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Statistical Fluid Mechanics and Statistical Mechanics of Fluid Turbulence

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Abstract

Turbulence in a fluid has two mutually contradictory aspects, that is, the apparent randomeness as a whole and the intrinsic determinicity due to the fluid-dynamical equations. In the traditional approach to this subject, the velocity field of turbulence has been represented by the mean products of the turbulent velocities at several points and a time, and the equations governing the dynamical system composed of the mean velocity products are solved to obtain the mean velocity products as the deterministic functions in time. The works along this approach constitute the main body of turbulence research as outlined by Monin and Yaglom (1975) under the title of 'Statistical Fluid Mechanics'. It should be noted, however, that this approach has the difficulty of unclosedness of the equations for the mean velocity products due to the nonlinearity of the Navier-Stokes equations and an appropriate closure hypothesis is still awaited.

In the statistical approach to turbulence, on the other hand, the random velocity field of turbulence is represented by the joint-probability distributions of the multi-point turbulent velocities. This approach has been taken by Lundgren (1967) and Monin (1967) independently, and the system of equations for the probability distributions of the multi-point velocities has been formulated. This system of equations is also unclosed, but in this case the problem is much easier to deal with since the unclosedness is not due to the nonlinearity but to the higher-order derivatives in the viscous term of the Navier-Stokes equations. Tatsumi (2001) proposed the cross-independence closure hypothesis for this purpose and proved its validity for homogeneous isotropic turbulence. More recently, the theory has extended by Tatsumi (2011) to general inhomogeneous turbulence and the closure is shown to provide an exact closure. This theory is outlined and discussed in the later part of this paper.

1 Introduction

The dual characters of turbulence, the randomness and the determinicity, are dealt with by means of statistics and mechanis respectively, and then these two elements are incorporated into 'Statistical Mechanics of Fluid Turbulence'.

1.1 Statistics of turbulence

For turbulence in an incompressible viscous fluid, the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ at all points $\mathbf{x}$ and time t is dealt with as the basic random function and the joint-probability distributions of the velocities $\mathbf{u}_m(t) = \mathbf{u}(\mathbf{x}_m, t)$ at the points $\mathbf{x}_m$ $(m = 1, \ldots, n)$ and a time t are assumed to exist and written as

$$f^{(n)}(\mathbf{v}_1, \ldots, \mathbf{v}_n; \mathbf{x}_1, \ldots, \mathbf{x}_n; t) = \langle \delta(\mathbf{u}_1(t) - \mathbf{v}_1) \cdots \delta(\mathbf{u}_n(t) - \mathbf{v}_n) \rangle, \tag{1}$$

where $\mathbf{v}_m$ $(m = 1, \ldots, n)$ represent the probability variables corresponding to the random velocities $\mathbf{u}_m(t)$ $(m = 1, \ldots, n)$, δ the three-dimensional delta function, and $\langle\ \rangle$ the average with respect to the distribution of the variable inside.

Then the statistical averages of common use are defined as follows.

$$\text{Mean velocity:} \qquad \overline{\mathbf{u}}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle = \int \mathbf{v}\ f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v}, \tag{2}$$

$$\text{Kinetic energy:} \qquad E(\mathbf{x}, t) = \frac{1}{2} \left\langle |\mathbf{u}(\mathbf{x}, t)|^2 \right\rangle = \frac{1}{2} \int |\mathbf{v}|^2\ f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v}, \tag{3}$$

1.2 Dynamics of turbulence

As a fluid motion, turbulence in an incompressible viscous fluid is subject to the Navier-Stokes equation for the velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure $p(\mathbf{x}, t)$ of the flow,

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{u} - \nu \left| \frac{\partial}{\partial \mathbf{x}} \right|^2 \mathbf{u} = -\frac{\partial}{\partial \mathbf{x}} \left(\frac{p}{\rho} \right), \tag{4}$$

and the non-divergence condition,

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{u} = 0, \tag{5}$$

where ρ and ν denote the density and kinetic viscosity of the fluid respectively.

The pressure term in Eq.(4) can be eliminated using Eq.(5) with the result,

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{u} - \nu \left| \frac{\partial}{\partial \mathbf{x}} \right|^2 \mathbf{u} = -\frac{\partial}{\partial \mathbf{x}} \frac{1}{4\pi} \int |\mathbf{x} - \mathbf{x}'|^{-1} \frac{\partial}{\partial \mathbf{x}'} \cdot \left(\mathbf{u}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \mathbf{u}' d\mathbf{x}', \tag{6}$$

where $\mathbf{u}'(t) = \mathbf{u}(\mathbf{x}', t)$, and the infinite point is assumed to be occupied by the fluid. Eq.(6) gives a deterministic equation for the velocity $\mathbf{u}(\mathbf{x}, t)$ so that it is used more conveniently than the original Eq.(4)

1.3 Mean flow and turbulent fluctuation

It is useful to decompose the velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure $p(\mathbf{x}, t)$ of the turbulent flow into their probability averages and the fluctuations around the averages as

$$\mathbf{u}(\mathbf{x}, t) = \overline{\mathbf{u}}(\mathbf{x}, t) + \widehat{\mathbf{u}}(\mathbf{x}, t), \quad p(\mathbf{x}, t) = \overline{p}(\mathbf{x}, t) + \widehat{p}(\mathbf{x}, t). \tag{7}$$

whith

$$\overline{\mathbf{u}}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle, \quad \overline{p}(\mathbf{x}, t) = \langle p(\mathbf{x}, t) \rangle, \tag{8}$$

$$\widehat{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \overline{\mathbf{u}}(\mathbf{x}, t), \quad \widehat{p}(\mathbf{x}, t) = p(\mathbf{x}, t) - \overline{p}(\mathbf{x}, t), \quad \langle \widehat{\mathbf{u}}(\mathbf{x}, t) \rangle = \langle \widehat{p}(\mathbf{x}, t) \rangle = 0. \tag{9}$$

denoting the averages and the fluctuations respectively.

The equations of motion for the *mean flow* $\overline{\mathbf{u}}(\mathbf{x},t)$ and the *turbulent fluctuation* $\widehat{\mathbf{u}}(\mathbf{x},t)$ are obtained by substituting the decomposition (7) into Eq.(4) and taking the mean of the resulting equations as follows:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \left(\overline{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\overline{\mathbf{u}} + \left\langle \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\widehat{\mathbf{u}} \right\rangle - \nu \left|\frac{\partial}{\partial \mathbf{x}}\right|^2 \overline{\mathbf{u}} = -\frac{\partial}{\partial \mathbf{x}}\left(\frac{\overline{p}}{\rho}\right), \tag{10}$$

for the *mean flow* $\overline{\mathbf{u}}(\mathbf{x},t)$, and

$$\frac{\partial \widehat{\mathbf{u}}}{\partial t} + \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\overline{\mathbf{u}} + \left(\overline{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\widehat{\mathbf{u}} + \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\widehat{\mathbf{u}} - \left\langle \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\widehat{\mathbf{u}} \right\rangle - \nu \left|\frac{\partial}{\partial \mathbf{x}}\right|^2 \widehat{\mathbf{u}} = -\frac{\partial}{\partial \mathbf{x}}\left(\frac{\widehat{p}}{\rho}\right), \tag{11}$$

for the fluctuation $\widehat{\mathbf{u}}(\mathbf{x},t)$, with the non-divergence conditions which follows from Eq.(5) as

$$\frac{\partial}{\partial \mathbf{x}} \cdot \overline{\mathbf{u}} = \frac{\partial}{\partial \mathbf{x}} \cdot \widehat{\mathbf{u}} = 0. \tag{12}$$

1.4 Unclosedness of statistical equations

Now the mean velocity $\overline{\mathbf{u}}$ can be obtained as the solutions of Eq.(10). In this context, however, we find that this equation is not closed since it includes the term,

$$\left\langle \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\widehat{\mathbf{u}} \right\rangle = \frac{\partial}{\partial x_j}\sum_{i,j=1}^{3}\langle \widehat{u}_i(\mathbf{x},t)\widehat{u}_j(\mathbf{x},t)\rangle = \frac{\partial}{\partial x_j}\sum_{i,j=1}^{3} R_{ij}(\mathbf{x},t), \tag{13}$$

which is known as the Reynolds stress ρR_{ij} $(i,j=1,2,3)$. Thus, in order to solve Eq.(10), we have to know this term represented by Eq.(13). This unclosedness is not resolved even if we proceed to the Reynolds stress R_{ij} since the equation for R_{ij} derived from Eq.(11) includes the triple mean product of $\widehat{\mathbf{u}}$. This is the 'closure problem' which constitutes the central difficulty of turbulence theory.

2 Statistical Fluid Mechanics

In order to deal with the closure problem seriously, we have to minimize the mathematical complexity of the equations as far as possible. For this purpose we consider *homogeneous isotropic turbulence* with no mean flow $\overline{\mathbf{u}} = 0$ which has been introduced by Taylor (1935).

2.1 Homogeneous isotropic turbulence

With no mean flow, the velocity $\mathbf{u}(\mathbf{x},t)$ and the pressure $p(\mathbf{x},t)$ of this turbulence are governed by Eqs.(4) and (5) directly. Among the mean products of various orders, those of first order, $\langle \mathbf{u}(\mathbf{x},t)\rangle$ and $\langle p(\mathbf{x},t)\rangle$, vanish according to the homogeneity and hence those of the second and third orders defined by

$$\begin{aligned} &B_{ij}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle\,, \\ &T_{ijk}(\mathbf{r},\mathbf{r}';t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)u_k(\mathbf{x}+\mathbf{r}',t)\rangle\,, \quad i,j,k=(1,2,3), \end{aligned} \tag{14}$$

play important role in this turbulence.

The equation for the velocity correlation B_{ij} is derived from Eq.(4) and written by Karmann-Howarth (1938) in the following nice scalar form:

$$\left[\frac{\partial}{\partial t} - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\right] B\left(r,t\right) = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) T\left(r,t\right), \tag{15}$$

where

$$B\left(r,t\right) = \left\langle u_{\parallel}(\mathbf{x},t)u_{\parallel}(\mathbf{x}+\mathbf{r},t)\right\rangle, \quad T\left(r,t\right) = \left\langle u_{\parallel}(\mathbf{x},t)^2 u_{\parallel}(\mathbf{x}+\mathbf{r},t)\right\rangle, \tag{16}$$

represent the longitudinal components of $B_{ij}\left(\mathbf{r},t\right)$ and $T_{ijk}(\mathbf{r},0;t)$ along the vector $\mathbf{r}$ and called the double and triple velocity correlations respectively.

2.2 Fourier analysis of turbulence

The velocity field $\mathbf{u}\left(\mathbf{x},t\right)$ of homogeneous isotropic turbulence can be expressed in its three-dimensional Fourier transform as

$$\mathbf{u}\left(\mathbf{x},t\right) = \int \widetilde{\mathbf{u}}\left(\mathbf{k},t\right) \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{x}\right)\right] d\mathbf{k}, \tag{17}$$

where $\mathrm{i} = \checkmark(-1)$ and $\widetilde{\mathbf{u}}\left(\mathbf{k},t\right)$ denotes the Fourier amplitude of the velocity $\mathbf{u}\left(\mathbf{x},t\right)$ at the wave number $\mathbf{k}$. In order that the Fourier transform $\widetilde{\mathbf{u}}\left(\mathbf{k},t\right)$ exists, the condition $\int\left|\mathbf{u}\left(\mathbf{x},t\right)\right| d\mathbf{x} < \infty$ should be satisfied,.but actually not in homogeneous turbulence. Then, equations should be written in the Fourier-Stieltjes integral, but the above notation is being used for simplicity.

The equations for the amplitude velocity $\widetilde{\mathbf{u}}\left(\mathbf{k},t\right)$ are obtained by applying the Fourier transform (17) to Eqs.(4) and (5) for the velocity $\mathbf{u}\left(\mathbf{x},t\right)$ as

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] \widetilde{\mathbf{u}}\left(\mathbf{k},t\right) = -i\int\left(\mathbf{k}\cdot\widetilde{\mathbf{u}}\left(\mathbf{k}-\mathbf{k}',t\right)\right)\left\{\widetilde{\mathbf{u}}\left(\mathbf{k}',t\right) - \frac{\mathbf{k}}{k^2}\left(\mathbf{k}\cdot\widetilde{\mathbf{u}}\left(\mathbf{k}',t\right)\right)\right\} d\mathbf{k}'. \tag{18}$$

where $k = |\mathbf{k}|$ and the pressure term has been eliminated using the non-divergence condition.

The velocity correlations defined by Eq.(14) are expressed in the Fourier space as

$$\begin{aligned} B_{ij}(\mathbf{r},t) &= \int \widetilde{B}_{ij}(\mathbf{k},t) \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r}\right)\right] d\mathbf{k}, \\ T_{ijk}(\mathbf{r},\mathbf{r}';t) &= \int\int \widetilde{T}_{ijk}(\mathbf{k},\mathbf{k}';t) \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r} + \mathbf{k}'\cdot\mathbf{r}'\right)\right] d\mathbf{k}d\mathbf{k}'. \end{aligned} \tag{19}$$

and the functions $\widetilde{B}_{ij}$ and $\widetilde{T}_{ijk}$ are called as the energy spectrum and the energy-transfer respectively according to their physical meaning.

The equation for the energy spectrum $\widetilde{B}_{ij}$ is obtained by taking the Fourier transform of Eq.(15) as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] \widetilde{B}_{ij}(\mathbf{k},t) = \mathrm{i}k_k \int \left\{\Delta_{il}\left(\mathbf{k}\right)\widetilde{T}_{ljk}(\mathbf{k},\mathbf{k}';t) - \Delta_{jl}\left(\mathbf{k}\right)\widetilde{T}_{lik}(-\mathbf{k},\mathbf{k}';t)\right\} d\mathbf{k}', \tag{20}$$

where $\Delta_{ij}\left(\mathbf{k}\right) = \delta_{ij} - k_i k_j / k^2$.

For isotropic turbulence, the spectral functions in Eq.(20) are expressed as

$$\widetilde{B}_{ij}(\mathbf{k},t)=\frac{1}{4\pi k^2}E(k,t)\Delta_{ij}\left(\mathbf{k}\right),\quad \int\widetilde{T}_{ijk}(\mathbf{k},\mathbf{k}';t)d\mathbf{k}'=\frac{\mathrm{i}}{8\pi k^4}W(k,t)\left\{k_i\Delta_{jk}\left(\mathbf{k}\right)+k_k\Delta_{ij}\left(\mathbf{k}\right)\right\},\tag{21}$$

and then Eq.(20) for the energy spectrum $\widetilde{B}_{ij}$ is written in a simple scalar form as

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]E(k,t)=W\left(k,t\right),\quad \int W(k,t)dk=0,\tag{22}$$

where $E(k,t)$ denotes the *energy spectrum function* representing the amount of the energy included in a sphere of the radius k, while $W\left(k,t\right)$ denotes the *energy-transfer function* representing the transfer of the energy between the different values of k.

2.3 Dynamical closure problem

So far, the closure problem mentioned in §1.4 has mostly been dealt with using *ad hoc* assumptions. For homogeneous isotropic turbulence, however, it has become possible to deal with everything in more systematic way. Hence, the closure problem also has become to be dealt with more systematically employing novel concepts and techniques in modern nonlinear physics.

2.4 Quasi-normal closure

Probably, the simplest closure relation may be given by the *quasi-normal* relation, which is symbolically written, to a few orders, as

$$\begin{aligned}
&\langle u_1u_2\rangle=\langle u_1\rangle\langle u_2\rangle,\\
&\langle u_1u_2u_3\rangle=\langle u_1u_2\rangle\langle u_3\rangle+\langle u_1u_3\rangle\langle u_2\rangle,\\
&\langle u_1u_2u_3u_4\rangle=\langle u_1u_2\rangle\langle u_3u_4\rangle+\langle u_1u_3\rangle\langle u_2u_4\rangle+\langle u_1u_4\rangle\langle u_2u_3\rangle.
\end{aligned}\tag{23}$$

These relations are exactly valid if $\langle u_m\rangle$ $(m\geqq 1)$ are all normal averages, but otherwise, they are valid only at large distances between the points of concern. In this sense, they are called the *quasi-normal* closure. For homogeneous turbulence, in which $\langle u_m\rangle=0$, the first two relations vanish, so that the third relation gives the first meaningful expression of this closure.

Tatsumi (1957) employed this closure for Eq.(22) and after some calculations, succeeded to express the function $W\left(k,t\right)$ in terms of the energy spectrum $E\left(k,t\right)$ as follows:

$$\begin{aligned}
W\left(k,t\right) \;=\;& \int_0^\infty\int_{-1}^1 W\left(k,k',\mu;0\right)\exp\left[-2\nu\left(k^2+k'^2+\mu kk'\right)t\right]k'^2dk'd\mu\\
&-\int_0^t dt'\int_0^\infty\int_{-1}^1\exp\left[-2\nu\left(k^2+k'^2+\mu kk'\right)(t-t')\right]\times\\
&\times\left\{E\left(k,t'\right)k'^2-E\left(k',t'\right)k^2\right\}E\left(k'',t'\right)\left(\frac{k^2k'^2}{k''^2}+\mu kk'\right)\left(1-\mu^2\right)k''^{-2}dk'd\mu,
\end{aligned}\tag{24}$$

where $\mu = (\mathbf{k} \cdot \mathbf{k}') / kk'$, and $W(k, k', \mu; 0)$ denotes the initial condition for $W(k, k', \mu; t)$ compatible with that for $E(k, t)$.

Eqs.(22) and (24) provide us with the closed set of equation for the energy spectrum $E(k, t)$. After publishing this result in a domestic congress, the author has learnt that another work along the same line of idea has been done by Proudman and Reid (1954) with the samel result but still in tensorial form. Thus, the closure problem of turbulence seemed to have been solved so far as homogeneous isotropic turbulence is concerned.

A few years later, Ogura (1963) carried out numerical computation of the energy spectrum $E(k, t)$ using Eqs.(22) and (24) by Tatsumi (1957). To the author's surprise, his result has shown that the function $E(k, t)$ is deformed largely in time to take negative values in some range of k for large Reynolds numbers. The reason of the failure was not clear at that time, but it seemed to be due to the lack of the inertial energy-dissipation, which will be discussed below, in the framework of quasi-normal closure.

2.5 Kolmogorov's theory of local turbulence

The non-zero *energy-dissipation rate* of turbulence defined by

$$\varepsilon(\mathbf{x}, t) = \nu \sum_{i,j=1}^{3} \left\langle \left(\frac{\partial u_i(\mathbf{x}, t)}{\partial x_j} \right)^2 \right\rangle, \tag{25}$$

for the limit of vanishing viscosity $\nu \to 0$ is the fundamental assumption in Kolmogorov's (1941).theory of local isotropic turbulence.

Under this premise, he has developed his theory of *local isotropic turbulence* in the wave-number space, assuming that there exists a local range of turbulence at large values of $k = |\mathbf{k}| = O\left((\varepsilon/\nu^3)^{1/4} \right)$ which is in a quasi-equilibrium between the energy inflow from the energy-containing range of $O(\varepsilon)$ and the energy outflow due to the molecular viscosity of $O(\nu)$. Then he expresses the the *energy spectrum* $E(k, t)$ at the local range of k in terms of two parameters ε and ν by dimensional analysis as

$$E(k, t) = (\nu^5 \varepsilon)^{1/4} F\left(k / (\varepsilon/\nu^3)^{1/4} \right), \tag{26}$$

where F denotes a non-dimensional function.

He further argues that at extremely large Reynolds numbers $\nu \to 0$, the spectrum $E(k, t)$ will become independent of the viscosity ν, so that the function F must be expressed as $F \propto \nu^{-5/4}$ and it immediately follows that:

$$E(k, t) = C \varepsilon^{2/3} k^{-5/3}, \tag{27}$$

with an universal constant C. This is known as Kolmogorov's $-5/3$ power energy spectrum .

Now it is apparent that the notion of the *inertial engergy dissipation,* $\varepsilon > 0$ for $\nu \to 0$, was out of scope in the quasi-normal theories of turbulence, and this is indeed the crucial reason for the failure of the theory. Various attempts for modification of the theory have been made by several people including the author (see for instance McComb (1990)), but the final answer to this question had to wait for Tatsumi (2001) which includes the inertial energy dissipation $\varepsilon > 0$ in its closure process.

2.6 Kraichnan's and related theories

At about the same time, the *direct-interaction approximation* (DIA) theory of Kraichnan (1959) has appeared. This theory seemed to be fairly similar to the quasi-normal theories, but being concerned with stationary turbulence according to Kolmogorov's idea of local turbulence, it has been free fom the trouble of the negaive energy spectrum. His theory on turbulence has been developed to a unique nonlinear theory of turbulence which has been nicely summarized in terms of the renormalization-group theory by McComb (1990). Probably, we cannot talk about mechanics of turbulence without mentioning his and his group's contributions.

3 Statistical Mechanics of Fluid Turbulence

Now, it seems adequate to move to statistical mechanical approach to turbulence in terms of the probability distributions of turbulent velocity, which constitutes the central part of 'statistical mechanics of fluid turbulence'.

3.1 Closure of equations for velocity distributions

The equations governing the multi-point probability distributions $f^{(n)}(\mathbf{v}_1,\ldots,\mathbf{v}_n;t)$ $(n\geq 1)$ have been obtained by Lundgren (1967) and Monin (1967), but any finite subset of these equations is not closed since each equation includes the higher-order velocity distributions as new unknowns. Then, in order to close these equations, we have to introduce a closure hypothesis. This 'statistical closure' problem, however, is much easier to deal with than the 'dynamical closure' problem, and it will be shown that this closure can be even carried out exactly.

3.2 Cross-Independence closure

If we consider the first two members f and $f^{(2)}$ of the velocity distributions $f^{(n)}$ represented by Eq.(1), the simplest relationship between these distributions may be given by the *independence* relation,

$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)=f(\mathbf{v}_1,\mathbf{x}_1,t)\,f(\mathbf{v}_2,\mathbf{x}_2,t)\,. \tag{28}$$

This relation corresponds to the first line of the quasi-normal closure (23) for the velocity products, which is known to be valid at large distances $r=|\mathbf{x}_2-\mathbf{x}_1|\to\infty$ but not otherwise.

On the other hand, if we define the sum and difference of the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ as $\mathbf{u}_{+\pm}=\frac{1}{2}(\pm\mathbf{u}_1+\mathbf{u}_2)$ and consider the distributions of these *cross-velocities* $\mathbf{u}_\pm$ as

$$\begin{aligned} g_\pm(\mathbf{v}_\pm;\mathbf{x}_1,\mathbf{x}_2;t) &= \langle\delta(\mathbf{u}_\pm(t)-\mathbf{v}_\pm)\rangle\,,\\ g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t) &= \langle\delta(\mathbf{u}_+(t)-\mathbf{v}_+)\,\delta(\mathbf{u}_-(t)-\mathbf{v}_-)\rangle\,, \end{aligned} \tag{29}$$

we can assume the *cross-independence* relation for the distributions $g^{(2)}$ and $g_\pm$ as

$$g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t)=g_+(\mathbf{v}_+;\mathbf{x}_1,\mathbf{x}_2;t)\,g_-(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t)\,. \tag{30}$$

The relation (30) gives, according to the identity,

$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_1d\mathbf{v}_2=g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_+d\mathbf{v}_-, \tag{31}$$

another closure relation for the distribution $f^{(2)}$ as

$$\begin{aligned} f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t) &= 2^{-3}g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t) \\ &= 2^{-3}g_+(\mathbf{v}_+;\mathbf{x}_1,\mathbf{x}_2;t)\,g_-(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t). \end{aligned} \tag{32}$$

where the Jacobian $\partial(d\mathbf{v}_+,d\mathbf{v}_-)/\partial(d\mathbf{v}_1,d\mathbf{v}_2)=2^{-3}$ has been used..

Unlike the ordinary independence relation (28), the *cross-independence* relation (32) is shown to be not only valid for large $r\to\infty$ but exactly satisfied at $r\to 0$ according to the coincidence conditions at $r=|\mathbf{x}_2-x_1|\to 0$ as follows:

$$\begin{aligned} \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t) &= f(\mathbf{v}_1,\mathbf{x}_1,t)\,\delta(\mathbf{v}_2-\mathbf{v}_1), \\ \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} g_+(\mathbf{v}_+;\mathbf{x}_1,\mathbf{x}_2;t) &= f(\mathbf{v}_1,\mathbf{x}_1,t), \\ \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} g_-(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t) &= \delta(\mathbf{v}_-)=2^3\delta(\mathbf{v}_2-\mathbf{v}_1). \end{aligned} \tag{33}$$

4 The Cross-Independence Theory

In the velocity distribution formalism of turbulence, the closed equations for the one- and two-point velocity distributions f and $f^{(2)}$ constitute the minimum deterministic dynamical system. Thus, let us briefly outline and discuss these equations.

4.1 Closure of One-Point Equation

The Lundgren-Monin equation for the one-point velocity distribution f is expressed for general inhomogeneous turbulence as follows:

$$\begin{aligned} \left[\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{x}_1}\right] f(\mathbf{v}_1,\mathbf{x}_1,t) &= -\nu \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0}\left|\frac{\partial}{\partial \mathbf{x}_2}\right|^2 \frac{\partial}{\partial \mathbf{v}_1}\cdot\int \mathbf{v}_2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_2 \\ &+\frac{\partial}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{x}_1}\frac{1}{4\pi}\int\int|\mathbf{x}_2-\mathbf{x}_1|^{-1}\left(\mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{x}_2}\right)^2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_2d\mathbf{x}_2, \end{aligned} \tag{34}$$

which includes the higher-order distributions $f^{(2)}$ on the right-hand side.

The closure of this equation by means of the *cross-independence closure hypothesis* (32) has been made by Tatsumi (2001) for homogeneous isotropic turbulence and by Tatsumi (2011) for general inhomogeneous turbulence as

$$\begin{aligned} &\left[\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{x}_1}+\nu\frac{\partial}{\partial \mathbf{v}_1}\cdot\mathbf{v}_1\left|\frac{\partial}{\partial \mathbf{x}_1}\right|^2+\alpha(\mathbf{x}_1,t)\left|\frac{\partial}{\partial \mathbf{v}_1}\right|^2-\frac{\partial}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{x}_1}\beta(\mathbf{v}_1;\mathbf{x}_1,t)\right]\times \\ &\times f(\mathbf{v}_1,\mathbf{x}_1,t)=0, \end{aligned} \tag{35}$$

where the parameters $\alpha(\mathbf{x}_1,t)$ and $\beta(\mathbf{v}_1,\mathbf{x}_1,t)$ are defined as

$$\begin{aligned}
\alpha\left(\mathbf{x}_1,t\right) &= \frac{2}{3}\nu \lim_{\left|\mathbf{x}_2-\mathbf{x}_1\right|\to 0}\left|\frac{\partial}{\partial \mathbf{x}_2}\right|^2 \int \left|\mathbf{v}_-\right|^2 g_-\left(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t\right)d\mathbf{v}_- \\
&= \frac{2}{3}\nu \lim_{\left|\mathbf{x}_2-\mathbf{x}_1\right|\to 0}\left|\frac{\partial}{\partial \mathbf{x}_2}\right|^2 \left\langle \left|\mathbf{u}_-\left(\mathbf{x}_1,\mathbf{x}_2;t\right)\right|^2\right\rangle \\
&= \frac{1}{3}\nu \sum_{i,j=1}^{3}\left\langle\left(\frac{\partial u_i\left(\mathbf{x}_1,t\right)}{\partial x_{1j}}\right)^2\right\rangle = \frac{1}{3}\varepsilon\left(\mathbf{x}_1,t\right), \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\beta\left(\mathbf{v}_1,\mathbf{x}_1,t\right) &= \frac{1}{4\pi}\int\int \left|\mathbf{x}_2-\mathbf{x}_1\right|^{-1}\left(\mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{x}_2}\right)^2\left(1+\mathbf{v}_-\cdot\frac{\partial}{\partial \mathbf{v}_1}\right)\times \\
&\quad \times g_-\left(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t\right)d\mathbf{v}_- d\mathbf{x}_2. \qquad (37)
\end{aligned}$$

where the equivalence of the parameter α with the energy dissipation rate ε follows from Eq.(25).

It should be noted that the *energy dissipation rate* $\alpha = \varepsilon/3$ has been expressed by Eq.(36) in terms of an integral of the *velocity-difference distribution* g_- which is mostly contributed from small turbulent fluctuations. This clearly manifests the "*fluctuation-dissipation theorem*" in non-equilibrium statistical mechanics. Another important point is that the first-order moment equation of Eq.(35) is identical with the version (6) of the Navier-Stokes equation. This gives an evidence proof to the exactness of the cross-independence closure.

4.2 Closure of Two-Point Equation

The closed equation for the two-point velocity distribution $f^{(2)}$ is obtained from the correponding Lundgren-Monin equation applying the *cross-independence closure hypothesis* (32) to the pairs of the velocities $(\mathbf{v}_1,\mathbf{v}_3)$ and $(\mathbf{v}_2,\mathbf{v}_3)$ of the distribution $f^{(3)}$. The result depends upon whether the distance $r = |\mathbf{x}_2 - \mathbf{x}_1|$ belongs to the *energy-containing* range larger than the distances $r' = |\mathbf{x}_3 - \mathbf{x}_1|$ and $r'' = |\mathbf{x}_3 - \mathbf{x}_2|$ or to the *local* range comparable to r' and r''.

If the distance r belongs to the energy-containing range, the closed equation for the distribution $f^{(2)}$ is obtained as follows (see Tatsumi (2011)):

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\sum_{m=1}^{2}\left\{\mathbf{v}_m\cdot\frac{\partial}{\partial \mathbf{x}_m}+\nu\frac{\partial}{\partial \mathbf{v}_m}\cdot\mathbf{v}_m\left|\frac{\partial}{\partial \mathbf{x}_m}\right|^2\right\}\right]f^{(2)}\left(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t\right) \\
&= -\sum_{m=1}^{2}\left\{\alpha\left(\mathbf{x}_m,t\right)\left|\frac{\partial}{\partial \mathbf{v}_m}\right|^2-\frac{\partial}{\partial \mathbf{v}_m}\cdot\frac{\partial}{\partial \mathbf{x}_m}\beta\left(\mathbf{v}_m,\mathbf{x}_m,t\right)\right\}f^{(2)}\left(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t\right) \qquad (38)
\end{aligned}$$

where the parameters α and β are defined in the similar manner as those in §**4.1**.

The double integral moment of Eq.(38) gives the equation for the *velocity correlation* $U^{(2)}\left(\mathbf{x}_1,\mathbf{x}_2;t\right) = \left\langle \mathbf{u}_1\left(\mathbf{x}_1,t\right)\cdot\mathbf{u}_2\left(\mathbf{x}_{2},t\right)\right\rangle$ as

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}-\nu\left(\left|\frac{\partial}{\partial \mathbf{x}_1}\right|^2+\left|\frac{\partial}{\partial \mathbf{x}_2}\right|^2\right)\right]U^{(2)}\left(\mathbf{x}_1,\mathbf{x}_2;t\right)-\left\langle \mathbf{u}_2\cdot\left(\mathbf{u}_1\cdot\frac{\partial}{\partial \mathbf{x}_1}\right)\mathbf{u}_1\right\rangle \\
&-\left\langle \mathbf{u}_1\cdot\left(\mathbf{u}_2\cdot\frac{\partial}{\partial \mathbf{x}_2}\right)\mathbf{u}\right\rangle = \left\langle \mathbf{u}_2\cdot\frac{\partial}{\partial \mathbf{x}_1}\left(\frac{p_1}{\rho}\right)\right\rangle+\left\langle \mathbf{u}_1\cdot\frac{\partial}{\partial \mathbf{x}_2}\left(\frac{p_2}{\rho}\right)\right\rangle, \qquad (39)
\end{aligned}$$

which is again identical with the corresponding equation derived from the Navier-Stokes equation, giving another confirmation to the exactness of the *cross-independence closure.*

4.3 Closure of Two-Point Local Equation

In the *local* range, we have to use the local variables normalized by Kolmogorov's length $\eta = (\nu^3/\varepsilon_0)^{1/4}$ and velocity $\upsilon = (\nu\varepsilon_0)^{1/4}$, ε_0 being an initial value of $\varepsilon(\mathbf{x}, t)$. In this case, it is also necessary to transform the velocities $(\mathbf{v}_1, \mathbf{v}_2)$ in the degenerate distributions $f^{(3)}$ of the Lundgren-Monin equation for the distribution $f^{(2)}$ into the cross-velocities $(\mathbf{v}_+, \mathbf{v}_-)$ in order to secure their mutual independence. Then, the closed two-point local equation is obtained following the procedure as for Eq.(38) as follows (see Tatsumi (2011)):

$$\begin{aligned} &\left[\frac{\partial}{\partial t^*} + \sum_{m=1}^{2}\left\{\mathbf{v}_m^* \cdot \frac{\partial}{\partial \mathbf{x}_m^*} + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}_m^*} \cdot \mathbf{v}_m^* \left(\left|\frac{\partial}{\partial \mathbf{x}_1^*}\right|^2 + \left|\frac{\partial}{\partial \mathbf{x}_2^*}\right|^2\right)\right\}\right] f^{(2)}\left(\mathbf{v}_1^*, \mathbf{v}_2^*; \mathbf{x}_1^*, \mathbf{x}_2^*; t^*\right) \\ = \; & -\left[\sum_{\pm} \alpha_\pm^*\left(\mathbf{x}_1^*, \mathbf{x}_2^*; t^*\right)\left|\frac{\partial}{\partial \mathbf{v}_\pm^*}\right|^2 + \sum_{m=1}^{2} \frac{\partial}{\partial \mathbf{v}_m^*} \cdot \frac{\partial}{\partial \mathbf{x}_m^*} \beta^*\left(\mathbf{v}_m^*, \mathbf{x}_m^*, t^*\right)\right] \\ & \times f^{(2)}\left(\mathbf{v}_1^*, \mathbf{v}_2^*; \mathbf{x}_1^*, \mathbf{x}_2^*; t^*\right), \end{aligned} \tag{40}$$

where the parameters $\alpha_\pm^*(\mathbf{x}_1^*, \mathbf{x}_2^*; t^*)$ and $\beta^*(\mathbf{v}_m^*, \mathbf{x}_m^*, t^*)$ are defined as

$$\alpha_\pm^*\left(\mathbf{x}_1^*, \mathbf{x}_2^*; t^*\right) = \frac{2}{3} \lim_{\left|\mathbf{x}_3^* - \mathbf{x}_1^*\right| \to 0} \left|\frac{\partial}{\partial \mathbf{x}_3^*}\right|^2 \int \left|\mathbf{v}_{\pm -}^{*\prime}\right|^2 g_{\pm -}\left(\mathbf{v}_{\pm -}^{*\prime}; \mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{x}_3^*; t^*\right) d\mathbf{v}_{\pm -}^{*\prime}, \tag{41}$$

$$\begin{aligned} \beta^*\left(\mathbf{v}_m^*, \mathbf{x}_m^*, t^*\right) \;=\; & \frac{1}{4\pi} \int\!\!\int \left|\mathbf{x}_3^* - \mathbf{x}_m^*\right|^{-1} \left(\mathbf{v}_3^* \cdot \frac{\partial}{\partial \mathbf{x}_3^*}\right)^2 \left(1 + \mathbf{v}_{-m}^* \cdot \frac{\partial}{\partial \mathbf{v}_m^*}\right) \times \\ & \times g_-\left(\mathbf{v}_-^{*\prime}; \mathbf{x}_m^*, \mathbf{x}_3^*; t^*\right) d\mathbf{v}_{-m}^* d\mathbf{x}_3^*; \end{aligned} \tag{42}$$

where $\mathbf{v}_{-m}^* = (\mathbf{v}_3^* - \mathbf{v}_m^*)/2$. The parameters $\alpha_\pm^*(\mathbf{x}_1^*, \mathbf{x}_2^*; t^*)$ represent the energy-dissipation rates in the *local* range of r^* corresponding to $\alpha(\mathbf{x}_1, t)$ and $\alpha(\mathbf{x}_2, t)$ for the *global* range of r^*.

The double integral moment of Eq.(40) gives the equation for the two-point *velocity correlation* in the *local range*, $U^{(2)*}(\mathbf{x}_1^*, \mathbf{x}_2^*; t^*) = \langle \mathbf{u}_1^*(\mathbf{x}_1^*, t^*) \cdot \mathbf{u}_2^*(\mathbf{x}_2^*, t^*)\rangle$ which is completely compatible with the establised equation (39) for $U^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t)$ (see Tatsumi (2011)).

5 Concluding Remarks

It has been established that the closed equations for the one-point velocity distribution f, the two-point velocity distribution $f^{(2)}$ and the two-point local velocity distribution $f^{(2)*}$ constitute the minimum deterministic dynamical system. Thus the present theory, which is based on the unclosed but exact Lundgren-Monin equations and the exactly valid cross-independence closure, provide us with a complete formulation of 'statistical mechanics of fluid turbulence'.

6 References

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