#5

Taylor Series: Expansions, Approximations and Error

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motivation

- All we can ever do is add and multiply with our Floating Point Unit (FPU)
- We can't directly evaluate e^x, $cos(x)$, $\sqrt{x}$
- What can we do? Use Taylor Series *approximation*

taylor series definition

The Taylor series expansion of $f(x)$ at the point $x = c$ is given by

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

an example

The Taylor series expansion of $f(x)$ about the point $x = c$ is given by

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Example (e^x)

We know $e^0 = 1$, so expand about $c = 0$ to get

$$\begin{aligned} f(x) = e^x &= 1 + 1 \cdot (x-0) + \frac{1}{2} \cdot (x-0)^2 + \ldots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \end{aligned}$$

taylor approximation

- So

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots$$

- But we can't evaluate an infinite series, so we truncate...

Taylor Series Polynomial Approximation

The Taylor Polynomial of degree n for the function $f(x)$ about the point c is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Example (e^x)

In the case of the exponential

$$e^x \approx p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

taylor approximation

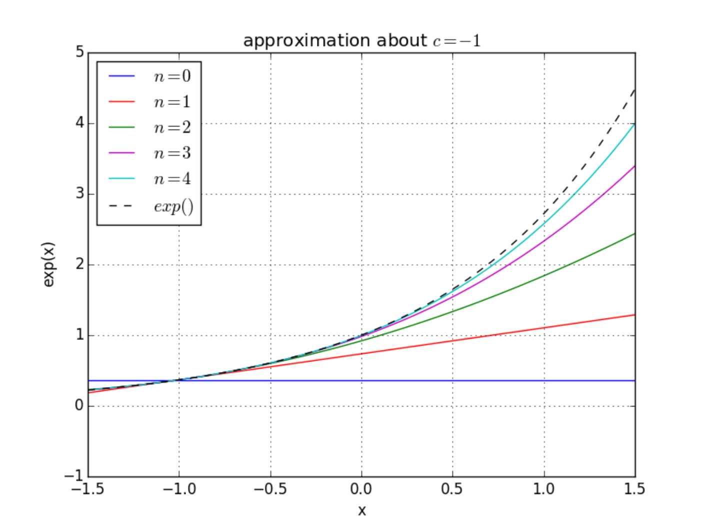
Evaluate e^2:

- Using 0^{th} order Taylor series: $e^x \approx 1$ does not give a good fit.
- Using 1^{st} order Taylor series: $e^x \approx 1 + x$ gives a better fit.
- Using 2^{nd} order Taylor series: $e^x \approx 1 + x + x^2/2$ gives a a really good fit.

```python
import numpy as np
x = 2.0
pn = 0.0
for k in range(15):
  pn += (x**k) / math.factorial(k)
  err = np.exp(2.0) - pn
```

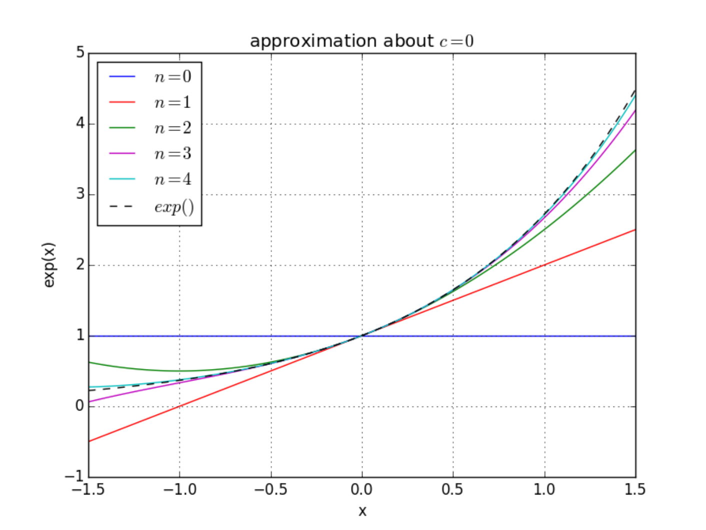
taylor approximation is local

Approximate e^x using $c = -1$:



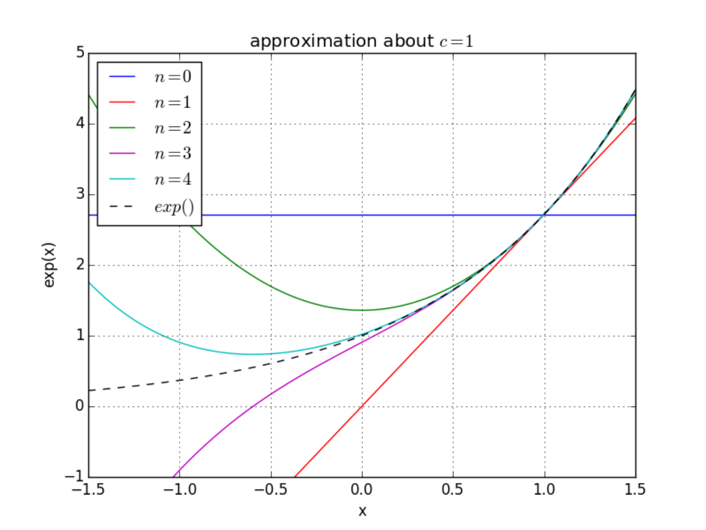
taylor approximation is local

Approximate e^x using $c = 0$:



taylor approximation is local

Approximate e^x using $c = 1$:



taylor approximation recap

Infinite Taylor Series Expansion (exact)

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

Finite Taylor Series Expansion (exact)

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-c)^n$$

but we don't know ξ.

Finite Taylor Series Approximation

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(x-c)^n$$

taylor approximation error

- How accurate is the Taylor series polynomial approximation?
- The *n* terms of the approximation are simply the first *n* terms of the *exact* expansion:

$$e^x = \underbrace{1 + x + \frac{x^2}{2!}}_{p_2 \text{ approximation to } e^x} + \underbrace{\frac{x^3}{3!} + \dots}_{\text{truncation error}} \tag{1}$$

- So the function $f(x)$ can be written as the Taylor Series approximation plus an error (truncation) term:

$$f(x) = f_n(x) + E_n(x)$$

where

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - c)^{n+1}$$

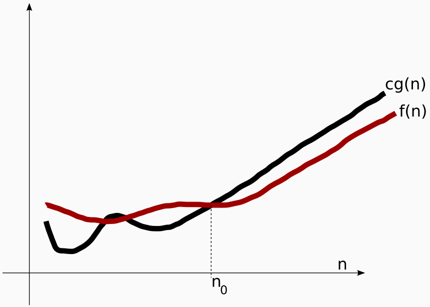
big-o (omicron)

Recall Big-O ”$\mathcal{O}$” notation

Let $g(n)$ be a function of n. Then define

$$\mathcal{O}(g(n)) = \{f(n) \mid \exists c, n_0 > 0 \, : \, 0 \leqslant f(n) \leqslant cg(n), \, \forall n \geqslant n_0\}$$

That is, $f(n) \in \mathcal{O}(g(n))$ if there is a constant c such that $0 \leqslant f(n) \leqslant cg(n)$ is satisfied.



truncation error

Using the Big "$\mathcal{O}$" notation,

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$
$$= \mathcal{O}\left(\frac{(x-c)^{n+1}}{(n+1)!}\right)$$

since we assume the $(n+1)^{th}$ derivative is bounded on the interval $[a, b]$.

Often, we let $h = x - c$ and we have

$$f(x) = p_n(x) + \mathcal{O}(h^{n+1})$$

truncation error

The Taylor series expansion of $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

If $x \ll 1$, then the remaining terms are small.
If we neglect these terms

$$\sin(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{\text{approximation to sin}} \underbrace{- \frac{x^7}{7!} + \frac{x^9}{9!} - \dots}_{\text{truncation error}}$$

another example: $f(x) = \frac{1}{1-x}$

- Evaluation of $f(x) = \frac{1}{1-x}$ using Taylor Series Expansion:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-c)^n$$

- Thus with c = 0

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

- Second order approximation:

$$\frac{1}{1-x} \approx 1 + x + x^2$$

taylor errors

- How many terms do I need to make sure my error is less than 2×10^{-8} for $x = 1/2$?

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \sum_{k=n+1}^{\infty} x^k$$

- so the error at $x = 1/2$ is

$$e_{x=1/2} = \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{(1/2)^{n+1}}{1-1/2}$$
$$= 2 \cdot (1/2)^{n+1} < 2 \times 10^{-8}$$

- then we need

$$n + 1 > \frac{-8}{\log_{10}(1/2)} \approx 26.6 \quad \text{or}$$
$$n > 26$$

some remarks

- can approximate infinite series; in particular analytic functions (those that have a power series representation).
- a local approximation (i.e. convergence can be slow far away from evaluation point c).
- Maclaurin is the special case when $c = 0$.
- useful for numerical approximation, differentiation, and integration