Fields Institute Communications
Volume **00**, 0000

On Arnold's variational principles in fluid mechanics

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This paper is dedicated to V.I.Arnold.

1 Introduction

In the 1960s in the series of pioneering papers V.I.Arnold obtained a number of fundamental results in the mathematical theory of the dynamics of an ideal incompressible fluid, especially in the area of hydrodynamic stability (see Arnold [1965a,b, 1966a,b]).

First, he has developed a new, very effective method in the hydrodynamic stability theory and proved the theorem on the nonlinear stability of steady two-dimensional flows that generalizes the well-known linear stability criterion of Rayleigh (Arnold [1965a,1966a]). Since that time this method, now known as the Arnold method (or Energy-Casimir method) has been successfully applied to a wide range of the problems in fluid mechanics, astrophysics, plasma physics etc. (for review see Holm *et al* [1985], Marsden and Ratiu [1994], Marchioro and Pulvirenti [1994], Arnold and Khesin [1998]). Remarkably, the Arnold method have found applications not only in theoretical sciences but also in such an applied field as geophysical fluid dynamics (see e.g. McIntyre and Shepherd [1987], Shepherd [1990], Cho *et al* [1993], Mu *et al* [1996]).

Second, V.I.Arnold [1966b] has discovered a close connection between the stability properties of an ideal incompressible fluid and the geometry of infinite dimensional Lie groups. In Arnold's theory, the configuration space of ideal incompressible hydrodynamics is identified with the Lie group $G = VDiff(D)$ of volume-preserving diffeomorphisms of the domain D, and fluid flows represent geodesics on G with respect to the metric given by the kinetic energy. One of the consequences of this theory is that any steady flow of an ideal fluid corresponds to a critical point of the energy functional restricted to the orbit of coajoint representation of G. In physical terms, this means that, on the set of all isovortical velocity fields, a

1991 *Mathematics Subject Classification.* Primary 76C05, 76E99; Secondary 76M30, 76W05.
The second author was supported by Hong Kong Research Grants HKUST701/96P and HKUST6169/97P..

steady flow is a distinguished one: it corresponds to a critical point of the energy functional (see Arnold [1965b, 1966b]). Another important consequence of the theory is that if the second variation of the energy evaluated in a given steady state on the same set of isovortical flows is definite in sign, this would imply at least linear stability of this steady state (see Arnold [1965b], [1966b], Arnold and Khesin [1998]). As has been shown by Rouchon [1991] and by Sadun and Vishik [1993], for three-dimensional flows of an ideal incompressible fluid the corresponding second variation is, in general, indefinite in sign, so that no conclusion about stability can be drawn. If however only flows with a symmetry (translational, rotational or helical) are considered then there exist non-trivial steady flows for which the second variation is definite in sign (Arnold [1965a,b]). Moreover, sometimes it is possible to obtain sufficient conditions for genuine nonlinear stability (see Arnold [1965a, 1966a], Holm *et al* [1985], Ovsiannikov *et al* [1985], Moffatt [1985,1986], Vladimirov [1985, 1986], Marchioro and Pulvirenti [1994], Davidson [1998]).

In this paper, we shall discuss a number of variational principles that are direct and natural generalizations of Arnold's principle [1965b] to more sophisticated hydrodynamic systems. These include:

1. a dynamical system 'rigid body + fluid', which may be either a body placed in an inviscid rotational flow or a body with a cavity containing a fluid;
2. flows of an ideal incompressible fluid with contact discontinuities and, in particular, flows with discontinuities of vorticity;
3. magnetohydrodynamic flows of an ideal, incompressible, perfectly conducting fluid.

We shall closely follow the original work of Arnold [1965b] (see also the paper by Sedenko and Yudovich [1978], who extended Arnold's principle for free-boundary flows of an ideal incompressible fluid, and the paper by Grinfeld [1984] who considered compressible barotropic flows). Our analysis will be based on simple physical arguments rather than on the general but highly abstract geometric theory. We shall not discuss the underlying Hamiltonian structures of the considered mechanical systems – they are all known and may be found in the literature (see e.g. Arnold [1966b], Holm et al [1985], Khesin and Chekanov [1989], Marsden and Ratiu [1994], Arnold and Khesin [1998]).

We believe that the construction of the variational principles, based on physically understandable ideas (similar to those of Arnold [1965b]) rather than on abstract machinery of the differential geometry, is interesting and important from several viewpoints:

1. the proposed theory may shed some light on the physical meaning of the related abstract theories as, for instance, in the case of ideal magnetohydrodynamics where the variational principle formulated in Section 4 of the present paper (see also Friedlander and Vishik [1994], Vladimirov and Ilin [1997b], Vladimirov *et al* [1998]) clarifies the physical meaning of coajoint orbits in the related semi-direct product theory of Marsden, Ratiu and Weinstein [1984] (see also Khesin and Chekanov [1989]);
2. in a very simple way it may, in some situations (such as ideal magnetohydrodynamics, see Friedlander and Vishik [1994], Vladimirov *et al* [1998]), result in stability criteria for general *three-dimensional* steady states;
3. it is applicable to the systems whose configuration space cannot be identified with any Lie group (the examples are: free-boundary flows of an ideal fluid,

see Yudovich and Sedenko [1978] and Section 3 of the present paper, and the system 'rigid body + fluid', see Section 2);

4. it may be modified so as to take account of the effects of dissipation (for example, for the system 'body + fluid' one may include dissipation in finite dimensional degrees of freedom corresponding to the rigid body).

For discussion of further generalizations and applications of the general geometric theory of V.I.Arnold in continuum mechanics we refer to the recent book by Arnold and Khesin [1998] (see also Holm *et al* [1985], Simo *et al* [1991a,b], Marsden and Ratiu [1994]).

We conclude this introduction with a statement of Arnold's original variational principle.

Variational principle for steady three-dimensional flows of an ideal fluid. Consider an ideal incompressible homogeneous fluid contained in a three-dimensional domain $\mathcal{D}$ with fixed rigid boundary $\partial\mathcal{D}$. Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $p(\mathbf{x},t)$ the pressure (divided by constant density). Then the governing equations are the Euler equations:

$$D\mathbf{u} = -\nabla p\,, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0\,, \quad D \equiv \partial/\partial t + \mathbf{u}\cdot\nabla\,. \tag{1.2}$$

The boundary condition for (1.1), (1.2) is the usual one of no normal flow through the rigid boundary

$$\mathbf{n}\cdot\mathbf{u} = 0 \quad \text{on} \quad \partial\mathcal{D}\,, \tag{1.3}$$

where $\mathbf{n}$ is a unit outward normal to the boundary.

In general, equations (1.1), (1.2) with boundary condition (1.3) have the only quadratic integral invariant [1], the energy, given by

$$E = \tfrac{1}{2}\int_{\mathcal{D}} \mathbf{u}^2 \mathrm{d}\tau\,, \quad \mathrm{d}\tau = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,. \tag{1.4}$$

Taking *curl* of equations (1.1) we obtain

$$\boldsymbol{\omega}_t = [\mathbf{u},\boldsymbol{\omega}] \tag{1.5}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity, subscript t denotes the partial derivative $\partial/\partial t$ and $[\mathbf{u},\boldsymbol{\omega}] = \nabla\times(\mathbf{u}\times\boldsymbol{\omega})$ denotes a commutator of divergence-free vector fields $\mathbf{u}$ and $\boldsymbol{\omega}$. Equation (1.5) implies that vortex lines are frozen in the fluid and, in particular, that circulation of velocity round any closed material contour $\gamma(t)$ is conserved (Kelvin's theorem), i.e.

$$\Gamma = \oint_{\gamma(t)} \mathbf{u}\cdot\mathrm{d}\mathbf{l} = const\,. \tag{1.6}$$

Steady flows. We now consider a steady solution of (2.1)-(2.3)

$$\mathbf{u} = \mathbf{U}(\mathbf{x})\,, \quad p = P(\mathbf{x})\,, \quad \boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{x}) \equiv \nabla\times\mathbf{U}\,. \tag{1.7}$$

From (1.1), (1.2) we have

$$\boldsymbol{\Omega}\times\mathbf{U} = -\nabla(P + \tfrac{1}{2}\mathbf{U}^2)\,, \quad \nabla\cdot\mathbf{U} = 0\,. \tag{1.8}$$

In addition to equation (1.8), $\mathbf{U}$ satisfies the boundary condition (1.3).

[1] For existence of another invariant, the helicity, one must assume that the vorticity $\boldsymbol{\omega} \equiv curl\mathbf{u}$ is everywhere tangent to the boundary $\partial\mathcal{D}$.

Arnold's isovorticity condition. Following Arnold [1965b], we consider a family of volume-preserving transformations $g^\epsilon : \mathbf{x} \mapsto \tilde{\mathbf{x}}$ of the domain $\mathcal{D}$ to itself which depend on a parameter ϵ and are defined by the solutions $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ of the equations

$$\mathrm{d}\tilde{\mathbf{x}}/\mathrm{d}\epsilon = \boldsymbol{\xi}(\tilde{\mathbf{x}}, \epsilon) \tag{1.9}$$

with the initial data $\tilde{\mathbf{x}}|_{\epsilon=0} = \mathbf{x}$. In equation (1.8), $\boldsymbol{\xi}$ is a divergence-free vector field tangent to the boundary $\partial\mathcal{D}$:

$$\tilde{\nabla} \cdot \boldsymbol{\xi} = 0 \quad \text{in} \quad \mathcal{D}\,, \quad \boldsymbol{\xi} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}\,.$$

For small ϵ, the explicit form of the map g^ϵ is $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon) = \mathbf{x} + \epsilon\boldsymbol{\xi}(\mathbf{x}, 0) + o(\epsilon)$.

The transformation g^ϵ may be interpreted as a 'virtual motion' of the fluid with ϵ playing the role of the 'virtual time', $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ being the position vector at the moment of time ϵ of the fluid particle whose position at the initial instant $\epsilon = 0$ was $\mathbf{x}$ and $\boldsymbol{\xi}$ representing the 'virtual velocity' of such a motion.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two velocity fields in $\mathcal{D}$ that are divergence-free and satisfy the boundary condition (1.3). Following Arnold [1965b], we say that these fields are *isovortical* if there exists a smooth, volume-preserving transformation g^ϵ of the domain $\mathcal{D}$ which sends every closed contour γ to a new one $g^\epsilon\gamma$ in such a way that the circulation of $\mathbf{u}_1$ round the original contour γ is equal to the circulation of $\mathbf{u}_2$ round its image $g^\epsilon\gamma$ under the transformation g^ϵ:

$$\oint_\gamma \mathbf{u}_1 \cdot \mathrm{d}\mathbf{l} = \oint_{g^\epsilon\gamma} \mathbf{u}_2 \cdot \mathrm{d}\mathbf{l}\,. \tag{1.10}$$

To find the general form of infinitesimal variations the field $\mathbf{u}$ satisfying this condition we introduce family of vector fields $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)$ such that the value $\epsilon = 0$ corresponds to the steady solution (1.7), i.e. $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)|_{\epsilon=0} = \mathbf{U}(\mathbf{x})$. For any ϵ, $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)$ is divergence-free and parallel to $\mathcal{D}$. Assuming that ϵ is small we define the first and the second variations of the field $\mathbf{u}$ as

$$\delta\mathbf{u} \equiv \tilde{\mathbf{u}}_\epsilon \Big|_{\epsilon=0}, \quad \delta^2\mathbf{u} \equiv \tfrac{1}{2}\tilde{\mathbf{u}}_{\epsilon\epsilon} \Big|_{\epsilon=0} \,.$$

For small ϵ the isovorticity condition (1.10) reduces to

$$\tilde{\Gamma} - \Gamma = \oint_{g^\epsilon\gamma} \tilde{\mathbf{u}} \cdot \mathrm{d}\mathbf{l} - \oint_\gamma \mathbf{U} \cdot \mathrm{d}\mathbf{l} = \epsilon \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \tilde{\Gamma} + \tfrac{1}{2}\epsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Big|_{\epsilon=0} \tilde{\Gamma} + o(\epsilon^2) = 0\,. \tag{1.11}$$

From this, on using the formula (see e.g. Batchelor [1967])

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \oint_{g^\epsilon\gamma} \tilde{\mathbf{u}} \cdot \mathrm{d}\mathbf{l} = \oint_{g^\epsilon\gamma} \Big(\tilde{\mathbf{u}}_\epsilon - \boldsymbol{\xi} \times \tilde{\boldsymbol{\omega}}\Big) \cdot \mathrm{d}\mathbf{l}$$

we obtain

$$\begin{aligned}\oint_\gamma \Big\{\epsilon\Big(\delta\mathbf{u} - \boldsymbol{\xi} \times \boldsymbol{\Omega}\Big) \quad &+ \quad \tfrac{1}{2}\epsilon^2\Big(\delta^2\mathbf{u} - \boldsymbol{\chi} \times \boldsymbol{\Omega} \\ &- \quad \boldsymbol{\xi} \times \delta\boldsymbol{\omega} - \boldsymbol{\xi} \times (\delta\boldsymbol{\omega} - [\boldsymbol{\xi}, \boldsymbol{\Omega}])\Big)\Big\} \cdot \mathrm{d}\mathbf{l} + o(\epsilon^2) = 0\,,\end{aligned}$$

where $\boldsymbol{\chi}(\mathbf{x}) \equiv \boldsymbol{\xi}_\epsilon|_{\epsilon=0}$. Since γ is an arbitrary closed material line, we arrive at conclusion that

$$\delta\mathbf{u} = \boldsymbol{\xi} \times \boldsymbol{\Omega} - \nabla\alpha \quad \text{or} \quad \delta\boldsymbol{\omega} = [\boldsymbol{\xi}, \boldsymbol{\Omega}]\,, \tag{1.12}$$

$$\delta^2\mathbf{u} = \boldsymbol{\xi} \times \delta\boldsymbol{\omega} + \boldsymbol{\chi} \times \boldsymbol{\Omega} - \nabla\beta\,. \tag{1.13}$$

where $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are scalar functions, which, in the case of singly-connected domain $\mathcal{D}$, are uniquely determined by the conditions

$$\nabla \cdot \delta\mathbf{u} = \nabla \cdot \delta^2\mathbf{u} = 0 \ \text{ in } \mathcal{D}\,, \quad \delta\mathbf{u}\cdot\mathbf{n} = \delta^2\mathbf{u}\cdot\mathbf{n} = 0 \ \text{ on } \partial\mathcal{D}\,.$$

Variational principle. Now we shall show that the first variation of energy (1.4) with respect to variations of the velocity field $\mathbf{u}$ of the form (1.12), (1.13) vanishes. We have

$$\begin{aligned}\delta E \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} &= \int_{\mathcal{D}} \mathbf{U}\cdot\delta\mathbf{u}\mathrm{d}\tau = \int_{\mathcal{D}} \mathbf{U}\cdot(\boldsymbol{\xi}\times\boldsymbol{\Omega} - \nabla\alpha)\mathrm{d}\tau \\ &= \int_{\mathcal{D}} \boldsymbol{\xi}\cdot(\boldsymbol{\Omega}\times\mathbf{U})\mathrm{d}\tau = -\int_{\mathcal{D}} \boldsymbol{\xi}\cdot\nabla(P + \tfrac{1}{2}\mathbf{U}^2)\mathrm{d}\tau = 0\,.\end{aligned}$$

We have thus proved the following.

Proposition 1.1 (Arnold, 1965) *On the set of all flows isovortical to a given steady flow (1.7) the energy functional (1.4) has a stationary value in this steady flow.*

The second variation. Let us now calculate the second variation of the energy at the stationary point. We have

$$\delta^2 E \equiv \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\bigg|_{\epsilon=0} E = \int_{\mathcal{D}} \Big(\tfrac{1}{2}(\delta\mathbf{u})^2 + \mathbf{U}\cdot\delta^2\mathbf{u}\Big)\mathrm{d}\tau\,.$$

After substitution of equation (1.13) and integration by parts, it may be shown that all the terms containing $\boldsymbol{\chi}$ vanish due to equations (1.8) and the boundary conditions on $\partial\mathcal{D}$ for the fields $\boldsymbol{\chi}$, and $\mathbf{U}$, and the second variation takes the form

$$\delta^2 E = \tfrac{1}{2}\int_{\mathcal{D}} \Big((\delta\mathbf{u})^2 + \delta\boldsymbol{\omega}\cdot(\mathbf{U}\times\boldsymbol{\xi})\Big)\mathrm{d}\tau\,. \tag{1.14}$$

$\delta^2 E$ is a quadratic functional of the field $\boldsymbol{\xi}(\mathbf{x})$. If for a given steady flow (1.7) this functional is definite in sign, it would mean that the energy (1.4) has a conditional extremum in this flow, and this would imply at least linear stability of the flow. Indeed, in the paper by Arnold [1966b] it has been shown that the second variation (1.14) is an integral invariant of the corresponding linearized problem, provided that $\delta\mathbf{u}$ is considered as an infinitesimal perturbation to the basic flow (1.7) that obeys appropriate linearized equations.

Unfortunately, as recently has been shown by Rouchon [1991] and Sadun and Vishik [1993], the second variation (1.14) is always indefinite in sign. There are only two exceptions: (i) the basic flow is irrotational, then $\delta\boldsymbol{\omega} = 0$, $\delta\mathbf{u} = -\nabla\alpha$, and $\delta^2 E$ is always positive definite; (ii) the basic flow is a rigid rotation of the fluid around a fixed axis, in this case one should consider a certain linear combination of the energy and the angular momentum of the fluid whose second variation is positive definite (see Arnold [1965b]). If however both the basic flow (1.7) and the perturbation has a symmetry (translational, rotational or helical) then there are known cases where $\delta^2 E$ is of definite sign (see Arnold [1965a,b], Holm et al [1985], Vladimirov [1986], Marchioro and Pulvirenti [1994], Arnold and Khesin [1998]).

Consider for example two-dimensional problem. Both the basic flow (1.7) and the perturbation have only two non-zero components and depend on only two coordinates in the plane of motion, i.e. $\mathbf{U} = (U_1(x,y), U_2(x,y), 0)$, $P = P(x,y)$ etc. In this case, we may introduce stream function Ψ such that $U_1 = \Psi_x$, $U_2 = -\Psi_y$.

Then, it follows from (1.8) that $\Omega = \Omega(\Psi)$ where $\Omega = -\nabla^2\Psi$, and (1.14) reduces to

$$\delta^2 E = \tfrac{1}{2}\int_{\mathcal{D}}\Big((\delta\mathbf{u})^2 - \frac{d\Omega}{d\Psi}(\boldsymbol{\xi}\cdot\nabla\Psi)^2\Big)\mathrm{d}\tau\,. \tag{1.15}$$

Evidently, the second variation (1.15) is positive definite provided that $d\Omega/d\Psi \le 0$.

2 Variational principle and stability of steady states of the dynamical system 'rigid body + inviscid fluid'

In this section we shall show how Arnold's principle can be generalized to the case of the dynamical system 'rigid body + inviscid fluid'.

We consider two general situations: (I) the system 'body + fluid' represents a rigid body with a cavity filled with a fluid, (II) it represents a rigid body surrounded by a fluid. In the first case the fluid is confined to an interior (for the body) domain. In the second case it occupies an exterior domain, the latter in turn may be bounded by some fixed rigid boundary or it may extend to infinity.

2.1 Governing equations. Consider a dynamical system consisting of an incompressible, homogeneous and inviscid fluid and a rigid body. Let $\mathcal{D}$ be a domain in three-dimensional space that contains both a fluid and a rigid body, and let $\mathcal{D}_b(t)$ be a domain (inside $\mathcal{D}$, i.e. $\mathcal{D}_b(t) \subset \mathcal{D}$) occupied by the body. The domain $\mathcal{D}_f(t) = \mathcal{D} - \mathcal{D}_b(t)$ is completely filled with a fluid; its boundary $\partial\mathcal{D}_f(t)$ consist of two parts: the inner boundary $\partial\mathcal{D}_b(t)$ representing the surface of the rigid body and the outer boundary $\partial\mathcal{D}$ which is fixed in the space.

In general, motion of the rigid body may be restricted by some geometric constraints or may be not. The number of degrees of freedom is denoted by N where necessarily $N \le 6$. Motion of the body is described by its generalized coordinates $q_\alpha(t)$ and velocities $v_\alpha(t) = \dot{q}_\alpha \equiv dq_\alpha/dt$ ($\alpha = 1, ..., N$). Fluid motion is described by velocity field $u_i(\mathbf{x}, t)$ ($i = 1, 2, 3$) and the pressure field $p(\mathbf{x}, t)$, here $\mathbf{x} \equiv (x_1, x_2, x_3)$ are Cartesian coordinates. From here on we shall use two types of indices, Greek and Latin. Greek indices take values from 1 to N and correspond to finite-dimensional degrees of freedom of the system 'body + fluid', while Latin indices take values from 1 to 3 and denote Cartesian components of vectors and tensors. In the rest of the paper the summation is implied over repeated both Greek and Latin indices.

We suppose that an external (with respect to the system 'body + fluid') force is applied to the rigid body. This force is characterized by potential energy $\Pi(q_\alpha)$.

The equations of motion for the fluid are the Euler equations (1.1), (1.2). Motion of the rigid body obeys the standard Lagrange equations of classical mechanics that may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial v_\alpha}\right] - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial \Pi}{\partial q_\alpha} + F_\alpha\,. \tag{2.1}$$

In equation. (2.1), $T(q_\alpha, v_\alpha)$ is the kinetic energy of the body given by the equation

$$T = \tfrac{1}{2}M w_i w_i + \tfrac{1}{2}I_{ik}\sigma_i\sigma_k\,, \tag{2.2}$$

where I_{ik} is the moment of inertia tensor; the velocity of the centre of mass $\mathbf{w} = d\mathbf{r}/dt$ and the angular velocity $\boldsymbol{\sigma}$ are considered as functions of the generalized velocities v_α and coordinates q_α (if the constraints on the body are holonomic and time-independent, as we shall always assume here, then kinetic energy T is a homogeneous quadratic form in the generalized velocities v_α (see e.g. Goldstein [1980]); F_α is given by the equation

$$F_\alpha = \int_{\partial\mathcal{D}_b} \left(\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial q_\alpha} + \left[\left(\mathbf{x} - \mathbf{r} \right) \times \mathbf{n} \right] \cdot \frac{\partial \boldsymbol{\sigma}}{\partial v_\alpha} \right) p \, dS \qquad (2.3)$$

and represents the α-component of the generalized force exerted on the body by the fluid. In eqn. (2.3), $\mathbf{n}$ is the unit normal to the surface $\partial\mathcal{D}_b$; throughout the paper, for all boundaries the direction of $\mathbf{n}$ is always taken to be outward with respect to the fluid domain $\mathcal{D}_f$.

*R*emark. An instantaneous angular velocity $\boldsymbol{\sigma}$ of the rigid body is defined by the equation

$$\sigma_i \equiv -\tfrac{1}{2} e_{ijk} \frac{\mathrm{d}P_{jl}}{\mathrm{d}t} P_{kl}$$

where e_{ijk} is the alternating tensor; $\left[P_{ik}\right]$ is an orthogonal matrix ($P_{il}P_{kl} = \delta_{ik}$) representing rotation from the axes $Ox_1x_2x_3$ of the coordinate system fixed in the space to the axes $O'x'_1x'_2x'_3$ of the coordinate system fixed in the body (with the origin in its center of mass), so that the position vector $\mathbf{x}$ of a point in the body relative to the space axes and the position vector $\mathbf{x}'$ of the same point measured by the body set of axes are related by the formula: $x_i = r_i + P_{ij}x'_j$. The rotation matrix $\left[P_{ik}\right]$ is a function of the generalized coordinates q_α; angular velocity $\boldsymbol{\sigma}$ can therefore be expressed in the form

$$\sigma_i = -\tfrac{1}{2} e_{ijk} \frac{\mathrm{d}P_{jl}}{\partial q_\alpha} P_{kl} v_\alpha \, . \qquad (2.4)$$

It is equation (2.4) that allows us to write the generalized force F_α in the form (2.3).

Boundary condition (1.3) for velocity field $\mathbf{u}(\mathbf{x}, t)$ is replaced by

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D} \, , \qquad \mathbf{u} \cdot \mathbf{n} = \Big(\mathbf{w} + \boldsymbol{\sigma} \times (\mathbf{x} - \mathbf{r}) \Big) \cdot \mathbf{n} \quad \text{on} \quad \partial\mathcal{D}_b \, . \qquad (2.5)$$

Equations (1.1), (1.2), (2.1)-(2.3) with boundary conditions (2.5) give us the complete set of equations governing the motion of the system 'body + fluid'.

The conserved total energy of the system is given by

$$\begin{aligned} E &= E_f + E_b = const \, , \quad E_b \equiv T + \Pi \, , \\ E_f &\equiv \tfrac{1}{2} \int_{\mathcal{D}_f} \mathbf{u}^2 \mathrm{d}\tau \, , \quad d\tau \equiv dx_1 dx_2 dx_3 \, . \end{aligned} \qquad (2.6)$$

Basic state. Steady solutions of the problem (1.1), (1.2), (2.1)-(2.3), (2.5) given by

$$\begin{aligned} v_\alpha &= 0 \, , \quad q_\alpha = Q_\alpha \, , \quad \mathbf{r} = \mathbf{R} = 0 \, , \quad \mathbf{u} = \mathbf{U}(\mathbf{x}) \, , \quad p = P(\mathbf{x}) \, , \\ \mathbf{w} &= \mathbf{W} = 0 \, , \quad \boldsymbol{\sigma} = \boldsymbol{\Sigma} = 0 \, , \quad P_{ij} = P_{0ij} = \delta_{ij} \end{aligned} \qquad (2.7)$$

satisfy the equations

$$\boldsymbol{\Omega} \times \mathbf{U} = -\nabla H \, , \quad H \equiv P + \tfrac{1}{2}\mathbf{U}^2 \, , \quad \nabla \cdot \mathbf{U} = 0 \quad \text{in} \quad \mathcal{D}_{f0} \, ; \qquad (2.8)$$

$$-\frac{\partial \Pi}{\partial Q_\alpha} + \int_{\partial\mathcal{D}_{b0}} \left(\mathbf{n} \cdot \frac{\partial \mathbf{R}}{\partial Q_\alpha} + \left[\left(\mathbf{x} - \mathbf{R} \right) \times \mathbf{n} \right] \cdot \frac{\partial \boldsymbol{\Sigma}}{\partial V_\alpha} \right) P \, dS = 0 \, ; \qquad (2.9)$$

and boundary conditions

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D} \quad \text{and on} \quad \partial\mathcal{D}_{b0} \, . \qquad (2.10)$$

This solution represents an equilibrium of the body in a steady rotational flow. In eqns. (2.8)-(2.10) boundary $\partial\mathcal{D}_{b0}$ corresponds to the equilibrium position of the rigid body. In obtaining equation (2.9) we used the fact that, according to (2.2), (2.7), $\partial T/\partial Q_\alpha = 0$.

2.2 Variational principle. We shall show that the total energy of the dynamical system 'body + fluid' has a stationary value at the steady solution (2.7) on the set of all possible fluid flows that are isovortical to the basic flow. The isovorticity condition is the same as in Arnold's principle: we admit only such variations of the velocity field $\mathbf{u}$ that preserve the velocity circulation over any material contour. It is however more convenient to reformulate Arnold's isovorticity condition in a form first proposed in Vladimirov [1987b].

Consider a family of transformations

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x}, \epsilon)\,, \quad \tilde{q}_\alpha = \tilde{q}_\alpha(\epsilon)\,. \tag{2.11}$$

depending on a parameter $\epsilon \geq 0$ where the functions $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ and $\tilde{q}_\alpha(\epsilon)$ are twice differentiable with respect to ϵ and the value $\epsilon = 0$ corresponds to the steady solution (2.7):

$$\tilde{\mathbf{x}}(\mathbf{x}, 0) = \mathbf{x}\,, \quad \tilde{q}_\alpha(0) = Q_\alpha\,. \tag{2.12}$$

The transformations defined by eqns. (2.11), (2.12) are similar to those introduced in Section 1 and can be interpreted as a 'virtual motion' of the system 'body + fluid' where ϵ plays the role of a 'virtual time', $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ is the position vector at the moment of 'time' ϵ of a fluid particle whose position at the initial instant $\epsilon = 0$ was $\mathbf{x}$ (in other words, $\mathbf{x}$ ($\mathbf{x} \in \mathcal{D}_{f0}$) serves as a label to identify the fluid particle, while $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ represents its trajectory) and where the functions $\tilde{q}_\alpha(\epsilon)$ determine the position and the orientation of the rigid body at the moment of 'time' ϵ. In such a 'motion', the domain $\mathcal{D}_{f0} = \tilde{\mathcal{D}}_f(0)$ evolves to a new one $\tilde{\mathcal{D}}_f(\epsilon)$ which is completely determined by the position and the orientation of the rigid body, i.e. by the generalized coordinates $\tilde{q}_\alpha(\epsilon)$.

Functions $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$, $\tilde{q}_\alpha(\epsilon)$ are specified through yet another set of functions $\boldsymbol{\xi}(\tilde{\mathbf{x}}, \epsilon)$, $h_\alpha(\epsilon)$ by the equations (cf (1.9))

$$\mathrm{d}\tilde{\mathbf{x}}/\mathrm{d}\epsilon = \boldsymbol{\xi}(\tilde{\mathbf{x}}, \epsilon)\,, \quad \mathrm{d}\tilde{q}_\alpha/\mathrm{d}\epsilon = h_\alpha(\epsilon)\,, \tag{2.13}$$

where $h_\alpha(\epsilon)$ are arbitrary differentiable functions, while $\boldsymbol{\xi}(\tilde{\mathbf{x}}, \epsilon)$ is an arbitrary divergence-free vector field differentiable with respect to ϵ and satisfying the conditions

$$\boldsymbol{\xi} \cdot \mathbf{n} = 0 \quad \text{on} \;\; \partial\tilde{\mathcal{D}}\,, \quad \boldsymbol{\xi} \cdot \mathbf{n} = \left[\tilde{\mathbf{r}}_\epsilon + \tilde{\boldsymbol{\varphi}}_\epsilon \times \left(\tilde{\mathbf{x}} - \tilde{\mathbf{r}}\right)\right] \cdot \mathbf{n} \quad \text{on} \;\; \partial\tilde{\mathcal{D}}_b(\epsilon)\,. \tag{2.14}$$

In (2.14),

$$\tilde{\mathbf{r}}_\epsilon \equiv \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{q}_\alpha} h_\alpha\,, \quad \tilde{\varphi}_{i\epsilon} \equiv -\tfrac{1}{2} e_{ijk} \frac{\partial \tilde{P}_{jl}}{\partial \tilde{q}_\alpha} \tilde{P}_{kl} h_\alpha\,. \tag{2.15}$$

In terms of 'virtual motions' the functions $\boldsymbol{\xi}(\tilde{\mathbf{r}}, \epsilon)$ and $h_\alpha(\epsilon)$ entering equations (2.13) have a natural interpretation as the 'virtual velocities' of the fluid and the rigid body. The conditions (2.14) mean that in the 'virtual motion' there is no fluid flow through the rigid boundaries.

The actual velocity field of the fluid and the actual generalized velocities of the rigid body in the 'virtual motion' are described by twice differentiable (with respect to ϵ) functions $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$ and $\tilde{v}_\alpha(\epsilon)$ such that the value $\epsilon=0$ corresponds to the steady state (2.7):

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)\Big|_{\epsilon=0} = \mathbf{U}(\mathbf{x}), \quad \tilde{v}_\alpha(\epsilon)\Big|_{\epsilon=0} = 0. \tag{2.16}$$

In addition, the field $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$ satisfies the conditions

$$\begin{aligned} \tilde{\nabla}\cdot\tilde{\mathbf{u}} &= 0 \quad \text{in } \tilde{\mathcal{D}}_f, \quad \tilde{\mathbf{u}}\cdot\mathbf{n}=0 \quad \text{on } \partial\tilde{\mathcal{D}}, \\ \tilde{\mathbf{u}}\cdot\mathbf{n} &= \Big(\tilde{\mathbf{w}}+\tilde{\boldsymbol{\sigma}}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{r}})\Big)\cdot\mathbf{n} \quad \text{on } \partial\tilde{\mathcal{D}}_b(\epsilon), \end{aligned} \tag{2.17}$$

where, as before, $\tilde{\mathbf{w}}$, $\tilde{\boldsymbol{\sigma}}$ are considered as functions of $\tilde{v}_\alpha(\epsilon)$ and $\tilde{q}_\alpha(\epsilon)$. The evolution with the 'time' ϵ of the generalized velocities $\tilde{v}_\alpha(\epsilon)$ is prescribed by the equation

$$\mathrm{d}\tilde{v}_\alpha/\mathrm{d}\epsilon = g_\alpha(\epsilon) \tag{2.18}$$

with some differentiable function $g_\alpha(\epsilon)$. Note that the functions $g_\alpha(\epsilon)$ and $h_\alpha(\epsilon)$ which determine the evolution in the 'virtual motion' of the generalized velocities and coordinates are both arbitrary, so that $\tilde{v}_\alpha(\epsilon)$ and $\tilde{q}_\alpha(\epsilon)$ vary independently.

The evolution of the field $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$ is defined through the evolution of vorticity $\tilde{\boldsymbol{\omega}}(\tilde{\mathbf{x}},\epsilon)\equiv\tilde{\nabla}\times\tilde{\mathbf{u}}$ by the equation

$$\tilde{\boldsymbol{\omega}}_\epsilon = [\boldsymbol{\xi},\tilde{\boldsymbol{\omega}}]. \tag{2.19}$$

Equation (2.19) means that the vorticity field $\tilde{\boldsymbol{\omega}}$ is considered as a passive vector advected by the 'virtual flow' rather than as a field related with the 'virtual velocity' $\boldsymbol{\xi}$ by *curl*-operator; in other words, the evolution of $\tilde{\boldsymbol{\omega}}$ is the same as that of a material line element $\delta\mathbf{l}$ or as the evolution of a frozen-in magnetic field in ideal MHD. Yet another meaning of the equation (2.19) is that the circulation of the velocity field $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$ round any closed material curve is conserved in the 'virtual motion', this, in turn, implies that equation (2.19) is equivalent to Arnold's original isovorticity condition (see Section 1).

On integrating equation (2.19) we obtain (cf (1.12))

$$\tilde{\mathbf{u}}_\epsilon = \boldsymbol{\xi}\times\tilde{\boldsymbol{\omega}} - \tilde{\nabla}\alpha \tag{2.20}$$

with a certain function $\alpha(\tilde{\mathbf{x}},\epsilon)$ which can be found from the conditions on $\tilde{\mathbf{u}}_\epsilon$ that follows from (2.17).

Remark. Though equation (2.20) also could be used as a primary condition for defining the evolution of the field $\tilde{\mathbf{u}}(\mathbf{x},\epsilon)$, from a view-point of physical interpretation equation (2.19) seems preferable.

Assuming that ϵ is small we define the first and the second variations of the velocity field of the fluid $\mathbf{u}$ and the generalized velocities and coordinates of the rigid body v_α, q_α as follows

$$\delta\mathbf{x}\equiv\boldsymbol{\xi}|_{\epsilon=0}, \quad \delta\mathbf{u}\equiv\tilde{\mathbf{u}}_\epsilon|_{\epsilon=0}, \quad \delta^2\mathbf{u}\equiv\tfrac{1}{2}\tilde{\mathbf{u}}_{\epsilon\epsilon}|_{\epsilon=0}, \quad \delta v_\alpha\equiv v_{\alpha\epsilon}|_{\epsilon=0} \quad \text{etc.} \tag{2.21}$$

In (2.21), $\delta\mathbf{x}$ is the Lagrangian displacement of the fluid element whose position at time t in the undisturbed flow was $\mathbf{x}$. The first and the second variations of the energy (2.6) considered as a functional of $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$, $\tilde{v}_\alpha(\epsilon)$, $\tilde{q}_\alpha(\epsilon)$ are, by definition,

$$\delta E\equiv \mathrm{d}E/\mathrm{d}\epsilon\Big|_{\epsilon=0}, \quad \delta^2 E\equiv\tfrac{1}{2}\mathrm{d}^2E/\mathrm{d}\epsilon^2\Big|_{\epsilon=0}.$$

The first variation of E is

$$\delta E = \delta E_f + \delta E_b.$$

From (2.2) it follows that

$$\delta E_b = MW_i\delta w_i + \tfrac{1}{2}\delta I_{ik}\Sigma_i\Sigma_k + I_{ik}\Sigma_i\delta\sigma_k + \frac{\partial\Pi}{\partial Q_\alpha}\delta q_\alpha\,, \tag{2.22}$$

where

$$\delta\mathbf{w} = \frac{\partial\mathbf{W}}{\partial Q_\alpha}\delta q_\alpha + \frac{\partial\mathbf{W}}{\partial V_\alpha}\delta v_\alpha\,, \quad \delta\boldsymbol{\sigma} = \frac{\partial\boldsymbol{\Sigma}}{\partial Q_\alpha}\delta q_\alpha + \frac{\partial\boldsymbol{\Sigma}}{\partial V_\alpha}\delta v_\alpha\,, \quad \delta I_{ik} = \frac{\partial I_{ik}}{\partial Q_\alpha}\delta q_\alpha\,.$$

Since in the basic state (2.7) $\mathbf{W} = \boldsymbol{\Sigma} = 0$, we obtain

$$\delta E_b = \frac{\partial\Pi}{\partial Q_\alpha}\delta q_\alpha\,. \tag{2.23}$$

To calculate δE_f we first note that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{\tilde{\mathcal{D}}_f(\epsilon)} F(\tilde{\mathbf{x}},\epsilon)\mathrm{d}\tau = \int_{\tilde{\mathcal{D}}_f(\epsilon)} F_\epsilon d\tau + \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)} F\big(\boldsymbol{\xi}\cdot\mathbf{n}\big)\mathrm{d}S$$

for any function $F(\tilde{\mathbf{x}},\epsilon)$ (see e.g. Batchelor [1967]). With help of this formula we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} E_f = \int_{\mathcal{D}_{f0}}\Big\{\boldsymbol{\xi}\cdot\Big(\boldsymbol{\Omega}\times\mathbf{U}\Big) + \mathbf{U}\cdot\nabla\alpha\Big\}\mathrm{d}\tau + \int_{\partial\tilde{\mathcal{D}}_{b0}} \tfrac{1}{2}\mathbf{U}^2\big(\boldsymbol{\xi}\cdot\mathbf{n}\big)\mathrm{d}S\,.$$

By using (2.8), Green's theorem and the boundary conditions (2.14), this can be transformed to

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} E_f = -\int_{\partial\tilde{\mathcal{D}}_b(0)} P\big[\delta\mathbf{r} + \delta\boldsymbol{\varphi}\times\big(\mathbf{x}-\mathbf{r}\big)\big]\cdot\mathbf{n}\mathrm{d}S\,. \tag{2.24}$$

Finally, from (2.23), (2.24) we have

$$\delta E = \frac{\partial\Pi}{\partial Q_\alpha}\delta q_\alpha - \int_{\partial\mathcal{D}_{b0}} P\Big(\delta\mathbf{r} + \delta\boldsymbol{\varphi}\times\big(\mathbf{x}-\mathbf{r}\big)\Big)\cdot\mathbf{n}\mathrm{d}S\,. \tag{2.25}$$

The comparison of (2.25) with (2.9) then shows that $\delta E = 0$. Thus, we have proved the following.

Proposition 2.1 *The energy of the system 'body + fluid' has a stationary value at any steady solution of the form (2.7) provided that we take account only of 'isovortical' fluid flows.*

This result is a natural generalization of Arnold's variational principle to the dynamical system 'body + fluid'.

2.3 The second variation. The second variation of the energy (2.6) evaluated at the stationary point is given by the expression Vladimirov and Ilin [1997a]

$$\begin{aligned}
\delta^2 E &= \delta^2 E_A + \delta^2 E_c + \delta^2 E_b\,,\\
\delta^2 E_A &\equiv \tfrac{1}{2}\int_{\mathcal{D}_{f0}}\Big\{\big(\delta\mathbf{u}\big)^2 + \mathbf{U}\cdot\big(\delta\mathbf{x}\times\delta\boldsymbol{\omega}\big)\Big\}\mathrm{d}\tau\,,\\
\delta^2 E_c &\equiv \tfrac{1}{2}\int_{\partial\mathcal{D}_{b0}}\Big\{2\mathbf{U}\cdot\delta\mathbf{u} - \delta\mathbf{y}\cdot\nabla P\Big\}\big(\delta\mathbf{y}\cdot\mathbf{n}\big)\mathrm{d}S + \tfrac{1}{2}\int_{\partial\mathcal{D}_{b0}}\big(\delta\mathbf{y}\cdot\mathbf{n}\big)\Big(\delta\mathbf{x}\cdot\nabla H\Big)\mathrm{d}S\\
&\quad -\tfrac{1}{2}\int_{\partial\mathcal{D}_{b0}} P\Big\{\mathbf{n}\cdot\big[\delta\mathbf{r}\times\delta\boldsymbol{\varphi}\big] + A_{\alpha\beta}\delta q_\alpha\delta q_\beta + B_{\alpha\beta}\delta q_\alpha\delta q_\beta\Big\}\mathrm{d}S\,,\\
\delta^2 E_b &\equiv \tfrac{1}{2}M\delta w_i\delta w_i + \tfrac{1}{2}I_{ik}\delta\sigma_i\delta\sigma_k + \tfrac{1}{2}\frac{\partial^2\Pi}{\partial Q_\alpha\partial Q_\beta}\delta q_\alpha\delta q_\beta\,,
\end{aligned} \tag{2.26}$$

where $\delta\mathbf{y} \equiv \delta\mathbf{r} + \delta\boldsymbol{\varphi} \times \mathbf{x}$ is the displacement of a point on the body surface and where

$$\begin{aligned} A_{\alpha\beta} &\equiv \mathbf{n}\cdot\mathbf{R}_{\alpha\beta}\,, \quad B_{\alpha\beta} \equiv \mathbf{n}\cdot\left[\boldsymbol{\Sigma}_{\alpha\beta}\times\mathbf{x}\right], \\ \mathbf{R}_{\alpha\beta} &\equiv \frac{\partial^2\mathbf{R}}{\partial Q_\alpha \partial Q_\beta}\,, \quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \frac{\partial^2\boldsymbol{\Sigma}}{\partial V_\alpha \partial Q_\beta}\,. \end{aligned} \tag{2.27}$$

In (2.26) $\delta^2 E_A$ is precisely Arnold's second variation of the energy of the fluid in the fixed domain $\mathcal{D}_{f0}$; $\delta^2 E_b$ involves only the variations of the generalized coordinates and velocities of the rigid body; $\delta^2 E_c$ depend on the variations of fluid variables and rigid body variables, so it may be interpreted as the part of $\delta^2 E$ appearing due to interaction between the body and the flow.

The remarkable fact about the second variation $\delta^2 E$ is that if we consider the variations $\delta\mathbf{x}$, $\delta\mathbf{u}$ and δq_α as the infinitesimal disturbances, whose evolution is governed by appropriate linearized equations, then $\delta^2 E$ is an invariant of these equations (see Arnold [1966b]). From this fact it immediately follows that the basic state (2.7) is linearly stable provided that $\delta^2 E$ is positive definite. The linear stability problem thus reduces to the analysis of the second variation.

Euler angles. Now consider the situation when no constraints are imposed on the motion of the rigid body. In this case it is natural to take as the generalized coordinates three Cartesian components of the radius-vector of the centre of mass of the body and three Euler angles ϕ, θ, ψ that characterize the orientation of the body in space. In defining the Euler angles we shall use the xyz-convention (as it described in the book by Goldstein [1980]), so that they are specified by an initial rotation about the original z axis through an angle ϕ, a second rotation about the intermediate y axis through an angle θ, and a third rotation about the final x axis through an angle ψ. With this choice the components of the angular velocity $\boldsymbol{\sigma}$ along the space axis are (see Goldstein [1980], p. 610)

$$\sigma_1 = \dot\psi\cos\theta\cos\phi - \dot\theta\sin\phi\,, \quad \sigma_2 = \dot\psi\cos\theta\sin\phi + \dot\theta\cos\phi\,, \quad \sigma_3 = \dot\phi - \dot\psi\sin\theta\,. \tag{2.28}$$

Now $q_\alpha = (\mathbf{r}, \boldsymbol{\phi})$, $v_\alpha = (\dot{\mathbf{r}}, \dot{\boldsymbol{\phi}})$ where we use the notation $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \equiv (\psi, \theta, \phi)$. The expression for the second variation given by eqns. (2.26) remains almost unchanged except that now $\delta\boldsymbol{\varphi} = \delta\boldsymbol{\phi}$, $\delta\mathbf{w} = \delta\dot{\mathbf{r}}$, $\delta\boldsymbol{\sigma} = \delta\dot{\boldsymbol{\phi}} = (\delta\dot\psi, \delta\dot\theta, \delta\dot\phi)$, $A_{\alpha\beta} = 0$ and $B_{\alpha\beta}\delta q_\alpha \delta q_\beta = \tilde B_{ik}\delta\phi_i\delta\phi_k$ where matrix $[\tilde B_{ik}]$ is given by

$$\left[\tilde B_{ik}\right] \equiv \begin{pmatrix} 0 & -\mathbf{e}_z\cdot(\mathbf{x}\times\mathbf{n}) & \mathbf{e}_y\cdot(\mathbf{x}\times\mathbf{n}) \\ -\mathbf{e}_z\cdot(\mathbf{x}\times\mathbf{n}) & 0 & -\mathbf{e}_x\cdot(\mathbf{x}\times\mathbf{n}) \\ \mathbf{e}_y\cdot(\mathbf{x}\times\mathbf{n}) & -\mathbf{e}_x\cdot(\mathbf{x}\times\mathbf{n}) & 0 \end{pmatrix}$$

Moreover, with help of the equilibrium condition (2.9) it can be shown that

$$-\tfrac{1}{2}\int_{\partial\mathcal{D}_{b0}} P\tilde B_{ik}\delta\phi_i\delta\phi_k \mathrm{d}S = \Pi_\psi\delta\theta\delta\phi - \Pi_\theta\delta\psi\delta\phi + \Pi_\phi\delta\psi\delta\theta$$

where $\Pi_{\phi_i} \equiv \partial\Pi/\partial\phi_i$ at $\mathbf{r} = 0$, $\boldsymbol{\phi} = 0$.

$\delta^2 E$ for a spherical body. Consider a particular case of the spherical body of radius a. Evidently, no torque is exerted on the spherical body by an inviscid fluid. We suppose that the potential $\Pi = \Pi(\mathbf{r})$ is independent of the Euler angles (i.e. no external moment of force is applied to the body). Then the Euler angles of the body are cyclic coordinates and can therefore be ignored. This means that in (2.26) all terms with the variations of the Euler angles can be discarded and the second

variation simplifies to

$$\begin{aligned} \delta^2 E &= \delta^2 E_A + \delta^2 E_c + \delta^2 E_b \,, \\ 2\delta^2 E_A &= \int_{\mathcal{D}_{f0}} \Big\{ \big(\delta \mathbf{u}\big)^2 + \mathbf{U} \cdot \big(\delta \mathbf{x} \times \delta \boldsymbol{\omega}\big) \Big\} \mathrm{d}\tau \,, \\ 2\delta^2 E_c &= \int_{\partial \mathcal{D}_{b0}} \Big\{ 2\mathbf{U} \cdot \delta \mathbf{u} - \delta \mathbf{r} \cdot \nabla P \Big\} \big(\delta \mathbf{r} \cdot \mathbf{n}\big) \mathrm{d}S + \int_{\partial \mathcal{D}_{b0}} \big(\delta \mathbf{r} \cdot \mathbf{n}\big) \Big(\delta \mathbf{x} \cdot \nabla H\Big) \mathrm{d}S \,, \\ 2\delta^2 E_b &= M \delta \dot{r}_i \delta \dot{r}_i + \frac{\partial^2 \Pi}{\partial R_i \partial R_k} \delta r_i \delta r_k \,. \end{aligned} \tag{2.29}$$

If, in addition, the basic flow is such that $\mathbf{\Omega} \cdot \mathbf{n} = 0$ on $\partial \mathcal{D}_{b0}$, then it can be shown from eqn. (2.8) that $H = const$ on $\partial \mathcal{D}_{b0}$, and $\delta^2 E_c$ in (2.29) reduces to the equation

$$2\delta^2 E_c = \int_{\partial \mathcal{D}_{b0}} \Big\{ 2\mathbf{U} \cdot \delta \mathbf{u} + \delta \mathbf{r} \cdot \nabla \big(\tfrac{1}{2} \mathbf{U}^2\big) \Big\} \big(\delta \mathbf{r} \cdot \mathbf{n}\big) \mathrm{d}S \,.$$

Rigid body with fluid-filled cavities. All the results described above were obtained for a rigid body placed in an arbitrary rotational inviscid flow. However it is easy to see that these results are equally valid for a rigid body with a cavity containing an ideal fluid. The only difference between these two problems lies in interpreting the boundary $\partial \mathcal{D}_b$, namely, for a body with a fluid-filled cavity we consider the surface $\partial \mathcal{D}_b$ as an internal (for the body) boundary which represents the boundary of the cavity, i.e. $\partial \mathcal{D}_b$ is an outer boundary of the fluid domain $\mathcal{D}_f$ which is completely filled with a fluid. With this interpretation the basic state given by equations (2.7)-(2.9) represents an equilibrium of a rigid body with a cavity containing a fluid which in turn is in a steady motion with velocity field $\mathbf{U}(\mathbf{x})$.

Remark. Evidently, the theory developed in the previous sections can be easily modified to cover the situation when there are n rigid bodies in a fluid or the situation when a cavity in the rigid body contains fluid and other rigid bodies.

For a general three-dimensional basic state (2.7) the second variation given by (2.26) (and by (2.29) for a spherical body) is indefinite in sign. Nevertheless, for some particular situations (such as a body in an irrotational flow, a force-free rotation of a body with fluid-filled cavity and some two-dimensional problems), it is possible to find sufficient conditions for sign-definiteness of $\delta^2 E$ and, hence, to prove the linear stability of corresponding steady states (see Vladimirov and Ilin [1994], Vladimirov and Ilin [1997a]).

3 Flows with contact discontinuities

In this section we shall discuss variational principles for steady flows of an ideal incompressible fluid with contact discontinuities. We shall consider two examples of such flows: steady flows of two-layer fluid and steady flows with discontinuities of vorticity.

3.1 Basic equations. Let $\mathcal{D}$ be a fixed in space three-dimensional domain containing two immiscible homogeneous fluids with (constant) densities ρ^+ and ρ^-, and let $\mathcal{D}^+(t) \subset \mathcal{D}$ and $\mathcal{D}^-(t) \subset \mathcal{D}$ ($\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$) be the domains occupied by each fluid and a smooth surface $S(t)$ be the surface of contact of these two fluids.

The velocity $\mathbf{u}^\pm(\mathbf{x}, t)$ of each fluid and the pressure $p^\pm(\mathbf{x}, t)$ obey the Euler equations

$$\begin{aligned} &\rho^\pm \big(\mathbf{u}^\pm_t + \boldsymbol{\omega}^\pm \times \mathbf{u}^\pm\big) = -\nabla H^\pm \,, \\ &\nabla \cdot \mathbf{u}^\pm = 0 \,, \quad H^\pm = \tfrac{1}{2} \rho^\pm |\mathbf{u}^\pm|^2 + p^\pm + \rho^\pm \Phi \quad \text{in } \ \mathcal{D}^\pm(t) \,, \end{aligned} \tag{3.1}$$

where $\Phi(\mathbf{x})$ is a given potential of an external body force.

Boundary conditions for equations (3.1) are

$$\mathbf{u}^{\pm}\cdot\mathbf{n}=0 \quad \text{on} \quad S^{\pm}=\partial\mathcal{D}^{\pm}\cap\partial\mathcal{D}\,; \tag{3.2}$$

$$\big[\mathbf{u}\cdot\mathbf{n}\big]=0\,, \quad \big[p\big]=0 \quad \text{on} \quad S(t)\,. \tag{3.3}$$

In (3.2), $\mathbf{n}$ is a unit outward normal to the fixed boundary $\partial\mathcal{D}$; in (3.3), $\mathbf{n}$ is a unit normal to the moving boundary $S(t)$, its direction being taken so that $\mathbf{n}$ is an outward normal for the domain $\mathcal{D}^{+}$; square bracket denotes a jump of the corresponding quantity on $S(t)$, e.g. $[p]=p^{+}-p^{-}$ on $S(t)$. Boundary condition (3.2) is the usual one of no normal flow through a fixed boundary and conditions (3.3) are the standard kinematic and dynamic conditions on a moving boundary.

We shall assume that the contact surface can be described by the equation

$$F(\mathbf{x},t)=0\,, \quad |\nabla F(\mathbf{x},t)|\neq 0\,.$$

Then the evolution of this surface is governed by the equation

$$\big(\partial/\partial t+\mathbf{u}^{\pm}\cdot\nabla\big)F=0 \quad \text{at} \quad F(\mathbf{x},t)=0\,. \tag{3.4}$$

Note that the condition $\big[\mathbf{u}\cdot\mathbf{n}\big]=0$ on $S(t)$ is a direct consequence of (3.4).

Steady flows of two-layer fluid. Consider a steady solution of the problem (3.1)-(3.3), given by

$$\mathbf{u}^{\pm}=\mathbf{U}^{\pm}(\mathbf{x})\,, \quad p^{\pm}=P^{\pm}(\mathbf{x})\,, \quad \boldsymbol{\omega}^{\pm}=\boldsymbol{\Omega}^{\pm}(\mathbf{x})\,,$$
$$H^{\pm}=H_0^{\pm}(\mathbf{x})=\tfrac{1}{2}\rho^{\pm}|\mathbf{U}^{\pm}|^2+P^{\pm}+\rho^{\pm}\Phi\,, \quad F(\mathbf{x},t)=F_0(\mathbf{x})\,. \tag{3.5}$$

In the steady flow (3.5),

$$\rho^{\pm}\boldsymbol{\Omega}^{\pm}\times\mathbf{U}^{\pm}=-\nabla H_0^{\pm}\,, \quad \nabla\cdot\mathbf{U}^{\pm}=0 \quad \text{in} \quad \mathcal{D}_0^{\pm}\,; \tag{3.6}$$

$$\mathbf{U}^{\pm}\cdot\mathbf{n}=0 \quad \text{on} \quad S^{\pm}\,, \quad \big[\mathbf{U}\cdot\mathbf{n}\big]=0\,, \quad \big[P\big]=0 \quad \text{on} \quad S_0\,. \tag{3.7}$$

Since in the steady flow (3.5) the contact surface is not moving ($F=F_0(\mathbf{x})$), it follows from (3.4) that

$$\mathbf{U}^{\pm}\cdot\mathbf{n}=0 \quad \text{on} \quad S_0\,. \tag{3.8}$$

3.2 Variational principle. Consider a one-parameter family of transformations of $\mathcal{D}$ defined via corresponding transformations of the domains $\mathcal{D}^{\pm}$

$$\mathbf{x}^{\pm}\mapsto\tilde{\mathbf{x}}^{\pm}=\tilde{\mathbf{x}}^{\pm}(\mathbf{x},\epsilon)\,, \quad \mathcal{D}^{\pm}\mapsto\tilde{\mathcal{D}}^{\pm}\,, \quad S\mapsto\tilde{S}\,, \tag{3.9}$$

such that

$$\tilde{\mathbf{x}}^{\pm}|_{\epsilon=0}=\mathbf{x}\,, \quad \tilde{\mathcal{D}}^{\pm}|_{\epsilon=0}=\mathcal{D}_0^{\pm}\,, \quad \tilde{S}|_{\epsilon=0}=S_0,. \tag{3.10}$$

Functions $\tilde{\mathbf{x}}^{\pm}(\mathbf{x},\epsilon)$ are the solutions of ordinary differential equations

$$\mathrm{d}\tilde{\mathbf{x}}^{\pm}/\mathrm{d}\epsilon=\boldsymbol{\xi}^{\pm}(\tilde{\mathbf{x}}^{\pm},\epsilon) \tag{3.11}$$

with initial data given by (3.10). In equation (3.11), $\boldsymbol{\xi}^{\pm}(\tilde{\mathbf{x}}^{\pm},\epsilon)$ are arbitrary divergence-free vector fields satisfying the following boundary conditions:

$$\boldsymbol{\xi}^{\pm}\cdot\mathbf{n}=0 \quad \text{on} \quad S^{\pm}\,, \quad \big[\boldsymbol{\xi}\cdot\mathbf{n}\big]=0 \qquad \text{on} \quad \tilde{S}\,. \tag{3.12}$$

As before (see Sections 1, 2), such a transformation may be viewed as a virtual motion of a two-layer fluid.

For the considered problem Arnold's isovorticity condition (1.10) remains almost the same. Only one correction is necessary, namely: we consider only such closed curves γ (see (1.10)) that do not intersect the contact surface S, or, in other words, that entirely lie either in $\mathcal{D}^+$ or in $\mathcal{D}^-$. Then, from (1.12), (1.13), we have

$$\delta\mathbf{u}^\pm = \boldsymbol{\xi}^\pm \times \boldsymbol{\Omega}^\pm - \nabla\alpha^\pm \quad \text{or} \quad \delta\boldsymbol{\omega}^\pm = [\boldsymbol{\xi}^\pm, \boldsymbol{\Omega}^\pm]\,, \tag{3.13}$$

$$\delta^2\mathbf{u}^\pm = \boldsymbol{\xi}^\pm \times \delta\boldsymbol{\omega}^\pm + \boldsymbol{\chi}^\pm \times \boldsymbol{\Omega}^\pm - \nabla\beta^\pm\,. \tag{3.14}$$

Scalar functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are determined by the conditions that $\nabla\cdot\delta\mathbf{u}^\pm = \nabla\cdot\delta^2\mathbf{u}^\pm = 0$ in $\mathcal{D}_0^\pm$, $\delta\mathbf{u}^\pm\cdot\mathbf{n} = \delta^2\mathbf{u}^\pm\cdot\mathbf{n} = 0$ on $S^\pm$ and and by the boundary conditions on S_0 that may be obtained by differentiating the condition $\big[\tilde{\mathbf{u}}\cdot\mathbf{n}\big] = 0$ on $\tilde{S}$ with respect to ϵ at $\epsilon = 0$.

Variational principle. Let us show that the first variation of the energy

$$E = \int_{\mathcal{D}^+} \rho^+\Big(\tfrac{1}{2}|\mathbf{u}^+|^2 + \Phi\Big)\mathrm{d}\tau + \int_{\mathcal{D}^-} \rho^-\Big(\tfrac{1}{2}|\mathbf{u}^-|^2 + \Phi\Big)\mathrm{d}\tau \tag{3.15}$$

with respect to variations of the form (3.13), (3.14) vanishes in the steady state (3.5).

We have

$$\delta E = \sum\int_{\mathcal{D}^\pm} \rho^\pm\mathbf{U}^\pm\cdot\delta\mathbf{u}^\pm\mathrm{d}\tau + \int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\big[\tfrac{1}{2}\rho\mathbf{U}^2 + \rho\Phi\big]\mathrm{d}S\,.$$

Here $\sum$ denotes the sum of the corresponding integrals over the domains $\mathcal{D}^\pm$. Substitution of (3.13) in this equation results in

$$\begin{aligned}\delta E &= \sum\int_{\mathcal{D}^\pm} \rho^\pm\mathbf{U}^\pm\cdot\Big(\boldsymbol{\xi}^\pm\times\boldsymbol{\Omega}^\pm - \nabla\alpha\Big)\mathrm{d}\tau + \int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\big[\tfrac{1}{2}\rho\mathbf{U}^2 + \rho\Phi\big]\mathrm{d}S \\ &= -\sum\int_{\mathcal{D}^\pm}\rho^\pm\boldsymbol{\xi}^\pm\cdot\nabla H^\pm\mathrm{d}\tau + \int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\big[\tfrac{1}{2}\rho\mathbf{U}^2 + \rho\Phi\big]\mathrm{d}S \\ &= -\int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\Big(\big[H\big] + \big[\tfrac{1}{2}\rho\mathbf{U}^2 + \rho\Phi\big]\Big)\mathrm{d}S = -\int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\big[P\big] = 0\,.\end{aligned}$$

Thus, the following assertion is valid.

Proposition 3.1 *With respect to variations isovortical to a given steady state (3.5) the energy functional (3.15) has a critical point in this steady state.*

The second variation. It can be shown that the second variation of the energy evaluated in the steady state (3.5) is given by the equation

$$\begin{aligned}\delta^2 E =& \tfrac{1}{2}\sum\int_{\mathcal{D}^\pm}\rho^\pm\Big((\delta\mathbf{u}^\pm)^2 + \delta\boldsymbol{\omega}^\pm\cdot\big(\mathbf{U}^\pm\times\boldsymbol{\xi}\big)\Big)\mathrm{d}\tau \\ &+ \tfrac{1}{2}\int_{S_0}(\boldsymbol{\xi}\cdot\mathbf{n})\Big(2\big[\rho\mathbf{U}\cdot\delta\mathbf{u}\big] + \big[\boldsymbol{\xi}\cdot\nabla\big(\tfrac{1}{2}\rho\mathbf{U}^2 + \rho\Phi\big)\big]\Big)\mathrm{d}S\,.\end{aligned} \tag{3.16}$$

In general, this second variation is indefinite in sign. There are however certain particular situations (including particular classes of variations) for which it is definite in sign. We shall not discuss all of them here. Instead, we shall concentrate our efforts on one important subclass of flows with contact discontinuities - on flows with discontinuous vorticity.

3.3 Flows with vorticity discontinuities. Consider a special subclass of flows with contact discontinuities, namely, flows with continuous density, pressure, and velocity and with contact discontinuities of vorticity. Evolution with time of such flows is governed by equations (3.1) with $\rho^+ = \rho^- = \rho$. Boundary conditions (3.2) on the fixed boundary $\mathcal{D}$ remains the same. The only difference from the general situation considered above is that, in addition to (3.3), we impose one more restriction: tangent to $S(t)$ components of velocity are also continuous, i.e.

$$\big[\mathbf{u}\cdot\boldsymbol{\sigma}_\alpha\big]=0\quad(\alpha=1,2)\quad\text{on}\quad S(t)\,, \tag{3.17}$$

where $\boldsymbol{\sigma}_\alpha$ $(\alpha=1,2)$ are independent unit vectors tangent to $S(t)$.

Steady flows. Consider now a steady solution (3.5) of the problem (3.1)-(3.3), (3.17) that satisfy (3.6)-(3.8) and, in addition, the following conditions

$$\big[\mathbf{U}\cdot\boldsymbol{\sigma}_\alpha\big]=0\quad(\alpha=1,2)\,,\quad\text{on}\quad S_0\,. \tag{3.18}$$

Boundary conditions (3.8) and (3.18) impose a certain restriction on possible discontinuities of vorticity. Note first that, in view of (3.7), (3.8) and (3.18), $[H_0]=0$ on S_0, and therefore $[\boldsymbol{\sigma}_\alpha\cdot\nabla H_0]=0$ on S_0. On taking scalar product of equation (3.6) with $\boldsymbol{\sigma}_\alpha$ and using (3.8), we obtain

$$\big(\mathbf{U}\cdot\boldsymbol{\sigma}_\beta\big)\rho\boldsymbol{\Omega}^\pm\cdot\big(\boldsymbol{\sigma}_\beta\times\boldsymbol{\sigma}_\alpha\big)=-\boldsymbol{\sigma}_\alpha\cdot\nabla H_0^\pm\,.$$

whence, with help of the formula

$$e_{\alpha\beta}\mathbf{n}=\frac{\boldsymbol{\sigma}_\alpha\times\boldsymbol{\sigma}_\beta}{|\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2|}$$

(where $e_{\alpha\beta}$ is a unit alternating tensor), we find that

$$e_{\beta\alpha}\big(\mathbf{U}\cdot\boldsymbol{\sigma}_\beta\big)\rho\big[\boldsymbol{\Omega}\cdot\mathbf{n}\big]/|\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2|=-\big[\boldsymbol{\sigma}_\alpha\cdot\nabla H_0\big]=0\,. \tag{3.19}$$

Therefore, in the steady flow (3.5) the vorticity field can have only tangent discontinuity on S_0:

$$\big[\boldsymbol{\Omega}\cdot\mathbf{n}\big]=0\,,\quad\big[\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}_\alpha\big]\neq0\quad(\alpha=1,2)\quad\text{on}\quad S_0\,. \tag{3.20}$$

Similarly, it can be shown that another consequence of (3.7), (3.8) and (3.18) is

$$\big[\mathbf{n}\cdot\nabla P\big]=0\quad\text{on}\quad S_0\,. \tag{3.21}$$

One more formula useful formula

$$\big[\mathbf{n}\cdot\nabla H_0\big]=-\rho|\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2|e_{\alpha\beta}\big[\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}_\alpha\big](\mathbf{U}\cdot\boldsymbol{\sigma}_\beta) \tag{3.22}$$

is obtained by taking scalar product of equation (3.6) with $\mathbf{n}$.

The second variation of the energy. Variational principle of previous subsection still holds for steady flows with vorticity discontinuities. But now we do not need to consider discontinuous fields $\boldsymbol{\xi}(\tilde{\mathbf{x}},\epsilon)$ and $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$, so that we assume that they are continuous

$$\big[\boldsymbol{\xi}\cdot\mathbf{n}\big]=\big[\boldsymbol{\xi}\cdot\boldsymbol{\sigma}_\alpha\big]=\big[\tilde{\mathbf{u}}\cdot\mathbf{n}\big]=\big[\tilde{\mathbf{u}}\cdot\boldsymbol{\sigma}_\alpha\big]=0\quad\text{on}\quad\tilde{S}\,,$$

and, hence,

$$\big[\boldsymbol{\xi}\cdot\mathbf{n}\big]\big|_{\epsilon=0}=\big[\boldsymbol{\xi}\cdot\boldsymbol{\sigma}_\alpha\big]\big|_{\epsilon=0}=\big[\delta\mathbf{u}\cdot\mathbf{n}\big]=\big[\delta\mathbf{u}\cdot\boldsymbol{\sigma}_\alpha\big]=0\quad\text{on}\quad S_0\,. \tag{3.23}$$

In view of (3.18), (3.20)-(3.23), the second variation (3.16) simplifies to

$$\begin{aligned}\delta^2E=&\tfrac12\sum\int_{\mathcal{D}^\pm}\rho\Big((\delta\mathbf{u})^2+\delta\boldsymbol{\omega}^\pm\cdot\big(\mathbf{U}\times\boldsymbol{\xi}\big)\Big)\mathrm{d}\tau\\&-\tfrac12\int_{S_0}\rho(\boldsymbol{\xi}\cdot\mathbf{n})^2|\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2|e_{\alpha\beta}\big[\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}_\alpha\big](\mathbf{U}\cdot\boldsymbol{\sigma}_\beta)\mathrm{d}S\,.\end{aligned} \tag{3.24}$$

If there is no discontinuity of vorticity then, evidently, (3.24) reduces to Arnold's second variation (1.14). The second variation (3.24) is, in general, indefinite in sign because of the volume integrals in (3.26). As in Arnold's case, if both the basic flow and the perturbation have a symmetry then there are situations when δ^2E is definite in sign.

Two-dimensional problem. Let both the basic steady flow and the variations be two-dimensional, i.e. the fields $\mathbf{U}$, $\boldsymbol{\xi}$, have only two non-zero components and depend only on two coordinates on the plane of motion, then

$$\mathbf{U} = (U_1(x,y), U_2(x,y), 0)\,, \quad \boldsymbol{\Omega}^{\pm} = \Omega_0^{\pm}\mathbf{e}_z\,, \quad F_0 = F_0(x,y)\,,$$
$$\boldsymbol{\sigma}_1 = \mathbf{e}_z\,, \quad \boldsymbol{\sigma}_2 = \mathbf{n}\times\mathbf{e}_z \quad \mathbf{n} = \nabla F_0/|\nabla F_0| \ \text{ at } F_0 = 0\,. \tag{3.25}$$

Let Ψ be stream function for $\mathbf{U}$ such that $U_1 = \Psi_x$, $U_2 = -\Psi_y$. Then, the vorticity $\Omega_0^{\pm} = -\nabla^2\Psi$ and stream function Ψ are functionally dependent $\Omega_0^{\pm} = \Omega^{\pm}(\Psi)$ and the second variation (3.24) takes the form

$$\delta^2 E = \tfrac{1}{2}\sum\int_{\mathcal{D}^{\pm}} \rho\Big((\delta\mathbf{u})^2 - \frac{\mathrm{d}\Omega_0^{\pm}}{\mathrm{d}\Psi}(\boldsymbol{\xi}\cdot\nabla\Psi)^2\Big)\mathrm{d}\tau$$
$$- \tfrac{1}{2}\big[\Omega_0\big]\int_{S_0} \rho(\boldsymbol{\xi}\cdot\mathbf{n})^2|\mathbf{U}|\mathrm{d}S\,. \tag{3.26}$$

It is clear that $\delta^2 E$ is positive definite provided that

$$\mathrm{d}\Omega_0^{\pm}/\mathrm{d}\Psi < 0 \quad \text{in} \ \ \mathcal{D}^{\pm}\,, \quad \big[\Omega_0\big] < 0 \quad \text{on} \ \ S_0\,. \tag{3.27}$$

Thus, we can formulate the following.

Proposition 3.2 *A two dimensional steady flow with discontinuity of vorticity along a contact line S_0 is linearly stable to two-dimensional isovortical perturbations provided that the conditions (3.27) are satisfied.*

In a particular case of a flow with piecewise constant vorticity ($\Omega_0^{+} = const$ in $\mathcal{D}^{+}$, $\Omega_0^{-} = const$ in $\mathcal{D}^{-}$), these sufficient conditions for stability reduce to only one condition on the sign of the vorticity jump across S_0: $\big[\Omega_0\big] < 0$.

More examples of stable two-dimensional flows with discontinuous vorticity, can be found in Vladimirov [1988].

4 Ideal magnetohydrodynamics

Here we discuss a variational principle for a steady three-dimensional magnetohydrodynamic flow of an ideal incompressible fluid which is a generalization of Arnold' principle for a steady three-dimensional inviscid flow. We formulate a certain 'generalized isovorticity condition' and then show that on the set of all possible velocity fields and magnetic fields satisfying this condition the energy has a critical point in a steady solution of the governing equations. The second variation of the energy is calculated. The 'modified vorticity field' introduced by Vladimirov and Moffatt [1995] and its connection with present analysis is also discussed.

4.1 Basic equations. Consider an incompressible, inviscid and perfectly conducting fluid contained in a domain $\mathcal{D}$ with fixed boundary $\partial\mathcal{D}$. Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $\mathbf{h}(\mathbf{x},t)$ the magnetic field (in Alfven velocity units), $p(\mathbf{x},t)$ the pressure (divided by density), and $\mathbf{j} = \nabla\times\mathbf{h}$ the current density. Then the governing equations are

$$D\mathbf{u} \equiv \Big(\partial/\partial t + \mathbf{u}\cdot\nabla\Big)\mathbf{u} = -\nabla p + \mathbf{j}\times\mathbf{h}\,, \tag{4.1}$$
$$\mathbf{h}_t = [\mathbf{u},\mathbf{h}] \equiv \nabla\times(\mathbf{u}\times\mathbf{h})\,, \tag{4.2}$$
$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0\,. \tag{4.3}$$

Equation (4.2) implies that $\mathbf{h}$ is frozen in the fluid, its flux through any material surface is conserved. We suppose that the boundary $\partial\mathcal{D}$ is perfectly conducting and therefore the magnetic field $\mathbf{h}$ does not penetrate through $\partial\mathcal{D}$. The boundary conditions are then

$$\mathbf{n}\cdot\mathbf{u}=0\,,\quad \mathbf{n}\cdot\mathbf{h}=0\quad \text{on}\ \ \partial\mathcal{D}\,. \tag{4.4}$$

We suppose further that at $t=0$, the fields $\mathbf{u}$ and $\mathbf{h}$ are smooth and satisfy (4.3) and (4.4), but are otherwise arbitrary.

The equations (4.1)-(4.3) with boundary conditions (4.4) have three quadratic integral invariants: the energy

$$E=\tfrac{1}{2}\int_{\mathcal{D}}\Big(\mathbf{u}^2+\mathbf{h}^2\Big)\mathrm{d}\tau\,, \tag{4.5}$$

the magnetic helicity

$$\mathcal{H}_M=\int_{\mathcal{D}}(\mathbf{h}\cdot curl^{-1}\mathbf{h})\mathrm{d}\tau\,, \tag{4.6}$$

and the cross-helicity

$$\mathcal{H}_C=\int_{\mathcal{D}}(\mathbf{u}\cdot\mathbf{h})\mathrm{d}\tau \tag{4.7}$$

(Woltjer 1958). By arguments of Moffatt [1969], the helicities $\mathcal{H}_M$ and $\mathcal{H}_C$ are both topological in character.

Taking *curl* of equation (4.1) we obtain

$$\boldsymbol{\omega}_t=[\mathbf{u},\boldsymbol{\omega}]+[\mathbf{j},\mathbf{h}]\,, \tag{4.8}$$

where $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ is the vorticity field. Equation (4.8) implies that vortex lines are *n*ot frozen in the fluid unless the Lorentz force $\mathbf{j}\times\mathbf{h}$ is irrotational. However, the flux of vorticity through any material surface bounded by a closed magnetic line (which, according to (4.2), is also a material line) is conserved. This fact has a consequence that the circulation of the velocity round any closed $\mathbf{h}$-line is conserved:

$$\Gamma_h=\oint_{\gamma_h(t)}\mathbf{u}\cdot\mathrm{d}\mathbf{l}=const\,. \tag{4.9}$$

In (4.9), $\gamma_h(t)$ is a closed $\mathbf{h}$-line. The invariants Γ_h will play the key role in the subsequent analysis.

Steady MHD flows. We now consider a steady solution of (4.1)-(4.4)

$$\mathbf{u}=\mathbf{U}(\mathbf{x})\,,\quad \mathbf{h}=\mathbf{H}(\mathbf{x})\,,\quad p=P(\mathbf{x})\,, \tag{4.10}$$

and the associated fields

$$\boldsymbol{\omega}=\boldsymbol{\Omega}(\mathbf{x})\equiv\nabla\times\mathbf{U}\,,\quad \mathbf{j}=\mathbf{J}(\mathbf{x})\equiv\nabla\times\mathbf{H}\,. \tag{4.11}$$

From (4.1), (4.2), we have

$$\boldsymbol{\Omega}\times\mathbf{U}-\mathbf{J}\times\mathbf{H}=-\nabla K\,,\quad \mathbf{U}\times\mathbf{H}=-\nabla I\,, \tag{4.12}$$

where $K\equiv P+\frac{1}{2}\mathbf{U}^2$ and I is an arbitrary scalar function. Note that, according to (4.4),(4.12), the function I is constant on the boundary $\partial\mathcal{D}$ provided that $\mathbf{U}$ is not parallel to $\mathbf{H}$ on $\mathcal{D}$.

4.2 Variational principle. We shall establish a variational principle for a steady MHD flow which is similar to Arnold's variational principle for a steady three-dimensional flow of an ideal incompressible fluid (see Section 1). First we shall define a set of MHD flows that are subject to a certain 'generalized isovorticity condition'. And then we shall show that the energy (4.5) restricted on such a set has a stationary value in the steady solution (4.10).

As in formulation of Arnold's principle (see Section 1), we introduce the family of volume-preserving transformations $g^\epsilon : \mathbf{x} \mapsto \tilde{\mathbf{x}}$ of the domain $\mathcal{D}$ to itself which depend on a parameter ϵ and are defined by the solutions $\tilde{\mathbf{x}}(\mathbf{x}, \epsilon)$ of the equations (1.9) with the same initial data $\tilde{\mathbf{x}}|_{\epsilon=0} = \mathbf{x}$.

Generalized isovorticity condition. Let $(\mathbf{u}_1,\ \mathbf{h}_1)$ and $(\mathbf{u}_2,\ \mathbf{h}_2)$ be two pairs of velocity fields and magnetic fields. We say that these pairs of the fields are *isovortical* in generalized sense if there is a transformation g^ϵ of the domain $\mathcal{D}$ which sends every closed contour γ to a new one $g^\epsilon\gamma$ in such a way that

1. the flux of the magnetic field $\mathbf{h}_2$ through the new contour is the same as the flux of the field $\mathbf{h}_1$ through the original one:
$$\int_S \mathbf{h}_1 \cdot \mathrm{d}\mathbf{S} = \int_{g^\epsilon S} \mathbf{h}_2 \cdot \mathrm{d}\mathbf{S}\,, \tag{4.13}$$
where S is any surface bounded by the curve γ and $g^\epsilon S$ is its image under the transformation g^ϵ;
2. the circulation of the velocity $\mathbf{u}_1$ round the original closed $\mathbf{h}$-line γ_h is equal to the circulation of $\mathbf{u}_2$ round its image $g^\epsilon\gamma_h$ under the transformation g^ϵ:
$$\oint_{\gamma_h} \mathbf{u}_1 \cdot \mathrm{d}\mathbf{l} = \oint_{g^\epsilon\gamma_h} \mathbf{u}_2 \cdot \mathrm{d}\mathbf{l}\,. \tag{4.14}$$

To find the general form of infinitesimal variations of the fields $\mathbf{u}$ and $\mathbf{h}$ that satisfy the 'generalized isovorticity condition' (expressed by (4.13), (4.14)) we introduce another family of transformations $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)$, $\tilde{\mathbf{h}}(\tilde{\mathbf{x}}, \epsilon)$ such that the value $\epsilon = 0$ corresponds to the steady solution (4.10):
$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)\Big|_{\epsilon=0} = \mathbf{U}(\mathbf{x})\,, \quad \tilde{\mathbf{h}}(\tilde{\mathbf{x}}, \epsilon)\Big|_{\epsilon=0} = \mathbf{H}(\mathbf{x})\,.$$
The fields $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \epsilon)$, $\tilde{\mathbf{h}}(\tilde{\mathbf{x}}, \epsilon)$ satisfy the following conditions:
$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0\,, \quad \tilde{\nabla} \cdot \tilde{\mathbf{h}} = 0 \ \text{ in } \mathcal{D}\,, \quad \tilde{\mathbf{u}} \cdot \mathbf{n} = 0\,, \quad \tilde{\mathbf{h}} \cdot \mathbf{n} = 0 \ \text{ on } \partial\mathcal{D}\,.$$
The first and the second variations of the fields $\mathbf{u}$ and $\mathbf{h}$ are given by
$$\delta\mathbf{u} \equiv \tilde{\mathbf{u}}_\epsilon\Big|_{\epsilon=0}\,, \quad \delta^2\mathbf{u} \equiv \tfrac{1}{2}\tilde{\mathbf{u}}_{\epsilon\epsilon}\Big|_{\epsilon=0}\,, \quad \delta\mathbf{h} \equiv \mathbf{h}_\epsilon\Big|_{\epsilon=0}\,, \quad \delta^2\mathbf{h} \equiv \tfrac{1}{2}\mathbf{h}_{\epsilon\epsilon}\Big|_{\epsilon=0}\,.$$
For small ϵ the generalized isovorticity conditions (4.13), (4.14) reduce to (cf (1.11))
$$\epsilon\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\int_{g^\epsilon S}\tilde{\mathbf{h}} \cdot \mathrm{d}\mathbf{S} + \tfrac{1}{2}\epsilon^2\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Big|_{\epsilon=0}\int_{g^\epsilon S}\tilde{\mathbf{h}} \cdot \mathrm{d}\mathbf{S} + o(\epsilon^2) = 0\,, \tag{4.15}$$
$$\epsilon\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\int_{g^\epsilon\gamma_h}\tilde{\mathbf{u}} \cdot \mathrm{d}\mathbf{l} + \tfrac{1}{2}\epsilon^2\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Big|_{\epsilon=0}\int_{g^\epsilon\gamma_h}\tilde{\mathbf{u}} \cdot \mathrm{d}\mathbf{l} + o(\epsilon^2) = 0\,. \tag{4.16}$$
From (4.15), using the formula (see e.g. Batchelor [1967])
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{g^\epsilon S}\tilde{\mathbf{h}} \cdot \mathrm{d}\mathbf{S} = \int_{g^\epsilon S}\Big(\tilde{\mathbf{h}}_\epsilon + (\boldsymbol{\xi} \cdot \nabla)\tilde{\mathbf{h}} - (\tilde{\mathbf{h}} \cdot \nabla)\boldsymbol{\xi}\Big) \cdot \mathrm{d}\mathbf{S}\,,$$

we obtain

$$\int_S \Big\{\epsilon\big(\delta\mathbf{h}-[\boldsymbol{\xi},\mathbf{H}]\big) \quad + \quad \tfrac{1}{2}\epsilon^2\Big(\delta\mathbf{h}-[\boldsymbol{\chi},\mathbf{H}] \\ - \quad [\boldsymbol{\xi},\delta\mathbf{h}]-\big[\boldsymbol{\xi},\delta\mathbf{h}-[\boldsymbol{\xi},\mathbf{H}]\big]\Big)\Big\}\cdot \mathrm{d}\mathbf{S}+o(\epsilon^2)=0\,.$$

Whence, using the fact that S is an arbitrary material surface, we deduce that

$$\delta\mathbf{h}=[\boldsymbol{\xi},\mathbf{H}]\,,\quad \delta^2\mathbf{h}=[\boldsymbol{\xi},\delta\mathbf{h}]+[\boldsymbol{\chi},\mathbf{H}]\,,\quad \boldsymbol{\chi}\equiv\boldsymbol{\xi}_\epsilon\Big|_{\epsilon=0}\,. \tag{4.17}$$

Note that the variations $\delta\mathbf{h}$, $\delta^2\mathbf{h}$ satisfy the conditions

$$\nabla\cdot\delta\mathbf{h}=\nabla\cdot\delta^2\mathbf{h}=0 \ \text{ in } \mathcal{D}\,,\quad \delta\mathbf{h}\cdot\mathbf{n}=\delta^2\mathbf{h}\cdot\mathbf{n}=0 \ \text{ on } \partial\mathcal{D}\,.$$

From (4.16), we obtain

$$\oint_{\gamma_h}\Big\{\epsilon\Big(\delta\mathbf{u}-\boldsymbol{\xi}\times\boldsymbol{\Omega}\Big)+\tfrac{1}{2}\epsilon^2\Big(\delta^2\mathbf{u}-\boldsymbol{\chi}\times\boldsymbol{\Omega} \\ -\boldsymbol{\xi}\times\delta\boldsymbol{\omega}-\boldsymbol{\xi}\times\big(\delta\boldsymbol{\omega}-[\boldsymbol{\xi},\boldsymbol{\Omega}]\big)\Big)\Big\}\cdot\mathrm{d}\mathbf{l}+o(\epsilon^2)=0\,. \tag{4.18}$$

Since γ_h is an arbitrary closed **h**-line we conclude that

$$\delta\mathbf{u}=\boldsymbol{\xi}\times\boldsymbol{\Omega}+\boldsymbol{\eta}\times\mathbf{H}-\nabla\alpha \quad \text{or, equivalently,}\quad \delta\boldsymbol{\omega}=[\boldsymbol{\xi},\boldsymbol{\Omega}]+[\boldsymbol{\eta},\mathbf{H}]\,, \tag{4.19}$$

where $\boldsymbol{\eta}$ is an arbitrary divergence-free, tangent to the boundary vector field that appears in (4.19) due to the fact that γ_h is a closed **h**-line, not an arbitrary material line [2]; and where the scalar function α is uniquely (in the case of singly-connected domain) determined by the fact that $\delta\mathbf{u}$ is a divergence-free and tangent to the boundary field.

Derivation of the formula for the second variation of **u** is somewhat more tricky. From (4.18), we obtain

$$2\delta^2\mathbf{u}=\boldsymbol{\chi}\times\boldsymbol{\Omega}+\boldsymbol{\xi}\times\delta\boldsymbol{\omega}+\boldsymbol{\xi}\times\big(\delta\boldsymbol{\omega}-[\boldsymbol{\xi},\boldsymbol{\Omega}]\big)+\mathbf{f}\times\mathbf{H}-\nabla\beta\,, \tag{4.20}$$

where **f** is an arbitrary divergence-free, tangent to the boundary vector field.

Further, we have

$$\begin{aligned}
\mathcal{J}\equiv\oint_{\gamma_h}\Big(\boldsymbol{\xi}\times\big(\delta\boldsymbol{\omega}-[\boldsymbol{\xi},\boldsymbol{\Omega}]\big)\Big)\cdot\mathrm{d}\mathbf{l} &= \oint_{\gamma_h}\Big(\boldsymbol{\xi}\times[\boldsymbol{\eta},\mathbf{H}]\Big)\cdot\mathrm{d}\mathbf{l} && \text{by (4.19)}\\
=\int_{S_h}\Big(\nabla\times\big(\boldsymbol{\xi}\times[\boldsymbol{\eta},\mathbf{H}]\big)\Big)\cdot\mathrm{d}\mathbf{S} &= \int_{S_h}\big[\boldsymbol{\xi},[\boldsymbol{\eta},\mathbf{H}]\big]\cdot\mathrm{d}\mathbf{S} && \text{by Stokes' formula}\\
=-\int_{S_h}\Big(\big[\mathbf{H},[\boldsymbol{\xi},\boldsymbol{\eta}]\big]+\big[\boldsymbol{\eta},[\mathbf{H},\boldsymbol{\xi}]\big]\cdot\mathrm{d}\mathbf{S} & && \text{by Jacobi identity}\\
=\int_{\gamma_h}\Big(\boldsymbol{\eta}\times\delta\mathbf{h}+[\boldsymbol{\xi},\boldsymbol{\eta}]\times\mathbf{H}\Big)\cdot\mathrm{d}\mathbf{l} & && \text{by Stokes' formula}
\end{aligned} \tag{4.21}$$

Noting that the last term in (4.21) vanishes since **H** is parallel to d**l** on γ_h (or, in other words, it can be absorbed in term $\mathbf{f}\times\mathbf{H}$ entering (4.20)), we find that

$$2\delta^2\mathbf{u}=\boldsymbol{\chi}\times\boldsymbol{\Omega}+\boldsymbol{\xi}\times\delta\boldsymbol{\omega}+\mathbf{f}\times\mathbf{H}+\boldsymbol{\eta}\times\delta\mathbf{h}-\nabla\beta\,, \tag{4.22}$$

---

[2] To satisfy the condition (4.18) it is not necessary for $\boldsymbol{\eta}$ to be a divergence-free field, so that this property is our assumption. We shall use it below while calculating the first variation of the energy functional.

Variational principle. Now we shall show that the first variation of the energy (4.5) vanishes with respect to variations of the fields **h**, **u** of the form (4.17), (4.19). We have

$$\begin{aligned}\delta E \equiv \frac{\mathrm{d}E}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} &= \int_{\mathcal{D}}\Big(\mathbf{U}\cdot\delta\mathbf{u}+\mathbf{H}\cdot\delta\mathbf{h}\Big)\mathrm{d}\tau \\ &= \int_{\mathcal{D}}\Big(\mathbf{U}\cdot(\boldsymbol{\xi}\times\boldsymbol{\Omega}+\boldsymbol{\eta}\times\mathbf{H}-\nabla\alpha)+\mathbf{H}\cdot(\nabla\times(\boldsymbol{\xi}\times\mathbf{H}))\Big)\mathrm{d}\tau \\ &= \int_{\mathcal{D}}\Big(\boldsymbol{\xi}\cdot(\boldsymbol{\Omega}\times\mathbf{H}-\mathbf{J}\times\mathbf{H})-\boldsymbol{\eta}\cdot(\mathbf{U}\times\mathbf{H})\Big)\mathrm{d}\tau \\ &= \int_{\mathcal{D}}\Big(-\boldsymbol{\xi}\cdot\nabla K+\boldsymbol{\eta}\cdot\nabla I\Big)\mathrm{d}\tau = 0\,. \end{aligned} \tag{4.23}$$

We have thus proved the following.

Proposition 4.1 *On the set of all possible fields* **h** *and* **u** *satisfying the generalized isovorticity conditions (4.13), (4.14) the energy functional (4.5) has a stationary value in the steady state (4.10).*

4.3 The second variation. . Let us now calculate the second variation of the energy at the stationary point. We have

$$\delta^2 E \equiv \frac{1}{2}\frac{\mathrm{d}^2E}{\mathrm{d}\epsilon^2}\bigg|_{\epsilon=0} = \int_{\mathcal{D}}\Big(\tfrac{1}{2}(\delta\mathbf{u})^2+\tfrac{1}{2}(\delta\mathbf{h})^2+\mathbf{U}\cdot\delta^2\mathbf{u}+\mathbf{H}\cdot\delta^2\mathbf{h}\Big)\mathrm{d}\tau\,.$$

After substitution of the equations (4.17), (4.22) and integration by parts, it may be shown that all the terms containing $\boldsymbol{\chi}$ and **f** vanish due to the equations (4.12) and the boundary conditions on $\partial\mathcal{D}$ for the fields $\boldsymbol{\chi}$, **U** and **H** and the second variation takes the form

$$\delta^2 E = \tfrac{1}{2}\int_{\mathcal{D}}\Big((\delta\mathbf{u})^2+(\delta\mathbf{h})^2+\delta\boldsymbol{\omega}\cdot(\mathbf{U}\times\boldsymbol{\xi})+\delta\mathbf{h}\cdot(\mathbf{U}\times\boldsymbol{\eta}+\mathbf{J}\times\boldsymbol{\xi})\Big)\mathrm{d}\tau\,. \tag{4.24}$$

Suppose now that $\delta\mathbf{u}$ and $\delta\mathbf{h}$ are identified with infinitesimal perturbations to the basic steady state (4.10) whose evolution is governed by the appropriate linearized equations. Then the following statement holds.

Proposition 4.2 *The second variation (4.24) is an integral invariant of the corresponding linearized problem.*

This proposition is an MHD counterpart of the corresponding result by Arnold [1966]. It follows from the general geometric theory of Khesin and Chekanov [1989]. For the direct proof which is nothing but the calculation of the time derivative of $\delta^2 E$ on a solution of the linearized problem we refer to Vladimirov and Ilin [1997b], Vladimirov, Moffatt and Ilin [1998].

It follows from this proposition that the steady state (4.10) is linearly stable provided that the second variation (4.24) is definite in sign. In contrast with ideal hydrodynamics, there are steady three-dimensional MHD flows for which $\delta^2 E$ is positive definite, and which, therefore, are linearly stable to small three-dimensional perturbations satisfying the generalized isovorticity condition. Examples of stable MHD flows may be found in Friedlander and Vishik [1995], Vladimirov, Moffatt and Ilin [1998].

4.4 Another form of variational principle for steady MHD flows. The theory developed above heavily uses the fact that the circulation of velocity round any closed **h**-line is conserved. It is known, however, that the situation when magnetic lines are all closed is very particular, usually even in steady MHD flows almost all magnetic lines are not closed. It is necessary therefore to modify our theory so as to cover such situations.

The field $\boldsymbol{\eta}$ turns out to be closely related with a certain generalization of the vorticity for MHD flows, namely with the 'modified vorticity field' **w** introduced by Vladimirov and Moffatt [1995]. We therefore start with a new approach (different from that of Vladimirov and Moffatt [1995]) to introducing the field **w**.

Modified vorticity field. The variational principle of section 4.2 was based on the fact that the circulation Γ_h of velocity round any closed **h**-line is conserved. It is easy to see that Γ_h is invariant with respect to transformations of the form

$$\mathbf{u} \to \mathbf{v} = \mathbf{u} + \mathbf{h} \times \mathbf{m} + \nabla c \tag{4.25}$$

where **m** is an arbitrary divergence-free, tangent to the boundary vector field and c is an arbitrary single-valued function. In what follows c will not play any role, so that we simply take $c = 0$.

It is natural to ask a question whether it is possible to find a field **m** such that the circulation of the 'modified velocity field' **v** ($\mathbf{v} = \mathbf{u} + \mathbf{h} \times \mathbf{m}$) round any material contour (not only round those ones which coincide with closed **h**-lines) is conserved. The answer to this question is affirmative. To show this, we define a 'modified vorticity field' **w**:

$$\mathbf{w} \equiv \nabla \times \mathbf{v} = \boldsymbol{\omega} + [\mathbf{h}, \mathbf{m}] \,. \tag{4.26}$$

The conservation of the circulation of **v** round any material contour is equivalent to the following equation for **w**:

$$\mathbf{w}_t = [\mathbf{u}, \mathbf{w}] \,. \tag{4.27}$$

According to (4.26), this implies that

$$\boldsymbol{\omega}_t + [\mathbf{h}_t, \mathbf{m}] + [\mathbf{h}, \mathbf{m}_t] = [\mathbf{u}, \boldsymbol{\omega}] + [\mathbf{u}, [\mathbf{h}, \mathbf{m}]] \,. \tag{4.28}$$

On substituting $\boldsymbol{\omega}_t$ from equation (4.8) and using the Jacobi identity we obtain

$$[\mathbf{j}, \mathbf{h}] + [\mathbf{h}_t, \mathbf{m}] + [\mathbf{h}, \mathbf{m}_t] = [\mathbf{h}, [\mathbf{u}, \mathbf{m}]] - [\mathbf{m}, [\mathbf{u}, \mathbf{h}]] \,,$$

whence, in view of (4.2),

$$[\mathbf{h}, \mathbf{m}_t] = [\mathbf{h}, \mathbf{j} + [\mathbf{u}, \mathbf{m}]] \,.$$

This means that up to an arbitrary field commuting with **h** the field **m** satisfies the equation

$$\mathbf{m}_t = [\mathbf{u}, \mathbf{m}] + \mathbf{j} \,, \tag{4.29}$$

which is exactly the same as that of Vladimirov and Moffatt [1995]. Thus, in our approach **m** appeared as a generator of transformations that leave the circulations Γ_h unchanged, while the equation (4.29) is a consequence of the requirement that the circulation of the 'modified velocity' **v** round any material contour is conserved.

Another form of the generalized isovorticity condition. Let, as in Section 4.2 ($\mathbf{u}_1$, $\mathbf{h}_1$) and ($\mathbf{u}_2$, $\mathbf{h}_2$) be two pairs of velocity field and magnetic field, and let $\mathbf{m}_1$, $\mathbf{m}_2$ be (associated with these pairs) fields satisfying (4.29). We say that the triplets of the fields ($\mathbf{u}_1$, $\mathbf{h}_1$, $\mathbf{m}_1$) and ($\mathbf{u}_2$, $\mathbf{h}_2$, $\mathbf{m}_2$) are *i*sovortical in generalized sense if there is a transformation g^ϵ of the domain $\mathcal{D}$ which sends every closed contour γ to a new one $g^\epsilon\gamma$ in such a way that

1. the flux of the magnetic field $\mathbf{h}_2$ through the new contour is the same as the flux of the field $\mathbf{h}_1$ through the original one, i.e. (4.13) holds;
2. the circulation of the modified velocity $\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{h}_1 \times \mathbf{m}_1$ round any closed material line γ is equal to the circulation of $\mathbf{v}_2 = \mathbf{u}_2 + \mathbf{h}_2 \times \mathbf{m}_2$ round its image $g^\epsilon\gamma$ under the transformation g^ϵ:

$$\oint_\gamma \mathbf{v}_1 \cdot \mathrm{d}\mathbf{l} = \oint_{g^\epsilon\gamma} \mathbf{v}_2 \cdot \mathrm{d}\mathbf{l}\,. \tag{4.30}$$

Now the generalized isovorticity condition given by (4.14) can be formulated precisely in the same way as it was done by Arnold [1965b]. This results in the replacement of (4.19), (4.22) by the equations (cf (1.12), (1.13))

$$\begin{aligned} &\delta\mathbf{v} = \boldsymbol{\xi} \times \mathbf{W} - \nabla\alpha \quad \text{or, equivalently,} \quad \delta\mathbf{w} = [\boldsymbol{\xi}, \mathbf{W}]\,, \\ &\delta^2\mathbf{v} = \boldsymbol{\xi} \times \delta\mathbf{w} + \boldsymbol{\chi} \times \mathbf{W} - \nabla\beta\,, \end{aligned} \tag{4.31}$$

where $\mathbf{W} = \boldsymbol{\Omega} + [\mathbf{H}, \mathbf{M}]$ is the 'modified vorticity field' in the basic state (4.10); $\mathbf{M}$ is a time-independent solution of (4.29) corresponding to the basic state. Note that in the basic state

$$\mathbf{U} \times \mathbf{W} = \nabla G\,, \quad \mathbf{J} = [\mathbf{M}, \mathbf{U}]\,, \tag{4.32}$$

with some function G. It follows from (4.26) that

$$\delta\mathbf{v} = \delta\mathbf{u} + \delta\mathbf{h} \times \mathbf{M} + \mathbf{H} \times \delta\mathbf{m}\,,$$

whence, in view of (4.31),

$$\delta\mathbf{u} = -\delta\mathbf{h} \times \mathbf{M} - \mathbf{H} \times \delta\mathbf{m} + \boldsymbol{\xi} \times \mathbf{W} - \nabla\alpha\,. \tag{4.33}$$

Similarly, we obtain

$$\begin{aligned} \delta^2\mathbf{u} = &- \delta^2\mathbf{h} \times \mathbf{M} - \delta\mathbf{h} \times \delta\mathbf{m} \\ &- \mathbf{H} \times \delta^2\mathbf{m} + \tfrac{1}{2}\big(\boldsymbol{\chi} \times \mathbf{W} + \boldsymbol{\xi} \times \delta\mathbf{w} - \nabla\beta\big)\,. \end{aligned} \tag{4.34}$$

Here $\delta\mathbf{h}$, $\delta^2\mathbf{h}$ are given by (4.17) and $\delta\mathbf{w}$ by (4.31).

4.4.1 *Variational principle.* Let us calculate the first variation of the energy (4.5) on the set of all possible flows satisfying (4.13) and (4.30). We nave

$$\begin{aligned} \delta E &= \int_{\mathcal{D}} \Big(\mathbf{U} \cdot \big(\mathbf{M} \times [\boldsymbol{\xi}, \mathbf{H}] + \delta\mathbf{m} \times \mathbf{H} + \boldsymbol{\xi} \times \mathbf{W} - \nabla\alpha\big) + \mathbf{H} \cdot [\boldsymbol{\xi}, \mathbf{H}]\Big)\mathrm{d}\tau \\ &= \int_{\mathcal{D}} \Big(\big(\boldsymbol{\xi} \times \mathbf{H}\big) \cdot \big([\mathbf{U}, \mathbf{M}] + \mathbf{J}\big) + \delta\mathbf{m} \cdot \big(\mathbf{H} \times \mathbf{U}\big) + \boldsymbol{\xi} \cdot \big(\mathbf{W} \times \mathbf{U}\big)\Big)\mathrm{d}\tau = 0\,. \end{aligned}$$

Here we used integration by parts and equations (4.12), (4.32).

Thus, we have proved the following.

Proposition 4.3 *The energy (4.5) has a critical point in a steady MHD flow (4.10) on the set of all possible flows satisfying the generalized isovorticity condition given by (4.13) and (4.30).*

The second variation. It can be shown by standard calculations that the second variation of the energy evaluated in the steady state (4.10) is given by

$$\delta^2 E = \tfrac{1}{2}\int_{\partial\mathcal{D}}\Big((\delta\mathbf{u})^2 + (\delta\mathbf{h})^2 + \delta\boldsymbol{\omega}\cdot(\mathbf{U}\times\boldsymbol{\xi}) + \delta\mathbf{h}\cdot\big(\mathbf{U}\times(\delta\mathbf{m} - [\boldsymbol{\xi},\mathbf{M}]) + \mathbf{J}\times\boldsymbol{\xi}\big)\Big)\mathrm{d}\tau\,. \qquad (4.35)$$

Comparing this formula with equation (4.24), we conclude that they coincide provided that

$$\boldsymbol{\eta} = \delta\mathbf{m} - [\boldsymbol{\xi},\mathbf{M}]\,. \qquad (4.36)$$

The relation between the fields $\boldsymbol{\eta}$ given by (4.36) is the same as obtained in Vladimirov and Ilin [1997b] from the analysis of corresponding linearized equations. Note that if we identify variations $\delta\mathbf{u}$, $\delta\mathbf{h}$, $\delta\mathbf{m}$ with infinitesimal perturbations to the basic state (4.10) that obey the corresponding linearized equations, then the relation (4.36) gives us an evolution equation for the field $\boldsymbol{\eta}$ (see Vladimirov and Ilin [1997b]).

5 Conclusion

We started with formulation of Arnold's variational principle for steady three-dimensional flows of an ideal incompressible fluid. Then we established the analogous variational principles for steady states of a system 'body + fluid', for steady flows of an ideal incompressible fluid with contact discontinuities and for steady magnetohydrodynamic flows of ideal, perfectly conducting fluid.

We should note that all these variational principles can be generalized so as to cover the situations when the basic state is unsteady provided that it is steady relative to coordinate system which either moves along a fixed axis with constant velocity or rotates around a fixed axis with constant angular velocity. For a system 'body + fluid' such principles have been established and exploited for obtaining stability conditions in Vladimirov and Ilin [1997a].

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