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Beta and Gamma Product of Fuzzy Graphs

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Abstract. In this paper, Beta product and Gamma product of two fuzzy graphs are introduced and we proved that the Beta product of two regular fuzzy graphs need not be regular and that if $G_1 \underset{\beta}{\times} G_2$ is regular, then G_1 (or) G_2 need not be regular. A necessary and sufficient condition for $G_1 \underset{\beta}{\times} G_2$ and $G_1 \underset{\gamma}{\times} G_2$ to be a regular fuzzy graph is determined. The degree of vertices in $G_1 \underset{\beta}{\times} G_2$ and $G_1 \underset{\gamma}{\times} G_2$ in terms of those in G_1 and G_2 are determined for some particular cases and regular property of $G_1 \underset{\beta}{\times} G_2$ and $G_1 \underset{\gamma}{\times} G_2$ are studied.

Keywords: Regular fuzzy graph, product of fuzzy graph, complete graph, regular graph, complement graph

AMS Mathematics Subject Classification (2010): 03E72, 05C72

1. Introduction

Fuzzy graph theory was introduced and developed by Rosenfeld in [8] and generalized standard results in graph theory [1,2]. Though it is very young, it has been growing fast and has numerous applications in various fields. During the same time Yeh and Bang [9] have also introduced various connectedness concepts in fuzzy graphs. The operations of union, join, Cartesian product and composition on two fuzzy graphs were defined by Mordeson and Peng [3]. Recently, new compositions on fuzzy graphs are introduced and studied in [4,6,7]. In this paper, we study about the regular property of the β- product and the γ-product of two fuzzy graphs. We determine necessary and sufficient conditions for the β- product and the γ-product of two fuzzy graphs to be regular under some restrictions are determined. Throughout this paper we assume that μ is reflexive and need not consider loops. Also, the underlying set V is assumed to be finite and σ can be chosen in any manner so as to satisfy the definition of a fuzzy graph in all the examples and all these properties are satisfied for all fuzzy graphs except null graphs.

2. Preliminaries

A *fuzzy graph* G: (σ,μ)G is a pair of functions $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$ where for all u,v$\in$ V, we have $\mu(uv) \le \sigma(u)\wedge\sigma(v)$.The *underlying crisp graph* of G: (σ,μ) is denoted by G*:(V, E) where E $\subseteq$V×V.
If $\mu(uv)=\sigma(u)\wedge\sigma(v)$ for all u,v$\in$ V, then G is called a *complete fuzzy graph*. The *complement* $\overline{G}$ *of a graph G* also has V(G) as its point set ,but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let G: (σ,μ) be a fuzzy graph. The *degree of a vertex u* is $d_G(u)=\sum_{uv\in E}\mu(uv)=\sum_{u\neq v}\mu(uv)$.Let G* :(V,E) be a graph. The *degree* $d_{G^*}(v)$ of a vertex v in G* is the number of edges incident with v.If all the vertices of G* have the same degree r, then G* is called a *regular graph* of degree r, here r is an integer.

Let G: (σ,μ) be a fuzzy graph on G* :(V,E).If $d_G(v)$=k for all v$\in$ V,that is, if each vertex has same degree k in G,then G is said to be a *regular fuzzy graph* of degree k or a k-regular fuzzy graph,here k need not be an integer[6]. The degree of vertices in fuzzy graphs have been studied in [5].

3. Beta Product of Fuzzy Graphs

Definition 3.1. The *β- product of two fuzzy graphs* $G_1$and G_2 is defined as a fuzzy graph, $G_1 \underset{\beta}{\times} G_2 = ((\sigma_1 \underset{\beta}{\times} \sigma_2),(\mu_1 \underset{\beta}{\times} \mu_2))$ on G* :(V,E) where V= $V_1 \underset{\beta}{\times} V_2$ and

E= $((u_1,u_2),(v_1,v_2))/\ u_1\neq v_1,\ u_2v_2\in E_2(or)u_2\neq v_2,\ u_1v_1\in E_1(or)u_1v_1\in E_1,\ u_2v_2\in E_2$

with $\sigma_1 \underset{\beta}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1)\wedge\sigma_2(u_2)$, $\forall (u_1,u_2)\in V_1 \underset{\beta}{\times} V_2$

$$\left(\mu_1 \underset{\beta}{\times} \mu_2\right)\left((u_1,u_2),(v_1,v_2)\right)$$
$$=\begin{cases}\mu_1(u_1v_1)\wedge\mu_2(u_2v_2), & \text{if } u_1v_1\in E_1, u_2v_2\in E_2\\ \sigma_2(u_2)\wedge\sigma_2(v_2)\wedge\mu_1(u_1v_1), & \text{if } u_2\neq v_2, u_1v_1\in E_1\\ \sigma_1(u_1)\wedge\sigma_1(v_1)\wedge\mu_2(u_2v_2), & \text{if } u_1\neq v_1, u_2v_2\in E_2\end{cases}$$

Example 3.2. Let V_1={ u_1,u_2 } and V_2 = { v_1, v_2 } such that

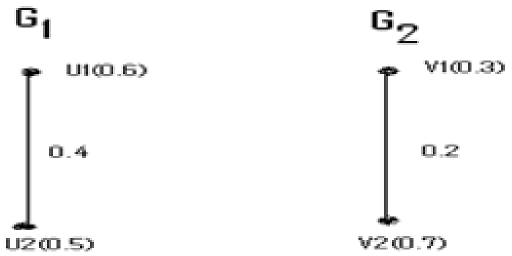

Figure 1:

β -product of two fuzzy graphs G_1 and G_2 is

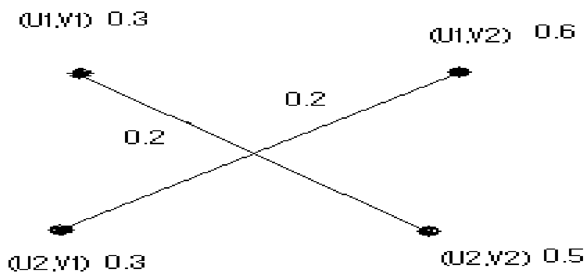

Figure 2:

Definition 3.3. The *β- product of two regular fuzzy graphs* $G_1$and G_2 is defined as a fuzzy graph , $G_1 \underset{\beta}{\times} G_2 = ((\sigma_1 \underset{\beta}{\times} \sigma_2),(\mu_1 \underset{\beta}{\times} \mu_2))$ on G* :(V,E) where

$V= V_1 \underset{\beta}{\times} V_2$ and

$E= ((u_1,u_2) , (v_1,v_2))/ u_1 \neq v_1, u_2v_2 \in E_2 (or) u_2 \neq v_2 , u_1v_1 \in E_1 (or) u_1v_1 \in E_1 , u_2v_2 \in E_2$

with $\sigma_1 \underset{\beta}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2) , \forall (u_1,u_2) \in V_1 \underset{\beta}{\times} V_2$

$$(\mu_1 \underset{\beta}{\times} \mu_2) ((u_1,u_2) , (v_1,v_2)) = \begin{cases} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) , & \text{if } u_1v_1 \in E_1, u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1) , & \text{if } u_2 \neq v_2, u_1v_1 \in E_1 \\ \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 \neq v_1, u_2v_2 \in E_2 \end{cases}$$

Remark 3.4. If G_1 and G_2 are regular fuzzy graphs then β -product of two fuzzy graphs G_1 and G_2 is need not be regular fuzzy graph.

Example 3.5. Let V_1={ u_1,u_2 } and V_2 = { v_1, v_2 , v_3 } such that

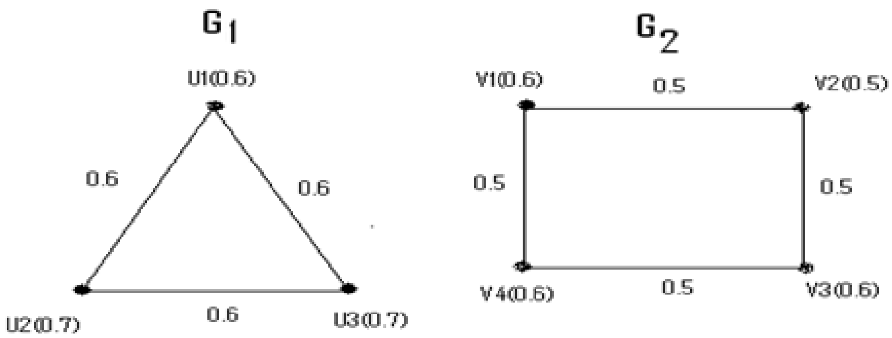

Figure 3:

Then β -product of two fuzzy graphs G_1 and G_2 is

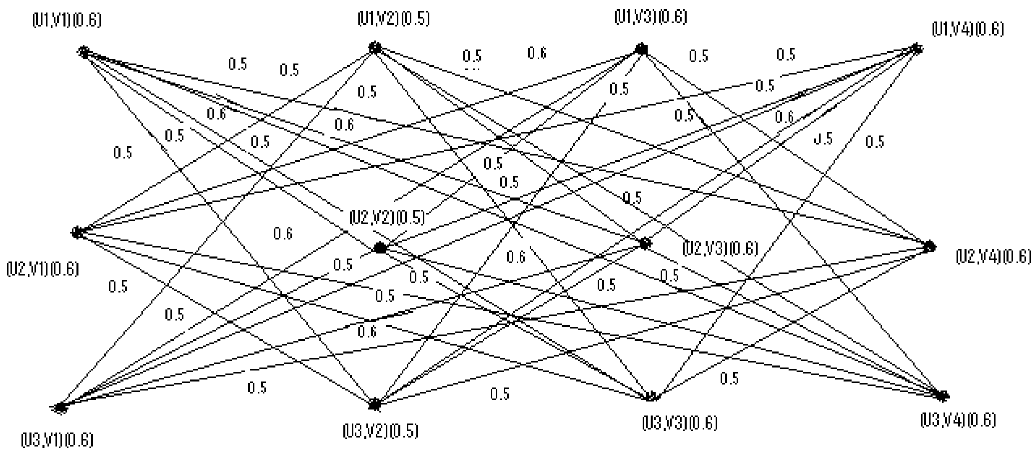

Figure 4:

Here both G_1 and G_2 are regular fuzzy graphs of degree 1.2 and 1.0. In $G_1 \times_\beta G_2$,$d_{G_1 \times_\beta G_2}(u_1,v_1)$ =3.2.$d_{G_1 \times_\beta G_2}(u_1,v_2)$ =3.0.Hence β -product of two fuzzy graphs G_1 and G_2 is not regular fuzzy graph.

Remark 3.6. If $G_1 \times_\beta G_2$ is a regular fuzzy graph ,then G_1 (or) G_2 need not be regular fuzzy graph.

Example 3.7. Let V_1={ u_1,u_2 } and V_2 = { v_1, v_2 , v_3 } such that

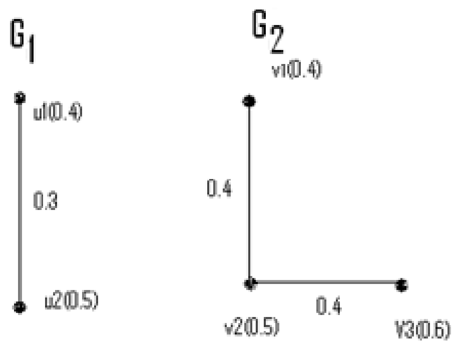

Figure 5:

$G_1 \times_\beta G_2$ is shown in Figure 6. Here both G_1 and G_2 are regular fuzzy graphs, since $d_{G_1 \times_\beta G_2}(u_i,v_j)$ =0.6, i=1,2; j=1,2,3. But, G_2 is not regular fuzzy graph.

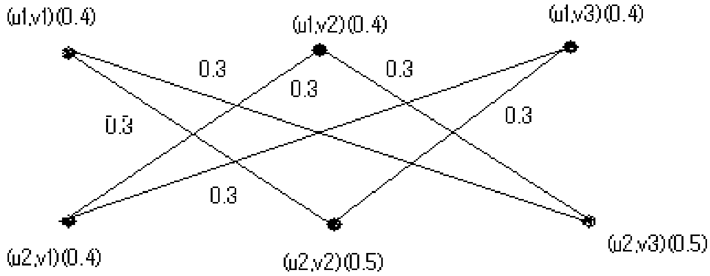

Figure 6:

4. Regular Properties of Beta Product of two Fuzzy Graphs

Theorem 4.1. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two fuzzy graphs such that underlying crisp graphs $G_1{}^*$ and $G_2{}^*$ are complete graphs, then $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph if and only if G_1 and G_2 are regular fuzzy graphs.*

Proof: Suppose that $G_1$and G_2 are regular fuzzy graphs of degrees k_1 and k_2 and $G_1{}^*$ and $G_2{}^*$ are complete graphs d_1 and d_2 respectively.

By definition for any $(u_1, u_2) \in V_1 \underset{\beta}{\times} V_2$,

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$$ [Since $G_1{}^*$ and $G_2{}^*$ are complete graphs.]

Case 1: If $\mu_1 \le \mu_2$, then

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1)$$

$$= d_{G_2}{}^*(u_2) d_{G_1}(u_1) \qquad (4.1.1)$$

$= d_2k_1$[since $d_{G_2}{}^*(u)= d_2, \forall u\in V_2, d_{G_1}(u) = k_1, \forall u\in V_1$]

This is true for all $(u_1, u_2) \in V_1 \underset{\beta}{\times} V_2$ .Hence $G_1 \underset{\beta}{\times} G_2$ is regular fuzzy graph.

Case 2: If $\mu_2 \le \mu_1$, then

From (4.1.1) $$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_2(u_2v_2)$$

$$= d_{G_1}{}^*(u_1) d_{G_2}(u_2) \qquad (4.1.2)$$

$= d_1k_2$[since $d_{G_1}{}^*(u)= d_1, \forall u\in V_1, d_{G_2}(u) = k_2, \forall u\in V_2$]

This is also true for all vertices of $V_1 \underset{\beta}{\times} V_2$.

Hence β -product of two fuzzy graphs G_1 and G_2 is regular fuzzy graph.

Conversely assume that $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph.

Then for any two points (u_1,u_2) & (v_1,v_2) in $V_1 \underset{\beta}{\times} V_2$

$$d_{G1 \underset{\beta}{\times} G2}(u_1, u_2) = d_{G1 \underset{\beta}{\times} G2}(v_1, v_2)$$

$$d_{G2^*}(u_2)d_{G1}(u_1) = d_{G2^*}(v_2)d_{G1}(v_1) \qquad \text{[using (4.1.1)]} \qquad (4.1.3)$$

Fix $u \in V_1$ and consider (u, u_2) and (u, v_2) in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

From (4.1.3), $d_{G2^*}(u_2)d_{G1}(u) = d_{G2^*}(v_2)d_{G1}(u)$

$\Rightarrow \qquad d_{G2^*}(u_2) = d_{G2^*}(v_2)$

This is true for all vertices $u_2, v_2 \in V_2$. Hence G_2^* is a regular graph. (4.1.4)

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (4.1.3), $d_{G2^*}(v)d_{G1}(u_1) = d_{G2^*}(v)d_{G1}(v_1)$

$\Rightarrow \qquad d_{G1}(u_1) = d_{G1}(v_1)$

This is true for all vertices $u_1, v_1 \in V_1$. Hence G_1 is a regular fuzzy graph. (4.1.5)

Similarly using (4.1.2) $d_{G1 \underset{\beta}{\times} G2}(u_1, u_2) = d_{G1 \underset{\beta}{\times} G2}(v_1, v_2)$

$$d_{G1^*}(u_1)d_{G2}(u_2) = d_{G1^*}(v_1)d_{G2}(v_2) \qquad (4.1.6)$$

Fix $u \in V_1$ and consider (u, u_2) and (u, v_2) in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

$$d_{G1^*}(u)d_{G2}(u_2) = d_{G1^*}(u)d_{G2}(v_2)$$

$\Rightarrow \qquad d_{G2}(u_2) = d_{G2}(v_2)$. This is true for all vertices $u_2, v_2 \in V_2$.

Hence G_2 is a regular fuzzy graph. (4.1.7)

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

$$d_{G1^*}(u_1)d_{G2}(v) = d_{G1^*}(v_1)d_{G2}(v)$$

$\Rightarrow \qquad d_{G1^*}(u_1) = d_{G1^*}(v_1)$

This is true for all vertices $u_1, v_1 \in V_1$. Hence G_1^* is a regular graph. (4.1.8)

From (4.1.5) and (4.1.7), if $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph, then G_1 and G_2 are regular fuzzy graphs of degree k_1 and k_2.

Theorem 4.2. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two fuzzy graphs and its underlying crisp graphs G_1^* is complete graph and G_2^* is regular graph. If $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$ and $\mu_1 = \mu_2$, then $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph if and only if G_1 is a regular fuzzy graph.*

Proof: Let G_2^* is a regular graph of degree d_2 and G_1^* is complete graph. Let $\mu_1 = \mu_2 = c$ for all E_1 and E_2, where c is a constant. We have $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Suppose that G_1 is a regular fuzzy graph of degree k_1.

By definition for any $(u_1,u_2) \in V_1 \underset{\beta}{\times} V_2$.

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) \quad [\text{since } \mu_1 = \mu_2]$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) \quad [\text{since } G_1^* \text{ is complete graph}]$$

$$= d_{G_1}(u_1) d_{G_2^*}(u_2) + \left|\overline{E2}\right| d_{G_1}(u_1)$$

$$= d_{G_1}(u_1)[d_{G_2^*}(u_2) + \left|\overline{E2}\right|]$$

Where $\left|\overline{E2}\right|$ is the degree of a vertex of complement graph G_2^*.

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = [d_{G_2^*}(u_2) + \left|\overline{E2}\right|] d_{G_1}(u_1) \qquad (4.2.1)$$

$$= [d_2 + \left|\overline{E2}\right|]k_1 [\text{since } d_{G_2}{}^*(u) = d_2, \forall u \in V_2 \ \& \ d_{G_1}(u) = k_1, \ \forall u \in V_1]$$

This is true for all vertices of $G_1 \underset{\beta}{\times} G_2$. Hence β -product of two fuzzy graphs G_1 and G_2 is regular fuzzy graph.

Conversely assume that $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph and G_2^* is a regular graph of degree $d_2$and G_1^* is complete graph.Then for any two points (u_1,u_2) & (v_1,v_2) in $V_1 \underset{\beta}{\times} V_2$,

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = d_{G_1 \underset{\beta}{\times} G_2}(v_1, v_2)$$

From (4.2.1)$[d_{G_2^*}(u_2) + \left|\overline{E2}\right|] d_{G_1}(u_1) = [d_{G_2^*}(v_2) + \left|\overline{E2}\right|] d_{G_1}(v_1)$

$$\Rightarrow \quad [d_2 + \left|\overline{E2}\right|] d_{G_1}(u_1) = [d_2 + \left|\overline{E2}\right|] d_{G_1}(v_1)$$

$\Rightarrow \quad d_{G_1}(u_1) = d_{G_1}(v_1)$ This is true for all $u_1, v_1 \in V_1$.

Hence G_1 is a regular fuzzy graph.

Theorem 4.3. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two fuzzy graphs and its underlying crisp graphs G_2^* is complete graph and G_1^* is regular graph .If $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$ and $\mu_1 = \mu_2$, then $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph if and only if G_2 is regular fuzzy graph.*

Proof: Let G_1^* is a regular graph of degree d_1 and G_2^* is complete graph .Let $\mu_1 = \mu_2 = c$ for all E_1 and E_2 ,where c is a constant. We have $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Suppose that G_2 is a regular fuzzy graph of degree k_2.

By definition for any $(u_1,u_2) \in V_1 \underset{\beta}{\times} V_2$.

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) \quad [\text{since } \mu_1 = \mu_2]$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) \quad [\text{since } G_2^* \text{ is complete graph}]$$

$$= d_{G_2}(u_2)\, d_{G_1^*}(u_1) + \left|\overline{E1}\right| d_{G_2}(u_2)$$

$$= d_{G_2}(u_2)[\, d_{G_1^*}(u_1) + \left|\overline{E1}\right|]$$

Where $\left|\overline{E1}\right|$ is the degree of a vertex of complement graph G_1^*.

$$d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = [\, d_{G_1^*}(u_1) + \left|\overline{E1}\right|]\, d_{G_2}(u_2) \qquad (4.3.1)$$

$$= [d_1 + \left|\overline{E1}\right|]k_2 [\text{since } d_{G_1^*}(u) = d_1, \forall\, u \in V_1 \,\&\, d_{G_2}(u) = k_2, \ \forall\, u \in V_2]$$

This is true for all vertices of $G_1 \underset{\beta}{\times} G_2$ .Hence β -product of two fuzzy graphs G_1 and G_2 is regular fuzzy graph.

Conversely assume that $G_1 \underset{\beta}{\times} G_2$ is a regular fuzzy graph and G_1^* is a regular graph of degree $d_1$and G_2^* is complete graph .Then for any two points (u_1,u_2) & (v_1,v_2) in $V_1 \underset{\beta}{\times} V_2$, $d_{G_1 \underset{\beta}{\times} G_2}(u_1, u_2) = d_{G_1 \underset{\beta}{\times} G_2}(v_1, v_2)$

From (4.3.1) $[\, d_{G_1^*}(u_1) + \left|\overline{E1}\right|]\, d_{G_2}(u_2) = [\, d_{G_1^*}(v_1) + \left|\overline{E1}\right|]\, d_{G_2}(v_2)$

$\Rightarrow$ $[d_1 + \left|\overline{E1}\right|]\, d_{G_2}(u_2) = [d_1 + \left|\overline{E1}\right|]\, d_{G_2}(v_2)$

$\Rightarrow$ $d_{G_2}(u_2) = d_{G_2}(v_2)$ This is true for all $u_2, v_2 \in V_2$.

Hence G_2 is a regular fuzzy graph.

Theorem 4.4. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two regular fuzzy graphs and its underlying crisp graphs G_1^* and G_2^* are regular but not complete graphs . If $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$,then β -product of two fuzzy graphs G_1 and G_2 is regular fuzzy graph,but converse is not true.*

Proof: Since G_1 and G_2 are regular fuzzy graphs, we have $d_{G1}(u) = k_1$, for every $u \in V_1$ and $d_{G2}(v) = k_2$, for every $v \in V_2$ and G_1^* and G_2^* are regular graphs of degree d_1 and d_2. Suppose that G_1^* and G_2^* are not complete graphs.

For any $(u_1,u_2) \in V_1 \times_\beta V_2$

$$d_{G1 \times_\beta G2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

Case(i): underlying crisp graphs G_1^* and G_2^* are isomorphic graphs and $\mu_1 = \mu_2$, say c.

Then we have $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$.

Therefore
$$d_{G1 \times_\beta G2}(u_1,u_2) = \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1)$$

$$= \left|\overline{E1}\right| d_{G2}(u_2) + \left|\overline{E2}\right| d_{G1}(u_1) + d_{G1}(u_1)\ d_{G2}^*(u_2)$$

where $\left|\overline{E1}\right|$ and $\left|\overline{E2}\right|$ is the degree of a vertex of a complement graphs G_1^* and G_2^*

$$d_{G1 \times_\beta G2}(u_1, u_2) = [d_{G2}^*(u_2) + \left|\overline{E2}\right|]\ d_{G1}(u_1) + \left|\overline{E1}\right| d_{G2}(u_2)$$

$$= [d_{G2}^*(u_2) + \left|\overline{E2}\right|]\ k_1 + \left|\overline{E1}\right|\ k_2$$

[since $d_{G1}(u_1) = k_1, \forall\ u_1 \in V_1$, $d_{G2}(u_2) = k_2, \forall\ u_2 \in V_2$]

$$= [d_2 + \left|\overline{E2}\right|]\ k_1 + \left|\overline{E1}\right|\ k_2$$

Since G_1^* and G_2^* are regular graphs of degree d_1 and d_2. G_1^* and G_2^* are isomorphic then $\left|\overline{E1}\right| = \left|\overline{E2}\right|$ .This is true for all vertices of $V_1 \times_\beta V_2$.

Hence β -product of two fuzzy graphs G_1 and G_2 is regular fuzzy graph.

Case (ii): Underlying crisp graphs G_1^* and G_2^* are not isomorphic and G_1^*, G_2^* are regular graphs of degrees d_1 and d_2. We have $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$.

Therefore
$$d_{G1 \times_\beta G2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) +$$

$$\sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1)\wedge\sigma_1(v_1)\wedge\mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1)\wedge\mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2) \qquad (4.4.1)$$

Suppose that $\mu_1\leq\mu_2$, then

$$d_{G_1\times_\beta G_2}(u_1, u_2) = \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2)$$

$$= d_{G_1}(u_1)\, d_{G_2}{}^*(u_2) + |\overline{E2}|\, d_{G_1}(u_1) + |\overline{E1}|\, d_{G_2}(u_2)$$

$$= d_{G_1}(u_1)[\,|\overline{E2}| + d_{G_2}{}^*(u_2)] + |\overline{E1}|\, d_{G_2}(u_2)$$

$$= k_1[\,|\overline{E2}| + d_{G_2}{}^*(u_2)] + |\overline{E1}|\, k_2$$

$$= k_1\,[d_2 + |\overline{E2}|\,] + |\overline{E1}|\, k_2$$

Clearly $G_1{}^*$ and $G_2{}^*$ are not isomorphic, then $|\overline{E1}| \neq |\overline{E2}|$,for each vertex.

Even though $G_1\times_\beta G_2$ is regular fuzzy graph. Suppose $\mu_2\leq\mu_1$,

Then, $$d_{G_1\times_\beta G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1\times\mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1)\wedge\mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \sigma_2(u_2)\wedge\sigma_2(v2)\wedge\mu_1(u_1v_1) +$$

$$\sum_{u_1\neq v_1, u_2v_2\in E_2} \sigma_1(u_1)\wedge\sigma_1(v_1)\wedge\mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_1(u_1v_1)\wedge\mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2)$$

$$= \sum_{u_1v_1\in E_1, u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_2\neq v_2, u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1\neq v_1, u_2v_2\in E_2} \mu_2(u_2v_2)$$

$$= d_{G_2}(u_2)\, d_{G_1}{}^*(u_1) + |\overline{E2}|\, d_{G_1}(u_1) + |\overline{E1}|\, d_{G_2}(u_2)$$

$$= d_{G_2}(u_2)[\,|\overline{E1}| + d_{G_1}{}^*(u_1)] + |\overline{E2}|\, d_{G_1}(u_1)$$

$$= k_2[\,|\overline{E1}| + d_{G_1}{}^*(u_1)] + |\overline{E2}|\, k_1$$

$$= k_2[\,|\overline{E1}| + d_1] + |\overline{E2}|\, k_1$$

Hence $G_1\times_\beta G_2$ is a regular fuzzy graph.

5. Gamma Product of Fuzzy Graphs

Definition 5.1. The **γ-** product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G_1\times_\gamma G_2 = (\sigma_1\times_\gamma\sigma_2, \mu_1\times_\gamma\mu_2)$ on $G^*:(V,E)$ where

$V = V_1 \underset{\gamma}{\times} V_2$ and

$E = ((u_1,u_2)\,,(v_1,v_2)) / u_1 = v_1,\ u_2v_2 \in E_2$ (or) $u_2 = v_2\ ,\ u_1v_1 \in E_1$ (or) $u_1 \neq v_1,\ u_2v_2 \in E_2$ (or) $u_2 \neq v_2\ ,\ u_1v_1 \in E_1$ (or) $u_1v_1 \in E_1\ ,\ u_2v_2 \in E_2$

with $\sigma_1 \underset{\gamma}{\times} \sigma_2(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)\ ,\ \forall\ (u_1,u_2) \in V_1 \underset{\gamma}{\times} V_2$

$$(\mu_1 \underset{\gamma}{\times} \mu_2)\ ((u_1,u_2)\,,(v_1,v_2))$$

$$= \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 = v_1, \quad u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu 1(u_1v_1), & \text{if } u_2 = v_2, \quad u_1v_1 \in E_1 \\ \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)\,, & \text{if } u_1 \neq v_1, \quad u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1), & \text{if } u_2 \neq v_2, \quad u_1v_1 \in E_1 \\ \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)\,, & \text{if } u_1v_1 \in E_1, \quad u_2v_2 \in E_2 \end{cases}$$

Example 5.2. The γ –product of two fuzzy graphs G_1 and G_2 have the vertex set $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3, v_4\}$ such that

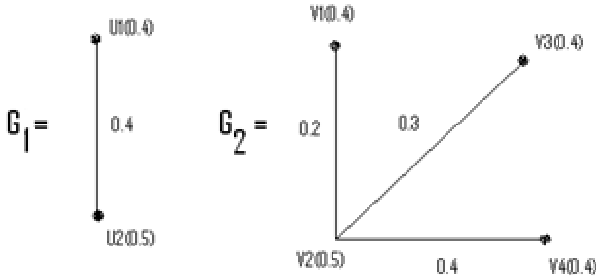

Figure 7:

Then $G_1 \underset{\gamma}{\times} G_2$ is

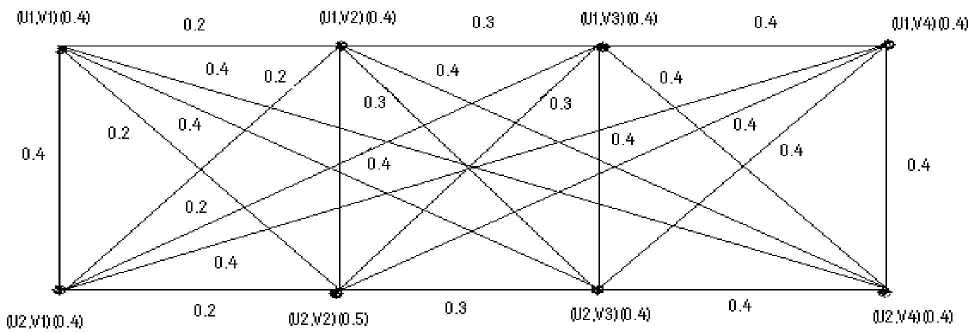

Figure 8:

6. Regular Properties of Gamma Product of Two Fuzzy Graphs

Theorem 6.1. *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1 \leq \mu_2$ and σ_1 is a constant. Then $G_1 \underset{\gamma}{\times} G_2$ is a regular fuzzy graph iff G_1 is a regular fuzzy graph and $G_2{}^*$ is a regular graph.*

Proof: Since σ_1 is a constant say c_1. Given $\sigma_1 \leq \mu_2$,then we have $\sigma_2 \geq \mu_1$.
Suppose that G_1 is a regular fuzzy graph of degree k_1 and G_2* is a regular graph of degree d_2 .By definition, for any $(u_1,u_2) \in V_1 \underset{\gamma}{\times} V_2$

$$d_{G_1 \underset{\gamma}{\times} G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1, u_2v_2 \in E_2} \sigma_1(u_1) \wedge \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \mu_1(u_1v_1) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2 \neq v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1 \neq v_1, u_2v_2 \in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1=v_1, u_2v_2 \in E_2} \sigma_1(u_1) + \sum_{u_2=v_2, u_1v_1 \in E_1} \mu_1(u_1v_1) + \sum_{u_1 \neq v_1, u_2v_2 \in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) + \sum_{u_2 \neq v_2, u_1v_1 \in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_1(u_1v_1)$$

$$= \sigma_1(u_1)\, d_{G_2}*(u_2) + d_{G_1}(u_1) + \sigma_1(u_1)\, d_{G_2}*(u_2)\left|\overline{E1}\right| + d_{G_1}(u_1)\left|\overline{E2}\right| + d_{G_1}(u_1) d_{G_2}*(u_2)$$

$$= d_{G_1}(u_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(u_2)] + \sigma_1(u_1)\, d_{G_2}*(u_2)[1+\left|\overline{E1}\right|] \qquad (6.1.1)$$

$= k_1[\, 1+\left|\overline{E2}\right| + d_2] + c_1\, d_2[1+\left|\overline{E1}\right|]$,since G_1 is a regular fuzzy graph of degree k_1& G_2 is a regular fuzzy graph of degree k_2 and σ_1 is a constant say c_1.
where $\left|\overline{E1}\right|$ and $\left|\overline{E2}\right|$ is the degree of the vertex of complement graphs G_1* and G_2* .
So γ –product of fuzzy graphs G_1 and G_2 is regular fuzzy graph.
Conversely assume that $G_1 \underset{\gamma}{\times} G_2$ is regular fuzzy graph. Then for any two points $(u_1,u_2)$and (v_1,v_2)in $V_1 \underset{\gamma}{\times} V_2$, $d_{G_1 \underset{\gamma}{\times} G_2}(u_1, u_2) = d_{G_1 \underset{\gamma}{\times} G_2}(v_1, v_2)$

$$d_{G_1}(u_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(u_2)] + \sigma_1(u_1)\, d_{G_2}*(u_2)[1+\left|\overline{E1}\right|] = d_{G_1}(v_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(v_2)] + \sigma_1(v_1) d_{G_2}*(v_2)[1+\left|\overline{E1}\right|] \qquad (6.1.2)$$

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1, v_1 \in V_1$are arbitrary.
From (6.1.2), $d_{G_1}(u_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(v)] + \sigma_1(u_1)\, d_{G_2}*(v)[1+\left|\overline{E1}\right|]$

$$= d_{G_1}(v_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(v)] + \sigma_1(v_1)\, d_{G_2}*(v)[1+\left|\overline{E1}\right|]$$

$$d_{G_1}(u_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(v)] + c_1\, d_{G_2}*(v)[1+\left|\overline{E1}\right|] = d_{G_1}(v_1)[\, 1+\left|\overline{E2}\right| + d_{G_2}*(v)] + c_1\, d_{G_2}*(v)[1+\left|\overline{E1}\right|]$$

[since $\sigma_1(u)=c_1$, $\forall\, u\in V_1$]

$\Rightarrow \quad d_{G1}(u_1)[\,1+|\overline{E2}| + d_{G2}{}^*(v)] = d_{G1}(v_1)[\,1+|\overline{E2}| + d_{G2}{}^*(v)]$

$\Rightarrow \quad d_{G1}(u_1) = d_{G1}(v_1)$

G_1 is regular fuzzy graph of degree k_1

Fix $u\in V_1$ and consider (u, u_2) and (u, v_2) in $V_1\times V_2$, where $u_2, v_2\in V_2$ are arbitrary.

$$d_{G1}(u)[\,1+|\overline{E2}| + d_{G2}{}^*(u_2)]+ \sigma_1(u)\, d_{G2}{}^*(u_2)[1+|\overline{E1}|\,]$$
$$= d_{G1}(u)[\,1+|\overline{E2}| + d_{G2}{}^*(v_2)]+\sigma_1(u) d_{G2}{}^*(v_2)[1+|\overline{E1}|\,] \quad (6.1.3)$$

Equation (6.1.3) can be modified in to the form,

$$d_{G2}{}^*(u_2)\{d_{G1}(u)+ \sigma_1(u)[1+|\overline{E1}|\,]\}+ d_{G1}(u)[\,1+|\overline{E2}|\,]=$$
$$d_{G2}{}^*(v_2)\{d_{G1}(u)+ \sigma_1(u)\,[1+|\overline{E1}|\,]\}+ d_{G1}(u)[\,1+|\overline{E2}|\,]$$

$\Rightarrow \quad d_{G2}{}^*(u_2)\{d_{G1}(u)+ \sigma_1(u)[1+|\overline{E1}|\,]\}= d_{G2}{}^*(v_2)\{d_{G1}(u)+ \sigma_1(u)\,[1+|\overline{E1}|\,]\}$

$\Rightarrow \quad d_{G2}{}^*(u_2)= d_{G2}{}^*(v_2)$.

This is true for all vertices. Hence $G_2{}^*$ is a regular graph of degree d_2.

Hence γ –product of two fuzzy graphs G_1 and G_2 are regular fuzzy graph.

Theorem 6.2. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two fuzzy graphs and its underlying crisp graphs $G_1{}^*$ and $G_2{}^*$ are completegraphs.If $\sigma_1\geq \mu_2$ and $\sigma_2\geq \mu_1$,then $G_1 \times_\gamma G_2$ is a regular fuzzy graph if and only if G_1 and G_2 are regular fuzzy graphs.*

Proof: Given $\sigma_1\geq \mu_2$ and $\sigma_2\geq \mu_1$ .Suppose that G_1 and G_2 are regular fuzzy graphs of degree k_1 and k_2 respectively.

For any vertex (u_1, u_2) in $V_1 \times_\gamma V_2$,

$$d_{G1\times_\gamma G2}(u_1, u_2)= \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1\times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1,u_2v_2\in E_2} \sigma_1(u_1) \wedge \mu_2(u_2v_2) + \sum_{u_2=v_2,u_1v_1\in E_1} \sigma_2(u_2) \wedge \mu_1(u_1v_1) +$$
$$\sum_{u_1v_1\in E_1,u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2\neq v_2,u_1v_1\in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) +$$
$$\sum_{u_1\neq v_1,u_2v_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1=v_1,u_2v_2\in E_2} \mu_2(u_2v_2) + \sum_{u_2=v_2,u_1v_1\in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1\in E_1,u_2v_2\in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$$

[since $G_1{}^*$ and $G_2{}^*$ are complete graphs]

Case (i): $\mu_1 \le \mu_2$

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = \sum_{u_1=v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_1(u_1v_1)$$

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_2}{}^*(u_2)\, d_{G_1}(u_1)$$

$$= d_{G_2}(u_2) + d_{G_1}(u_1)[1 + d_{G_2}{}^*(u_2)] \qquad (6.2.1)$$

$$= k_2 + k_1 [1 + d_{G_2}{}^*(u_2)]$$

[since $d_{G_1}(u_1)=k_1, \forall\, u_1 \in V_1$, $d_{G_2}(u_2)=k_2, \forall\, u_2 \in V_2$]

$= k_2 + k_1 [1 + d_2]$, since $G_2{}^*$ is complete graph of degree d_2

Thus $G_1 \times_\gamma G_2$ is regular fuzzy graph.

Case (ii): $\mu_2 \le \mu_1$

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = \sum_{u_1=v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \mu_1(u_1v_1) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_2(u_2v_2)$$

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_1}{}^*(u_1)\, d_{G_2}(u_2)$$

$$= d_{G_1}(u_1) + d_{G_2}(u_2)[1 + d_{G_1}{}^*(u_1)] \qquad (6.2.2)$$

$= k_1 + k_2 [1 + d_1]$ [since $G_1{}^*$ is complete graph of degree d_1]

Thus $G_1 \times_\gamma G_2$ is regular fuzzy graph.

Conversely assume that $G_1 \times_\gamma G_2$ is a regular fuzzy graph.

For any two vertices (u_1, u_2) and (v_1, v_2) in $V_1 \times_\gamma V_2$, we have

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = d_{G_1 \times_\gamma G_2}(v_1, v_2)$$

For $\mu_1 \le \mu_2$, from (6.2.1)

$d_{G_2}(u_2) + d_{G_1}(u_1)[1 + d_{G_2}{}^*(u_2)] = d_{G_2}(v_2) + d_{G_1}(v_1)[1 + d_{G_2}{}^*(v_2)]$

Fix $u \in V_1$ and consider (u, u_2) and (u, v_2) in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

$d_{G_2}(u_2) + d_{G_1}(u)[1 + d_{G_2}{}^*(u_2)] = d_{G_2}(v_2) + d_{G_1}(u)[1 + d_{G_2}{}^*(v_2)]$

Since $G_1{}^*$ and $G_2{}^*$ are complete graphs, we have $d_{G_2}{}^*(u)=d_2$, for every $u \in V_2$ and $d_{G_1}{}^*(u)=d_1$, for every $u \in V_1$.

Thus $d_{G_2}(u_2) + d_{G_1}(u)[1 + d_2] = d_{G_2}(v_2) + d_{G_1}(u)[1 + d_2]$

$\Rightarrow \quad d_{G_2}(u_2) = d_{G_2}(v_2)$

This is true for all vertices $u_2, v_2 \in V_2$. Thus G_2 is a regular fuzzy graph.

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary.

From (6.2.1), $d_{G_2}(v) + d_{G_1}(u_1)[1 + d_{G_2}{}^*(v)] = d_{G_2}(v) + d_{G_1}(v_1)[1 + d_{G_2}{}^*(v)]$

$\Rightarrow \quad d_{G_1}(u_1)[1 + d_{G_2}{}^*(v)] = d_{G_1}(v_1)[1 + d_{G_2}{}^*(v)]$

$\Rightarrow \quad d_{G_1}(u_1)[1 + d_2] = d_{G_1}(v_1)[1 + d_2]$

[since $d_{G_2}{}^*(u)=d_2$, for every $u \in V_2$]

$\Rightarrow \quad d_{G_1}(u_1) = d_{G_1}(v_1)$

This is true for all vertices V_1. Thus G_1 is a regular fuzzy graph.

For $\mu_2 \leq \mu_1$: For any two vertices (u_1, u_2) and (v_1, v_2) in $V_1 \times_\gamma V_2$,we have

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = d_{G_1 \times_\gamma G_2}(v_1, v_2)$$

From (6.2.2) ,$d_{G_1}(u_1)+ d_{G_2}(u_2)[1+ d_{G_1^*}(u_1)] = d_{G_1}(v_1)+ d_{G_2}(v_2)[1+ d_{G_1^*}(v_1)]$

Fix $u \in V_1$ and consider (u, u_2) and (u, v_2) in $V_1 \times V_2$, where $u_2, v_2 \in V_2$ are arbitrary.

$$d_{G_1}(u)+ d_{G_2}(u_2)[1+ d_{G_1^*}(u)] = d_{G_1}(u)+ d_{G_2}(v_2)[1+ d_{G_1^*}(u)]$$

$\Rightarrow$ $$d_{G_2}(u_2)[1+ d_1] = d_{G_2}(v_2)[1+ d_1]$$

[since $d_{G_1^*}(u)=d_1$, for every $u \in V_2$]

$\Rightarrow$ $$d_{G_2}(u_2) = d_{G_2}(v_2)$$

This is true for all vertices V_2.Thus G_2 is a regular fuzzy graph.

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1, v_1 \in V_1$ are arbitrary

Then From (6.2.2) ,we have

$$d_{G_1}(u_1)+ d_{G_2}(v)[1+ d_{G_1^*}(u_1)] = d_{G_1}(v_1)+ d_{G_2}(v)[1+ d_{G_1^*}(v_1)]$$

$\Rightarrow$ $d_{G_1}(u_1)+ d_{G_2}(v)[1+ d_1] = d_{G_1}(v_1)+ d_{G_2}(v)[1+ d_1]$, since G_1^* is complete graph of degree d_1.

Hence $d_{G_1}(u_1)= d_{G_1}(v_1)$. Thus G_1 isa regular fuzzy graph.

Theorem 6.3. *Let $G_1=(\sigma_1, \mu_1)$ and $G_2=(\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_2 \leq \mu_1$ and σ_2 is a constant. Then $G_1 \times_\gamma G_2$ is a regular fuzzy graph iff G_2 is a regular fuzzy graph and G_1^* is a regular graph.*

Proof: Given σ_2 is a constant say c_2 and $\sigma_2 \leq \mu_1$,then we have $\sigma_1 \geq \mu_2$.

By definition, for any $(u_1,u_2) \in V_1 \times_\gamma V_2$

$$d_{G_1 \times_\gamma G_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E} (\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1, u_2v_2 \in E_2} \sigma_1(u_1) \wedge \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \mu_1(u_1v_1) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) + \sum_{u_2 \neq v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) \wedge \mu_1(u_1v_1) + \sum_{u_1 \neq v_1, u_2v_2 \in E_2} \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2)$$

$$= \sum_{u_1=v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \sigma_2(u_2) + \sum_{u_1 \neq v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) + \sum_{u_2 \neq v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \sigma_2(v2) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$$

Clearly $\mu_2 \leq \mu_1$,we have

$$d_{G_1 \times_\gamma G_2}(u_1,u_2) = \sum_{u_1=v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) + \sum_{u_2=v_2, u_1v_1 \in E_1} \sigma_2(u_2) + \sum_{u_1 \neq v_1, u_2v_2 \in E_2} \mu_2(u_2v_2) +$$

$$\sum_{u_2 \neq v_2, u_1v_1 \in E_1} \sigma_2(u_2) \wedge \sigma_2(v_2) + \sum_{u_1v_1 \in E_1, u_2v_2 \in E_2} \mu_2(u_2v_2)$$

$= d_{G2}(u_2)+ \sigma_2(u_2)\, d_{G1}^*(u_1)+ d_{G2}(u_2)\left|\overline{E1}\right| + \left|\overline{E2}\right| \sigma_2(u_2)\, d_{G1}^*(u_1)+ d_{G2}(u_2) d_{G1}^*(u_1)$

$= d_{G2}(u_2)[1+\left|\overline{E1}\right| + d_{G1}^*(u_1)]+ d_{G1}^*(u_1)\, \sigma_2(u_2)[1+\left|\overline{E2}\right|]$ (6.3.1)

Assume that G_2 is a regular fuzzy graph of degree k_2 and G_1^* is a regular graph of degree d_1 .

Then (6.3.1) becomes $d_{G1 \times_\gamma G2}(u_1 , u_2)=k_2[1+\left|\overline{E1}\right| + d_1]+ d_1 c_2[1+\left|\overline{E2}\right|]$ (6.3.2)

where σ_2 is a constant ,say c_2 and $\left|\overline{E1}\right|$ and $\left|\overline{E2}\right|$ are the degree of a vertex of complement graphs G_1^* and G_2^* .

Thus from (6.3.2) , γ –product of two fuzzy graphs G_1 and G_2 is a regular fuzzy graph.

Conversely assume that $G_1 \times_\gamma G_2$ is a regular fuzzy graph.

Then for any two points (u_1,u_2) and (v_1,v_2) in $V_1 \times_\gamma V_2$,we have

$$d_{G1 \times_\gamma G2}(u_1 , u_2) = d_{G1 \times_\gamma G2}(v_1 , v_2)$$

$d_{G2}(u_2)[1+\left|\overline{E1}\right| +d_{G1}^*(u_1)]+d_{G1}^*(u_1)\sigma_2(u_2)[1+\left|\overline{E2}\right|]$

$= d_{G2}(v_2)[1+\left|\overline{E1}\right| +d_{G1}^*(v_1)]+ d_{G1}^*(v_1)\sigma_2(v_2)[1+\left|\overline{E2}\right|]$ (6.3.3)

Now fix $v \in V_2$ and consider (u_1, v) and (v_1, v) in $V_1 \times V_2$, where $u_1 , v_1 \in V_1$ are arbitrary.

From (6.3.3), $d_{G2}(v)[1+\left|\overline{E1}\right| +d_{G1}^*(u_1)]+d_{G1}^*(u_1)\sigma_2(v)[1+\left|\overline{E2}\right|]$

$= d_{G2}(v)[1+\left|\overline{E1}\right| +d_{G1}^*(v_1)]+d_{G1}^*(v_1)\sigma_2(v)[1+\left|\overline{E2}\right|]$

The above equation can be modified in to

$d_{G1}^*(u_1)[d_{G2}(v)+\sigma_2(v)[1+\left|\overline{E2}\right|]+d_{G2}(v)[1+\left|\overline{E1}\right|]=$

$d_{G1}^*(v_1)[d_{G2}(v)+\sigma_2(v)[1+\left|\overline{E2}\right|]+ d_{G2}(v)[1+\left|\overline{E1}\right|]$

$\Rightarrow$ $d_{G1}^*(u_1)[d_{G2}(v)+\sigma_2(v)[1+\left|\overline{E2}\right|]= d_{G1}^*(v_1)[d_{G2}(v)+\sigma_2(v)[1+\left|\overline{E2}\right|]$

$\Rightarrow$ $d_{G1}^*(u_1)= d_{G1}^*(v_1)$

This is true for all vertices of $u_1, v_1 \in V_1$. Hence G_1^* is a regular graph of degree d_1 .

Now fix $u \in V_1$ and consider (u, u_2) and (u, v_2) in $V_1 \times V_2$, where u_2 , $v_2 \in V_2$ are arbitrary.

From (6.3.3), $d_{G2}(u_2)[1+\left|\overline{E1}\right| +d_{G1}^*(u)]+d_{G1}^*(u)\sigma_2(u_2)[1+\left|\overline{E2}\right|]$

$= d_{G2}(v_2)[1+\left|\overline{E1}\right| +d_{G1}^*(u)]+ d_{G1}^*(u)\sigma_2(v_2)[1+\left|\overline{E2}\right|]$

$d_{G2}(u_2)[1+\left|\overline{E1}\right| +d_{G1}^*(u)]+d_{G1}^*(u)c_2[1+\left|\overline{E2}\right|]$

$$= d_{G2}(v_2)[1+\left|\overline{E1}\right| + d_{G1}^*(u)] + d_{G1}^*(u)c_2[1+\left|\overline{E2}\right|]$$

(since $\sigma_2(v)=c_2, \forall\, v\in V_2$)

$$d_{G2}(u_2)[1+\left|\overline{E1}\right| + d_{G1}^*(u)] = d_{G2}(v_2)[1+\left|\overline{E1}\right| + d_{G1}^*(u)]$$

$$\Rightarrow \qquad d_{G2}(u_2) = d_{G2}(v_2)$$

This is true for all vertices of V_2. Hence G_2 is a regular fuzzy graph of degree k_2.

7. Conclusion

It is convenient to consider large fuzzy graph as a combination of small fuzzy graphs and to derive its properties from those of the small ones .Operation on fuzzy graph is a great tool that can be used for this purpose. We made a step in that direction through this paper. Much more work can be done to investigate the structure of Beta and Gamma product which would have applications in communication networks, Information technology and so on.

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