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Coordinating the Study of Learning Theories and Linear Algebra

Research Objectives

The Linear Algebra Project is developing, implementing, and disseminating curriculum and pedagogy for parallel courses in (a) undergraduate mathematics content and (b) learning theory as applied to the study of mathematics. The purpose of the research, partially funded by the National Science Foundation, is to investigate how parallel study of learning theories and advanced mathematics influences the thinking of individuals in both domains. We conjecture that strengthened understanding of mathematics and learning theory will be an outcome of the reflection promoted by this parallel study, and that the deeper insights will contribute to more effective instruction by those who become high school mathematics teachers and, consequently, better learning by their students in secondary mathematics. These courses are appropriate for mathematics majors, pre-service secondary mathematics education majors, and practicing mathematics teachers.

The initial focus of the project is on *Topics in Linear Algebra* and on *Theories for the Learning of Mathematics*. We plan to adapt this approach to other undergraduate mathematics content areas. The learning theory course focuses most heavily on constructivist theories, though it also examines sociocultural and historical perspectives. A particular theory, APOS (Asiala et al., 1996), is directly related to their study of linear algebra. APOS (Action-Process-Object-Schema) has already been used in a variety of research studies focusing on the understanding of undergraduate mathematics. The study of topics in linear algebra focuses on standard material that is found in many advanced undergraduate linear algebra courses. This study of linear algebra is designed to highlight connections between collegiate linear algebra and secondary mathematics from an advanced perspective. This paper reports on the results of two studies piloting the implementation of this approach, as well as the plans for implementation of dual courses in Linear Algebra and the Learning of Linear Algebra.

Research questions

This project investigates three questions about the coordinated study of linear algebra and learning theories in mathematics:

1. Do participants make any connections between their study of linear algebra content and their study of learning theories?
2. Do participants reflect upon and evaluate their own learning in terms of their study of learning theories?
3. Do participants connect what they study about linear algebra or the learning theories to their planned mathematics content or pedagogy for their own high school mathematics teaching?

Research Perspective

Research at the National Center for Research in Teacher Education found that teachers who majored in the subject they taught often were not able to explain fundamental concepts in their discipline more clearly than other teachers. (McDiarmid & Wilson, 1991, p.i). Their investigations led to the conclusion that "Teachers need explicit disciplinary focus, but few positive results can be expected by merely requiring teachers to major in an academic subject. Studying subject matter in relation to subject matter pedagogy helps teachers be more effective. Teacher education programs that emphasize the underlying nature of the subject matter . . . more often result in knowledgeable, dynamic teachers with transformed dispositions and understandings of subject matter and pedagogy.” It appears that in addition to knowledge of advanced mathematics, effective teachers need mathematical knowledge organized for teaching deep understanding of the subject; awareness of conceptual barriers to learning; and knowledge of the historical, cultural, and scientific roots of mathematical ideas and techniques (Ma, 1999). In Dorier (1995, 2000), it is suggested that particularly the learning of linear algebra necessitates concepts to be unified and generalized, and thus needs to be supported by meta-cognitive activities.

The APOS framework utilizes qualitative methods for research and is based on a specific theoretical perspective that has been developed through attempts to understand the ideas of Piaget (1972) and Piaget & Garcia (1989) concerning reflective abstraction in the context of college level mathematics (Asiala et al., 1996). The approach has three components. It begins with an initial theoretical analysis of what it means to understand a concept and how that understanding could be constructed by the learner. This leads to the design of instructional treatment focused on these mental constructions. Instruction leads to gathering of data, which is analyzed in the context of the theoretical perspective. The three components are cycled and both the theory and instructional treatments are revised as needed. This project supports an instructional approach that assists in reflective abstraction, helping to shape the mathematical mental structures needed for building linear algebra concepts. The study of mathematics will be at a much higher, deeper or more conceptual level than is common for undergraduate linear algebra. The project incorporates a range of pedagogical activities including group activities, open writing, projects, and technology.

Methodology

While listed as separate courses, our vision is that the mathematics content and the theory of learning courses are closely integrated, to the extent of shared time rather than strictly delimited schedules. The courses are co-taught by a mathematics faculty member and an education faculty member so that both professors participate in both the content and the learning theory courses. The primary goal is to study important mathematics–with some clear ties to high school content and teaching–in a way that leads to a deeper understanding into how it is learned. The curriculum and classroom organization is designed to provide rich opportunities for students to use and integrate linear algebra concepts with related topics from secondary mathematics. The instructional practices used in these courses model approaches that reflect the learning theories being studied. Discussion is critical to success in this mode of instruction; it provides a rich opportunity to raise questions, share insights, clarify understanding, and express confusion. It provides a more natural setting to negotiate meaning and understanding with greater personal involvement; interactions can be lively and intense. We have implemented this plan in two pilot settings: (a) three-day weekend workshop for secondary mathematics teachers, and (b) a parallel set of undergraduate and graduate courses taught during Spring 2005.

Three-day Weekend Workshop for Secondary Teachers

The weekend pilot study began two weeks before the actual workshop. Participants were sent readings about concept maps and also about APOS Theory (Appendix A) and were asked to read them before coming to the workshop. The workshop began with a viewing of the video *A Private Universe* in which Harvard graduates at commencement are asked what causes the changing of the seasons. This was followed by a group discussion about learning and the value of conceptual understanding. The facilitators and teachers then had a joint discussion about the reading on APOS theory followed by a joint discussion about concept maps. The workshop leaders and participants also jointly developed a concept map for *parabola* in order to practice the technique. They then worked in pairs to develop a concept map for *vector*.

The participants next worked on activities designed in *Maple* and *Geometer's Sketch Pad* (*GSP*). The first day ended with a joint discussion about how the activities they had completed in *Maple* and *GSP* related to APOS theory. They were also asked "How would you interpret the kind of thinking that these activities may promote in your students' minds?" For Saturday, they were asked to write how the activities we completed Friday developed their own thinking and how they may relate their own thinking in terms of APOS theory. This was also the basis of the first discussion on Saturday.

The opening discussion on Saturday was followed by a discussion of "What is a vector?" They followed this with an application on the computer that illustrated the range of a transformation. Following a lunch break, the group was asked to sketch by hand a tissue box (from 3-D to 2-D). Following this effort, the group viewed a *Pixar* clip and discussed the role of the vectors on the 2-dimensional image. This was followed by a discussion about the application of projections to computer graphics.

The group then discussed how all the activities that had been done could be incorporated into the high school curriculum. They were asked to reflect on how they learned these concepts and to also describe the day's activities and learning in terms of APOS theory. On Sunday morning, the participants were put in the same pairs as Friday evening and asked to again construct a concept map of *vector*. Once completed, the concept maps from Friday were hung on the walls alongside the ones from Sunday morning. Participants then reflected on the changes and wrote about their reflections.

The workshop took place in March 2005. Three project leaders led the activities for a group of eight high school mathematics teachers (2 male, 6 female). Their experience ranged from just graduating from a teacher education program to ten years of classroom experience. Considerable data were collected during the workshop. Participants (a) prepared two concept maps about the vector concept-one at the start and the second at the end of the workshop; (b) a short reflective short between the first and second day, and (c) a reflective essay to self-analyze their own growth in understanding of vector, based upon their analyses of their own concept maps. Many of the workshop activities and discussions were videotaped by an undergraduate technician and a graduate assistant took extensive observational notes during the workshop.

Parallel Courses in Linear Algebra and Learning of Linear Algebra

The parallel courses were taught at another site during the Spring 2005 semester. Linear algebra was offered as a 3-credit dual-listed upper level undergraduate and graduate course required of mathematics majors. The text book for the linear algebra course was Serge Lang's *Linear Algebra (3rd edition)* (2004). Twenty-five students, including three graduate students, enrolled in the course. Students who enrolled in the regular linear algebra course were invited to participate in a special two-credit seminar on learning theories in mathematics education. The education seminar met once per week for two hours. It was taken by 3 graduate students who were also enrolled in Linear Algebra and had from one to three years of high school mathematics or science teaching experience

Students in the seminar read and discussed a variety of papers dealing with mathematics education, relating the ideas whenever possible to their concurrent study of linear algebra. The readings were drawn from *How People Learn: Brain, Mind, Experience and School* (NRC, 2000), *Handbook of Research on Teaching* (Grouws, 1986), *Handbook of Research on Mathematics Teaching and Learning* (Wittrock, 1992) along with several papers dealing specifically with APOS (see Asiala et al., 1996; Baker, Cooley and Trigueros, 2000). Students enrolled in the seminar also worked with concept maps and videos as described in the following section on the weekend workshop for secondary teachers.

The courses that will be offered again in Spring 2006 at two universities will have a similar structure: a traditional linear algebra course and a 2 hour, 2 credit elective in the learning theories of mathematics with a focus on APOS and Linear Algebra.

Results

The pilot studies produced data in the form of written work produced by participants, video of workshop sessions, pre- and post-workshop concept maps, instructor reflective notes, and transcriptions of selected sessions.

For these initial pilot studies, we reviewed a final assessment from the learning theory seminar and all the materials collected during the workshop. Our three research questions were:

1. Do participants make any connections between their study of linear algebra content and their study of learning theories?
2. Do participants reflect upon and evaluate their own learning in terms of their study of learning theories?
3. Do participants connect what they study about linear algebra or the learning theories to their planned mathematics content or pedagogy for their own high school mathematics teaching?

We found evidence of all forms of connections in the data. However, the strongest evidence in the participants' responses was the evaluation of their own learning and the growth of understanding from the mix of activities they worked through. They also demonstrated in their writing and discussions reflective thinking about their learning and how they could apply what they had learned back to their own mathematics classrooms. Finally, while they used the language of APOS, for most there was very little clear connection between their analysis of the content of linear algebra and the theory. There was more evidence of the ability to analyze the learning of linear algebra by the three graduate students participating in the semester-long pilot study who had more opportunities to examine the theories and their own learning. The weekend pilot study participants had more difficult in understanding and applying APOS Theory in terms of the linear algebra content that they had learned. The following discussion of the three research questions uses analysis of students' work to help illustrate how the coordinated study of learning theory and linear algebra seemed to influence student thinking.

Research Question 1. Do participants make any connections between their study of linear algebra content and their study of learning theories?

In the learning theory seminar, participants were able to prepare genetic decompositions of several topics they had studied in linear algebra, including (a) finding determinants via cofactor analysis, (b) linear dependence and independence, and (c) linear transformations. These participants also indicated in a final assessment that the learning theory seminar had influenced their understanding of linear algebra material, and that the two courses had influenced their own intentions for future instruction of high school mathematics and science.

For example, during one lesson in the learning theory course one participant asked a question about calculating the determinant of a 4x4 matrix, which had been assigned as an exploration problem in the linear algebra course prior to being taught an algorithm. The question was specifically triggered by a student who was trying to make sense of the computation described in the textbook in which the author stated that the determinant of a 4x4 matrix was equal to the determinant of an associated 3x3 matrix. This equivalence was due to a cofactor simplification expanding on a column with just one non-zero entry. The simplification was not explicitly explained at this point in the text's narrative.

During a lengthy class discussion, students came to see the connections between determinants of matrices with different dimensions. The discussion then turned to how APOS theory could be applied to this situation. Participants described their initial understanding as being at the action level; they were only able to compute determinants of specific matrices. When they came to understand the example given in the text, they suggested this required at least a process level understanding to recognize the simplification that had initially triggered the discussion. The discussion had moved beyond a process level because they had come to recognize a determinant as a property of a matrix that could be found for any square matrix, regardless of dimension, by generalizing the cofactor process.

The students suggested that this recognition required an object level understanding of determinant. The APOS learning theory ideas helped the students recognize the increasing sophistication of their understanding of the computation of determinants, from an initial stage when they could only follow a computational algorithm for a specific dimension, to an understanding that gave them confidence that they could devise a strategy to compute any determinant, even one with variable rather than numeric elements. Participants recognized that they had gained a more sophisticated understanding of these computations by making sense for themselves of written material, discussing the ideas with fellow students and the instructor, and then reflecting on and describing their own thought processes. They believed that the understanding they had achieved could not come from direct instruction alone. Instead, their students–like they themselves–would need time and instructional experiences that also allowed and encouraged them to wrestle with mathematical ideas and generate their own explanations, while interacting with other students and the teacher.

Students in the seminar were asked for written responses to questions during the final course meeting. Two questions referred directly to APOS: (a) Describe the components of the APOS theory of learning, and (b) Choose a concept from Linear Algebra, then give a *genetic* *decomposition* of the concept according to APOS. The following genetic decomposition of linear independence was given in Jill's response:

> *Linear Independence*
>
> Action *Given two or more vectors, one can set up a linear combination, solve for the unknown coefficients, and tell whether the vectors are lin. indep or dep.*
>
> Process*: One can describe the process of determining independence without actually having a set of vectors. Vectors could also be thought of geometrically, and one could picture a set of lin indep or dep vectors*
>
> Object*: Linear independence is thought of as an object when the concept can be conceptualized beyond the process of determining independence or picturing vectors. Linear independence connects to the ideas of basis and span*
>
> Schema*: One can see linear indep as a concept beyond vectors in* R^n*, but as a concept relating to basis and span in any sort of vector space.*

The other two students gave written examples of genetic decompositions that more superficially described the components of the APOS framework. Since students were not interviewed about their papers, we are unable to say more about their understanding of and ability to apply APOS theory to specific topics in linear algebra based on their final written assessment.

Research Question 2. Do participants reflect upon and evaluate their own learning in terms of their study of learning theories?

The major themes that teachers raised across their discussions included the role of visualization in helping them to develop multiple representations as well as making connections between representations. They also included that the activities they were engaged in promoted higher-level thinking and sense-making in them.

For example, in a discussion between the two concept maps about vector, Ellen[1] wrote:

> *In the first map we connected other concepts directly to vectors. We basically showed some ideas of what vectors represent. We showed relationships with the arrows pointing away from vectors, so you could start out most sentences by saying "vectors are…" or "vectors represent…" […] In the second map we showed a lot of different connections between vectors and related concepts, and also among the related concepts. Because of the exercises and discussions during the workshop, we were able to talk about vectors in a more meaningful way.*

The teacher recognized change in her own understanding and that she had more ideas, more connections, and more structure to her knowledge.

This next brief interchange between a workshop leader and participant illustrates evidence of a participant making connections between learning theory and linear algebra studied during the workshop:

> Leader: *Does anyone else have a particular example of action, process, or object?*

[1] All participant names are pseudonyms.

> Participant*: I think on the action level I was having trouble visualizing the three vectors. You know, with the third axis. And I think that I was really working on the action level at that point. Even though, I think you know with that whole spiraling idea. Because with the two-dimensional vectors it was, I wasn't on the action level. It was easier for me to visualize that. But then with the third dimension, and maybe it didn't have anything to do with vectors so much as with the visual part. So I spent some time with that on that level.*

In comparing the two concept maps she had prepared at the start and end of the workshop, Lynn wrote:

> *The first concept map was created by brainstorming. Dawn and I were thinking about anything that we had heard or could remember about vectors. The links between the nodes were vague. We were aware of the connections between certain ideas but were unsure of the exact ways in which topics connected. […]*
>
> *The second concept map was created with the idea of putting all of our knowledge from the last days into an organized map. We had much more information and the connections were more specific. This new knowledge made creating the map more difficult because we wanted to make sure the links were accurate and thorough enough. When creating the second map the key topics that came out were Matrices, Transformations, and Physics. Then we realized that there were both*

> *algebraic and geometric representations of vectors. […] Under Transformations, we had the animation ideas with projection and 2D-3D. Though physics is another subheading it is still not completely defined in the concept map which is why all the properties (motion, projectile, velocity, speed) are in one big node. We felt as though we knew much more about vectors, though as the concept map was being created we became aware that there are still many holes.*
>
> *Another difference between the two maps is that the first map is much more circular, everything comes off of Vector. The second map is laid out in parts. The connections are more elaborate in the second map.*

Lynn recognized that in the first concept map, the information was less meaningful. There was more recall than connections. In the second map, she saw more complexity from her specific connections. She was able to recognize as well that the map is not complete, but still more sophisticated with the development of connections between concepts that were related to vector and not all stemming directly from vector.

Finally, Carrie explained the changes in her concept maps along with reason why she believed those changes took place:

> *The differences in the two maps stem from a deeper understanding of the concept the second time around. While we began with a basis for understanding vectors,*

our knowledge base grew after more interaction, discussion, and participation with vector activities.

The second map provides more detail, and more connections of ideas and concepts. It became easier to link and relate the concepts, causes, results, dependency, and applications. The second map shows in detail a greater understanding of the links and math within the concept of vectors. We were also better able to ***describe*** *relationships and connections of ideas.* (Emphasis given by the teacher.)

Research Question 3. Do participants connect what they study about linear algebra or the learning theories to their planned mathematics content or pedagogy for their own high school mathematics teaching?

Many teachers were able to reflect not only on the content, but also on the pedagogy and how these approaches could be incorporated into their mathematics classrooms. For example, Carrie expressed the multiple methods that were used:

The activities today were very useful. They can be incorporated in daily math classroom planning by using technology and GSP software to stimulate higher level thinking and reinforce mathematical concepts and ideas. The experience of using multiple strategies to teach math concepts for mastery is important for students. We used lectures, group collaboration, and technology to assist us in understanding the concept of vectors.

The hands-on interactive experience is especially useful. This allows students to be participatory learners. They will be better able to retain ideas and concepts taught because they were active participants in seeking, clarifying, and imagining the knowledge and its concepts.

Learning?? APOS Theory relates in this situation because we used the learning environment to develop mathematical knowledge. Through out this activity, we were able to focus on what our object was, what actions needed to be performed, and with several hands-on discoveries, we were better able to conceptualize ideas and outcomes without even doing them. It was great to then see our ideas and conjectures come alive on through the software program.

The following teacher commented on the power of the visualizations, as well as using the concept map. She further considered how she might incorporate concept maps into her own classroom:

The design of the exercises started out introducing concepts through application on the action level, which then allowed room to discuss results and generate a deeper understanding of the concepts. Each time I started working on an exercise, I tried to relate it to the previous concepts and also thought back to the concept map we developed yesterday. The visualization factor was extremely helpful to me in connecting the concepts to form processes and even objects. The structure of

> *the workshop made me think that it would be useful to have a concept wall in a classroom, which students would continually add to as they developed both new concepts and developed richer understanding of previous ones. Students could continually refer to the concept wall and really interrelate the concepts throughout the school year.*

Conclusions and Implications for Further Research

The data gathered demonstrated the teachers reflecting on their own learning while also putting it in the context of APOS theory. Their self-assessment shows some misunderstanding of the theory, but still clearly shows a level of engagement that is promoting self-awareness as well as awareness to the learning process. Furthermore, they were able to take both the content and the pedagogical methods employed and relate them back to their own classrooms and how they might be utilized in that setting.

These pilot studies have given preliminary support to the notion that teachers gain deeper insights to both mathematical content and learning theories through their coordinated studies. Because the workshop and pilot courses involved a small number of pre-service and in-service teachers and because we did not conduct individual interviews to probe participant thinking, conclusions about the efficacy of this approach are limited. Still, we found that even during a short weekend workshop participants are able to describe interactions of ideas from the two domains.

Building on this preliminary work, we have designed and are implementing a seminar that is listed concurrently as an undergraduate and graduate seminar for the learning of linear algebra (Spring, 2006). Two groups of students enrolled in the class: (a) undergraduates who either have already studied linear algebra or are enrolled in linear algebra and who are planning on teaching secondary mathematics, and (b) secondary mathematics teachers pursuing a master's degree. The seminar is designed to give students an overview of some of the theories of learning mathematics and then apply those theories as they reflect on their own understanding of linear algebra. The seminar incorporates activities, such as writing, discussion, and use of technologies, to explore linear algebra concepts. At the end of the semester, the pre-service and in-service math teachers enrolled in the seminar will participate in clinical interviews designed to elicit information about whether they make any connections between their study of linear algebra content and their study of learning theories. The interviews will also probe (a) whether they reflect upon and evaluate their own learning in terms of their study of learning theories and (b) whether they connect what they study about linear algebra, or the learning theories, to their planned mathematics content or pedagogy for their own high school mathematics teaching. This methodology is described by Vidakovic and Martin (2004).

There are several areas for further study. One is an implementation issue: If the concurrent study of content and educational theory—with deliberate examination of the specific interactions of ideas involved—is seen as a worthwhile endeavor, how can this strategy be incorporated in existing teacher education programs? In these preliminary studies, we have depended on the interest and goodwill of participants—and also discovered that many teachers have such full schedules already that it is difficult for them to choose to take another course that simply counts as an elective. Another area for study is whether this approach can be expanded to other content areas, not only within mathematics, but even to other areas such as science and the humanities.

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