FUNDAMENTALS OF FLUID MECHANICS

Chapter 6 Flow Analysis Using Differential Methods

MAIN TOPICS

I. Fluid Element Motion

II. Conservation of Mass (Continuity equation) and Conservation of Linear Momentum (Navier-Stokes Equation)

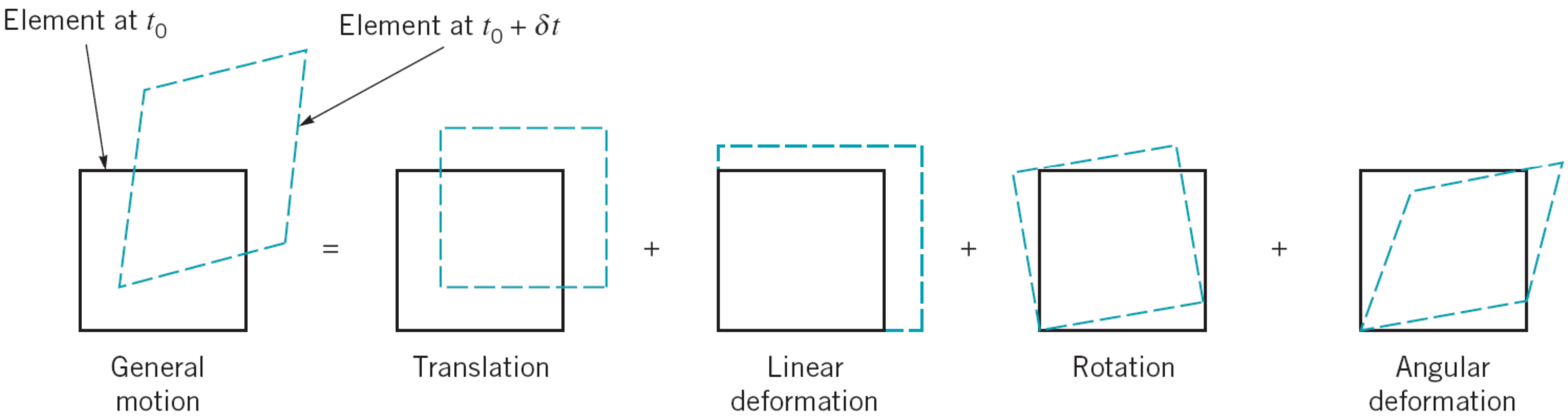
III. Inviscid Flow (Bernoulli equation) and Potential Flow (Stream function)

- Incompressible – Compressible
- Inviscid – Viscous
- Steady – Unsteady
- Rotational - Irrotational

} Mathematical equation?

Motion of a Fluid Element

- 1. Fluid **Translation**: The element moves from one point to another.
- 3. Fluid **Rotation**: The element rotates about any or all of the x,y,z axes.
- Fluid **Deformation**:
  - ⇨4. Angular Deformation:The element's angles between the sides change.
  - ⇨2. Linear Deformation:The element's sides stretch or contract.



1. Fluid Translation velocity and acceleration

❖ The velocity of a fluid particle can be expressed

$$\vec{V} = \vec{V}(x, y, z, t) = u\vec{i} + v\vec{j} + w\vec{k}$$ Velocity field

❖ The total acceleration of the fluid particle is given by

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt}$$ Acceleration field

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w$$

$$\Rightarrow \vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$\vec{a} = \frac{D\vec{V}}{Dt}$ is called the material, or substantial derivative.

$$\neq \frac{d\vec{V}}{dt}$$

Physical Significance

$$\vec{a} = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

Total Acceleration of a particle

Convective Acceleration -> Space

Local Acceleration -> Time

$$\vec{a} = \frac{D\vec{V}}{Dt} = (\vec{V} \cdot \nabla)\vec{V} + \frac{\partial \vec{V}}{\partial t}$$

Scalar Component (외울필요 없음)

$$a_x = \vec{i}\cdot\vec{a} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \vec{j}\cdot\vec{a} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \vec{k}\cdot\vec{a} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Rectangular coordinates system

$$a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V^2{}_\theta}{r} + V_z\frac{\partial V_r}{\partial z}$$

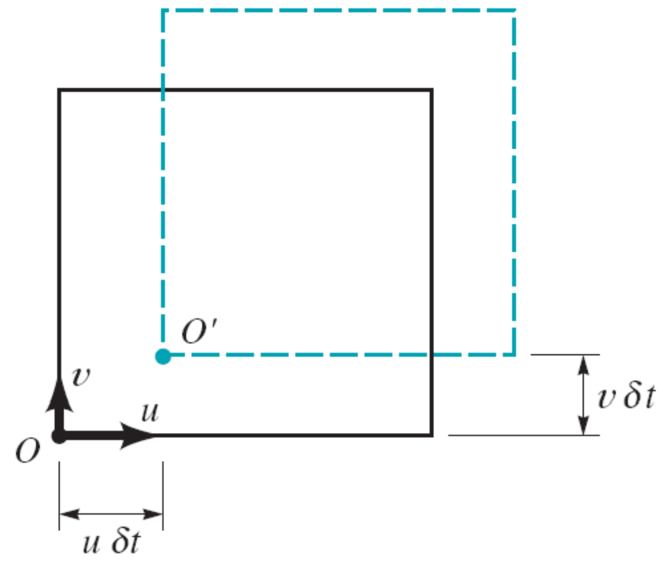
$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}$$

$$a_z = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$$

Cylindrical coordinates system

Translation

❖ All points in the element have the same velocity, then the element will simply translate from one position to another.

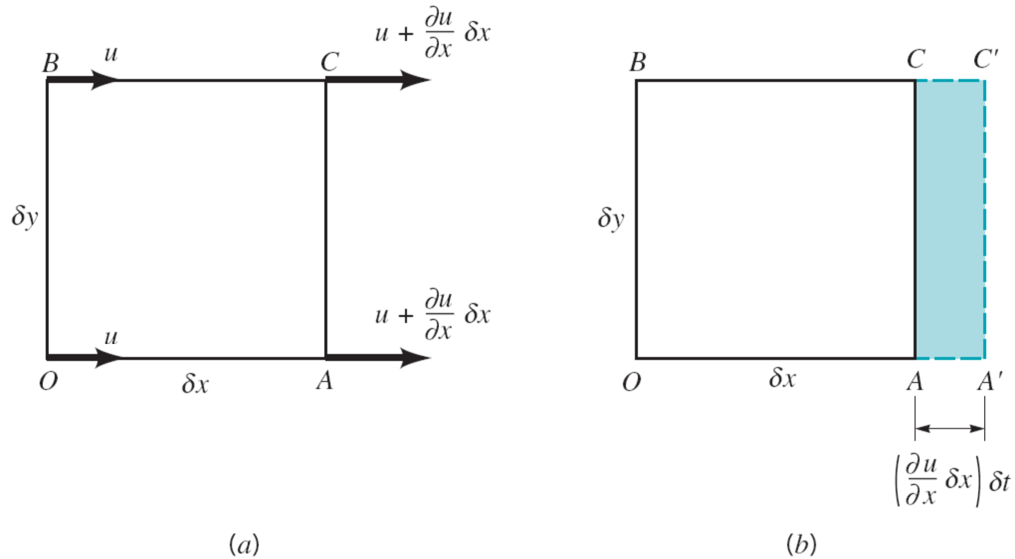


2. Linear Deformation [1/2]

- The shape of the fluid element, described by the angles at its vertices, remains unchanged, since all right angles continue to be right angles.
- A change in the x dimension requires a nonzero value of

  $\partial u / \partial x$

- A .... y $\partial v / \partial y$
- A .... z $\partial w / \partial z$



Linear Deformation 2/2

- The change in length of the sides may produce change in volume of the element.

The change in $\delta \forall = \left(\frac{\partial u}{\partial x} \delta x \right) (\delta y \delta z)(\delta t) = \frac{\partial u}{\partial x} (\delta x \delta y \delta z)(\delta t)$

The rate at which the δV is changing per unit volume due to gradient ∂u/ ∂x $\frac{1}{\delta \forall} \frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x}$

If ∂v/ ∂y and ∂w/ ∂z are involved as in 2-D or 3D cases,

$$d(\delta \forall) \cong \left(\frac{\partial u}{\partial x} \delta x \right) (\delta y \delta z)(\delta t) + \left(\frac{\partial v}{\partial y} \delta y \right) (\delta x \delta z)(\delta t) + \left(\frac{\partial w}{\partial z} \delta z \right) (\delta x \delta y)(\delta t) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) (\delta x \delta y \delta z)(\delta t)$$

Volumetric dilatation rate $\frac{1}{\delta \forall} \frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V}$

Divergence of V

For an *incompressible* fluid (constant density), the volumetric dilatation rate is zero.

3. Angular Rotation 1/4

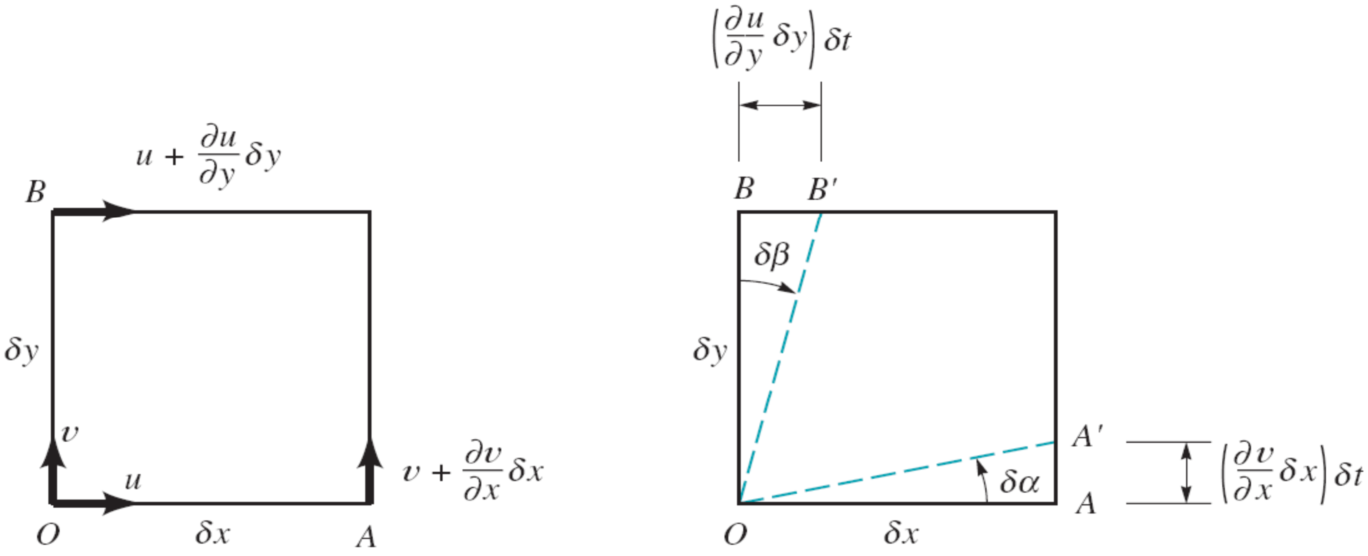
The angular velocity (각속도) of line OA $\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t}$

For small angles $\tan \delta\alpha \doteq \delta\alpha = \frac{\left(\frac{\partial v}{\partial x}\right)\delta x \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$

$\omega_{OA} = \frac{\partial v}{\partial x}$ **CCW** (시계반대방향)

$\omega_{OB} = \frac{\partial u}{\partial y}$ **CW**

$\left(\omega_{OB} = \frac{\partial u}{\partial y}\right) < 0 \rightarrow$ **CCW**



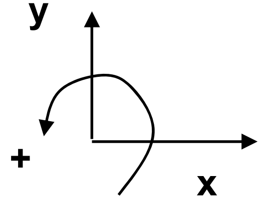
Positive direction

(a) (b)

Angular Rotation 2/4

The rotation of the element about the z-axis is defined as the average of the angular velocities of the two mutually perpendicular lines OA and OB about the z-axis.

$$\omega_z = \frac{1}{2}\lim_{\delta t \to 0}\frac{[\delta\alpha + (-\delta\beta)]}{\delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



CCW
(시계반대방향)

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

In vector form $\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$

Angular Rotation 3/4

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right]$$

$$\Rightarrow \quad \vec{\omega} = \frac{1}{2} curl\vec{V} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]\vec{i} + \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]\vec{j} + \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\vec{k}$$

Defining **vorticity** $\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V}$ -> Angular **rotation**

Defining irrotation $\boxed{\nabla \times \vec{V} = 0}$

Angular Rotation 4/4

$$\vec{\omega} = \frac{1}{2} curl\vec{V} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]\vec{i} + \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]\vec{j} + \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\vec{k}$$

Vorticity

❖ Defining **Vorticity** ζ which is a **measurement of the rotation of a fluid element** as it moves in the flow field:

$$\vec{\zeta} = 2\vec{\omega} = \text{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right] = \frac{1}{2}\nabla \times \vec{V}$$

❖ In cylindrical coordinates system:

$$\nabla \times \vec{V} = \vec{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \vec{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \vec{e}_z\left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

4. Angular Deformation [1/2]

❖ Angular deformation of a particle is given by the sum of the two angular deformation

$$\delta\gamma = \delta\alpha + \delta\beta$$

$$\delta\xi = \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta t - u\delta t = \frac{\partial u}{\partial y}\delta y\delta t \qquad \delta\eta = \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta t - v\delta t = \frac{\partial v}{\partial x}\delta x\delta t$$

$$\delta\alpha = \delta\eta / \delta x \qquad \delta\beta = \delta\xi / \delta y$$

ξ（Xi）η（Eta）

Rate of shear strain or the rate of angular deformation

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\alpha + \delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{\left(\frac{\partial v}{\partial x}\delta t + \frac{\partial u}{\partial y}\delta t \right)}{\delta t} = \ldots = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

Angular Deformation 2/2

❖ The rate of angular deformation in xy plane

$$\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

❖ The rate of angular deformation in yz plane

$$\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

❖ The rate of angular deformation in zx plane

$$\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

Example 6.1 Vorticity

- For a certain two-dimensional flow field the velocity is given by

$$\vec{V} = 4xy\,\vec{i} + 2(x^2 - y^2)\,\vec{j}$$

Is this flow irrotational?

Example 6.1 Solution

$$u = 4xy \qquad v = x^2 - y^2 \qquad w = 0$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

This flow is irrotational

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Conservation Equations

- Continuity equation
  - Conservation of Mass
- Momentum equation (Navier-Stokes Eq.)
  - Conservation of Linear Momentum
- Angular momentum equation
  - Conservation of Angular Momentum
- Energy equation
  - Conservation of Energy

- Representation
  - Integral (control volume) representation
  - Differential representation

Conservation of Mass 1/5

- To derive the differential equation for conservation of mass in rectangular and in cylindrical coordinate system.
- The derivation is carried out by applying conservation of mass to a differential control volume.

With the control volume representation of the conservation of mass

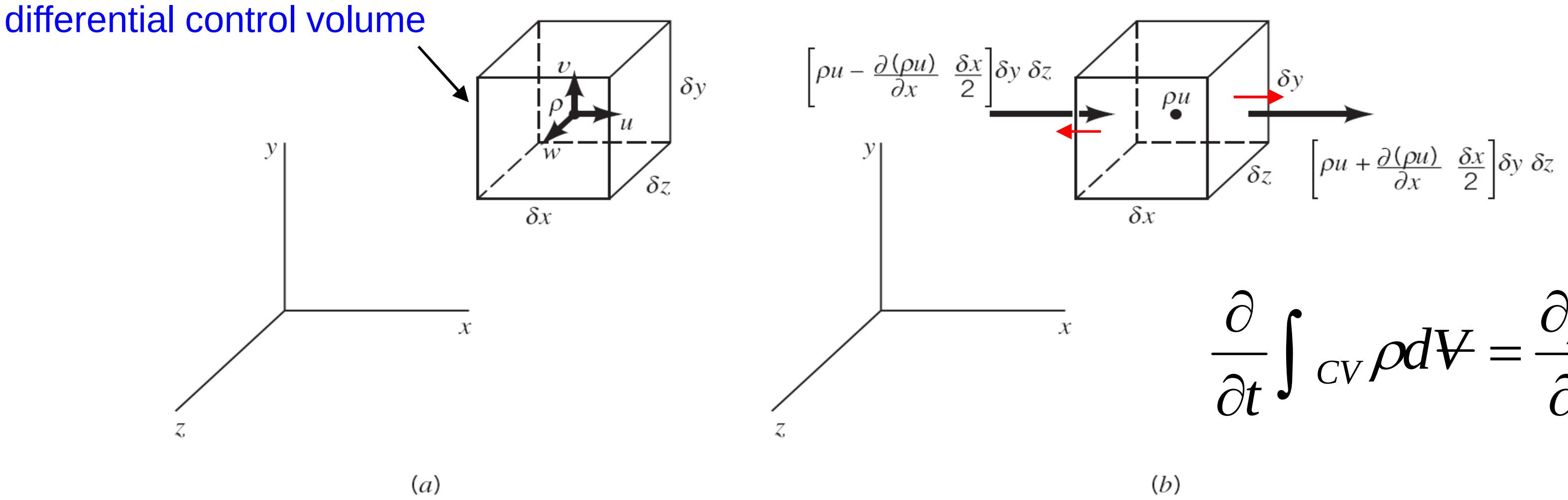
$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot \vec{n} dA = 0$$

The differential form of continuity equation???

Conservation of Mass 2/5

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot \vec{n} dA = 0$$

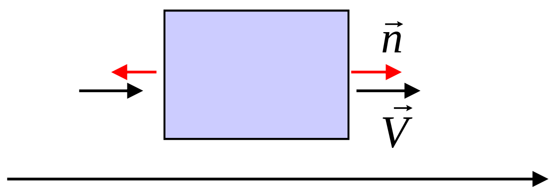
❖ The CV chosen is an infinitesimal cube with sides of length δx, δy, and δz.



$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

$$\rho u \Big|_{x+\left(\frac{dx}{2}\right)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \qquad \rho u \Big|_{x-\left(\frac{\delta x}{2}\right)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}$$

Taylor series



$\vec{n} \cdot \vec{V} = \vec{i} \cdot V\vec{i} = V$ on the right surface

$\vec{n} \cdot \vec{V} = -\vec{i} \cdot V\vec{i} = -V$ on the left surface

Conservation of Mass 3/5

Net rate of mass
Outflow in x-direction

$$=\left[\rho u+\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\delta z-\left[\rho u-\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\delta z=\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z$$

Net rate of mass
Outflow in y-direction

$$=\cdots\cdots=\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z$$

Net rate of mass
Outflow in z-direction

$$=\cdots\cdots=\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

Conservation of Mass 4/5

Net rate of mass Outflow → $$\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\delta x\delta y\delta z$$

The differential equation for Continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\frac{\partial \rho}{\partial t}+\nabla\cdot\left(\rho\vec{V}\right)=0$$

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall+\int_{CS}\rho\vec{V}\cdot\vec{n}dA=0$$

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall=\frac{\partial \rho}{\partial t}\delta x\delta y\delta z$$

$$\int_{CS}\rho\vec{V}\cdot\vec{n}dA=\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\delta x\delta y\delta z$$

Conservation of Mass 5/5

❖ Incompressible fluid (density is constant and uniform)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{V}=0$$

❖ Steady flow

$$\frac{\partial(\bullet)}{\partial t}=0\rightarrow\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\nabla\cdot\rho\vec{V}=0$$

Example 6.2 Continuity Equation

- The velocity components for a certain incompressible, steady flow field are

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine the form of the z component, w, required to satisfy the continuity equation.

Example 6.2 Solution

The continuity equation $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial x}=2z$$

$$\frac{\partial v}{\partial y}=x+z$$

$$\frac{\partial w}{\partial z}=-2x-(x+z)=-3x-z$$

$$\Rightarrow\ w=-3xz-\frac{z^2}{2}+f(x,y)$$

Conservation of Linear Momentum

❖ Applying Newton's second law to control volume

$$\vec{F} = \left.\frac{D\vec{P}}{Dt}\right|_{SYS} \qquad \vec{P}_{system} = \int_{M(system)} \vec{V}dm = \int_{\forall(system)} \vec{V}\rho d\forall$$

$$\delta\vec{F} = \frac{D\left(\vec{V}\delta m\right)}{Dt} = \delta m\left(\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right)$$

$$= \delta m\frac{D\vec{V}}{Dt} = \delta m\vec{a}$$

Newton's 2nd law

For a ***infinitesimal system of mass dm****, what's the* ***the <u>differential form of linear momentum equation</u>?***

Forces Acting on Element 1/2

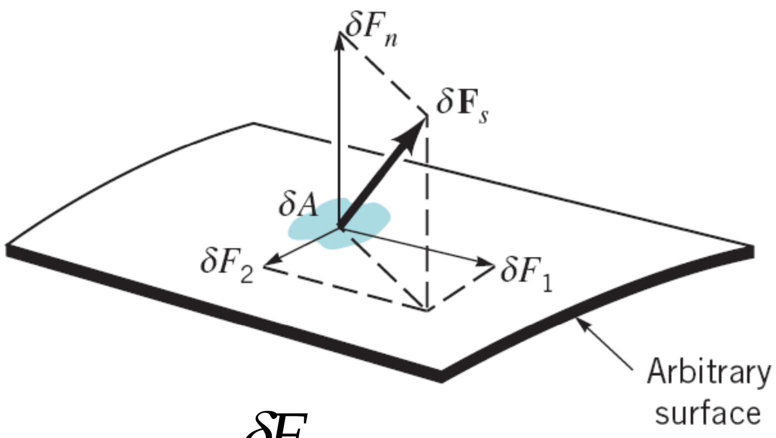
❖ The forces acting on a fluid element may be classified as body forces and surface forces; surface forces include **normal forces and tangential** (shear) **forces**.

$$\delta\vec{F} = \delta\vec{F}_S + \delta\vec{F}_B$$

$$= \delta F_{sx}\vec{i} + \delta F_{sy}\vec{j} + \delta F_{sz}\vec{k}$$

$$+ \delta F_{bx}\vec{i} + \delta F_{by}\vec{j} + \delta F_{bz}\vec{k}$$

Surface forces acting on a fluid element can be described in terms of normal and shear stresses.



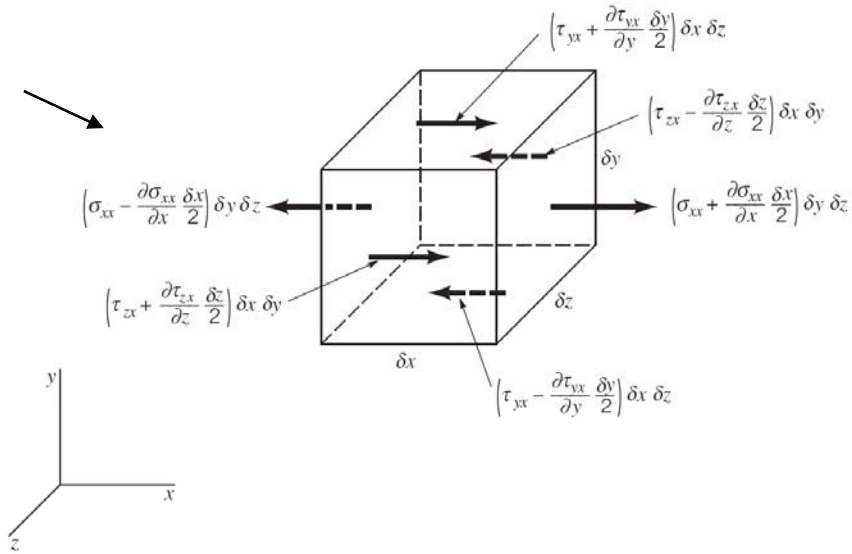
$$\sigma_n = \lim_{\delta t \to 0} \frac{\delta F_n}{\delta A} \qquad \tau_1 = \lim_{\delta t \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta t \to 0} \frac{\delta F_2}{\delta A}$$

Forces Acting on Element 2/2

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \delta x \delta y \delta z$$



$$\delta F_{bx} = \rho g_x \delta x \delta y \delta z$$

$$\delta F_{by} = \rho g_y \delta x \delta y \delta z$$

$$\delta F_{bz} = \rho g_z \delta x \delta y \delta z$$

Equation of Motion

$$\sigma_{xx} = -p + \tau_{xx}$$

$$\sigma_{yy} = -p + \tau_{yy}$$

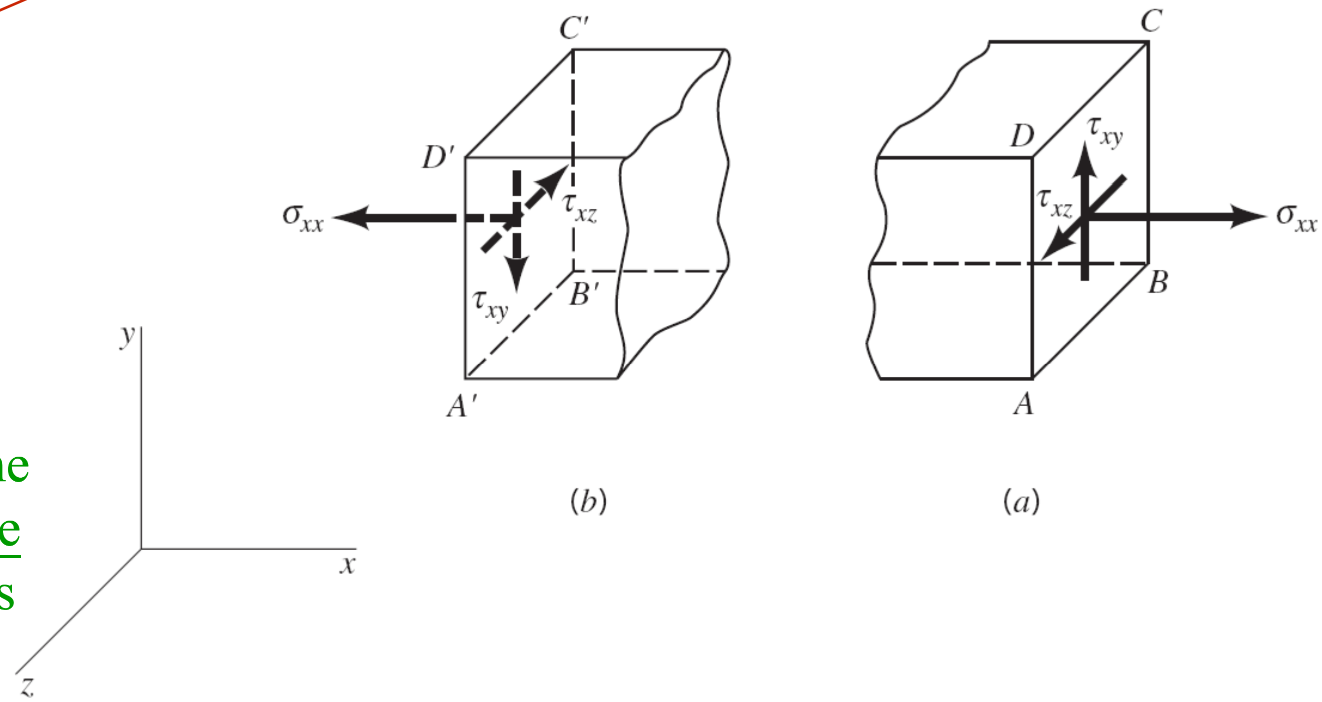
$$\sigma_{zz} = -p + \tau_{zz}$$

Double Subscript Notation for Stresses

$$\tau_{xy}$$

The direction of the stress

The direction of the normal to the plane on which the stress acts



Equation of Motion

$$\delta F_x = \delta m a_x \qquad \delta F_y = \delta m a_y \qquad \delta F_z = \delta m a_z$$

General equation of motion

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

These are the differential equations of motion for any fluid. *How to solve u,v,w ?*

-> **These can't be solved because of more variables than equations, which requires more equations called "constitutive equations" to solve the equations in the case of "Newtonian fluids"** ⇨⇨⇨

Stress-Deformation Relationship: constitutive equations 1/2

❖ The stresses must be expressed in terms of the velocity and pressure field.

Cartesian coordinates in **Newtonian and Compressible fluids**

$$\sigma_{xx} = -p + \tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\sigma_{yy} = -p + \tau_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\sigma_{zz} = -p + \tau_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$-p = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

Stress-Deformation Relationship: constitutive equations 2/2

- The stresses must be expressed in terms of the velocity and pressure field.

Cartesian coordinates in **Newtonian and Incompressible fluids**

$$
\begin{cases}
\sigma_{xx} = -p + \tau_{xx} = -p + 2\mu \dfrac{\partial u}{\partial x} \\
\sigma_{yy} = -p + \tau_{yy} = -p + 2\mu \dfrac{\partial v}{\partial y} \\
\sigma_{zz} = -p + \tau_{zz} = -p + 2\mu \dfrac{\partial w}{\partial z} \\
\tau_{xy} = \tau_{yx} = \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) \\
\tau_{xz} = \tau_{zx} = \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) \\
\tau_{yz} = \tau_{zy} = \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right)
\end{cases}
$$

$$-p = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

The Navier-Stokes Equations 1/2

- These obtained equations of motion are called the Navier-Stokes Equations.
- Under **incompressible Newtonian fluids**, the Navier-Stokes equations are reduced to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

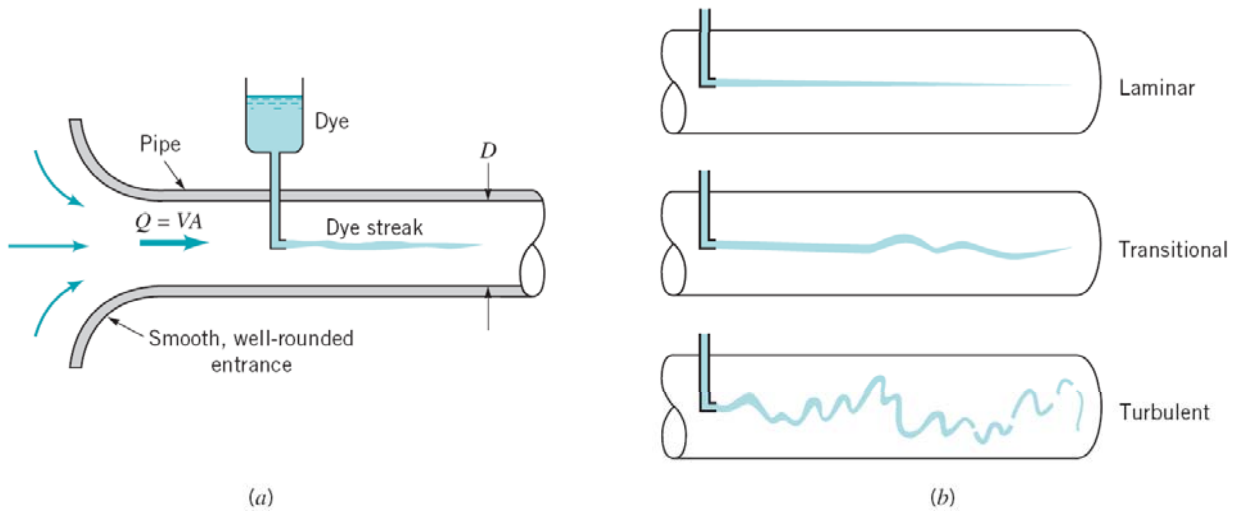
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

The Navier-Stokes Equations 2/2

- The Navier-Stokes equations apply to both laminar and turbulent flow, but for turbulent flow each velocity component fluctuates randomly with respect to time and this added complication makes an analytical solution intractable.
- The exact solutions referred to are for laminar flows in which the velocity is either independent of time (steady flow) or dependent on time (unsteady flow) in a well-defined manner.

Laminar or Turbulent Flow 1/2

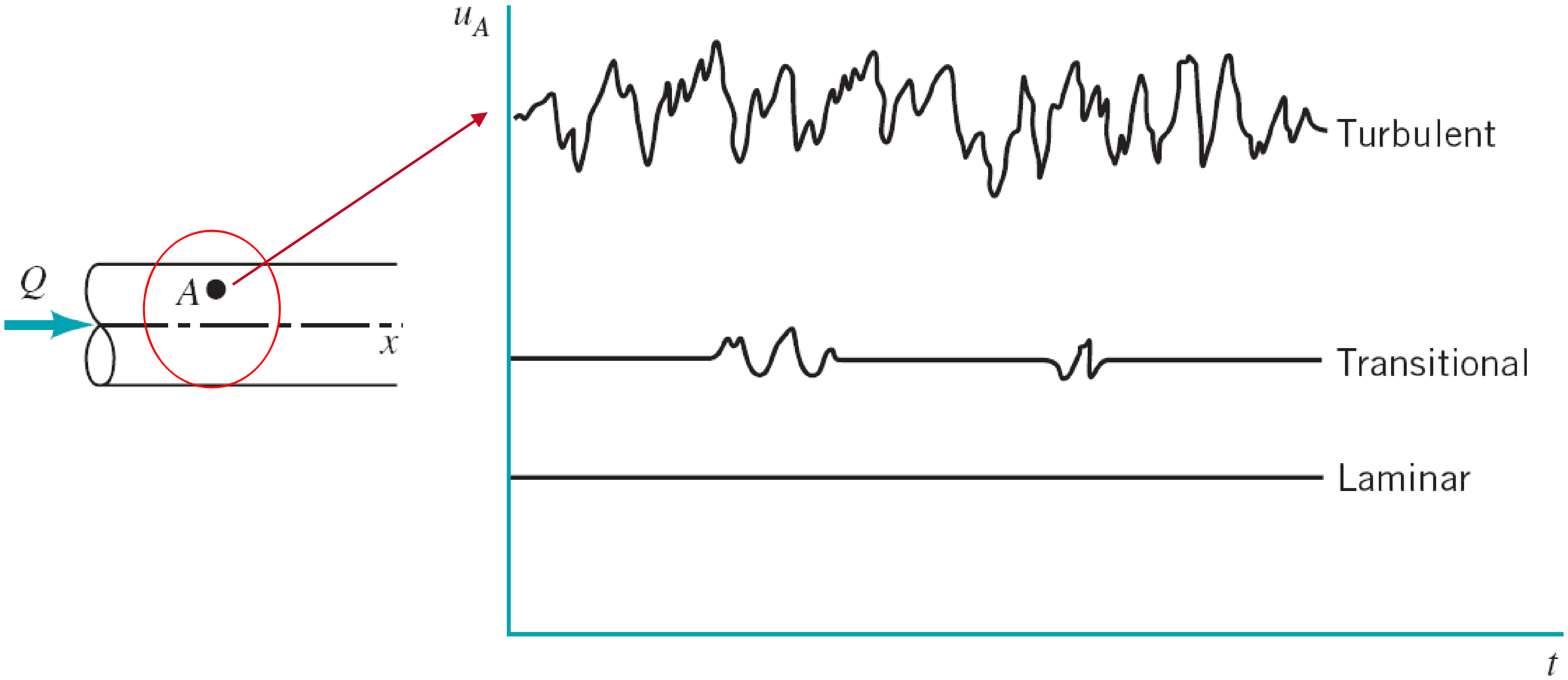
- The flow of a fluid in a pipe may be **Laminar ? Or Turbulent ?**
- **Osborne Reynolds**, a British scientist and mathematician, was the first to distinguish the difference between these classification of flow by using a **simple apparatus** as shown.



Laminar or Turbulent Flow 2/2

- ⇨ For "**small enough flowrate**" the dye streak will remain as a well-defined line as it flows along, with only slight blurring due to molecular diffusion of the dye into the surrounding water.
- ⇨ For a somewhat larger "**intermediate flowrate**" the dye fluctuates in time and space, and intermittent bursts of irregular behavior appear along the streak.
- ⇨ For "**large enough flowrate**" the dye streak almost immediately become blurred and spreads across the entire pipe in a random fashion.

Time Dependence of Fluid Velocity at a Point



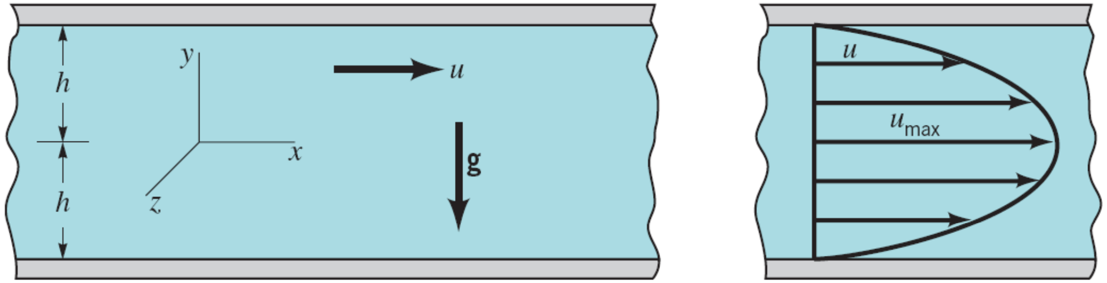
Indication of Laminar or Turbulent Flow

- The term **flowrate** should be replaced by Reynolds number, $R_e = \rho VL / \mu$,where *V* is the average velocity in the pipe, and *L* is the characteristic dimension of a flow. *L* is usually *D (diameter)* in a pipe flow. -> a measure of inertial force to the viscous force.
- It is **not only the fluid velocity** that determines the character of the flow – its density, viscosity, and the pipe size are of equal importance.
- For general engineering purpose, the flow in a **round pipe**
  - ⇨**Laminar** $R_e < 2100$
  - ⇨**Transitional**
  - ⇨**Turbulent** $R_e > 4000$

Some Simple Solutions for Viscous, Incompressible Fluids

- A principal difficulty in solving the Navier-Stokes equations is because of their **nonlinearity** arising from the **convective acceleration terms**.
- There are no general analytical schemes for solving nonlinear partial differential equations.
- There are a few special cases for which the convective acceleration vanishes. In these cases exact solution are often possible.

Steady, Laminar Flow between Fixed Parallel Plates 1/4



(a) (b)

1. Schematic:
2. Assumptions: Incompressible, Newtonian, Steady, One dimensional flow
3. Continuity equation $\nabla \cdot \vec{V} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial u}{\partial x} = 0 \rightarrow u = u(y)$
4. The Navier-Stokes equations

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) \qquad 0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right) \qquad 0 = -\frac{\partial p}{\partial y} - \rho g$$

$$\rho\left(\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \cancel{\rho g_z} + \mu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right) \qquad 0 = -\frac{\partial p}{\partial z}$$

Steady, Laminar Flow between Fixed Parallel Plates 2/4

5. Boundary conditions (B.C.) *u=0 at y=-h* *u=0 at y=h* (no-slip boundary condition)
6. Solve the equations with B.C.

$$0 = -\frac{\partial p}{\partial y} - \rho g, \quad 0 = -\frac{\partial p}{\partial z} \quad \xrightarrow{\text{Integrating}} \quad p = -\rho g y + f_1(x)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \quad \xrightarrow{\text{Integrating}} \quad u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2 \qquad c_1\,? \qquad c_2\,?$$

$$c_2 = 0, c_1 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - h^2\right)$$

Steady, Laminar Flow between Fixed Parallel Plates 3/4

❖ Shear stress distribution

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} = \left(\frac{\partial p}{\partial x}\right) y$$

❖ Volume flow rate per unit depth (z direction)

$$q = \int_{-h}^{h} u dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2) dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial x} = \text{constant} = \frac{p_2 - p_1}{(x_2 - x_1) = \ell} = -\frac{\Delta p = (p_1 - p_2)}{\ell} < 0$$

$$>> q = \frac{2h^3 \Delta p}{3\mu\ell}, \text{ where } p_1 \text{ is the inlet pressure and } p_2 \text{ is the outlet pressure}$$

$$q = \frac{2h^3 \Delta p}{3\mu\ell} \rightarrow q = \frac{2h^3}{3\mu\ell}\Delta p \approx i = \frac{1}{R}V \quad \text{Flow resistance} \propto \mu \frac{\ell}{h^3}$$

Steady, Laminar Flow between Fixed Parallel Plates 4/4

- Average velocity per unit depth

$$V_{average} = \frac{q}{2h} = \frac{h^2 \Delta p}{3\mu\ell}$$

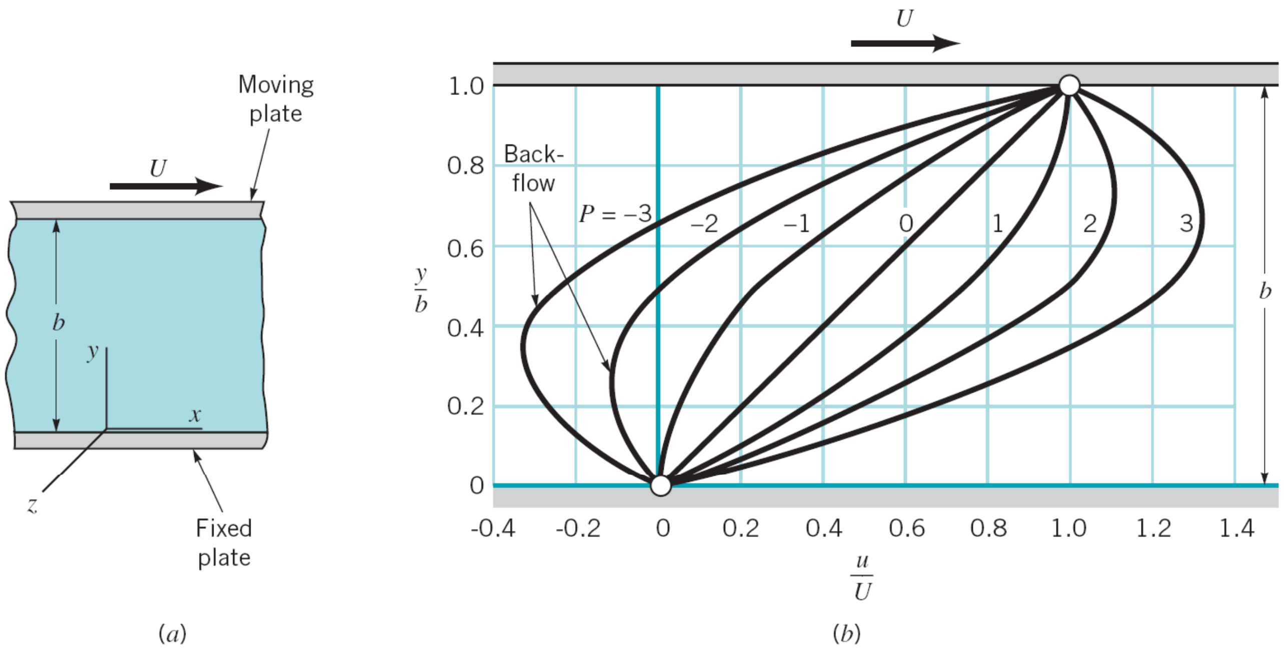
- Point of maximum velocity

$$\frac{\mathrm{du}}{\mathrm{dy}} = 0$$ **at y=0**

$$u = u_{\max} = U = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{3}{2} V_{average}$$

Couette Flow 1/3 (HW)

❖ Since only the **boundary conditions have changed**, there is **no need to repeat the entire analysis** of the "both plates stationary" case.



Couette Flow 2/3

❖ The boundary conditions for the moving plate case are

u=0 at y=0

u=U at y=b

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2 \qquad c_1\,? \qquad c_2\,?$$

$$\Rightarrow c_1 = \frac{U}{b} - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)b \qquad c_2 = 0$$

Velocity distribution

$$u = \frac{Uy}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)by$$

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left[1-\left(\frac{b}{y}\right)\right]$$

$$P = \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

Couette Flow 3/3

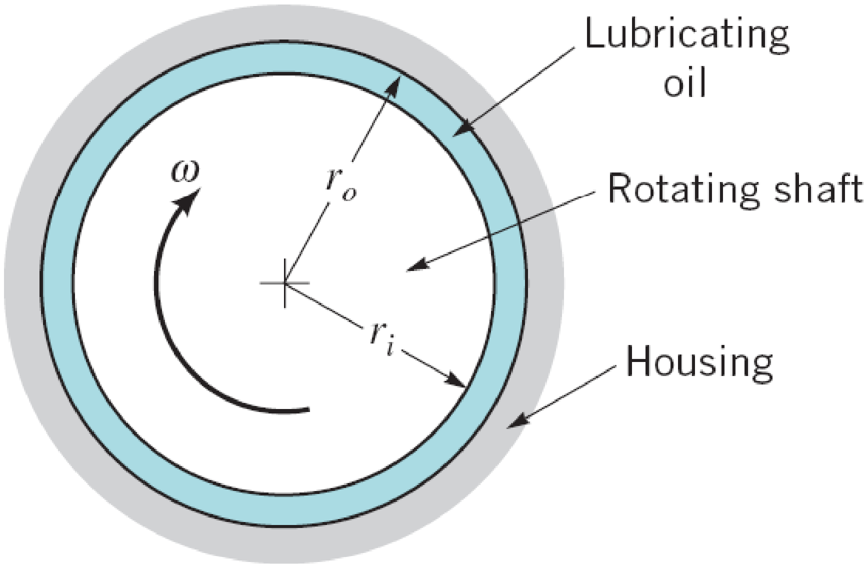
❖ **Simplest type of Couette flow**

$$\frac{\partial p}{\partial x} = 0 \Rightarrow u = U\frac{y}{b}$$

This flow can be approximated by the flow between closely spaced concentric cylinder is fixed and the other cylinder rotates with a constant angular velocity.

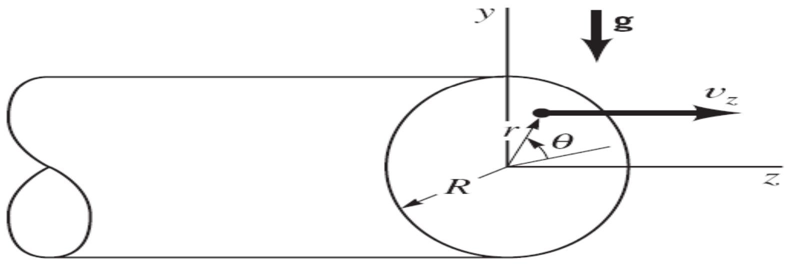
$$U = r_i\omega \qquad b = r_o - r_i$$

$$\tau = \mu r_i \omega / (r_o - r_i)$$



Flow in the narrow gap of a journal bearing.

Steady, Laminar Flow (Hagen-Poiseuille Flow) in Circular Tubes 1/5



1. Schematic:
2. Assumptions: Incompressible, Newtonian, Steady, Laminar, One dimensional flow

$$v_r = 0, v_\theta = 0, v_z \neq 0$$

3. Continuity equation $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \rightarrow \frac{\partial v_z}{\partial z} = 0 \rightarrow v_z = v_z(r)$
4. The Navier-Stokes equations
5. Boundary Conditions: At r=0, the velocity v_z is finite. At r=R, the velocity v_z is zero.
6. Solve the equation with B.C.

From the Navier-Stokes Equations in Cylindrical coordinates

- General motion of an incompressible Newtonian fluid is governed by the continuity equation and the momentum equation

Mass conservation →

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Navier-Stokes Equation in a cylindrical coordinate

r-Direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] \quad \text{(a)}$$

θ-Direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] \quad \text{(b)}$$

z-Direction

Acceleration

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \quad \text{(c)}$$

Steady, Laminar Flow in Circular Tubes 2/5

Navier – Stokes equation reduced to

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r}$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

Integrating

$$p = -\rho g\left(r\sin\theta\right) + f_1\left(z,\theta\right)$$

$$\rightarrow p = -\rho g y + f_1\left(z\right)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right)$$

Integrating

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2 \qquad c_1\,? \qquad c_2\,?$$

Steady, Laminar Flow in Circular Tubes 3/5

At r=0, the velocity v_z is finite. At r=R, the velocity v_z is zero.

$$c_1 = 0, c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$$

Velocity distribution

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r^2 - R^2\right)$$

Steady, Laminar Flow in Circular Tubes 4/5

❖ The shear stress distribution

$$\tau_{rz} = \mu \frac{dv_z}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial z}\right)$$

❖ Volume flow rate

$$Q = \int_0^R u_z 2\pi r dr = \ldots\ldots = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\partial p}{\partial z} = \text{constant} = \frac{p_2 - p_1}{\ell} = -\Delta p / \ell$$

$$>> Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{\pi R^4}{8\mu}\left(\frac{\Delta p}{\ell}\right) = \frac{\pi \mathrm{D}^4}{128\mu\ell}\Delta p$$

Steady, Laminar Flow in Circular Tubes 5/5

- Average velocity

$$V_{average} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{R^2 \Delta p}{8 \mu \ell}$$

- Point of maximum velocity

$$\frac{dv_z}{dr} = 0 \qquad \textbf{at r=0}$$

$$v_{max} = -\frac{R^2 \Delta p}{4 \mu \ell} = 2V_{average} \Rightarrow \frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

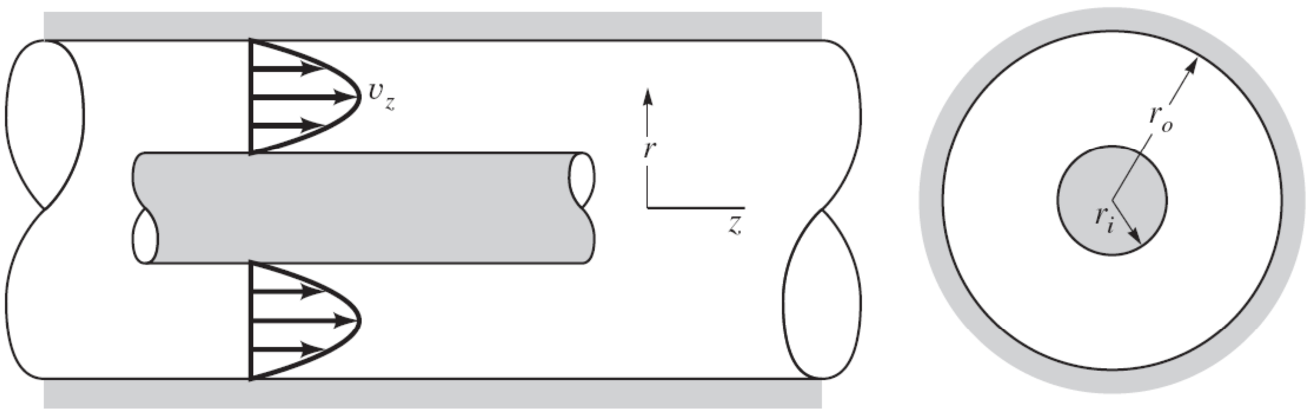
Steady, Axial, Laminar Flow in an Annulus 1/2 (HW)

For steady, laminar flow in annular tubes

Boundary conditions

$v_z = 0 \ , \ \text{at} \ r = r_o$

$v_z = 0 \ , \ \text{at} \ r = r_i$



Steady, Axial, Laminar Flow in an Annulus 2/2

The velocity distribution

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o / r_i)}\ln\frac{r}{r_o}\right]$$

The volume rate of flow

$$Q = \int_{r_i}^{r_o} v_z(2\pi r)dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^{\ 4} - r_i^{\ 4} - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o / r_i)}\right]$$

The maximum velocity occurs at $r=r_m$

$$\frac{\partial v_z}{\partial r} = 0 \rightarrow r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o / r_i)}\right]^{1/2}$$

Inviscid Flow

- Shear stresses develop in a moving fluid because of the viscosity of the fluid.
- For some common fluid, such as air, the viscosity is small, and therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity.
- Flow fields in which the shear stresses are assumed to be negligible are said to be inviscid, or frictionless.

Define the pressure, p, as the negative of the normal stress

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

Euler's Equation of Motion

- Under **inviscid flows: frictionless condition**, the equations of motion are reduced to **Euler's Equation**:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y$$

Euler's Equation

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z$$

$$\Rightarrow \quad \rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p$$

Bernoulli Equation 1/3

❖ Euler's equation for steady flow along a streamline is

$$\rho\vec{g} - \nabla p = \rho(\vec{V} \cdot \nabla)\vec{V}$$

Selecting the coordinate system with the z-axis vertical so that the acceleration of gravity vector can be expressed as

$$\vec{g} = -g\nabla z$$

$$(\vec{V} \cdot \nabla)\vec{V} = \frac{1}{2}\nabla(\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V})$$ **Vector identity ….**

$$-\rho g \nabla z - \nabla p = \frac{\rho}{2}\nabla(\vec{V} \cdot \vec{V}) - \rho\left(\vec{V} \times \left(\nabla \times \vec{V}\right)\right)$$

Bernoulli Equation 2/3

$$\frac{\nabla p}{\rho}+\frac{1}{2}\nabla\left(V^2\right)+g\nabla z=\vec{V}\times\left(\nabla\times\vec{V}\right)=\vec{V}\times\vec{\zeta}$$

$\vec{V}\times\vec{\zeta}$ perpendicular to $\vec{V}$

$\cdot d\vec{s}$ →

$$\frac{\nabla p}{\rho}\cdot d\vec{s}+\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{s}+g\nabla z\cdot d\vec{s}=\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]\cdot d\vec{s}$$

With $d\vec{s}=dx\vec{i}+dy\vec{j}+dz\vec{k}$ tangential vector on a streamline

$$\nabla p\cdot d\vec{s}=\left(\frac{\partial p}{\partial x}\vec{i}+\frac{\partial p}{\partial y}\vec{j}+\frac{\partial p}{\partial z}\vec{k}\right)\cdot\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz=dp$$

$$\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{s}=\frac{1}{2}\left(\frac{\partial V^2}{\partial x}\vec{i}+\frac{\partial V^2}{\partial y}\vec{j}+\frac{\partial V^2}{\partial z}\vec{k}\right)\cdot\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=\frac{1}{2}\left(\frac{\partial V^2}{\partial x}dx+\frac{\partial V^2}{\partial y}dy+\frac{\partial V^2}{\partial z}dz\right)=\frac{1}{2}\left(dV^2\right)$$

$$g\nabla z\cdot d\vec{s}=g\left(\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}+\frac{\partial z}{\partial z}\vec{k}\right)\cdot\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=g\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy+\frac{\partial z}{\partial z}dz\right)=gdz$$

Bernoulli Equation 3/3

$$\frac{\nabla p}{\rho}\cdot d\vec{s}+\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{s}+g\nabla z\cdot d\vec{s}=0$$

$$\Rightarrow \frac{dp}{\rho}+\frac{1}{2}d\left(V^2\right)+g\,dz=0$$

Integrating … $\int\frac{dp}{\rho}+\frac{V^2}{2}+gz=\text{constant}$

For steady, inviscid, incompressible fluid (commonly called ideal fluids) **along a streamline** Bernoulli equation is given by

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$ **Bernoulli equation**

Irrotational Flow 1/2

❖ Irrotation ? The irrotational condition is

$$\nabla \times \vec{V} = 0$$

⇨ In rectangular coordinates system

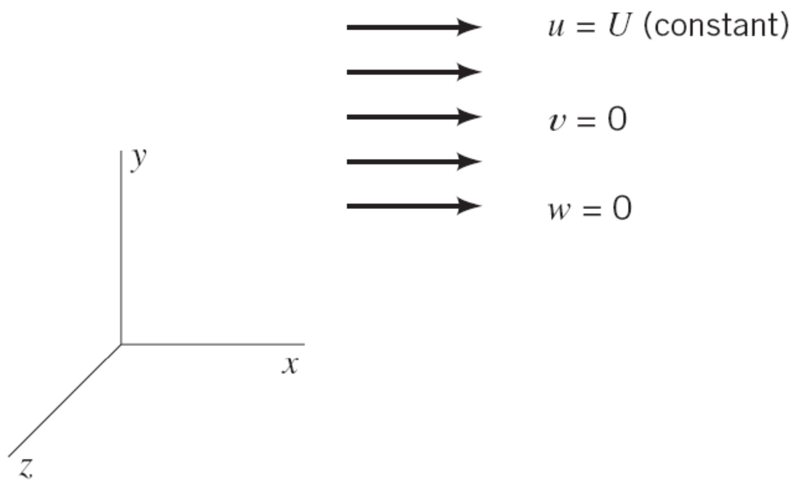
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

⇨ In cylindrical coordinates system

$$\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} = 0$$

Irrotational Flow 2/2

- A general flow field would not be irrotational flow.
- A special uniform flow field is an example of an irrotational flow



Bernoulli Equation for Irrotational Flow 1/3

- The Bernoulli equation for **steady, incompressible, and inviscid flow** is

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

- The equation can be applied between **any two points on the same streamline**. In general, **the value of the constant will vary from streamline to streamline**.
- Under additional **irrotational condition**, *the Bernoulli equation ?* Starting with Euler's equation in vector form $\rho\vec{g}-\nabla p=\rho\left(\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right)$

$$(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-g\vec{k}=\frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)-\vec{V}\times\left(\nabla\times\vec{V}\right)$$

ZERO Regardless of the direction of ds

Bernoulli Equation for Irrotational Flow 2/3

❖ With **irrotaional** condition

$$\nabla \times \vec{V} = 0$$

$$(\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p - g\vec{k} = \frac{1}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \vec{V} \times \left(\nabla \times \vec{V}\right)$$

$$\Rightarrow \frac{1}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) = \frac{1}{2}\nabla\left(V^2\right) = -\frac{1}{\rho}\nabla p - g\vec{k} \quad \cdot d\vec{r}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

Not a streamline

$$\Rightarrow \frac{1}{2}\nabla\left(V^2\right) \cdot d\vec{r} = -\frac{1}{\rho}\nabla p \cdot d\vec{r} - g\vec{k} \cdot d\vec{r}$$

$$>> \frac{1}{2}d\left(V^2\right) = -\frac{dp}{\rho} - gdz >> \frac{dp}{\rho} + \frac{1}{2}d\left(V^2\right) + gdz = 0$$

Bernoulli Equation for Irrotational Flow 3/3

Integrating for incompressible flow

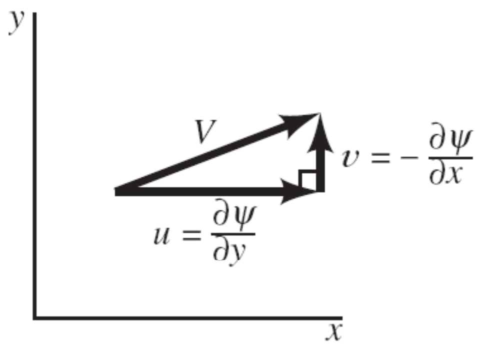
$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{contant} \quad \Rightarrow \quad \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

This equation is valid between <u>any two points in a steady, incompressible, inviscid, and irrotational flow irrespective of streamlines</u>.

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

Stream Function 1/6

- **Streamlines:** Lines tangent to the instantaneous velocity vectors at every point.
- Stream function Ψ(x,y) [Psi] ? Used to represent the velocity component u(x,y,t) and v(x,y,t) of a "two-dimensional incompressible" flow.
- Define a function Ψ(x,y), called the stream function, which relates the velocities shown by the figure in the margin as



$$u \equiv \frac{\partial \psi}{\partial y} \qquad v \equiv -\frac{\partial \psi}{\partial x}$$

Stream Function 2/6

- The stream function Ψ(x,y) satisfies the two-dimensional form of the incompressible continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

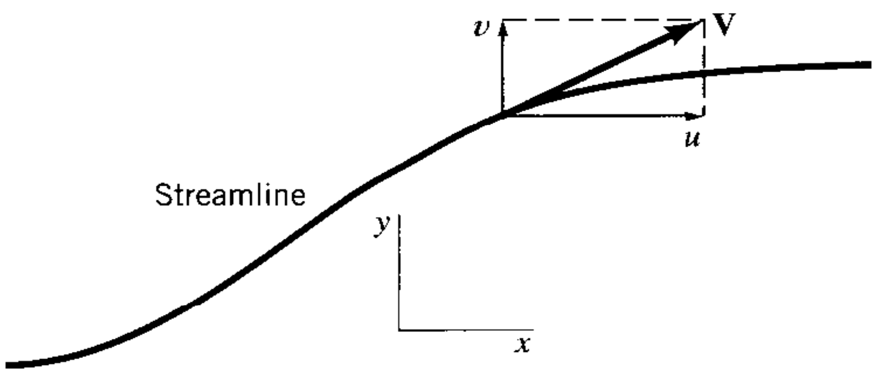
- **Ψ(x,y)** is still unknown for a particular problem, but at least we have simplify the analysis by having to determine only one unknown, **Ψ(x,y)** , rather than the two unknown function u(x,y) and v(x,y).

Stream Function 3/6

❖ Another advantage of using stream function is related to the fact that line along which **Ψ(x,y) =constant** are streamlines.

❖ How to prove ? From the definition of the streamline that the slope at any point along a streamline is given by

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u}$$



Velocity and velocity component along a streamline

Stream Function 4/6

- The change of Ψ(x,y) as we move from one point (x,y) to a nearly point (x+dx,y+dy) is given by

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

$$>> d\psi = 0 \qquad >> -vdx + udy = 0$$

Along a line of constant Ψ

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u}$$

This is the definition for a streamline. Thus, if we know the stream function Ψ(x,y) we can plot lines of constant Ψ to provide the family of streamlines that are helpful in visualizing the pattern of flow. There are an infinite number of streamlines that make up a particular flow field, since for each constant value assigned to Ψ a streamline can be drawn.

Stream Function 5/6

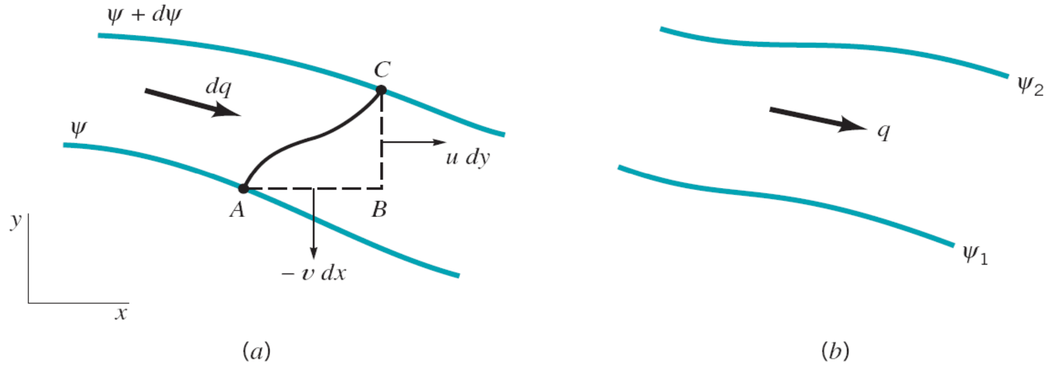
- The actual numerical value associated with a particular streamline is not of particular significance, but the change in the value of Ψ is related to the volume rate of flow.
- dq : volume rate of flow passing between the two streamlines. Flow never crosses streamlines by definition.

$$dq = udy - vdx = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

If $q > 0$, the flow is from left to right.

If $q < 0$, the flow is from right to left.



Stream Function 6/6

- Thus the volume flow rate between any two streamlines can be written as the difference between the constant values of Ψ defining two streamlines.
- The velocity will be relatively high wherever the streamlines are close together, and relatively low wherever the streamlines are far apart.

Example 6.3 Stream Function

- The velocity component in a steady, incompressible, two dimensional flow field are

$$u = 2y \qquad v = 4x$$

Determine the corresponding stream function and show on a sketch several streamlines. Indicate the direction of glow along the streamlines.

Example 6.3 Solution

From the definition of the stream function

$$u = \frac{\partial \psi}{\partial y} = 2y \qquad v = -\frac{\partial \psi}{\partial x} = 4x$$

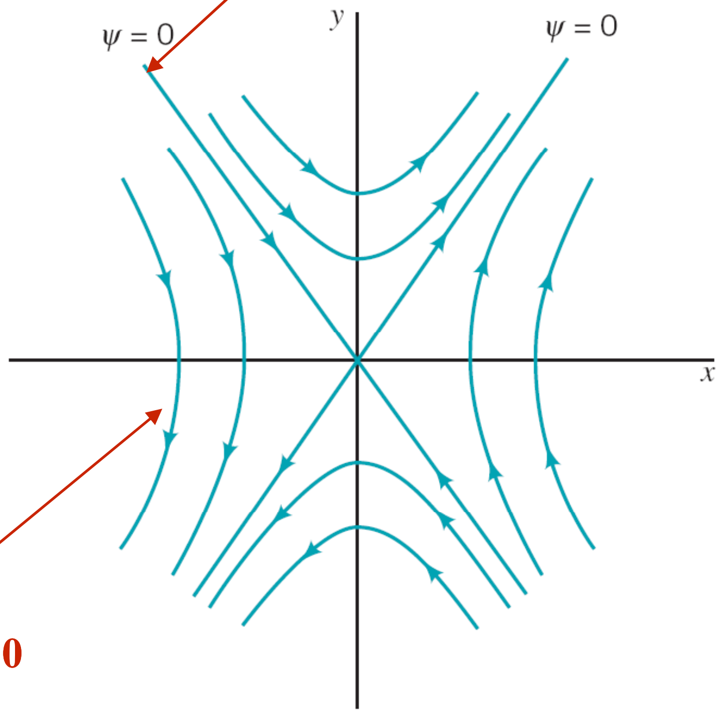
$$\psi = y^2 + f_1(x) \qquad \psi = -2x^2 + f_2(y)$$

$$\psi = -2x^2 + y^2 + C$$

For simplicity, we set C=0

$$\psi = -2x^2 + y^2$$

$\Psi=0$



$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

$\Psi \neq 0$

Velocity Potential $\Phi(x,y,z,t)$ 1/3

- The stream function for two-dimensional incompressible flow is $\Psi(x,y)$
- For an irrotational flow, the velocity components can be expressed in terms of a scalar function **$\Phi(x,y,z,t)$** as

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

where $\Phi(x,y,z,t)$ is called the velocity potential.

$$\nabla \times \vec{V} = 0 \rightarrow \vec{V} = \nabla \phi$$

Velocity Potential Φ(x,y,z,t) 2/3

- In vector form

$$\vec{V} = \nabla\phi$$

- For an incompressible flow

$$\nabla \cdot \vec{V} = 0$$

Also called a potential flow

For incompressible, irrotational flow

$$\vec{V} = \nabla\phi \Rightarrow \nabla \cdot \vec{V} = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Laplace's equation

Laplacian operator

Velocity Potential $\Phi(x,y,z,t)$ 3/3

- Inviscid, incompressible, irrotational fields are governed by Laplace's equation.
- This type flow is commonly called a potential flow.
- To complete the mathematical formulation of a given problem, boundary conditions have to be specified. These are usually velocities specified on the boundaries of the flow field of interest.