + \dots + d_m = 2g-2$ with integer d_i for $i = 1, \dots, n$, we get a *family* $\mathcal{H}(d_1, \dots, d_m)$ *of translation surfaces*. Vectors

$\vec{v}_1, \ldots, \vec{v}_n$ can be viewed as complex coordinates in this space, called *period coordinates*. These coordinates define a structure of a *complex orbifold* (manifold with moderate singularities) on each space $\mathcal{H}(d_1, \ldots, d_m)$. The geometry and topology of spaces of translation surfaces is not yet sufficiently explored.

Readers preferring algebro-geometric language may view a family of translation surfaces with fixed conical singularities $2\pi(d_1+1), \ldots, 2\pi(d_m+1)$ as the stratum $\mathcal{H}(d_1, \ldots, d_m)$ in the moduli space $\mathcal{H}_g$ of pairs (Riemann surface C; holomorphic 1-form ω on C), where the stratum is specified by the degrees $d_1, \ldots, d_m$ of zeroes of ω, where $d_1 + \cdots + d_m = 2g-2$. Note that while the moduli space $\mathcal{H}_g$ is a holomorphic $\mathbb{C}^g$-bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces, individual strata are not. For example, the minimal stratum $\mathcal{H}(2g-2)$ has complex dimension $2g$, while the moduli space $\mathcal{M}_g$ has complex dimension $3g-3$. The very existence of a holomorphic form with a single zero of degree $2g-2$ on a Riemann surface C is a strong condition on C.

To complete the description of the space of translation surfaces we need to present one more very important structure: the action of the group $\mathrm{GL}(2,\mathbb{R})$ on $\mathcal{H}_g$ preserving strata. The description of this action is particularly simple in terms of our polygonal model Π of a translation surface S. A linear transformation $g \in \mathrm{GL}(2,\mathbb{R})$ of the plane maps the polygon Π to a polygon $g\Pi$. The new polygon again has all sides arranged into pairs, where the two sides in each pair are parallel and have equal length. We can glue a new translation surface and call it $g \cdot S$. It is easy to see that unwrapping the initial surface into different polygons would not affect the construction. Note also, that we explicitly use the choice of the vertical direction: any polygon is endowed with an embedding into $\mathbb{R}^2$ defined up to a parallel translation.

The subgroup $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$ preserves the flat area. This implies, that the action of $\mathrm{SL}(2,\mathbb{R})$ preserves the real hypersurface $\mathcal{H}_1(d_1, \ldots, d_m)$ of translation surfaces of area one in any stratum $\mathcal{H}(d_1, \ldots, d_m)$. The codimension-one subspace $\mathcal{H}_1(d_1, \ldots, d_m)$ can be compared to the unit sphere (or rather to the unit hyperboloid) in the ambient stratum $\mathcal{H}(d_1, \ldots, d_m)$.

Recall that under appropriate assumption on the permutation π, the n vectors

$$\vec{v}_1 = \begin{pmatrix} v_{1,x} \\ v_{1,y} \end{pmatrix} \ldots, \vec{v}_n = \begin{pmatrix} v_{n,x} \\ v_{n,y} \end{pmatrix}$$

as in Figure 2 define local coordinates in the embodying family $\mathcal{H}(d_1, \ldots, d_m)$ of translation surfaces. Let $d\nu := dv_{1x} dv_{1y} \ldots dv_{nx} dv_{ny}$ be the associated volume element in the corresponding coordinate chart $U \subset \mathbb{R}^{2n}$. It is easy to verify, that $d\nu$ does not depend on the choice of "coordinates" $\vec{v}_1, \ldots, \vec{v}_n$, so it is well-defined on $\mathcal{H}(d_1, \ldots, d_m)$. Similarly to the case of Euclidian volume element, we get a natural induced volume element $d\nu_1$ on the unit hyperboloid $\mathcal{H}_1(d_1, \ldots, d_m)$. It is easy to check that the action of the group $\mathrm{SL}(2,\mathbb{R})$ preserves the volume element $d\nu_1$.

The following Theorem proved independently and simultaneously by H. Masur [Ma1] and W. Veech [Ve] is the keystone of this area.

Theorem (H. Masur; W. A. Veech)**.** *The total volume of every stratum* $\mathcal{H}_1(d_1, \ldots, d_m)$ *is finite.*

The group $\mathrm{SL}(2,\mathbb{R})$ *and its diagonal subgroup act ergodically on every connected component of every stratum* $\mathcal{H}_1(d_1, \ldots, d_m)$.

Here "ergodically" means that any measurable subset invariant under the action of the group has necessarily measure zero or full measure. Ergodic theorem claims that in such situation the orbit of almost every point homogeneously fills the ambient connected component. In plain terms, the ergodicity of the action of the diagonal subgroup can be interpreted as follows. Having almost any polygon as above, we can choose appropriate sequence of times t_i such that contracting the polygon horizontally with a factor e^{t_i} and expanding it vertically with the same factor e^{t_i} and modifying the resulting polygonal pattern of the resulting translation surface by an appropriate sequence of cut-and-paste transformations (see Figure 4) we can get arbitrary close to, say, regular octagon rotated by any angle chosen in advance.

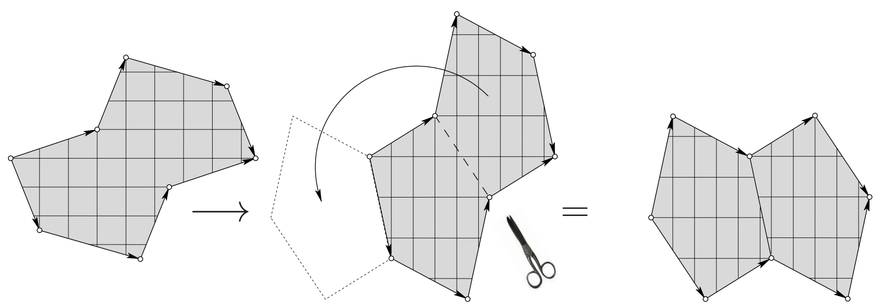

FIGURE 4. Note that expansion-contraction (action of the diagonal group) changes the translation surface, white cut-and-paste transformations change only the polygonal pattern, and do not change the flat surface.

Now everything is prepared to present the first marvel of this story. Suppose that we want to find out some fine properties of the vertical flow on an individual translation surface. Applying a homothety we can assume that the translation surface has area one. Howard Masur and William Veech suggest the following approach. Consider our translation surface (endowed with a vertical direction) as a point S in the ambient stratum $\mathcal{H}_1(d_1,\dots,d_m)$. Consider the orbit $\mathrm{SL}(2,\mathbb{R})\cdot S$ (or the orbit of S under the action of the diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ depending on the initial problem). Numerous important properties of the initial vertical flow are encoded in the geometrical properties of the closure of the corresponding orbit. This approach placed the problem of finding the orbit closures under the action of $\mathrm{SL}(2,\mathbb{R})$, and studying their geometry at the center of the studies in this area for the last three decades. To give at least one example of this approach we present Masur's criterion of *unique ergodicity* of the vertical flow.

In the late 60's and 70's several authors (including W. Veech, M. Keane, A. Katok, and others) discovered in different terms and in various contexts the phenomenon which can be illustrated by the following example. Take a translation surface as in Figure 5, where the rectangle has size 1×2, and the slits have any irrational length. (We are not restricted to polygons as in Figure 2 to construct translation surfaces; we could also glue translation surfaces from triangles imbedded into the Eucledian plane, provided that all identifications of sides are parallel translations.) For uncountable number of directions all trajectories of the straight line flow would be dense, but it would not be ergodic: some trajectories would spend most of the time in the middle part of the translation surface, while the other ones — mostly in the complementary part (see [Ma2] for details). Clearly, applying an appropriate rotation to the translation surface, one can make a nonergodic direction vertical.

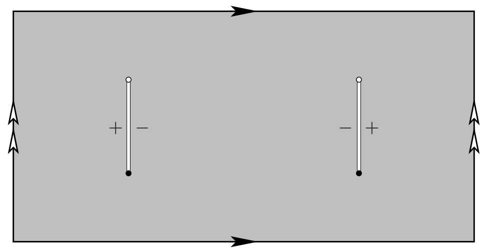

FIGURE 5. For appropriate directions of the straight line flow on this translation surface all trajectories are dense, but the flow is not ergodic: trajectories are *not* uniformly distributed on the translation surface.

Having presented the phenomenon, we can state Masur's criterion of unique ergodicity of the vertical flow.

Theorem (H. Masur)**.** *Suppose S is a translation surface. Suppose the flow in the vertical direction on S is not uniquely ergodic. Then for any compact set $K \subset \mathcal{H}(d_1, \dots, d_m)$ in the ambient space of translation surfaces the trajectory $g_t \cdot S$ of S would never visit K for t sufficiently large, $t > t_0(K)$. In other words, the trajectory of S under the action of the diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ eventually leaves any compact set; it "escapes to a multidimensional cusp" of $\mathcal{H}(d_1, \dots, d_m)$. (Actually, the statement is even stronger: the projection of the trajectory $g_t \cdot S$ to the moduli space $\mathcal{M}_g$ of Riemann surfaces eventually leaves any compact set in $\mathcal{M}_g$.)*

The vertical flow on the surface in Figure 5 is periodic, so extremely nonergodic. And indeed, the action of the diagonal subgroup makes our flat torus with slits become more and more narrow and long. In this case the family of deformed translation surfaces leaves any compact set in a very simple way: there is a closed flat geodesics which gets pinched. To construct minimal nonergodic examples mentioned above one should show that for certain angles the vertical flow applied to rotated translation surface makes the deformed translation surface leave eventually any compact set in the family of flat surfaces and never return back.

At the first glance, we have just reduced the study of a rather simple dynamical system, namely, of the vertical flow on a translation surface, to a really complicated one — to the study of the action of the group $\mathrm{GL}(2, \mathbb{R})$ on the space $\mathcal{H}(d_1, \dots, d_m)$. Nevertheless, this approach proved to be extremely fruitful and powerful, despite the fact that geometry and topology of spaces of translation surfaces is not sufficiently explored yet.

However, as it would be explained in the next section, successful implementations of this approach were often based on a chain of happy coincidences (like reduction to translation surfaces of genera one or two which are very special). The Magic Wand Theorem of A. Eskin and M. Mirzakhani and of A. Eskin, M. Mirzakhani, A. Mohammadi transforms this approach from art to science.

4. "Almost all" versus "all"

Even for those classes of dynamical systems which are sufficiently well understood, the only kind of predictions of "what would happen to a particle after sufficiently long time" always contain some version of the word "typically" usually meaning "for a full measure set of initial data". The trouble (which, depending on the taste, might be considered as an advantage: do not get distracted by details) is that even for those dynamical systems which are very well studied and understood, and where one knows, basically, everything about "typical behavior" of trajectories, one can say almost nothing about behavior of any concrete particular trajectory: there is no way to tell, whether your particular starting data are "typical" or not. If you repeat thousands of experiments with random starting data and you want to establish some statistics, you do not care about rare nontypical fluctuations. But if you are interested in the future of some very special asteroid B 612, and only by this, most of the methods of dynamical systems become completely useless for you.

The difficulty is conceptual; it is neither related to lack of knowledge at the current state of development of mathematics, nor to the presence of noise, or friction, etc in realistic dynamical systems. Even for absolutely deterministic systems, and even assuming all necessary mathematical abstractions like absence of any noise or friction, the trouble persists. The reason is that for the vast majority of dynamical systems (in particular, for very smooth and nice ones) certain individual trajectories might be extremely sophisticated. For example, they can cover extremely fractal sets on a large scale of time.

For example, the map $f : x \mapsto \{2x\}$ homogeneously twisting the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ twice around itself has orbits filling Cantor sets of, basically, arbitrary Hausdorff dimension between zero and one; has nonclosed orbits avoiding certain arcs of the circle, etc. In other words, this extremely nice map, clearly, has trajectories with very peculiar properties.

All these properties become much more visible using the binary representation of a real number $x \in [0, 1[$ instead of the usual decimal one. If

$$x = \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} + \cdots ,$$

where all binary digits n_k are zeroes or ones, then the map f acts on the sequence $(n_1, n_2, \ldots, n_k, \ldots)$ by erasing the first digit. (This operation on the space of semi-infinite sequences of zeroes and ones is called the *Bernoulli shift*).

The geodesic flow on any compact Riemann surface of constant negative curvature has similar behavior. It was observed long ago by H. Furstenberg and B. Weiss that the closures of individual trajectories might have arbitrary (or almost arbitrary) Hausdorff dimension in the range from 1 (closed trajectories) to 3 (typical trajectories).

A straight-line flow on a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is an example of a (very rare) dynamical system, where the closure of *any* orbit is a nice submanifold. Let $\vec{V} = (V_1, \ldots, V_n)$ denote the direction of the flow. The closure of *any* trajectory in direction $\vec{V}$ is a sub-torus $\mathbb{T}^k$, where $1 \le k \le n$ is the degree of irrationality $k = \dim_{\mathbb{Q}}\{V_1, \ldots, V_n\}$ of the direction. Say, in the particular case of a two-dimensional torus, when $n = 2$, all trajectories of the flow in a rational direction are already closed — they are circles $S^1 = \mathbb{T}^1$, and the closure of any trajectory of the flow in an irrational direction is the entire torus $\mathbb{T}^2$. Actually, this is not surprising at all: torus is a homogeneous space, and the group of automorphisms of the torus preserving the flow acts transitively on the torus.

In a sense, up to know, there was only one known class of dynamical systems, for which one could find the closure of any single trajectory, and for which all possible closures were described by a short list of possible simple cases like in the example above. It happens for very special dynamical systems in homogeneous spaces. One of the key statements in this theory was proved by Marina Ratner; extremely important contributions to this theory as well as fantastic applications to the number theory, were developed by S. G. Dani, G. Margulis, and by other great mathematicians, including A. Eskin, S. Mozes, and N. Shah. The scale of applications of this theory to different areas of mathematics continues to extend. Indeed, homogeneous spaces naturally appear in various domains of mathematics. (Both the theory and the list of major contributors merit a separate paper rather than a short paragraph.)

5. Magic Wand Theorem

And now comes the result of Alex Eskin and Maryam Mirzakhani [EMi] (incorporating the joint results of these authors and of Amir Mohammadi [EMiMh]).

It is known that the moduli space *is not* a homogeneous space. Nevertheless, the orbit closures of $\mathrm{GL}(2,\mathbb{R})$ in the space of translation surfaces are as nice as one can only hope: they are complex manifolds possibly with very moderate singularities (so-called "orbifolds"). In this sense the action of the $\mathrm{GL}(2,\mathbb{R})$ and of $\mathrm{SL}(2,\mathbb{R})$ on the space of translation surfaces described above mimics certain properties of the dynamical systems in homogeneous spaces mentioned at the end of the previous section.

Magic Wand Theorem. *The closure of any* $\mathrm{GL}(2,\mathbb{R})$*-orbit is a complex suborbifold (possibly with self-intersections); in period coordinates* $\vec{v}_1,\dots,\vec{v}_n$ *in the corresponding space* $\mathcal{H}(d_1,\dots,d_m)$ *of translation surfaces it is locally represented by an affine subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$*-invariant measure is supported on a suborbifold. In coordinates* $\vec{v}_1,\dots,\vec{v}_n$ *this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.*

As a vague conjecture (or, better say, as a very optimistic dream) this property was discussed since long ago, and since long ago, there was not a slightest hint for a general proof. The only exception is the case of surfaces of genus two, for which ten years ago C. McMullen proved a very precise statement [Mc1] classifying all possible orbit closures. He used, in particular, the hard artillery of Ratner's results which are applicable here. However, the theorem of McMullen is based on the very special properties of surfaces of genus two, which do not generalize to higher genera.

The proof of Alex Eskin and Maryam Mirzakhani is a titanic work which took many years. It absorbed numerous fundamental developments in dynamical systems which do not have any direct relation to moduli spaces. To mention only a few, it incorporates certain ideas of low entropy method of M. Einsiedler, A.Katok, E. Lindenstrauss; results of G. Forni and of M. Kontsevich on Lyapunov exponents of the Teichmüller geodesic flow; the ideas from the works of Y. Benoit and J.-F. Quint on stationary measures; iterative improvement of the properties of the invariant measure inspired by the approach of G. Margulis and G. Tomanov to the actions of unipotent flows on homogeneous spaces; some fine ergodic results due to Y. Guivarch and A. Raugi.

What can we do now, when this theorem is proved? For example, in certain situations this theorem works like a Magic Wand, which allows to touch *any* given billiard in certain class of billiards and in theory (and more and more often in practice) find the corresponding orbit closure in the moduli space of translation surfaces. The geometry of this orbit closure tells you, basically, everything you want to know about the initial billiard.

Suppose you want to study the billiard in the plane filled periodically with the obstacles as in Figure 6 (see the paper of P. Hubert and R. Krikorian [HuKr] in this issue discussing such a *windtree model*). A trajectory might go far away, then return relatively close back to the starting point, then make other long trips. The *diffusion rate* ν describes the average rate T^ν with which the trajectory expands in the plane on a long range of time $T \gg 1$. More formally,

$$\nu := \limsup_{T\to\infty} \frac{\log(\text{diameter of part of trajectory for interval of time } [0,T])}{\log T}\,.$$

For the usual random walk in the plane, or for a billiard with periodic circular obstacles the diffusion rate is known to be 1/2: the most distant point of a piece of trajectory corresponding to segment of time $[0,T]$ would be located roughly at a distance $\sqrt{T}$. (We do not discuss at what time $t_0 \in [0,T]$ the trajectory would be that far.)

The Magic Wand Theorem and companion results allow to show that for *any* obstacle as in Figure 6 the diffusion rate ν is one and the same for all starting points and for almost all starting directions. Moreover, one can even compute the diffusion rate! To perform this task one has to proceed as follows. Find the associated translation surface S; it is really easy. Using the Magic Wand theorem (and its development by A. Wright [W]) find the corresponding orbit closure $\mathcal{L}$. By a very recent theorem of A. Eskin and J. Chaika [ECh] almost all directions for *any* translation surface are Lyapunov-generic, so it is sufficient to find the appropriate Lyapunov exponent of the Teichmüller geodesic flow on $\mathcal{L}$ and you are done! (See details on the windtree billiard and on Lyapunov exponents in the paper [HuKr] in this issue.)

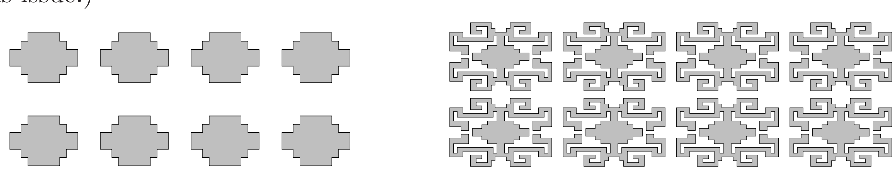

FIGURE 6. The diffusion rate depends only on the number of the angles of almost any symmetric obstacle as on the picture .

Say, for almost any symmetric obstacle with $4m-4$ angles $3\pi/2$ and $4m$ angles $\pi/2$ V. Delecroix and the author showed that the diffusion rate is

$$\frac{(2m)!!}{(2m+1)!!} \quad \sim \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as} \quad m \to \infty\,.$$

(This answers a question by J.-C. Yoccoz whether for certain shapes of the obstacles the diffusion rate can be arbitrary small and develops the original answer for $m = 1$ obtained in [DHL]).

Another application of the Magic Wand Theorem which is easy to describe is the following advance in *Illumination Problem* asking, whether a room with mirrored walls can always be illuminated by a single point light source. R. Penrose designed in 1958 a planar room with walls made from flat and elliptic mirrors that always has dark regions no matter where you place a candle in this room. In 1995 G. Tokarsky constructed a polygonal room with a similar property: it has one dark point if the idealized candle is placed at the correct point. Using the Magic Wand Theorem, S. Lelièvre, T. Monteil, and B. Weiss, proved in [LMtW] that for any translation surface M, and any point $x \in M$, the set of points y which are not illuminated by x is always finite.

One should not have an impression that the theory developed by A. Eskin, M. Mirzakhani, A. Mohammadi and other researchers in this area is designed to serve billiards. A billiard in a polygon is just a cute way to describe certain class of dynamical systems; we have seen that same kind of dynamical systems appear in solid state physics, in conductivity theory, and so on.

The result of A. Eskin and M. Mirzakhani opens a new way to study moduli spaces, which in the last several decades became a central object both in mathematics and in theoretical physics. We do not know yet all possible applications of the Magic Wand Theorem which might be obtained in this direction.

The integral calculus was partly developed by Kepler (a century before Newton and Leibniz) in order to measure the volume of wine barrels. Who could imagine at that time that volume of a solid of revolution would be discussed in any textbook of mathematics for beginners and that the integral calculus would become an essential part of all contemporary engineering. The theorem proved by Alex Eskin and Maryam Mirzakhani is so beautiful and powerfull that, personally, I have no doubt that it would find numerous applications far beyond our current imagination.

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