# MANDELBROT ISO-SETS: ISO-UNIT IMPACT ASSESSMENT

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**Abstract**

In this introductory paper, we use Santilli's iso-topic lifting as a platform to explore Mandelbrot's set. The objective is to upgrade Mandelbrot's complex quadratic polynomial with iso-multiplication and then probe the effects on this revolutionary fractal. For this, we define the "iso-complex quadratic polynomial" and engage it to generate an array of "Mandelbrot iso-sets" by varying the iso-unit. The computational results indicate two general topological effects: scale-deformation and boundary-deformation, which are consequently connected to dynamic iso-spaces. In total, these new and preliminary developments spark further insight into the emerging realm of iso-fractals.

**Keywords:** Geometry and topology; Chaos theory; Santilli iso-number; Fractal; Iso-fractal; Mandelbrot set; Mandelbrot iso-set.

## 1 Introduction

The Mandelbrot set is often considered to be the most famous fractal. It is a mathematical set of points in a Euclidean complex space $\mathbb{C}$, with a distinctive boundary that characterizes a fractal structure with self-similarity [1, 2]. The set is closely related to Julia sets [3] and is named after the French mathematician Benoit Mandelbrot, the pioneer who analyzed and popularized it [1, 2]. Images of Mandelbrot's set are created by iteratively sampling complex numbers and determining, for each one, if the result tends towards infinity when a particular mathematical operation is iterated on it [1, 2]. For each complex number, the real and imaginary components serve as 2D image coordinates in $\mathbb{C}$ [4], where the pixels are colored to encode the sequence divergence rate [1, 2]. In particular, the Mandelbrot set is the set of values of $c \in \mathbb{C}$ for which the orbit of 0 under iteration of Mandelbrot's complex quadratic polynomial [1, 2]

$$z_{n+1} = z_n^2 + c, \tag{1}$$

remains bounded, where $z_n, z_{n+1}, c \in \mathbb{C}$ are complex numbers. That is, $c$ is part of the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of $z_n$ remains bounded however large $n$ gets [1, 2]. Beyond the discipline of mathematics, the Mandelbrot set has become prominent in various art forms due to its aesthetic appeal [5, 6, 7] and, moreover, because it is an emergent complex structure that arises from the application of *simple* rules [1, 2].

So why are the Mandelbrot set and other such fractals an important subject to study in science and mathematics? Well, it turns out that *fractal geometry* is the language of chaos theory [4, 8], and fractal/chaotic patterns are *abundant* in the physical, chemical, and biological expressions of nature [9, 10, 6]. Moreover, fractal geometry and chaos theory are a relatively new discipline [4]. Chaos theory examines the behavior of dynamical systems that are *highly sensitive* to initial conditions [11, 9]. In a chaotic dynamical system, *miniscule* differences in initial conditions yield *widely diverging* outcomes, thereby generally rendering long-term predictions impossible [11, 9]. For this, additional examples of chaos and fractals are also observed in lightning discharges [12, 13, 14, 15], weather patterns [16, 17, 18], aquatic ecosystems [19, 20], population biology [21], the biological allometric scaling laws [22, 23, 24, 25, 26], cancers and genetics [27, 28], viruses and pathogens [29, 30], the human brain [31, 32, 33], earthquakes [34, 35, 36], volcanoes [37, 38, 39], the global stock market [40, 41], and more. Certainly, fractals such as the Mandelbrot set must play a fundamental role in classifying and demystifying such *complex* systems—but how?

In this paper, we resume the iso-fractal developments of [4] and attack this complex problem by utilizing the power of Santilli's new iso-topic lifting [42, 43, 44, 45, 46] to probe Mandelbrot's set [1, 2]. For this, we launch with Section 2, where we deploy Santilli's iso-numbers [4, 42, 43, 44, 45, 46] to upgrade Mandelbrot's complex quadratic polynomial—eq. (1)—with iso-multiplication to construct the *iso-complex quadratic polynomial*, which is used to construct a *Mandelbrot iso-set*. For this, we identify the procedure and results for a computational experiment that assesses the impact of Santilli's iso-unit [4, 42, 43, 44, 45, 46] for various Mandelbrot iso-sets. Finally, we conclude with Section 3, where we briefly recapitulate this mode of research and suggest future actions to take.

## 2 Experiment

Here, motivated by the iso-fractal initiation of [4], we engage Santilli's iso-numbers [42, 43, 44, 45, 46] to explore Mandelbrot's set [1, 2] in Euclidean complex space. In the procedure of Section 2.1, we attack our objective by upgrading Mandelbrot's complex quadratic polynomial—eq. (1)—with Santilli's iso-multiplication [4, 42, 43, 44, 45, 46] to construct the iso-complex quadratic polynomial, which is used to construct a Mandelbrot iso-set. Afterwards, in Section 2.2, we examine the computational results for an array of Mandelbrot iso-sets with distinct iso-topic liftings to assess the impact of varying the iso-units.

### 2.1 Procedure

In this section, the iso-complex quadratic polynomial for the experiment is assembled as follows:

1. First, in accordance to Santilli's iso-number methodology [4, 42, 43, 44, 45, 46], we select the positive-definite iso-unit $\hat{r} > 0$ with the corresponding inverse $\hat{\kappa} = \frac{1}{\hat{r}} > 0$.

2. Second, given that $\mathbb{C}$ is the set of all complex numbers, then we demonstrate that $\mathbb{C}$ is iso-topically lifted via $\mathbb{C} \to \mathbb{C}_{\hat{r}}$ to establish $\mathbb{C}_{\hat{r}}$, which is the set of all iso-complex numbers [42, 43, 44, 45, 46, 4]. Thus, if $z_1, z_2 \in \mathbb{C}$ are complex numbers, then the corresponding iso-complex numbers $\hat{z}_1, \hat{z}_2 \in \mathbb{C}_{\hat{r}}$ are directly related via [4, 42, 43, 44, 45, 46]

$$\begin{array}{rcl} \hat{z}_1 & = & z_1 \times \hat{r} \\ \hat{z}_2 & = & z_2 \times \hat{r} \end{array}, \ \forall z_1, z_2 \in \mathbb{C} \to \forall \hat{z}_1, \hat{z}_2 \in \mathbb{C}_{\hat{r}}, \tag{2}$$

where the conventional complex multiplication $\hat{z}_1 \times \hat{z}_2$ is upgraded with the iso-multiplication [4, 42, 43, 44, 45, 46]

$$\hat{z}_1 \ \hat{\times} \ \hat{z}_2 = \hat{z}_1 \times \hat{\kappa} \times \hat{z}_2 = \hat{z}_1 \times \frac{1}{\hat{r}} \times \hat{z}_2. \tag{3}$$

3. Third, given the iso-multiplication of eq. (3), we deduce the iso-square via the expansion

$$\begin{array}{rcl} \hat{z}_n^2 & = & \hat{z}_n \ \hat{\times} \ \hat{z}_n \\ & = & (z_n \times \hat{r}) \times \hat{\kappa} \times (z_n \times \hat{r}) \\ & = & (z_n \times \hat{r}) \times \frac{1}{\hat{r}} \times (z_n \times \hat{r}) \\ & = & z_n \times z_n \times \hat{r}. \end{array} \tag{4}$$

4. Fourth, we prove that the axiom of the multiplicative units of eqs. (2–4) is confirmed by the expressions [4, 42, 43, 44, 45, 46]

$$1 \ \hat{\times} \ \hat{z}_n = 1 \times \hat{\kappa} \times \hat{z}_n = \hat{z}_n \times \frac{1}{\hat{r}} \times 1 = \hat{z}_n \ \hat{\times} \ 1, \ \forall \hat{z}_n \in \mathbb{C}_{\hat{r}}. \tag{5}$$

5. Fifth, we establish that eqs. (2–5) are characterized by the iso-topic lifting and its inverse [4, 42, 43, 44, 45, 46]

$$\begin{array}{rccc} f(\hat{r}): & \mathbb{C} & \to & \mathbb{C}_{\hat{r}} \\ f^{-1}(\hat{r}): & \mathbb{C}_{\hat{r}} & \to & \mathbb{C}, \end{array} \tag{6}$$

respectively.

6. Finally, we engage eqs. (2–6) to upgrade eq. (1) to define the iso-complex quadratic polynomial as

$$\hat{z}_{n+1} \equiv \hat{z}_n^2 + \hat{c} \equiv (\hat{z}_n \hat{\times} \hat{z}_n) + \hat{c} \equiv (z_n \times z_n \times \hat{r}) + (c \times \hat{r}) \equiv z_{n+1} \times \hat{r}, \quad (7)$$

where $\hat{z}_n, \hat{z}_{n+1}, \hat{c} \in \mathbb{C}_{\hat{r}}$ are iso-complex numbers and $z_n, z_{n+1}, c \in \mathbb{C}$ are the corresponding complex numbers. Hence, we can computationally generate a Mandelbrot iso-set by systematically iterating eq. (7)!

At this point, we've successfully upgraded Mandelbrot's complex quadratic polynomial [1, 2] of eq. (1) with Santilli's iso-multiplication [4, 42, 43, 44, 45, 46] to construct the iso-complex quadratic polynomial of eq. (7), which are used to construct Mandelbrot iso-sets.

### 2.2 Results

In total, we computationally experimented with the 5 distinct iso-units:

$$\hat{r} \in \{\frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, 2\}. \quad (8)$$

In eq. (8), we observe that $\frac{1}{2}$ is the inverse of 2, 1 is the inverse of 1, and $\frac{3}{4}$ is the inverse of $\frac{4}{3}$, so these iso-unit values are in fact *dual*. Our objective is to insert the various iso-units of eq. (8) into the iso-complex quadratic polynomial of eq. (7) to observe the effect of Santilli's iso-topic lifting [4, 42, 43, 44, 45, 46] on Mandelbrot's set [1, 2].

For our control, we started with $\hat{r} = 1$ and generated the Mandelbrot set—see the *middle* graphic in Figure 1. Afterwards, we varied the iso-unit for $\hat{r} \neq 1$, such that $\hat{r} = \frac{1}{2}, \frac{3}{4}, \frac{4}{3}, 2$, to generate the Mandelbrot iso-sets—see the *non-middle* graphics in Figure 1. In this preliminary assessment, we observe that the iso-unit variation impact results of Figure 1 indicate that Santilli's iso-topic lifting [4, 42, 43, 44, 45, 46] yields—at minimum—*two* general topological effects:

1. **scale-deformation**, where the fractal is magnified ("zoom-in") or de-magnified ("zoom-out"); and

2. **boundary-deformation**, where the relative position of the fractal boundaries and sequence divergence rates are restructured.

Effects 1 and 2 prove the iso-mathematical existence of the proposed Mandelbrot iso-sets, which are indeed *locally iso-morphic* to the Mandelbrot set. Moreover, these computational results are an experimental implementation of the discrete dynamic iso-spaces in [47], where the iso-unit is treated as a dynamic iso-unit function of a parameter that varies by taking on discrete values.

## 3 Conclusion

The outcomes of this investigation reveal and assess the preliminary impact of Santilli's iso-unit [42, 43, 44, 45, 46] on Mandelbrot's set [1, 2]. More precisely, we were inspired by the iso-fractal developments of [4] and deployed iso-topic liftings [42, 43, 44, 45, 46] to transform Mandelbrot's complex quadratic polynomial into an iso-complex quadratic polynomial, which thereby enabled us to forge the new Mandelbrot iso-set—the Mandelbrot set and a given Mandelbrot iso-set are locally iso-morphic. Subsequently, the initial results of the computational experiment revealed that varying the iso-units causes two general topological effects: scale-deformation and boundary-deformation. For this, we noted that this experiment is an implementation of discrete dynamic iso-spaces [47].

In our opinion, the said examination and results indicate an exciting and promising future for this mode of cutting-edge research: the territory of iso-fractals is a vast, uncharted frontier. Ultimately, the implications of this venture are significant because they advance the borderland of iso-mathematics to new trajectories of thought, inquiry, and experimentation. Hence, with the objective of further implementing these developments in the disciplines of science, technology, and engineering, we propose that additional rigorous iso-mathematical investigations should be conducted along this pattern to challenge, upgrade, and generalize these emerging iso-fractals.

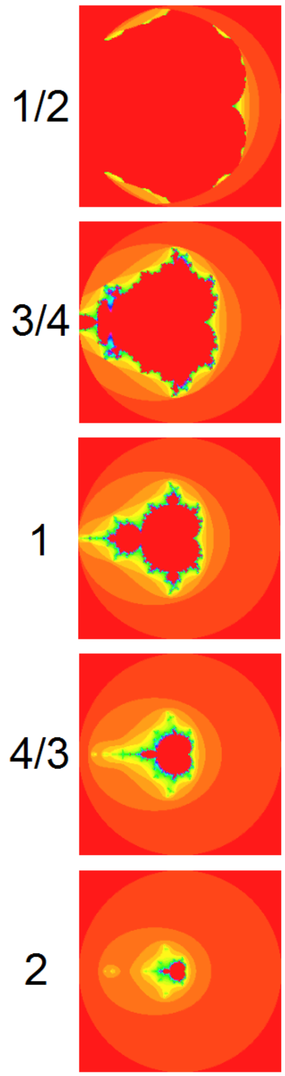


Fig. 1: A depiction of the iso-unit impact, where the varying iso-units are listed in the left column. In the right column, the middle graphic is the Mandelbrot set and the non-middle graphics are the Mandelbrot iso-sets. Observe that Santilli's iso-topic lifting yields two general topological effects: scale-deformation and boundary-deformation.

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