Fluid Mechanics

Jim Emery

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1 Bernoulli's Law

Consider flow in a pipe of variable diameter. Assume the fluid is not compressible, there is zero viscosity and friction, and the internal energy and temperature of the fluid is constant.

Consider the volume of fluid between two cross sections of the pipe. At one end let the cross sectional area be A_1, the pressure p_1, the height h_1, and the velocity v_1. At the other end let the cross sectional area be A_2, the pressure p_2, the height h_2, and the velocity v_2. Suppose the surface boundary of the volume moves positive distance dx_1 in time dt at the first end. Then in effect a volume $dV = dx_1 A_1$, is moved to the other end with a new velocity v_2 with $dV = dx_2 A_2$. The work done on the volume V is

$$dW = dx_1 A_1 p_1 - dx_2 A_2 p_2 = dV(p_1 - p_2).$$

This is equal to the change dE of the kinetic and potential energy of the transferred volume dV.

$$dE = \frac{dV\rho}{2}(v_2^2 - v_1^2) + dVg\rho(h_2 - h_1).$$

Hence

$$dW - dE = 0,$$

and so

$$\frac{dW - dE}{dV} = (p_1 - p_2) - \left[\frac{\rho}{2}(v_2^2 - v_1^2) + g\rho(h_2 - h_1)\right] = 0.$$

So

$$p_1 + \frac{\rho}{2}v_1^2 + g\rho h_1 = p_2 + \frac{\rho}{2}v_2^2 + g\rho h_2.$$

That is

$$\frac{p}{g\rho} + \frac{v^2}{2g} + h,$$

is constant anywhere in the pipe.

This is Bernoulli's Law.

2 Torricelli's Law

Suppose we have a tank of relatively large diameter and height h. Suppose there is an exit hole of area A at the bottom of the tank. We may apply Bernoulli's law where we take both the pressure and velocity at the top of the tank to be zero. And we assume that the pressure at the bottom of the tank, in the exit hole, is also zero. Thus we have

$$h = \frac{v^2}{2g}$$

So the exit velocity is

$$v = \sqrt{2gh}.$$

This is Torricelli's law.

3 Time to Empty a Tank

From Torricelli's law the change of volume due to flow out of a bottom hole of area A is

$$\frac{dV}{dt} = -A\sqrt{2gh},$$

where

$$\frac{dV}{dt}$$

might be measured in cubic meters per second.

If the cylindrical tank has radius r then

$$V(h) = h\pi r^2$$

So

$$dV = \pi r^2 dh$$

Thus

$$dt = -\frac{\pi r^2}{A\sqrt{2gh}} dh.$$

Thus the time to drain the tank is

$$t = -\frac{\pi r^2}{A\sqrt{2g}} \int_{h_0}^{0} \frac{dh}{\sqrt{h}},$$

where h_0 is the full height of the tank. Thus

$$\begin{aligned} t &= -\frac{\pi r^2}{A\sqrt{2g}}[2\sqrt{h}]_{h_0}^{0} \\ &= \frac{2\pi r^2}{A\sqrt{2g}}\sqrt{h_0} \\ &= \frac{\pi r^2}{A}\sqrt{2h_0/g} \\ &= \frac{A_t}{A_h}\sqrt{2h_0/g}, \end{aligned}$$

where A_t is the cross sectional area of the tank, and A_h is the cross sectional area of the hole at the bottom of the tank. This is assuming zero viscosity of the fluid.

Notice that if $A_h = A_t$ the time is

$$\sqrt{2h_0/g},$$

which is the time it takes a mass to fall a distance h_0. However, we are using Torricelli's law, whose derivation assummed that the cross sectional area of the tank to be very much larger than the exit hole.

4 Viscosity

Newton first introduced the concept of viscosity using a idea analogous to a shear strain, replacing static displacement by velocity. Thus in the simplest case, if a fluid is flowing between two parallel plates, one moving with velocity V and the other fixed, then in a time δt the fluid contacting the upper plate moves a distance $V\delta t$ while the fluid attached to the lower plate is at rest. So the fluid is being sheared. If it were a solid this displacement would be accompanied by a shear stress equal to a force per unit area on the upper plate. Assuming that the velocity between the plates varies linearly with height, the shear strain would be

$$\frac{du}{dy}\delta t$$

If the fluid were a solid this shear strain and shear stress would be proportional. In the case of a fluid the force between layers of fluid is determined by the change in velocity rather than the displacement. Hence the shear stress τ is proportional to the gradient of the velocity rather than to the gradient of the displacement as is the case in elasticity. Thus the shear stress is

$$\tau = \mu \frac{dv}{dy}$$

where v is the horizontal velocity and y is the distance perpendicular to the flow and to the plates. μ is called the viscosity. Newton first introduced this equation and hence a fluid that obeys this law is called a Newtonian fluid. From this equation the units of viscosity are FT/L^2 where F is force, T is time, and L is distance. So in the SI system the unit of viscosity is Newton second per meter squared,

$$\frac{Ns}{m^2}.$$

In the cgs system it is in dyne seconds per square centimeter

$$\frac{ds}{cm^2},$$

a unit called the Poise.

5 The Acceleration

Acceleration in fluid mechanics can be confusing. This is because there are two different velocities, a particle velocity attached to a moving particle, or to an infinitesimal volume in the fluid, and a flow velocity at a fixed point in the fluid. As a particle p moves in the fluid with a particle velocity, which we write as $\mathbf{v}_p(t)$, it may experience an acceleration, as any particle can in mechanics. This is the acceleration of Newton's second law, which is proportional to the force on the particle. However, at the same time we may consider the flow velocity at a fixed point $\mathbf{x}$ in space , which we call the field velocity, which we write as $\mathbf{v}(t, \mathbf{x})$. As time passes this velocity will be the velocity of different particles as they pass this fixed point. We consider this to be the fluid velocity at that point. This fluid velocity, the field velocity, is written as

$$\mathbf{v}(t, \mathbf{x}).$$

The rate of change of this field velocity at the fixed point $\mathbf{x}$ is not the acceleration of a particle that happens to be at $\mathbf{x}$ at time t. However, acceleration of the particle velocity, sometimes called the material velocity, can be computed by differentiating the field velocity with respect to time. The result is obtained by partial differentiation as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}.$$

Indeed, suppose a particle p has coordinates

$$(x_p(t), y_p(t), z_p(t)),$$

at time t. Then the particle velocity at time t is

$$\left(\frac{dx_p(t)}{dt}, \frac{dy_p(t)}{dt}, \frac{dz_p(t)}{dt} \right) = \mathbf{v}_p(t),$$

which has the same value as the field velocity at the point $(x_p(t), y_p(t), z_p(t))$ at time t. That is,

$$\mathbf{v}_p(t) = \mathbf{v}(t, x_p(t), y_p(t), z_p(t)).$$

The two velocities are distinguished as function, being functions of different variables. The x component of particle acceleration is

$$\frac{d^2 x_p(t)}{dt^2} = \frac{dv_{px}(t)}{dt} = \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}\frac{dx_p}{dt} + \frac{\partial v_x}{\partial y}\frac{dy_p}{dt} + \frac{\partial v_x}{\partial z}\frac{dz_p}{dt}$$

$$= \frac{\partial v_x}{\partial t} + \nabla v_x \cdot \mathbf{v}_p(t).$$

We have written v_x for the x component of $\mathbf{v}$. Similarly,

$$\frac{dv_{py}(t)}{dt} = \frac{\partial v_y}{\partial t} + \nabla v_y \cdot \mathbf{v}_p(t),$$

and

$$\frac{dv_{pz}(t)}{dt} = \frac{\partial v_z}{\partial t} + \nabla v_z \cdot \mathbf{v}_p(t).$$

The particle velocity has the same value as the field velocity at time t,

$$\mathbf{v}_p(t) = \mathbf{v}(t, x_p(t), y_p(t), z_p(t)),$$

so we can replace $\mathbf{v}_p$ by $\mathbf{v}$, and then obtain

$$\mathbf{a} = \frac{d\mathbf{v}_p}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v},$$

as written above.

6 The Equation of Motion Due to Cauchy

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T}$$

where the left side is the acceleration, **f** is an external force, and **T** is the stress tensor. (Serrin, p 136)

7 The Deformation Tensor

In the case of elasticity, the strain tensor is defined in terms of the derivatives of the displacement u. Let u_1, u_2, u_3 be the coordinates of a displacement. The strain tensor is defined by

$$e_{ij} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})/2.$$

See Solkolnikoff, **The Mathematical Theory of Elasticity**, p25, for comments about alternate notation for the strain coefficients. The strain is a Cartesian tensor, as one sees by looking at the properties of the gradient and the Jacobian. In the usual case the stress tensor is a linear function of the strain tensor.

In the case of fluid mechanics we have not just displacement but continuous motions, so the analog of the strain tensor is called the deformation tensor **D**. It is defined similarly to the strain tensor, replacing the displacement **u** by the velocity **v**,

$$d_{ij} = (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})/2.$$

8 The Navier-Stokes Equation for Incompressible Flow

In the case where the stress tensor is a linear function of the deformation tensor **D**, the case of linear viscosity, the Cauchy equation of motion become the Navier-Stokes equation,

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \nabla \cdot (2\mu \mathbf{D}).$$

(Serrin, p 217)

9 Flow in Pipes, Poiseuille Flow

The Navier-Stokes equation is solvable in closed form for only a few special cases. One of these is the steady laminar flow in cylindrical pipes (Munson et. al., page 379). This flow is called Hagan-Poiseuille flow. Poiseuille's law (pwah-zweez)

$$Q = \frac{\pi R^4 \Delta p}{8\mu\ell},$$

where Q is the volume rate of flow, R is the pipe radius, Δp, is the pressure drop, μ is the viscosity, and ℓ is the pipe length. At any cross section the velocity distribution is parabolic.

Notice that if the viscosity were zero, then the flow rate would be infinite. This seems a bit disturbing. However, if the viscosity were zero, then there would be no resistance to flow, and the pressure difference would accelerate the velocity to infinity.

10 Flow in River Channels

11 Ground Water Flow

12 Diffusion

Various kinds of diffusion processes are governed by a diffusion equation, which is similar to the heat equation.

13 Dimensional Analysis and the Reynolds Number

The Reynolds Number is a dimensionless number which determines the boundary between laminar flow and turbulent flow.

$$\frac{\rho v d}{\mu},$$

where ρ is the density, v is the velocity, d is the pipe diameter (or some other characteristic length in a non-pipe problem), and μ is the viscosity. The dimension is

$$\begin{aligned}&(m\ell^{-3})(\ell t^{-1})(\ell)(\ell^2 f^{-1} t^{-1})\\=\ &(f\ell^{-1}t^2)(\ell^{-3})(\ell t^{-1})(\ell)(\ell^2 f^{-1} t^{-1})\\=\ &f^0\ell^0 t^0,\end{aligned}$$

where m is mass, f is force, ℓ is length, and t is time.

14 Hydrology

Hydrology is an earth science that studies the movement of water and water vapor on the earth.

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