Fluid Mechanics

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Abstract

This is a script made with the help of Landau Lifshitz, Book VI [1] on fluid mechanics, that gives a short introduction to basic fluid mechanics. This short script includes, various equations of continuity, Eulers equation for motion of nonviscous fluid, gravitational waves in nonviscous fluids, Navier-Stokes Equation for viscous fluids, viscous flow within a pipe, some turbulence and laminar wake (can be used to calculate the lift of a wing), all along with some basic thermodynamics and vector calculus.

Plagiarism:

For creation of this script we have closely oriented ourselves towards Laundau Lifschitz Course in theoretical physics Vol 6 Fluid Mechanics. This is no original work! Even though we have tried to write everyting in our own words (in order to have to understand everything), most of the formulas are similar or equal to the book, while some of the steps which seemed too fast are supplemented by own "fill in" calculations. I will not cite anything properly, since this is not an official paper or homework or term paper, and since I basically cite every line of calculation and from the content also every line of text.

I. INTRODUCTION

This short script gives an introduction to fluid mechanics, the physics of moving fluids. Among others we will discuss how to derive from first principle the flow profile through pipes. This is however only the short end of fluid dynamics. Landau & Lifschitz, Volume 6 [1], the reference we are strongly oriented towards, starts with this introduction. However the field of fluids in motion applies to many things in current research, e. g. the physics of combustion, shock and supersonic motion is of importance for supernova simulation and physics (see e. g. the research of Prof. Alan Calder), physics of superfluids, like the phonon spectra of superfluids in various dimensions (c. f. e. g. [2]), combustion physics, which is needed for blowing up stuff (and honestly, who in their right mind doesn't like blowing up stuff). Sadly, there will not be any time to get to the really interesting physics, but we can hope to set the foundation to start the readers off on this interesting topic, Landau Lifschitz Volume 6 treats all of these interesting phenomena, and we refer the interested reader to that excellent book.

Part I

Nonviscous Fluids

Friction is always a big bother, since it usually does not enter equations of motion from first principle or nice heuristic assumption, but it is mostly an added term, to mimic friction what is measured ($\vec{F}_{\text{fric}} \propto \vec{v}$ or $\vec{F}_{\text{fric}} \propto \hat{v}\,(\vec{v}^2)$), since friction originates almost directly from microscopic properties of matter in combination with thermodynamics (microscopic coulomb attraction which create phonons, i. e. heat, hence seemingly the energy dissipates away, which is a sloppy way of saying the directed energy is converted to inner energy or heat). Hence, in the first chapter we will treat frictionless fluids, and add the artificial friction term in the second part about viscous fluids.

II. EQUATION OF CONTINUITY FOR MATTER

Since fluids do not just disappear (just like charge and matter) it is the foremost thing to get hold of a continuity equation. the flow of fluid out of a volume V bounded by the surface ∂V is given by

$$\mathrm{Flow}_{\mathrm{fluid}} = \oint_{\partial V} \rho\left(\vec{r}, t\right) \vec{v}\left(\vec{r}, t\right) d\vec{A}\,, \tag{1}$$

where ρ is the density of the fluid and v is its velocity. The amount of fluid inside this volume is

$$\mathrm{Mass}_{\mathrm{fluid}} = \int_V \rho\left(\vec{r}, t\right) dV\,. \tag{2}$$

The amount of flow out of the volume at any given time has to be the same as the decrease of in mass within the volume i. e.

$$-\partial_t \int_V \rho\left(\vec{r}, t\right) dV = \oint_{\partial V} \rho\left(\vec{r}, t\right) \vec{v}\left(\vec{r}, t\right) d\vec{A} \tag{3}$$

$$= \int_V \vec{\nabla} \cdot \left(\rho\left(\vec{r}, t\right) \vec{v}\left(\vec{r}, t\right)\right) dV\,, \tag{4}$$

or equivalently

$$\vec{\nabla} \cdot \left(\rho\left(\vec{r}, t\right) \vec{v}\left(\vec{r}, t\right)\right) + \partial_t \rho\left(\vec{r}, t\right) = 0\,. \tag{5}$$

This is the equation of continuity, $\vec{j}\left(\vec{r}, t\right) = \rho\left(\vec{r}, t\right) \vec{v}\left(\vec{r}, t\right)$ is the current flow of fluid out of an inifinitesimally small unit volume $\partial_t \rho\left(\vec{r}, t\right)$ is the rate of change of mass within the volume. It is simply the statement, that the total amount of matter (i. e. fluid) is conserved.

Stated like this it is clear that this continuity equation is exactly the same as the electromagnetic continuity equation

$$\vec{\nabla} \cdot \left(\vec{j}_{\mathrm{curr}}\left(\vec{r}, t\right)\right) + \partial_t \rho_{\mathrm{charge}}\left(\vec{r}, t\right) = 0\,, \tag{6}$$

where $\vec{J}_{\mathrm{curr}}$ is the current density and ρ_{charge} is the charge density. This again is nothing else than the statement of the conservation of charge, in the approximation of continuousdistributions.

III. EQUATION OF MOTION FOR NONVISCOUS FLUIDS

Next we will derive the equation of motion of a fluid, that has no friction. This means, that we can imagine two layers of fluid with different velocities moving side by side to each other, touching at a boundary plane. This process will for now not involve dissipation, later, when talking about the Navier-Stokes equation, we will include the friction force between two adjecent layers of fluid moving with different velocities (friction forces in fluid mechanics are usually called viscosity, hence now we are dealing with nonviscous fluids).

The force on a fluid volume V (with boundary ∂V) through pressure (I will use the symbol P for pressure) is given by (remember $P = \vec{F} \cdot \vec{A}$)

$$\vec{F}_V(\vec{r}, t) = -\oint_{\partial V} P(\vec{r}', t)\, d\vec{A}' \tag{7}$$

$$= -\int_V \vec{\nabla}\left(P(\vec{r}', t)\right) dV', \tag{8}$$

where I have can "handwave" that the above transformation actually makes sense, by writing out explicitly the integrands

$$P d\vec{A} = \hat{x} P n_x dydz + \hat{y} P n_y dxdy + \hat{z} P n_z dxdy \tag{9}$$

$$\vec{\nabla} P dV = \hat{x}\left(\partial_x P\right) dxdydz + \hat{y}\left(\partial_y P\right) dydxdz + \hat{z}\left(\partial_z P\right) dzdxdy \tag{10}$$

$$= \hat{x} P dydz + \hat{y} P dxdz + \hat{z} P dxdy\,. \tag{11}$$

It is relatively easy to understand, that Gauss Theorem (or Divergence Theorem) also applies to Tensors of with more than one index (Gauss theorem as used in electromagnetism is $\oint_{\partial V} d\vec{A} \cdot \vec{V} = \int_V dV \vec{\nabla} \cdot \vec{V}$) and it can be expanded to tensors of rank two and higher by simply taking $T_{i,j,..,q,..,l}$ and using q as the vector index for the Gauss theorem and keeping all other indices constant. The maybe more surprising fact, is that this also has a meaningful extension to scalars, i. e. to tensors of rank zero, which is the case given above.

The total mass of the fluid in V is given by

$$M = \int_V \rho(\vec{r}', t)\, dV'\,. \tag{12}$$

Hence using Newtons second law, I get

$$\int_V \rho(\vec{r}', t) \frac{d\vec{v}(\vec{r}', t)}{dt} dV' + \int_V \vec{\nabla}\left(P(\vec{r}', t)\right) dV' = \int_V \left\{\rho(\vec{r}', t) \frac{d\vec{v}(\vec{r}, t)}{dt} + \vec{\nabla}\left[P(\vec{r}', t)\right]\right\} dV' = 0 \tag{13}$$

or

$$\rho(\vec{r}, t) \frac{d\vec{v}(\vec{r}, t)}{dt} = -\vec{\nabla}\left[P(\vec{r}, t)\right]\,. \tag{14}$$

This is not a very convenient way of writing things, since now we are tracing a fluid particle through space (remember how we said we are using Newtons laws) hence we still have to treat $\vec{r}$ as $\vec{r}(t)$. The really nice thing to do eventually is to treat the fluid as a field of densities ρ and velocities v as opposed to tracing single fluid particles through space and applying newtons laws to get the time evolution.

To arrive at a rather field like description of the fluid we can expand the total time derivative in terms of the chain rule

$$\begin{aligned} \frac{d\vec{v}\left(\vec{r}\left(t\right),t\right)}{dt} &= \partial_t \vec{v}\left(\vec{r}\left(t\right),t\right) + \left(\sum_i \frac{\partial r_i}{\partial t}\partial_{r_i}\right)\vec{v}\left(\vec{r}\left(t\right),t\right) \qquad (15) \\ &= \partial_t \vec{v}\left(\vec{r},t\right) + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\,. \qquad (16) \end{aligned}$$

At this point we stopped tracing fluid particles through space. We are simply asking what is the rate of change of density and velocity of the fluid at a fixed point in space. We have started at a system of classical mechanics (particles moving through space due to Newtons second law) and arrived at an expression for the density and velocity field of the fluid.

The final expression is

$$\rho\left(\vec{r},t\right)\partial_t \vec{v}\left(\vec{r},t\right) + \rho\left(\vec{r},t\right)\left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}P\left(\vec{r},t\right)\ . \qquad (17)$$

If we now have additional external forces on the fluid, like e. g. the acceleration due to gravity $m\vec{g} = -\hat{z}m\left(9.81ms^{-2}\right)$, we can easily modify our equation to account for this change in physical setting. We came from $F = ma$, where the force was due to pressure. Now we have

$$\vec{F} = \vec{F}_{\text{pressure}} + m\vec{g} \qquad (18)$$

which gives

$$\rho\left(\vec{r},t\right)\partial_t \vec{v}\left(\vec{r},t\right) + \rho\left(\vec{r},t\right)\left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}P\left(\vec{r},t\right) + \rho\left(\vec{r},t\right)\vec{g}\,, \qquad (19)$$

or equivalently

$$\partial_t \vec{v}\left(\vec{r},t\right) + \left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla}P\left(\vec{r},t\right)}{\rho\left(\vec{r},t\right)} + \vec{g}\,, \qquad (20)$$

which finally gives Eulers equation, which determines the motion of nonviscous fluids.

IV. THERMODYNAMIC BASICS OF FLUIDS

We have already used a thermostatic property in the previous calculation, namely the pressure within the fluid. Opposed to usual statistical mechanics (which is usually called thermodynamics, but is much rather thermostatics or equilibirium thermostatics), our system is not assumed to be in thermal equilibirum since we assumed that there may be a pressure gradient within the fluid.

A. Thermostatic Basics

As we will talk about entropy soon, we want to remind ourselves how all of these strange quantities are defined. Usually ther is a kind of "God-Rule" that defines all thermodynamic quantities, and in combination with statistical mechanics and Legendre Transformations yields basically all of equilibrium thermostatics (Maxwell-Boltzmann, Gibbs-Enthalpy, Helmholtz Free Energy, you name it). This ultimately useful equation is the differential of the internal energy of the system under consideration

$$dU = (\partial_S U)\, dS + (\partial_V U)\, dV + (\partial_N U)\, dN \,. \tag{21}$$

The equation above is pretty blunt and stupid, all we did is write the total differential of the internal energy, assuming that it can only depend on the entropy of the system, the volume of the system and the number of particles in the system. It is obvious what we mean by volume and number of particles. When saying entropy, we mean

$$S\,(E) = k_B \log\left(\#_{\text{states}}\,(E)\right) \tag{22}$$

the logarithmic number of states, that have a certain energy. It can be generalized to the quantum case (von Neuman Entropy), but that is not terribly important, all that counts, is that Entropy is something easy to grasp, if we look at it at a microscpic level, it simply amounts to the question of how many states are there for a given energy.

The actual "God-Rule" of thermodynamics is then

$$dU = TdS - PdV + \mu dN \,, \tag{23}$$

which of course now defines the partial derivatives of U (the pressure has a negative sign since it is the work done by the system to change volume). Doing a Legendre Transformation of U in S gives the Helmholz free energy

$$dF = -SdT - PdV + \mu dN\,, \tag{24}$$

and now we have a slightly more measurable expression for the entropy as

$$-\frac{\partial F}{\partial T} = S\,, \tag{25}$$

or if we have for some reason a measure of the total internal energy,

$$\delta S = \frac{\delta U}{T}\,. \tag{26}$$

The rest of equilibrium thermostatics can pretty much be derived from the rule

$$\text{Maximize the Entropy !} \tag{27}$$

which for the correct boundary conditions amounts to minimizing various Legendre Transformation of the total internal energy, e. g. $F = U - TS$ (the helmoltz free energy) or $G = F + PV$ (Gibbs free Enthalpy) or $H = U + PV$ (Enthalpy) and so on.

Furthermore a clarification for notification. All thermodynamic extensive quantities (i. e. Entropy S, Internal Energy U, Gibbs Enthalpy G, Enthalpy H, Gibbs Free Energy F) are defined as the total overall quantity (e. g. total Entropy is S) if a capital letter is used, and are defined as the quantity per unit mass is a small letter is used (e. g. Entropy per unit mass is s).

B. Adiabatic Motion of a Nonviscous Fluid & Continuity Equation of Entropy

Now if the motion of fluid is adiabatic, i. e. there is no transfer of energy between the fluid particles due to friction, then the entropy within the fluid is conserved, i. e.

$$\frac{ds\left(\vec{r}\left(t\right),t\right)}{dt} = 0\,, \tag{28}$$

where s is the entropy per unit mass ($S = sdM = sdM\frac{dV}{dV} = \frac{s}{\rho}dV$). If we again would like to see s rather as a field like variable, then we would again apply the chain rule, and get

$$\partial_t s\left(\vec{r},t\right) + \left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)s\left(\vec{r},t\right) = 0\,. \tag{29}$$

Adding a zero yields

$$\rho\partial_t s + s\partial_t\rho + s\vec{\nabla}\cdot(\rho\vec{v}) + \rho\left(\vec{v}\cdot\vec{\nabla}\right)s = \partial_t(\rho s) + \vec{\nabla}\cdot(\rho s\vec{v}) = 0\,, \tag{30}$$

which is a continuity equation of entropy, where $\rho s\vec{v}$ is the entropy flux density and ρs is the entropy density.

If now entropy is constant throughout the considered volume for some particular time, then

$$s(\vec{r},t) = \text{const} \tag{31}$$

for all times and all places under consideration.

Using the thermodynamics, we can see that (compare subsection A.)

$$dH := dH_{N=\text{const}} = dU_{N=\text{const}} + d(PV) = TdS + VdP\,. \tag{32}$$

Using that s and hence S is constant (i. e. $dS = ds = 0$), we see, that

$$dH = VdP\,. \tag{33}$$

Then, dividing by the mass $V\rho$, we get (h is the enthalpy or heat function per unit mass at constant number of particles)

$$dh := \frac{dH}{(V\rho)} = \frac{dP}{\rho}\,. \tag{34}$$

Hence $\vec{\nabla}h = \left(\vec{\nabla}p\right)/\rho$, such that (if there is no additional force such as gravitation)

$$\partial_t\vec{v}(\vec{r},t) + \left(\vec{v}(\vec{r},t)\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}h\,. \tag{35}$$

Using vector calculus, one can derive (use relation of for the complete antisymmetric tensor of rank three $\sum_k \epsilon_{klm}\epsilon_{kij} = \delta_{ki}\delta_{mj} - \delta_{lj}\delta_{mi}$)

$$\begin{aligned}
\frac{1}{2}\left(\vec{\nabla}\left(\vec{v}^2\right)\right)_i &= \frac{1}{2}\partial_i \sum_j v_j^2 && (36)\\
&= \sum_j v_j \partial_i v_j && (37)\\
&= \sum_j \left(v_j\partial_i v_j - v_j\partial_j\partial_i\right) + \left(\sum_j v_j\partial_j\right) v_i && (38)\\
&= \sum_{j,l,m} v_j \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_l v_m + \left(\sum_j v_j\partial_j\right) v_i && (39)\\
&= \sum_{j,k,l,m} v_j \epsilon_{kij}\epsilon_{klm}\partial_l v_m - \left(\sum_j v_j\partial_j\right) v_i && (40)\\
\frac{1}{2}\left(\vec{\nabla}\left(\vec{v}^2\right)\right)_i &= \left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right)_i - \left(\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right)_i , && (41)
\end{aligned}$$

and hence

$$\partial_t \vec{v} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = -\vec{\nabla}\left(h - \frac{1}{2}\vec{v}^2\right) . \tag{42}$$

If we now take the curl of this, we get an equation of motion for $\vec{v}$, that only involves $\vec{v}$

$$\partial_t\left(\vec{\nabla}\times\vec{v}\right) = \vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right) . \tag{43}$$

We do not quite get where the point of all of this is, but we wrote it down hoping that this would make more sense in the progress of doing so. All that we take from this, is that we are left with an equation, that determines the velocity field, without using neither density nor pressure, in the case of a frictionless fluid that moves adiabatically, so kind of neat, but we still lack deeper physical insight into this.

V. HYDROSTATICS

In the static case (i. e. nonmoving fluid) we have

$$\partial_t \vec{v} = \vec{v} = 0 , \tag{44}$$

which gives for the equation of motion ($\partial_t \vec{v}\left(\vec{r},t\right) + \left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla}P(\vec{r},t)}{\rho(\vec{r},t)} + \vec{g}$)

$$\vec{g}\rho\left(\vec{r}\right) = \vec{\nabla}P\left(\vec{r}\right) . \tag{45}$$

Without gravity (or without an external force) the density would be constant everywhere, as would be the pressure.

If we assume incompressibility, i. e. $\rho(\vec{r}) = \text{const}$, then

$$P(\vec{r}) = -\rho g z + \text{const}\,. \tag{46}$$

Incompressibility is not always a good assumption, especially, when the ρ is small, i. e. for low densities (i. e. gases) and for high pressures (i. e. high masses, because when the pressure becomes too large the state of the matter will change).

VI. STEADY FLOW AND STEAMLINES

The notion of steady flow means, that there is a nonzero flow of fluid, while the flow is constant in time,

$$\partial_t \vec{v} = 0\,,\ \vec{v} \neq 0\,. \tag{47}$$

In the case of steady flow, the equation of motion for the fluid reads (using Eq. (42), we can assume nonchanging entropy, since we are using considering steady (timeindependent) flow in a frictionless environment)

$$\vec{\nabla}\frac{\vec{v}^2}{2} - \vec{v} \times \left(\vec{\nabla} \times \vec{v}\right) = -\vec{\nabla} h\,, \tag{48}$$

where $dh = \frac{dP}{\rho}$.

As is usually done for electric and magnetic fields, field lines, which are tangential to the flow velocity are a helpful tool for depicting flows, especially in the case of steady flow. Projecting the equation of motion along the direction of velocity $\hat{l} = \vec{v}/|\vec{v}|$, gives ($\hat{l} \cdot \vec{\nabla} = \partial/\partial l$, i. e. derivative along direction of $\hat{l}$, and $\vec{v} \times \vec{a} \cdot \hat{l} = 0$)

$$\frac{\partial}{\partial l}\left(\frac{\vec{v}^2}{2} + h\right) = 0\,, \tag{49}$$

this means, that along a streamline, the quantity

$$\frac{\vec{v}^2}{2} + h = \frac{\vec{v}^2}{2} + \frac{P}{\rho} \tag{50}$$

is constant. In a gravitational field this of course becomes

$$\frac{\partial}{\partial l}\left(\frac{\vec{v}^2}{2}+h\right)+g\hat{z}\cdot\hat{l} = \frac{\partial}{\partial l}\left(\frac{\vec{v}^2}{2}+h\right)+g\frac{dz}{dl} \tag{51}$$

$$= \frac{\partial}{\partial l}\left(\frac{\vec{v}^2}{2}+h+gz\right)=0 \tag{52}$$

$$\frac{\vec{v}^2}{2}+h+gz = \text{const}\,. \tag{53}$$

These are the equations, that determine the flux lines, i. e. the field lines of the velocity field.

VII. CONTINUITY EQUATION FOR ENERGY

Next we want to derive a continuity equation for energy. It will turn out that this contains some futher thermodynamic insight into the matter at hand.

The energy in a unit volume is

$$\varepsilon = \frac{1}{2}\rho\left(\vec{r},t\right)\vec{v}^2\left(\vec{r},t\right)+\rho u\,, \tag{54}$$

where $u = U/\left(\rho V\right) = U/M$ is the internal energy per unit mass, or the specific internal energy. The time derivative of the first term is

$$\frac{1}{2}\partial_t\left(\rho\vec{v}^2\right) = \frac{1}{2}\vec{v}^2\partial_t\rho+\rho\vec{v}\cdot\partial_t\vec{v}\,. \tag{55}$$

The equation of continuity for matter is $\partial_t\rho = -\vec{\nabla}\cdot\left(\rho\vec{v}\right)$, and gives

$$\frac{1}{2}\partial_t\left(\rho\vec{v}^2\right) = -\frac{1}{2}\vec{v}^2\vec{\nabla}\cdot\left(\rho\vec{v}\right)+\rho\vec{v}\cdot\partial_t\vec{v}\,. \tag{56}$$

The time derivative of $\vec{v}$ is given by $\rho\partial_t\vec{v} = -\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}-\vec{\nabla}P$, hence

$$\frac{1}{2}\partial_t\left(\rho\vec{v}^2\right) = -\frac{1}{2}\vec{v}^2\vec{\nabla}\cdot\left(\rho\vec{v}\right)-\rho\vec{v}\cdot\left[\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right]-\vec{v}\cdot\vec{\nabla}P\,. \tag{57}$$

Now

$$\vec{v}\cdot\left[\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right] = \sum_{j,k} v_j v_k \partial_k v_j \quad (58)$$

$$= \sum_{j,k} v_j v_k \partial_k v_j \quad (59)$$

$$= \sum_{j,k} v_j v_k \partial_j v_k \quad (60)$$

$$= \frac{1}{2}\sum_{j,k} v_j \partial_j v_k^2 \quad (61)$$

$$= \frac{1}{2}\vec{v}\cdot\vec{\nabla}\left(\vec{v}^2\right)\,, \quad (62)$$

where in the third step we rename $k \to j$ and $j \to k$. As we have already seen

$$dh = \frac{dP}{\rho} + Tds\,, \quad (63)$$

or

$$dP = \rho dh - \rho T ds\,, \quad (64)$$

or

$$\vec{\nabla}P = \rho\vec{\nabla}h - \rho T\vec{\nabla}s\,. \quad (65)$$

This finally gives

$$\frac{1}{2}\partial_t\left(\rho\vec{v}^2\right) = -\frac{1}{2}\vec{v}^2\vec{\nabla}\cdot(\rho\vec{v}) - \rho\frac{1}{2}\vec{v}\cdot\vec{\nabla}\left(\vec{v}^2\right) - \rho\vec{v}\cdot\vec{\nabla}h + \rho T\vec{v}\cdot\vec{\nabla}s \quad (66)$$

$$= -\frac{1}{2}\vec{v}^2\vec{\nabla}\cdot(\rho\vec{v}) - \rho\vec{v}\cdot\vec{\nabla}\left(\frac{1}{2}\vec{v}^2 + h\right) + \rho T\vec{v}\cdot\vec{\nabla}s\,, \quad (67)$$

pretty nice.

Next up is the time derivative of ρu. As already seen,

$$du = Tds - PdV/M \quad (68)$$

$$= Tds - Pd\left(\rho^{-1}\right) \quad (69)$$

$$= Tds + \rho^{-2}Pd\rho\,. \quad (70)$$

Then, with $h = u + \rho^{-1}P$ (from $h = \mathcal{L}_\rho(u) = u + \rho^{-1}P$, wher $\mathcal{L}_\rho$ is the legendre transformation of u in ρ.)

$$d\left(\rho u\right) = \rho du + u d\rho \tag{71}$$

$$= \rho T ds + \rho^{-1} P d\rho + u d\rho \tag{72}$$

$$= \rho T ds + \left(\rho^{-1} P + u\right) d\rho \tag{73}$$

$$= \rho T ds + h d\rho\,. \tag{74}$$

This gives for (using continuity of matter and conservation of entropy $ds/dt = \partial_t s + \vec{v} \cdot \vec{\nabla} s = 0$)

$$\partial_t \left(\rho u\right) = \rho T \partial_t s + h \partial_t \rho \tag{75}$$

$$= \rho T \partial_t s - h \vec{\nabla} \cdot \left(\rho \vec{v}\right) \tag{76}$$

$$= -\rho T \vec{v} \cdot \vec{\nabla} s - h \vec{\nabla} \cdot \left(\rho \vec{v}\right)\,. \tag{77}$$

The complete time derivative of the energy in a unit volume then gives

$$\partial_t \left(\frac{1}{2}\rho \vec{v}^2 + \rho u\right) = -\left(\frac{1}{2}\vec{v}^2 + h\right) \vec{\nabla} \cdot \left(\rho \vec{v}\right) - \rho \vec{v} \cdot \vec{\nabla} \left(\frac{1}{2}\vec{v}^2 + h\right)\,. \tag{78}$$

This is of the form

$$a \vec{\nabla} \cdot \vec{b} + \vec{b} \cdot \vec{\nabla} a = \sum_k \left(a \partial_k b_k + b_k \partial_k a\right) \tag{79}$$

$$= \sum_k \left(\partial_k a b_k\right) \tag{80}$$

$$= \vec{\nabla} \cdot \left(a \vec{b}\right)\,, \tag{81}$$

hence we get finally

$$\partial_t \left(\frac{1}{2}\rho \vec{v}^2 + \rho u\right) = -\vec{\nabla} \cdot \left[\rho \vec{v} \left(\frac{1}{2}\vec{v}^2 + h\right)\right] = -\vec{\nabla} \cdot \left[\rho \vec{v} \left(\frac{1}{2}\vec{v}^2 + u + \frac{P}{\rho}\right)\right]\,. \tag{82}$$

This is almost what one would naively expect: the change in internal and kinetic energy should to be equal to the flux of internal and kinetic energy out of the medium. But if that were the case, one would expect the internal energy u at the place where one finds the enthalpy h. The interesting thing now is, that the pressure in the unit volume, also does work, and hence we also have to correct for this loss mechanism of energy, which then yields h instead of u in the energy current density.

VIII. CONTINUITY EQUATION FOR MOMENTUM

Naturally, as the next point on our agenda we also will attempt to get a continuity equation for momentum, just as we did for the energy.

The time derivative of momentum per unit volume $\rho\vec{v}$ is (as usual using continuity of matter and $\rho\partial_t\vec{v} = -\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} - \vec{\nabla}P$)

$$\begin{aligned}
\partial_t\left(\rho v_i\right) &= v_i\partial_t\rho + \rho\partial_t v_i && (83)\\
&= -v_i\sum_j \partial_j\left(\rho v_j\right) + \rho\partial_t v_i && (84)\\
&= -v_i\sum_j \partial_j\left(\rho v_j\right) - \rho\sum_k v_k\partial_k v_i - \partial_i P && (85)\\
&= -\partial_i P - \sum_j \partial_j\left(\rho v_i v_j\right)\,. && (86)
\end{aligned}$$

Using a definition of the right hand side, this can be put into a more compact form (DUH!, I know right?)

$$\begin{aligned}
-\partial_i P - \sum_j \partial_j\left(\rho v_i v_j\right) &= -\sum_j \partial_j\left(\delta_{ij}P + \left(\rho v_i v_j\right)\right) && (87)\\
&=: -\sum_j \partial_j \Pi_{ij}\,, && (88)
\end{aligned}$$

where we have defined $\Pi_{ij} = \delta_{ij}P + \rho v_i v_j$. Now the equation of continuity for momentum reads !where of course one has to apply usual matrix multiplication rules, aswell as interpreting Π_{ij} as the entries of a matrix $\overleftrightarrow{\Pi}$.

$\overleftrightarrow{\Pi}$ is the momentum flux tensor (it has to be a second rank tensor, since it has to contain the direction of the momentum as well as the direction of flow, unlike as for the energy, where the energy flow vector has only to contain the direction of flow, since energy is a scalar).

IX. GRAVITATIONAL WAVES IN A DEEP TANK

We are going to examine how waves at the surface of a fluid propagate. We are going to assume that the fluid is under the influence of gravity and has a plane surface if it is not perturbed, and that the fluid is incompressible, hence

$$\partial_t\rho = \vec{\nabla}\cdot\vec{v} = 0\,. \tag{89}$$

Starting from eulers equation

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} h \,, \tag{90}$$

we see that for gravity waves of fluid particle velocity u and amplitude a made by a body of length l (equivalently of wavelength l), that the term is of order

$$\left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} \approx \frac{u^2}{l} \,, \tag{91}$$

while

$$\partial_t \vec{v} \approx u\omega = u\frac{u}{a} = \frac{u^2}{a} \,. \tag{92}$$

Assuming that the wavelength is much greater than the amplitude of the waves, we get

$$\partial_t \vec{v} = -\vec{\nabla} h \,, \tag{93}$$

and hence

$$\partial_t \left(\vec{\nabla} \times \vec{v}\right) = 0 \tag{94}$$

$$\vec{\nabla} \times \vec{v} = \text{const} \tag{95}$$

Since we look at oscillations the average of $\vec{v}$ is zero, and hence the time average of $\vec{\nabla} \times \vec{v}$ is zero. Since the curl of the velocity is however zero we have

$$\vec{\nabla} \times \vec{v} = 0 \,, \tag{96}$$

i. e. we have potential flow, which means that we can calculate the velocity using

$$\vec{v} = \vec{\nabla} \Phi \,, \tag{97}$$

where Φ is the "velocity potential".

The continuity equation then gives

$$\vec{\nabla} \cdot \vec{v} = \Delta \Phi = 0 \,. \tag{98}$$

Moreover using eulers equation, and also inserting that we have potential flow it follows that

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} P/\rho - g \tag{99}$$

$$\partial_t \vec{v} + \frac{1}{2} \vec{\nabla} \left(\vec{v}^2\right) = -\vec{\nabla} P/\rho - g \tag{100}$$

$$\vec{\nabla} \left(\partial_t \Phi + \frac{1}{2} \vec{v}^2 + \frac{P}{\rho} + gz\right) = 0 \tag{101}$$

$$\left(\partial_t \Phi + \frac{1}{2} \vec{v}^2 + \frac{P}{\rho} + gz\right) = f(t) \,. \tag{102}$$

Assuming $f(t) = 0$ we get

$$\rho\partial_t\Phi + \rho\frac{1}{2}\vec{v}^2 + P + \rho g z = 0\,. \tag{103}$$

Neglecting $\vec{v}^2$ we get

$$P = -\rho g z - \rho\partial_t\Phi\,. \tag{104}$$

We will next introduce the coordinate $\zeta = \zeta(x, y, t)$ which denotes where the surface of the fluid is. In equilibrium, the unperturbed fluid hence has $\zeta = 0$ everywhere, everytime. Furthermore assume, that at the surface, a costant pressure P_0 is applied. We then get

$$P_0 = -\rho g\zeta - \rho\partial_t\Phi\,, \tag{105}$$

where P_0 is a constant. Redefining $\Phi' = \Phi - P_0 t/\rho$ eliminates P_0, hence

$$g\zeta + (\partial_t\Phi)_{z=\zeta} = 0\,. \tag{106}$$

Now we know that $v_z = \partial_t\zeta$, however $\partial_z\Phi = v_z = \partial_t\zeta$, which yields if we derive Eq. (106) w. r. t. (with respect to) time

$$\left(\partial_z\Phi + g^{-1}\partial_t^2\Phi\right)_{z=\zeta} = 0\,. \tag{107}$$

Since we assumed small oscillations, we can also take the derivatives at $z = 0$, such that the oscillatory motion of a noviscous incompressible fluid in time is given by the equations $(\Delta = \vec{\nabla}^2)$

$$\Delta\Phi \;=\; 0 \tag{108}$$

$$\left(\partial_z\Phi + g^{-1}\partial_t^2\Phi\right)_{z=0} \;=\; 0\,. \tag{109}$$

Lets assume a wave that propagates, w. l. o. g. (without loss of generality), along the x-axis, hence the function is periodic in x and has some dependence on z and is independent of y

$$\Phi(x, z, t) = f(z)\cos(kx - \omega t)\ . \tag{110}$$

The laplace equation demands

$$\cos(kx - \omega t)\,\partial_z^2 f(z) + f(z)\,\partial_x^2\cos(kx - \omega t) \;= \tag{111}$$

$$\cos(kx - \omega t)\,\partial_z^2 f(z) - k^2\cos(kx - \omega t)\,f(z) \;=\; 0 \tag{112}$$

$$\partial_z^2 f(z) - k^2 f(z) \;=\; 0\,. \tag{113}$$

This has solutions

$$f(z) = A\exp(kz) + B\exp(-kz) \ , \tag{114}$$

imposing that the wavefunction should be finite for all $z \leq 0$, we see that

$$\Phi(x, z, t) = A\exp(kz)\cos(kx - \omega t) \ . \tag{115}$$

The boundary condition at the surface simply is

$$\left(\partial_z\Phi + g^{-1}\partial_t^2\Phi\right)_{z=0} = k\Phi - \frac{\omega^2}{g}\Phi = 0 \tag{116}$$

$$kg = \omega^2 \, . \tag{117}$$

Thats already it, we simply take the gradient of Φ to get the velocity field

$$v_x = \partial_x\Phi = -Ak\exp(kz)\sin(kx - \omega t) \tag{118}$$

$$v_z = \partial_z\Phi = Ak\exp(kz)\cos(kx - \omega t) \ . \tag{119}$$

If we interpret $v_x = \frac{dx}{dt}$ and $v_z = \frac{dy}{dt}$, we get the motion which the fluid particles make in space

$$x - x_0 = -A\frac{k}{\omega}\exp(kz)\cos(kx - \omega t) \tag{120}$$

$$z - z_0 = -A\frac{k}{\omega}\exp(kz)\sin(kx - \omega t) \ . \tag{121}$$

This exhibits clearly, that the fluid particles perform a circular motion in the fluid, which exponentially decreases when we go into the fluid.

Note that we have still assumed, that the fluid is “deep”, i. e. that we are operating in a tank with infinite depth, the wave solutions change, when dealing with tanks of finite depth.

Part II

Viscous Fluids

Viscosity or friction, concerning the motion of fluids is a very important ingredient in actual flow problems, like the flow of water through a pipe, the flow of air past a wing. In this next part we will be concerned with how viscous fluids move.

X. MOTION OF A VISCOUS FLUID

We have seen in great detail how ideally nonviscous fluids move. We will now attempt to derive from heuristics how an equation of motion in that case has to look.

In the chapter about continuity of momentum, we have seen that

$$\partial_t (\rho \vec{v}) = -\vec{\nabla} \cdot \bar{\Pi} \,, \tag{122}$$

which is basically newtons second law for fluids ($\dot{\vec{p}} = \vec{F}$, $\rho\vec{v}$ is the momentum per unit volume and the force is due to pressure). It is equivalent to Eulers equation of motion for nonviscous fluids.

It is clear, that we will now add a suitable fritction force to the right hand side, in order to get the full equation of motion for a viscous fluid. This will look as follows, we will first take the momentum flux tensor alone, which gives the reversible momentum transfer. We then add a friction term to the momentum flux tensor, that gives the viscous or non-reversible momentum flux

$$\bar{\Pi}_{\text{visc}} = \bar{\Pi} - \bar{\sigma} \tag{123}$$

$$\Pi^{\text{visc}}_{ij} = \delta_{ij} P + \partial_j (\rho v_i v_j) - \sigma_{ij} \,. \tag{124}$$

The first condition has to be, that friction in the fluid can only appear, when different velocities are present. If two layers of fluid flow along each other with the same velocity, we will never expect friction, if we however do have two layers of fluid which do not have the same velocity, flow along each other, we will expect friction. Hence σ_{ij} has to be a function of spacial derivatives of the velocity $\partial_k^n v$. For small gradients, as is usual for physicists, we will use a leading order approximation and say, that σ_{ij} is only linear in $\partial_k^n v$, and furthermore, that higher differentials (i. e. $n > 1$) also are negligible, hence

$$\sigma_{ij} = \sum_{kl} a_{ijkl} \partial_k v_l \,. \tag{125}$$

Furthermore, if the whole fluid is rotating, adjecent fluid particles do not move past each other, which implies, that we do not expect friction, where in this case there will be a nonvanishing velocity gradient. In formulas, if the velocity is (we rotate about an axis

$\hat{\Omega} = \vec{\Omega}/\left|\vec{\Omega}\right|$, with an angular velocity of $\left|\vec{\Omega}\right|$)

$$v_i = \left(\vec{\Omega} \times \vec{r}\right)_i \tag{126}$$

$$= \sum_{jk} \epsilon_{ijk} \Omega_j r_k \,. \tag{127}$$

The gradient of one velocity component is

$$\partial_l v_i = \partial_l \sum_{jk} \epsilon_{ijk} \Omega_j r_k \tag{128}$$

$$= \sum_{jk} \delta_{lk} \epsilon_{ijk} \Omega_j \tag{129}$$

$$= \sum_{j} \epsilon_{ijl} \Omega_j \tag{130}$$

$$= -\sum_{j} \epsilon_{lji} \Omega_j \tag{131}$$

$$= -\sum_{jk} \delta_{ik} \epsilon_{ljk} \Omega_j \tag{132}$$

$$= -\partial_i \sum_{jk} \epsilon_{ljk} \Omega_j r_k \tag{133}$$

$$= -\partial_i v_l \,. \tag{134}$$

Such that terms, that are of the form

$$\partial_l v_i + \partial_i v_l \tag{135}$$

vanish for rotations, as has to be the case. σ_{ij} may hence only be made up of these symmetrical combinations of spatial derivatives of $\vec{v}$. For isotropic fluids the most general tensor that fulfills these quantities is ($a_{ijkl} \neq 0$ only if $i = k$ and $j = l$ or $i = l$ and $j = k$)

$$\sigma_{ij} = \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \sum_k \partial_k v_k \right) + \zeta \delta_{ij} \sum_k \partial_k v_k \,. \tag{136}$$

The expression in parantheses has, as is not very hard to guess a vanishing trace. As is also not hard to guess η and ζ are coefficients of viscosity, ζ is often called the second viscosity.

What is now eventually left to do is to add the gradient of the tensor that describes the viscosity of the fluid to the right hand side of $\partial_t (\rho \vec{v}) = -\vec{\nabla} \cdot \bar{\Pi}$, which is equivalent ot adding the gradient of σ_{ij} to the right hand side of Eulers equation $\rho \partial_t \vec{v} + \rho \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} P$, i. e.

$$\rho \partial_t v_i + \rho \sum_j (v_j \partial_j) v_i = -\partial_i P + \sum_k \partial_k \left[\eta \left(\partial_i v_k + \partial_k v_i - \frac{2}{3} \delta_{ik} \sum_l \partial_l v_l \right) + \zeta \delta_{ik} \sum_l \partial_l v_l \right] . \tag{137}$$

If the viscosities are taken to be constant (they of course do depend on temperature perssure and for high velocities probably even on velocity, but for many cases the viscosities are constant to good accuracy), we get

$$\left\{\rho\partial_t\vec{v} + \rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right\}_i = -\partial_i P + \eta\left[\sum_k \partial_k\partial_k\right]v_i + \eta\partial_i\left[\sum_k \partial_k v_k - \frac{2}{3}\sum_l \partial_l v_l\right] + \zeta\partial_i\sum_l \partial_l v_l \tag{138}$$

$$= -\left\{\vec{\nabla}P - \eta\Delta\vec{v} - \left(\zeta + \frac{1}{3}\eta\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right\}_i . \tag{139}$$

Where now we have finally arrived at the navier stokes equation. For incompressible fluids $\partial_t\rho = 0 = \rho\vec{\nabla}\cdot\vec{v}$, the second viscosity drops out of the equation, and the navier stokes equation becomes

$$\rho\partial_t\vec{v} + \rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}P + \eta\Delta\vec{v}\,. \tag{140}$$

We can also apply the same procedure as in Section IV B. to eliminate pressure in the incompressible case. For the nonviscous fluid we got

$$\partial_t\left(\vec{\nabla}\times\vec{v}\right) = \vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right) . \tag{141}$$

We simply add the viscous term to the right hand side to get

$$\left\{\partial_t\left(\vec{\nabla}\times\vec{v}\right)\right\}_i = \left\{\vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right) + \vec{\nabla}\times(\Delta\vec{v})\right\}_i \tag{142}$$

$$= \left\{\vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right)\right\}_i + \sum_{jkl}\epsilon_{ijk}\partial_j\partial_l\partial_l v_k \tag{143}$$

$$= \left\{\vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right)\right\}_i + \sum_l \partial_l\partial_l\sum_{jk}\epsilon_{ijk}\partial_j v_k \tag{144}$$

$$= \left\{\vec{\nabla}\times\left(\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right) + \Delta\left(\vec{\nabla}\times\vec{v}\right)\right\}_i . \tag{145}$$

XI. LAMINAR FLOW WITHIN A PIPE

One of the most immediate things that one would like to know from hydrodynamics is stationary flow of an incompressible fluid through a pipe, which is what we can caluculate now.

Let us first clarify what exactly we mean by "flow through a pipe". The fluid has some boundary, at which the fluid velocity vanishes, the fluid moves parallel to this boundary. There is a pressure gradient "pushing" the fluid through the pipe.

Lets consider two parallel planes with with $z = \text{const}$. The pressure gradient is provided externally, along the x-axis, while there is no pressure gradient along y or z. There will be no gravity, the pressure gradient will be uniform across the whol fluid, hence the velocity will not be a function of x, y or t, only of z. Furthermore boundary conditions imply, that $v_z = 0$, and stationarity in combination with friction implies, that $v_y = 0$.

The navier stokes equation is

$$\rho \partial_t \vec{v} + \rho \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} P + \eta \Delta \vec{v} \,, \tag{146}$$

and it simplifies to ($\partial_t \vec{v} = 0$)

$$\rho \left(\sum_j v_j \partial_j \right) v_i = -\vec{\nabla} P + \eta \Delta \vec{v} \,. \tag{147}$$

The term on the left hand side gives

$$\left(v_x \partial_x + v_y \partial_y + v_z \partial_z \right) \vec{v} = \left(v_x \partial_x + 0 \partial_y + 0 \partial_z \right) \vec{v} \tag{148}$$

$$= v_x \partial_x \begin{pmatrix} v_x(z) \\ 0 \\ 0 \end{pmatrix} = 0 \,, \tag{149}$$

hence

$$\partial_z \partial_z v_x = \frac{1}{\eta} \partial_x P =: \frac{1}{\eta} p = \text{const} \,. \tag{150}$$

(Note that I defined $p := \frac{\partial P}{\partial x}$)The pressure gradient can not depend on x

This equation can then be integrated up directly, to

$$v_x(z) = \frac{p}{2\eta} z^2 + az + b \,. \tag{151}$$

The natural boundary condition to give the constants a and b are, that the velocity has to vanish at the plates (assume we put the plates at $z = 0$ and $z = d$), then

$$v_x(z) = \frac{p}{2\eta} z \left(z - d \right) \,. \tag{152}$$

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[2] A. Recati, P. O. Fedichev, W. Zwerger, J. v. Delft, P. Zoller, Atomic Quantum Dots Coupled to a Reservoir of a Superfluid Bose-Einstein Condensate, Phys. Rev. Lett. 94, 040404 (2005)