Logarithmic vector fields and multiplication table.

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Dedicated to the 61st birthday of Kyoji Saito

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Abstract

This is a review article on the Gauss-Manin system associated to the complete intersection singularities of projection. We show how the logarithmic vector fields appear as coefficients to the Gauss-Manin system (Theorem 2.3). We examine further how the multiplication table on the Jacobian quotient module calculates the logarithmic vector fields tangent to the discriminant and the bifurcation set (Proposition 3, Proposition 7). As applications, we establish signature formulae for Euler characteristics of real hypersurfaces (Theorem 4.1) and real complete intersections (Theorem 5.1) by means of these fields.

1 Introduction

This is a review article on the Gauss-Manin system associated to the isolated complete intersection singularities (i.c.i.s.) of projection and objects tightly related with them. The notion of i.c.i.s. of projection has been picked up among general i.c.i.s. by Viktor Goryunov [1, 2] as good models to which many arguments on the hypersurface singularities can be applied (see for example Theorem 2.1, Lemma 1). All isolated hypersurface singularities can be considered as a special case of the i.c.i.s. of projection. Many of important quasihomogeneous i.c.i.s. are also i.c.i.s. of projection.

The main aim of this article is to transmit the message that the multiplication tables defined on different quotient rings calculate important data both on analytic and topological characterisation of the i.c.i.s. of projection. We show that the multiplication table on the Jacobian quotient module in $(\mathcal{O}_{\tilde{X}\times S})^k$ calculates the logarithmic vector fields (i.e. the coefficients to the Gauss-Manin system defined for the period integrals) tangent to the discriminant and the bifurcation set (Proposition 3, Proposition 7) of the i.c.i.s. of projection. This idea is present already in the works by Kyoji Saito [3] and James William Bruce [4] for the case of hypersurface singularities (i.e. $k = 1$).

On the other hand, as applications, we establish signature formulae for Euler characteristics of real hypersurfaces (Theorem 4.1) and real complete intersections (Theorem 5.1) by means of logarithmic vector fields. These are paraphrase of results established by Zbigniew Szafraniec [5].

It is well known in the study of real algebraic geometry, Oleg Viro's patchworking method ([6]) furnishes us with a relatively simple and effective method to construct various nonsingular real plane projective algebraic curves of a given degree m with different isotopy types. As this method is based on perturbations of singular curves with quasihomogeneous singularities, our study on the versal deformation of hypersurface singularities fits into the context of real algebraic geometry. We shall notice that Viro's patch working method does not describe all possible curves corresponding to the full deformation parameter values outside the real discriminant.

The deformation parameter values $s \in \mathbf{R}^{\mu}$ that can be treated by Viro's method are located (on a quasihomogeneous curve) in certain specially selected real components of the complement to the discriminant. This situation is explained by the essential use of regular triangulation of the Newton polyhedron of the defining equation $F(x, s)$ in his construction. At the end of §6, Example 2, we indicate cases of real curves with different Euler characteristics that are impossible to distinguish after patch working method. We hope that this approach would give a new complementary tool to the topological study of real algebraic curves.

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2 Complete intersection of projection

Let us consider a $k-$tuple of holomorphic germs

$$\vec{f}(x, u) = (f_1(x, u), \cdots, f_k(x, u)) \in (\mathcal{O}_X)^k \tag{2.1}$$

in the neighbourhood of the origin for $X = (\mathbf{C}^{n+1}, 0)$. This is a 1- parameter deformation of the germ

$$\vec{f}^{(0)}(x) = (f_1(x, 0), \cdots, f_k(x, 0)) \in (\mathcal{O}_{\tilde{X}})^k \tag{2.2}$$

for $\tilde{X} = (\mathbf{C}^n, 0)$.

After [1] we introduce the notion of R_+equivalence of projection. Let $p : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ be a non degenerate linear projection i.e. $dp \neq 0$.

Definition 1 *We call the diagram*

$$Y \hookrightarrow \mathbf{C}^{n+1} \longrightarrow^p \mathbf{C},$$

the projection of the variety $Y \hookrightarrow \mathbf{C}^{n+1}$ on the line. Two varieties Y_1, Y_2 belong to the same R_+ equivalence class of projection if there exists a bi-holomorphic mapping from $\mathbf{C}^{n+1}$ to $\mathbf{C}^{n+1}$ that preserves the projection and induces a translation $p \to p + const$ on the line.

In this way, we are led to the definition of an equivalence class up to the following ideal,

$$T_f = \mathcal{O}_X \langle \frac{\partial \vec{f}}{\partial x_1}, \cdots, \frac{\partial \vec{f}}{\partial x_n} \rangle + \vec{f}^* \left(m_{\mathbf{C}^k,0} \right) \cdot (\mathcal{O}_X)^k \tag{2.3}$$

and the sub-ring,

$$T_f^+ := T_f + \mathbf{C} \frac{\partial \vec{f}}{\partial u} \tag{2.4}$$

that is nothing but the tangent space to the germ of R_+ equivalence class of projection. We introduce the spaces

$$Q_f := (\mathcal{O}_X)^k / T_f, \tag{2.5}$$

$$Q_f^+ := (\mathcal{O}_X)^k / T_f^+. \tag{2.6}$$

We remark that though T_f^+ is not necessarily an ideal the quotient Q_f^+ can make sense. Assume that Q_f is a finite dimensional $\mathbf{C}$ vector space. In this case, we call the number $\tau := dim_{\mathbf{C}} Q_f^+$ the R_+- co-dimension of projection. We denote by $\langle \vec{e}_1(x,u), \cdots, \vec{e}_\tau(x,u) \rangle$ the basis of the $\mathbf{C}$-vector space Q_f^+. If $\tau < \infty$, it is easy to see that $\vec{f}(x,u) = 0$ (resp. $\vec{f}(x,0) = 0$) has isolated singularity at $0 \in X$ (resp. $0 \in \tilde{X}$). We shall denote by μ the multiplicity of the critical point $(x,u) = 0$ of the height function u on $X_0 := \{(x,u) \in X; f_1(x,u) = \cdots = f_k(x,u) = 0\}$. Let us consider a R_+- versal deformation of $\vec{f}^{(0)}(x)$

$$\vec{F}(x,u,t) = \vec{f}^{(0)}(x) + \vec{e}_0(x,u) + t_1 \vec{e}_1(x,u) + \cdots + t_\tau \vec{e}_\tau(x,u) = \begin{pmatrix} F_1(x,s) \\ F_2(x,s) \\ \vdots \\ F_k(x,s) \end{pmatrix}, \tag{2.7}$$

with $\vec{e}_0(x,u) = \vec{f}(x,u) - \vec{f}(x,0)$. We consider the deformation of X_0 as follows

$$X_t := \{(x,u) \in X; \vec{F}(x,u,t) = \vec{0}\}, \tag{2.8}$$

that is also a $(\tau+1)$-dimensional deformation of the germ $\tilde{X}_0 := \{x \in \tilde{X}; f_1(x,0) = \cdots = f_k(x,0) = 0\}$. The following fact is crucial for further arguments.

Theorem 2.1 *([1], Theorem 2.1) For the k-tuple of holomorphic germs* (2.1) *with* $0 < \mu < +\infty$*, we have the equality* $\mu = \tau + 1$.

As a consequence we have $\mu = dim_{\mathbf{C}} Q_f$.

Recently a conceptual understanding in terms of homological algebra of this phenomenon appeared. See [7], §3.

Further, in view of the Theorem 2.1 we make use of the notation, $S = (\mathbf{C}^{\tau+1},0) = (\mathbf{C}^{\mu},0)$,$s = (u,t) \in S$, $s_0 = u, s_i = t_i, 1 \leq i \leq \tau$. We will denote the deformation parameter space $t \in T = (\mathbf{C}^{\tau},0)$.

Let $I_{C_0} \subset \mathcal{O}_X$ be the ideal generated by $k \times k$ minors of the matrix $(\frac{\partial \vec{f}(x,u)}{\partial x_1}, \cdots, \frac{\partial \vec{f}(x,u)}{\partial x_n})$.

Proposition 1 *([1], Proposition 1.2) We have the equality*

$$\mu = dim_{\mathbf{C}} Q_f = dim \frac{\mathcal{O}_X}{\mathcal{O}_X(f_1(x,u), \cdots, f_k(x,u)) + I_{C_0}}.$$

Let us denote by $Cr(\vec{F})$ the set of critical locus of the projection $\pi : \bigcup_{t \in T} X_t \rightarrow S$. That is to say

$$Cr(\vec{F}) = \{(x,u,t); (x,u) \in X_t, rank(\frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}) < k\}. \tag{2.9}$$

We denote by $D \subset S$ the image of projection $\pi(Cr(\vec{F}))$ which is usually called discriminant set of the deformation X_t of projection. It is known that for the R_+-versal deformation, D is defined by a principal ideal in $\mathcal{O}_S$ generated by a single defining function $\Delta(s)$ [8], (4.8). Under this situation we define $\mathcal{O}_S-$ module of vector fields tangent to the discriminant D which is a sub-module of Der_S the vector fields on S with coefficients from $\mathcal{O}_S$.

Definition 2 *We define the* logarithmic vector fields *associated to D as follows,*

$$Der_S(log\ D) = \{\vec{v} \in Der_S; \vec{v}(\Delta) \in \mathcal{O}_S \cdot \Delta\}.$$

We call that a meromorphic $p-$form ω with a simple pole along D belongs to the $\mathcal{O}_S$module of the logarithmic differential forms $\Omega^p_S(log\ D)$ *associated to D iff the following two conditions are satisfied*

$$1)\Delta \cdot \omega \in \Omega^p_S,$$

$$2) d\Delta \cdot \omega \in \Omega_S^{p+1},$$

or equivalently

$$\Delta \cdot d\omega \in \Omega_S^{p+1}.$$

For the $\mathcal{O}_S$-module of the logarithmic differential forms the following fact is known.

Theorem 2.2 *(See [9] for the case $k = 1$, [8, 10] for the case k general) The module $Der_S(log\ D)$ is a free $\mathcal{O}_S$-module of rank μ. Furthermore there exists a μ-tuple of vectors $\vec{v}_1, \cdots, \vec{v}_\mu \in Der_S(log\ D)$ such that*

$$\Delta(s) = det(\vec{v}_1, \cdots, \vec{v}_\mu).$$

Proposition 2 *(see [11] for the case $k = 1$, [1] for general k)*

For every $\vec{v}_j \in Der_S(log\ D)$, $1 \le j \le \mu$, there exists its lifting $\hat{\vec{v}}_j \in Der_{\tilde{X}\times S}$ tangent to the critical set $Cr(\vec{F})$. More precisely, the following decomposition holds,

$$\vec{v}_j(F_q(x,s)) = \sum_{p=1}^{n} h_{j,p}(x,s)\frac{\partial F_q}{\partial x_p} + \sum_{r=1}^{k} a_{jq}^{(r)}(x,s)F_r + b_{j,q}(x,s,\vec{F}),\ 1 \le q \le k$$

for some $h_{s,j}(x,s) \in \mathcal{O}_{\tilde{X}\times S}$, $b_{j,q}(x,s,\vec{F}) \in \mathcal{O}_{\tilde{X}\times S} \otimes_{\mathcal{O}_{\tilde{X}\times S}} m_S^2$. In this notation,

$$\hat{\vec{v}}_j = \vec{v}_j - \sum_{p=1}^{n} h_{j,p}(x,s)\frac{\partial}{\partial x_p}.$$

Conversely, to every vector field $\hat{\vec{v}}_j \in Der_{\tilde{X}\times S}$ tangent to the critical set $Cr(\vec{F})$ we can associate a vector field $\vec{v}_j \in Der_S(log\ D)$ as its push down.

This is a direct consequence of the preparation theorem (see [12]). Further on in this article we denote by $\vec{v}(F(x,s))$ the action of a vector field $\vec{v} \in Der_{\tilde{X}\times S}$ on a function $F(x,s)$.

Lemma 1 *([1]) The discriminant $\Delta(s)$ defined in Theorem 2.2 can be expressed by a Weierstrass polynomial,*

$$\Delta(s) = u^\mu + d_1(t)u^{\mu-1} + \cdots + d_\mu(t),$$

with $d_1(0) = \cdots = d_\mu(0) = 0$.

This can be deduced by another way by making use of (5.5) for the case of CI (5.1). Namely we have $\Delta(s) = detP(s)$. From this lemma we deduce immediately the existence of an "Euler" vector field even for non-quasihomogeneous $\vec{f}(x,u)$ that plays essential rôle in the construction of the higher residue pairing by K.Saito[13].

Lemma 2 *(For $k=1$, see [13] (1.7.5)) There is a vector field $\vec{v}_1 = (u+\sigma_1^0(t))\frac{\partial}{\partial u} + \sum_{i=1}^{\tau}\sigma_1^i(t)\frac{\partial}{\partial s_i} \in Der_S(log\ D)$ such that*

$$\vec{v}_1(\Delta(s)) = \mu\Delta(s).$$

Proof. It is clear that for a vector field $\vec{v}_1 \in Der_S(log\ D)$ with the component $(u+\sigma_1^0(t))\frac{\partial}{\partial u}$ whose existence is guaranteed by Theorem 3,1 [1] , the expression $\vec{v}_1(\Delta(s))$ must be divisible by $\Delta(s)$. In calculating the term of $\vec{v}_1(\Delta(s))$ that may contain the factor u^μ, we see that

$$\vec{v}_1(\Delta(s)) = \mu u^\mu + \tilde{d}_1(t)u^{\mu-1} + \cdots + \tilde{d}_\mu(t).$$

Thus we conclude that $\tilde{d}_i(t) = \mu d_i(t)$, $1 \le i \le \mu$. ∎

Now we introduce the filtered $\mathcal{O}_S$-module of fibre integrals $\mathcal{H}^{(\vec{\lambda})}$ for a multi-index of negative integers $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k) \in (\mathbf{Z}_{<0})^k$.

$$I_\phi^{\vec{\lambda}}(s) = \int_{t(\gamma)} \phi(x,s)F_1(x,s)^{\lambda_1}\cdots F_k(x,s)^{\lambda_k}dx,$$

for $\phi(x,s) \in \mathcal{O}_{\tilde{X}\times S}$. Let us denote by $X^{(q)} := \{x \in \tilde{X}; F_q(x,s) = 0\}$ a smooth hypersurface defined for $s \notin D$. In this situation we define the Leray's tube operation isomorphism (see [15, 14]),

$$\begin{array}{cccc} t: & H_{n-k}(\cap_{q=1}^k X^{(q)}) & \rightarrow & H_n(\tilde{X} \setminus \cup_{q=1}^k X^{(q)}), \\ & \gamma & \mapsto & t(\gamma). \end{array}$$

The concrete construction of the operation t can be described as follows. First we consider the co-boundary isomorphism of the homology groups,

$$\delta : H_{n-k}(\cap_{q=1}^k X^{(q)}) \rightarrow H_{n-k+1}(\cap_{q=2}^k X^{(q)} \setminus X^{(1)}).$$

A cycle γ in $\cap_{q=1}^k X^{(q)}$ is mapped onto a cycle $\delta(\gamma)$ of one higher dimension that is obtained as a S^1 bundle over γ. Repeated application of δ yields an iterated co-boundary homomorphism,

$$H_{n-k}(\cap_{q=1}^k X^{(q)}) \rightarrow^{\delta} H_{n-k+1}(\cap_{q=2}^k X^{(q)} \setminus X^{(1)}) \rightarrow^{\delta} \cdots$$

$$\cdots \to^{\delta} H_{n-1}(X^{(k)} \setminus \cup_{q=1}^{k-1} X^{(q)}) \to^{\delta} H_n(\tilde{X} \setminus \cup_{q=1}^{k} X^{(q)}).$$

The Leray's tube operation is a $k-$time iterated δ homomorphism i.e. $t = \delta^m$. The Froissart decomposition theorem ([14], §6-3) shows that the collection of all cycles of $H_n(\tilde{X} \setminus \cup_{q=1}^{k} X^{(q)})$ are obtained by the application of iterated δ homomorphism operations to the cycles from $H_{n-p}(\tilde{X} \cap X^{(q_1)} \cap X^{(q_2)} \cdots \cap X^{(q_p)})$, $p = 0, \cdots, k$.

Further without loss of generality, we consider the situation where a versal deformation of a mapping $\vec{f}^{(0)}(x)$ can be written down in the following special form for $s = (u, t) \in S$,

$$\vec{F}(x,s) = \vec{f}^{(0)}(x) + \sum_{\ell=1}^{\tau} t_\ell \vec{e}_\ell(x) + u\vec{e}_0(x) = \begin{pmatrix} \tilde{F}_1(x,t) + u \\ F_2(x,t) \\ \vdots \\ F_k(x,t) \end{pmatrix}$$

for

$$\{\vec{e}_0(x), \cdots, \vec{e}_\tau(x)\} \in Q_f,$$

where $\vec{e}_0(x) = {}^t(1, 0, \cdots, 0)$. We adopt the notation $\tilde{F}_1(x,t) + u = F_1(x,s)$. One may consult [8] (6.7) to see that $\vec{F}(x,s)$ really gives a versal deformation of $\vec{f}^{(0)}(x)$ by virtue of the definitions (2.3), (2.5).

Let us recall the Brieskorn lattice defined for the singularity $\tilde{X}_0$

$$\Psi := \frac{\Omega^n_{\tilde{X}}}{dF_1(x,s) \wedge \cdots \wedge dF_k(x,s) \wedge d\Omega^{n-k-1}_{\tilde{X}} + \langle F_1(x,s), \cdots, F_k(x,s) \rangle \Omega^n_{\tilde{X}}},$$

that is known to be a $\mathcal{O}_S$ free module of rank $\tilde{\mu}$:the Milnor number of the singularity $\tilde{X}_0$ ([17], Proposition 2.6). We denote its basis by $(\psi_1(x)dx, \cdots, \psi_{\tilde{\mu}}(x)dx)$.

Now let us introduce a notation of the multi-index $-\mathbf{1} = (-1, \cdots -1) \in (\mathbf{Z}_{<0})^k$. We consider a vector of fibre integrals $\mathbf{I}_\Psi :={}^t (I^{(-\mathbf{1})}_{\psi_1}(s), \cdots, I^{(-\mathbf{1})}_{\psi_{\tilde{\mu}}}(s))$. The following theorem for $k = 1$ has been announced in [3] (4.14) without proof.

Theorem 2.3 *For every $\vec{v} \in Der_S(log\ D)$, we have the following inclusion relation*

$$\vec{v} : \mathcal{H}^{(-\mathbf{1})} \hookrightarrow \mathcal{H}^{(-\mathbf{1})}.$$

That is to say for every $\vec{v}_j \in Der_S(log\ D)$, there exists a $\mu \times \tilde{\mu}$ matrix with holomorphic entries $B_j(s) \in Mat(\mu, \tilde{\mu}) \otimes \mathcal{O}_S$ such that

$$\vec{v}_j(\mathbf{I}_\Psi) = B_j(s)\mathbf{I}_\Psi, 1 \le j \le \mu.$$

Proof. First we remark the following equality that yields from Proposition 2,

$$\vec{v}_j\left(\int_{t(\gamma)} \phi(x)F_1(x,s)^{-1}\cdots F_k(x,t)^{-1}dx\right) =$$

$$= \int_{t(\gamma)} \vec{F}^{-\mathbf{1}} d(\phi(x)\sum_{p=1}^{n}(-1)^{p-1}h_{j,p}(x,s)dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n)+$$

$$+\int_{t(\gamma)} \vec{F}^{-\mathbf{1}}\phi(x)(\sum_{q=1}^{k}\sum_{r=1}^{k} a_{j,q}^{(r)}F_rF_q^{-1})dx + \int_{t(\gamma)} \vec{F}^{-\mathbf{1}}\sum_{q=1}^{k}F_q^{-1}b_{j,q}(x,u,t,\vec{F})dx$$

$$= \int_{t(\gamma)} \vec{F}^{-\mathbf{1}} d(\phi(x)\sum_{p=1}^{n}(-1)^{p-1}h_{j,p}(x,s)dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n)$$

$$+\int_{t(\gamma)} \vec{F}^{-\mathbf{1}}\phi(x)(\sum_{r=1}^{k} a_{j,r}^{(r)}(x,s))dx.$$

The last equality can be explained by the vanishing of the integral

$$\int_{t(\gamma)} F_1^{-1}\cdots 1 \overset{r}{\overset{\vee}{\cdots}} F_q^{-2} \overset{q}{\overset{\vee}{\cdots}} F_k^{-1}\phi(x,u)(a_{j,q}^{r})dx = 0,$$

because of the lack of the residue along $F_r(x,s) = 0$ and

$$\int_{t(\gamma)} \vec{F}^{-\mathbf{1}}F_{q_1}F_{q_2}F_q^{-1}\phi(x,u)(b_{j,q}^{0}(x,s))dx = 0,$$

in view of the lack of at least one of residues either along $F_{q_1} = 0$ or along $F_{q_2} = 0$. These equalities are derived from the property of the Leray's tube $t(\gamma)$ which needs co-dimension k residue to give rise to a non-zero integral. We consider the class in Ψ of an $n-$form $d(\phi(x)\sum_{p=1}^{n}(-1)^{p-1}h_{j,p}(x,s)dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n)$ which permits a representation like

$$\sum_{\ell=1}^{\tilde{\mu}} A_\ell(s)\psi_\ell(x)dx + dF_1(x,s)\wedge\cdots\wedge dF_k(x,s)\wedge d\omega + \sum_{q=1}^{k}F_q(x,s)\gamma_q,$$

for $\omega \in \Omega_{\tilde{X}}^{n-k-1}$, $\gamma_q \in \Omega_{\tilde{X}}^{n}$. Thus we get the following equality

$$\int_{t(\gamma)} \vec{F}^{-\mathbf{1}} d(\phi(x)\sum_{p=1}^{n}(-1)^{p-1}h_{j,p}(x,s)dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n) = \sum_{\ell=1}^{\tilde{\mu}} A_\ell(s)I_{\psi_\ell}^{-\mathbf{1}}(s),$$

which evidently belongs to $\mathcal{H}^{(-\mathbf{1})}$. This can be seen from the fact that both $\int_{t(\gamma)} F_q(x,s)\vec{F}^{-\mathbf{1}}\gamma_q$ and $\int_{t(\gamma)} \vec{F}^{-\mathbf{1}} dF_1(x,s) \wedge \cdots \wedge dF_k(x,s) \wedge d\omega$ vanish. The first because of lack of the residue along $F_q(x,s) = 0$ and the second by virtue of the Stokes theorem. ■

As will be indicated below, the versal deformation (2.7) is not mini-versal deformation of $\vec{f}^{(0)}(x)$ and we have in general $\mu \geq \tilde{\mu}$ (See [17], §5.1). Let us restrict the deformation (2.7) to a subspace of a versal deformation $T' \subseteq S$ with a coordinate system $(t'_1, \cdots, t'_{\tilde{\mu}}) \in T'$ so that every t'_i coincides with some s_{j_i}. This can be done in view of the fact that the Milnor number is larger than the Tjurina number of $\vec{f}^{(0)}(x)$. Let us denote the restriction of $\mathbf{I}_\Psi$ to T' by $\mathbf{I}_\Psi(t')$ and that of D to T' by D'.

Corollary 1 *The vector of fibre integrals $\mathbf{I}_\Psi(t')$ satisfies the following Pfaff system of Fuchsian type*

$$d\mathbf{I}_\Psi(t') = \Omega \cdot \mathbf{I}_\Psi(t'),$$

for some $\Omega \in End(\mathbf{C}^{\tilde{\mu}}) \otimes_{\mathcal{O}_{T'}} \Omega^1_{T'}(log\ D')$.

Proof. In modifying all arguments above into the case for the deformation parameter space T', we get the following version of the above Theorem 2.3, That is to say for every $\vec{v}'_j \in Der_{T'}(log\ D')$, there exists a $\tilde{\mu} \times \tilde{\mu}$ matrix with holomorphic entries $B'_j(t') \in Mat(\tilde{\mu}, \tilde{\mu}) \otimes \mathcal{O}_{T'}$ such that

$$\vec{v}'_j(\mathbf{I}_\Psi(t')) = B'_j(t')\mathbf{I}_\Psi(t'), 1 \leq j \leq \tilde{\mu}.$$

Let us rewrite the relation obtained above into the form,

$$dI^{(-\mathbf{1})}_{\psi_q}(t') = \sum_{r=1}^{\mu} \omega_{q,r} I^{(-\mathbf{1})}_{\psi_r}(t'),$$

for some $\omega_{q,r} \in \Omega^1_{T'}(-D')$ meromorphic 1-forms with poles along D'. These $\omega_{q,r}$ satisfy the following relations,

$$\vec{v}_j(I^{(-\mathbf{1})}_{\psi_q}(t')) = \langle \vec{v}'_j, dI^{(-\mathbf{1})}_{\psi_q}(t')\rangle = \langle \vec{v}'_j, \sum_{r=1}^{\tilde{\mu}} \omega_{q,r} I^{(-\mathbf{1})}_{\psi_r}(t')\rangle \ \ 1 \leq j, q \leq \tilde{\mu}.$$

If $\langle \vec{v}'_j, \omega_{q,r}\rangle \in \mathcal{O}_{T'}$ for all $\vec{v}'_j \in Der_{T'}(log\ D')$ $1 \leq j \leq \tilde{\mu}$ then $\omega_{q,r} \in \Omega^1_{T'}(log\ D')$ in view of the modified version of Theorem 2.2 which requires only the deformation by T' of $\vec{f}^{(0)}(x)$ be versal. ■

Let us introduce a filtration as follows $\mathcal{H}^{(\lambda)} = \bigoplus_{\lambda_1+\cdots+\lambda_k=\lambda} \mathcal{H}^{(\vec{\lambda})}$. For this rough filtration we have the following generalisation of the Griffiths' transversality theorem ([16] Theorem 3.1).

Corollary 2 *For every $\vec{v} \in Der_S(log\ D)$, we have the following inclusion relation*

$$\vec{v} : \mathcal{H}^{(\lambda)} \hookrightarrow \mathcal{H}^{(\lambda)}.$$

Proof. For $\partial_{s_j} I_\Phi \in \mathcal{H}^{(-k-1)}$ and $\vec{v}_\ell \in Der_S(log\ D)$ we have

$$\vec{v}_\ell(\partial_{s_j} I_\Phi) = [\vec{v}_\ell, \partial_{s_j}] I_\Phi + \partial_{s_j} \vec{v}_\ell(I_\Phi)$$

$$= [\vec{v}_\ell, \partial_{s_j}] I_\Phi + \partial_{s_j}(B_\ell(s) I_\Phi) = [\vec{v}_\ell, \partial_{s_j}] I_\Phi + (\partial_{s_j} B_\ell(s)) I_\Phi + B_\ell(s)(\partial_{s_j} I_\Phi).$$

As the commutator $[\vec{v}_\ell, \partial_{s_j}]$ is a first order operator, the term above $[\vec{v}_\ell, \partial_{s_j}] I_\Phi$ belongs to $\mathcal{H}^{(-k-1)}$. The term $\partial_{s_j} B_\ell(s) I_\Phi \in \mathcal{H}^{(-k)}$ again belongs to $\mathcal{H}^{(-k-1)}$. Thus we see $\vec{v}_\ell(\partial_{s_j} I_\Phi) \in \mathcal{H}^{(-k-1)}$. In an inductive way, for any $\lambda \leq -k$ we prove the statement. ■

3 Multiplication table and the logarithmic vector fields

Let us consider the **C** vector space

$$\Phi := \frac{\mathcal{O}_X}{I_{C_0} + \mathcal{O}_X(f_1(x) + u, f_2(x), \cdots, f_k(x))}. \tag{3.1}$$

and fix its basis. We remark here that the basis of Φ can be represented by functions from $\mathcal{O}_{\tilde{X}}$ as we can erase the variable u by the relation $f_1(x) = u$ in Φ. It turns out that we can regard $\{\phi_0(x), \cdots, \phi_\tau(x)\}$ as a free basis of the $\mathcal{O}_S$ module $\Phi(s)$ treated in the Proposition 6 below. Under these circumstances, we introduce holomorphic functions $\tau^\ell_{i,j}(s) \in \mathcal{O}_S$ in the following way.

$$\phi_i(x)\vec{e}_j(x) \equiv \sum_{\ell=0}^{\tau} \tau^\ell_{i,j}(s)\vec{e}_\ell(x) \tag{3.2}$$

$$mod(\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}, F_1(x,s), \cdots, F_k(x,s)\rangle).$$

The functions $\tau^\ell_{i,j}(s) \in \mathcal{O}_S$ exist due to the versality of the deformation $\vec{F}(x,s)$. We denote by

$$T_j(s) = \left(\tau^\ell_{i,j}(s)\right)_{0\leq j,\ell\leq\tau}, \tag{3.3}$$

a $\mu \times \mu$ matrix which is called the matrix of **multiplication table**. We denote the discriminant associated to this deformation by $D \subset S$.

Further on we will make use of the abbreviation $mod(d_x\vec{F}(x,s))$ instead of making use of the expression $mod(\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}, F_1(x,s), \cdots, F_k(x,s)\rangle)$.

After Proposition 2 the vector field $\vec{v}_1$ constructed in Lemma 2 has its lifting $\hat{\vec{v}}_1 \in Der_{\tilde{X}\times S}$. Let us denote by $\check{\vec{v}}_1 = \hat{\vec{v}}_1 - \vec{v}_1 \in \mathcal{O}_{\tilde{X}\times S} \otimes Der_{\tilde{X}}$.

$$\hat{\vec{v}}_1(\vec{F}(x,s)) \cdot \phi_i(x) =$$

$$= \check{\vec{v}}_1(\vec{f}^{(0)}(x)) \cdot \phi_i(x) + \sum_{\ell=0}^{\tau} \vec{v}_1(s_\ell)\vec{e}_\ell(x)\phi_i(x) + \sum_{\ell=0}^{\tau} s_\ell(\check{\vec{v}}_1 e_\ell(x))\phi_i(x)$$

$$\equiv \sum_{\ell=0}^{\tau} \vec{v}_1(s_\ell)\vec{e}_\ell(x)\phi_i(x) \quad mod(d_x\vec{F}(x,s)).$$

Lemma 3 *There exists a vector valued function* $M(x,\vec{F}(x,s)) \in (\mathcal{O}_{\tilde{X}\times \mathbf{C}^k})^k$ *such that*

$$\hat{\vec{v}}_1(\vec{F}(x,s)) \equiv M(x,\vec{F}(x,s)) \quad mod(d_x\vec{F}(x,s)),$$

with

$$M(x,\vec{F}(x,s)) = M^0 \cdot \vec{F}(x,s) + M^1(x,\vec{F}(x,s)),$$

where $M^0 \in GL(k,\mathbf{C})$*: a non-degenerate matrix and* $M^1(x,\vec{F}(x,s)) \in (\mathcal{O}_{\tilde{X}} \otimes m_S^2)^k$*. Especially the first row of* $M^0 = (1,0,\cdots,0)$.

Proof. First of all we remember a theorem due to [17] §1.1, [3] Proposition 2.3.2 which states that the Krull dimension of the ring of holomorphic functions on the critical set $Cr(\vec{F})$ is equal to $\mu - 1$ and this ring is a Cohen- Macaulay ring. Let us denote by $L = {}_nC_k$. We have $(k+L)$tuple of $k \times k-$ minors $j_{k+1}(x,s) \cdots j_{k+L}(x,s)$ of the matrix $(\frac{\partial}{\partial x_1}\vec{F}(x,s), \cdots, \frac{\partial}{\partial x_n}\vec{F}(x,s))$ such that

$$Cr(\vec{F}) = V(\langle F_1(x,s), \cdots, F_k(x,s), j_{k+1}(x,s), \cdots, j_{k+L}(x,s)\rangle).$$

The lemma 2 yields that the lifting $\hat{\vec{v}}_1$ of the vector field $\vec{v}_1$ satisfies the relations,

$$\langle F_1(x,s), \cdots, F_k(x,s), j_{k+1}(x,s), \cdots, j_{k+L}(x,s)\rangle$$

$$= \langle \hat{\vec{v}}_1(F_1(x,s)), \cdots, \hat{\vec{v}}_1(F_k(x,s)), \hat{\vec{v}}_1(j_{k+1}(x,s)), \cdots, \hat{\vec{v}}_1(j_{k+L}(x,s))\rangle.$$

As it has been seen from the above Proposition 2, the vector $\hat{\vec{v}}_1$ is tangent to $Cr(\vec{F})$. If the above equality does not hold, it would entail the relation

$$\{s \in S; \Delta(s) = 0\}$$

$$\subsetneq \pi(V(\langle \hat{\vec{v}}_1(F_1(x,s)), \cdots, \hat{\vec{v}}_1(F_k(x,s)), \hat{\vec{v}}_1(j_{k+1}(x,s)), \cdots, \hat{\vec{v}}_1(j_{k+L}(x,s))\rangle)),$$

after elimination theoretical consideration. This yields

$$\hat{\vec{v}}_1(F_q(x,s)) = \sum_{\ell=1}^{k} C_q^\ell F_\ell(x,s) + m_q(x,\vec{F}) + \sum_{\ell=k+1}^{k+L} C_q^\ell j_\ell(x,s), 1 \leq q \leq k,$$

$$\hat{\vec{v}}_1(j_p(x,s)) = \sum_{\ell=k+1}^{k+L} C_p^\ell j_\ell(x,s) + m_p(x,\vec{F}), k+1 \leq p \leq k+L,$$

for $m_r(x,\vec{F}) \in \mathcal{O}_{\tilde{X}} \otimes m_S^2$, $1 \leq r \leq k+L$ and some constants $C_q^\ell, 1 \leq \ell \leq k$. First we see that the expression $\hat{\vec{v}}_1(j_p(x,s))$ cannot contain terms of $F_q(x,s)$ like $F_q(0,s)$ in view of the situation that the versality of the deformation makes all linear in x variable terms dependent on some of deformation parameters. Secondly the non-degeneracy of the matrix $M^0 := (C_q^\ell)_{1\leq q,\ell \leq k}$ is necessary so that the above equality among ideals holds. From this relation and the preparation theorem, we see

$$\hat{\vec{v}}_1(\vec{F}(x,s)) = M^0 \cdot \vec{F}(x,s) + M^1(x,\vec{F}(x,s)) + \sum_{j=1}^{n} h_{1,j}(x,s) \frac{\partial \vec{F}(x,s)}{\partial x_j},$$

with $M^1(x,\vec{F}(x,s)) = {}^t(m_1(x,\vec{F}), \cdots, m_k(x,\vec{F})) \in (\mathcal{O}_{\tilde{X}} \otimes m_S^2)^k$.

More precisely we can state that $C_1^1 = 1$, $C_1^\ell = 0, 2 \leq \ell \leq k$. The dependence of some coefficients of $\hat{\vec{v}}_1$ on $F_i(x,t)$ is necessary so that $C_1^\ell \neq 0$ for some $2 \leq \ell \leq k$. But this is impossible because if not it would mean that some of the coefficients of $\hat{\vec{v}}_1$ contains factor $F_2(x,s), \cdots, F_k(x,s)$ that contradicts the construction of $\hat{\vec{v}}_1$ in Proposition 2. This can be seen from the fact that the expressions $\frac{\partial F_1(x,s)}{\partial x_1}, \cdots,$ $\frac{\partial F_1(x,s)}{\partial x_n}, \frac{\partial F_1(x,s)}{\partial s_1}, \cdots, \frac{\partial F_1(x,s)}{\partial s_\mu}$ do not contain the deformation parameters present in the polynomials $F_2(x,s), \cdots, F_k(x,s)$. ■

Lemma 4 *A basis of logarithmic vector fields $\vec{v}_0, \cdots, \vec{v}_\tau \in Der_S(log\ D)$ can be produced from the functions $\sigma_i^\ell(s)$ defined as follows,*

$$\hat{\vec{v}}_1(\vec{F}(x,s)) \cdot \phi_i(x) = M(x,\vec{F}(x,s)) \cdot \phi_i(x) = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) \vec{e}_\ell + \check{\vec{v}}_i(\vec{F}(x,s))$$

$$\equiv \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) \vec{e}_\ell \ \ mod(d_x \vec{F}(x,s)),$$

where the vector valued function $M(x,\vec{F}(x,s))$ denotes the one defined in the Lemma 3 and $\check{\vec{v}}_j = \sum_{p=1}^n h_{j,p}(x,s) \frac{\partial}{\partial x_p}$ is a certain vector field with holomorphic coefficients.

Proof. We remark the following relation,

$$\hat{\vec{v}}_1(\vec{F}(x,s))\phi_i(x)$$

$$= \check{\vec{v}}_1(\vec{f}^{(0)}(x))\phi_i(x) + \sum_{j=0}^{\tau} \vec{v}_1(s_j)\vec{e}_j(x)\phi_i(x) + \sum_{j=0}^{\tau} s_j \check{\vec{v}}_1(\vec{e}_j(x))\phi_i(x)$$

$$\equiv \sum_{j=0}^{\tau} \vec{v}_1(s_j)\vec{e}_j(x)\phi_i(x) \;\; mod(d_x\vec{F}(x,s)).$$

The relation (3.2) above entails,

$$M(x,\vec{F}(x,s)) \cdot \phi_i(x) \equiv \sum_{\ell=0}^{\tau}\sum_{j=0}^{\tau} \vec{v}_1(s_j)\tau^{\ell}_{i,j}(s)\vec{e}_\ell(x) \;\; mod(d_x\vec{F}(x,s)).$$

As $\phi_i(x)$ can be considered to be a basis of $\mathcal{O}_S$ module $\Phi(s)$ above (see Proposition 6), vectors $(\sigma_i^0(s), \cdots, \sigma_i^{\tau}(s))$, $0 \leq i \leq \tau$ are $\mathcal{O}_S$ linearly independent at each generic point $S \setminus D$. If we put

$$\sigma_i^{\ell}(s) = \sum_{j=0}^{\tau} \vec{v}_1(s_j)\tau^{\ell}_{i,j}(s),$$

then the vector field $\hat{\vec{v}}_i \in Der_{\tilde{X}\times S}$

$$\hat{\vec{v}}_i = \sum_{\ell=0}^{\tau} \sigma_i^{\ell}(s)\frac{\partial}{\partial s_\ell} + \phi_i(x)\check{\vec{v}}_1,$$

is tangent to $Cr(\vec{F})$. The only non-trivial relations that may arise between $\check{\vec{v}}_i$ and $\check{\vec{v}}_{i'}$ $i \neq i'$ is

$$\phi_i(x)\check{\vec{v}}_{i'} = \phi_{i'}(x)\check{\vec{v}}_i.$$

These vectors give rise to the same push down vector field in $Der_S(log\ D)$. Namely,

$$\pi_*(\phi_i(x)\hat{\vec{v}}_{i'}) = \pi_*(\phi_{i'}(x)\hat{\vec{v}}_i) = \sum_{j=0}^{\tau}\sum_{\ell=0}^{\tau} R^{\ell}_{i,i',j}(s)\frac{\partial}{\partial s_\ell},$$

for the coefficients $R^{\ell}_{i,i',j}(s)$ determined by

$$\sum_{j=0}^{\tau} \vec{v}_1(s_j)\phi_i(x)\phi_{i'}(x)\vec{e}_j(x) \equiv \sum_{j=0}^{\tau}\sum_{\ell=0}^{\tau} R^{\ell}_{i,i',j}(s)\vec{e}_\ell(x) mod(d_x\vec{F}(x,s)).$$

This means that $\hat{\vec{v}}_0, \cdots, \hat{\vec{v}}_\tau$ form a free basis of $Der_{\tilde{X}\times S}(Cr(\vec{F}))$ hence $\vec{v}_0, \cdots, \vec{v}_\tau$ that of the module $Der_S(log\ D)$. ■

This lemma gives us a correspondence between $\phi_i(x) \in \Phi$ and $\vec{v}_i \in Der_S(log\ D)$, therefore it is quite natural to expect that the mixed Hodge structure on Φ would induce that on $Der_S(log\ D)$, and would hence contribute to describe $B_i(s)$ of Theorem 2.3, 1 in a precise manner. A good understanding of this situation is indispensable to characterise the rational monodromy of solutions to the Gauss-Manin system in terms of the mixed Hodge structure on Φ. Confer to Proposition 8 below.

We formulate the lemma 4 into the following form (see [4] Theorems A2, A4, [9] (3.19), [3] (4.5.3) Corollary 2 for $k = 1$ and [8] (6.13), [1] Theorem 3.2 for k general).

Proposition 3 *There exist holomorphic functions $w_j(s) \in \mathcal{O}_S$, $0 \le j \le \tau$ such that the components of the matrix*

$$\Sigma(s) := \sum_{j=0}^{\tau} w_j(s) T_j(s), \tag{3.4}$$

give rise to a basis of logarithmic vector fields $\vec{v}_0, \cdots, \vec{v}_\tau \in Der_S(log\ D)$. Namely, if we write $\Sigma(s) = \left(\sigma_i^\ell(s)\right)_{0\le i,\ell\le\tau}$, then the expression

$$\vec{v}_i = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) \frac{\partial}{\partial s_\ell}, \tag{3.5}$$

consists a base element of the $\mathcal{O}_S$ module $Der_S(log\ D)$.

Especially in the case of quasihomogeneous singularity $\vec{f}(x,u)$ we have the following simple description of the vector field that can be deduced from Lemma 4. To do this, it is enough to remark that the vector field $\vec{v}_1$ is the Euler vector field by definition and $\vec{v}_1(s_r) = \frac{w(s_r)}{w(s_0)} s_r$, where $w(s_j)$ denotes the quasihomogeneous weight of the variable s_j.

Proposition 4 *([2] Theorem 2.4) In the case of quasihomogeneous singularity* (2.1)*, the basis* (3.5) *of $Der_S(log\ D)$ can be calculated by*

$$\sigma_i^\ell(s) = \sum_{j=0}^{\tau} w(s_j) s_j \tau_{i,j}^\ell(s).$$

Furthermore, the vector valued function $M(x, \vec{F}(x,s))$ of Lemma 3 has the expression,

$$M(x, \vec{F}(x,s)) = M^0 \cdot \vec{F}(x,s) = diag\ (w(f_1), \cdots, w(f_k)) \cdot \vec{F}(x,s).$$

4 Multiplication table and the topology of real hypersurfaces

In this section we continue to consider the situation where $\mu = \tau + 1$ for $k = 1$ in (2.5). We associate to the versal deformation of the hypersurface singularity

$$F(x,s) = f(x) + \sum_{i=0}^{\tau} s_i e_i(x), \tag{4.1}$$

the following matrix $\Sigma(s) = (\sigma_i^\ell(s))_{0\le i,\ell\le\tau}$ after the model (3.2),

$$F(x,s)e_i(x) = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) e_\ell(x) \;\; mod(d_x F(x,s)). \tag{4.2}$$

$$e_i(x)e_j(x) \equiv \sum_{\ell=0}^{\tau} \tau_{i,j}^\ell(t) e_\ell(x) \;\; mod(d_x F(x,s)). \tag{4.3}$$

Further on we make use of the convention $e_0(x) = 1$ and $s = (s_0, t)$. We denote the deformation parameter space $t \in T = (\mathbf{C}^\tau, 0)$.

We recall the Milnor ring for $k = 1$ whose analogy has been introduced in (2.5) (and in the case k general, $\Phi(s)$ will be introduced in Proposition 6),

$$Q_F := \frac{\mathcal{O}_{\tilde{X}\times S}}{\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial F(x,s)}{\partial x_1}, \cdots, \frac{\partial F(x,s)}{\partial x_n}\rangle}.$$

We introduce the Bezoutian matrix $B^F(s)$ whose (i,j) element is defined by the trace of the multiplication action $F(x,s)e_i(x)e_j(x)\cdot$ on the Milnor ring Q_F,

$$F(x,s)e_i(x)e_j(x) \equiv (\sum_{c=0}^{\tau} \sigma_i^c(s) e_c(x)) e_j(x)$$

$$\equiv \sum_{c=0}^{\tau} \sigma_i^c(s) (\sum_{r=0}^{\tau} \tau_{c,j}^r(t) e_r(x)) mod(d_x F(x,s)).$$

For the sake of simplicity we will use the following notation,

$$\tau^r(t) = (\tau_{c,b}^r(t))_{0\le c,b\le\tau}. \tag{4.4}$$

To clarify the structure of the Bezoutian matrix $B^F(s)$ we introduce a matrix

$$T(t) = \left(\sum_{r=0}^{\tau} \zeta_r(t) \tau^r(t)\right), \tag{4.5}$$

with the notation

$$\zeta_r(t) = tr(e_r(x)\cdot) = \sum_{\ell=0}^{\tau} \tau^{\ell}_{r,\ell}(t). \tag{4.6}$$

The (i,j) element of the matrix $T(t)$ (4.5) equals to $tr(e_i(x)e_j(x)\cdot)$ on the Milnor ring Q_F. It is possible to show that $\{t \in T; det(T(t)) = 0\}$ coincides with the bifurcation set of $F(x,s)$ outside the Maxwell set (see Proposition 7 below). Thus we get the Bezoutian matrix

$$B^F(s) = \Sigma(s) \cdot T(t). \tag{4.7}$$

Following statement is a simple application of Morse theory to the multiplication table see [5] Theorem 2.1. From here on we assume that $|s|$ is small enough and denote by $\tilde{X} = \{x \in \mathbf{C}^n; |x| \leq \delta\}$ a closed ball such that all critical points of $F(x,s)$ are located inside $\tilde{X}$.

Proposition 5 *sign* $\Sigma(s) \cdot T(t)$ =\{ *number of real critical points in* $F(x,s) > 0$, $x \in \tilde{X} \cap \mathbf{R}^n$\} -\{ *number of real critical points in* $F(x,s) < 0$, $x \in \tilde{X} \cap \mathbf{R}^n$\}. *Here* $sign(A)$ *denotes the signature of a symmetric matrix* A *i.e. the difference between the number of positive and negative eigenvalues.*

Let us denote by $h(x,t)$ the determinant of the Hessian

$$h(x,t) := det\left(\frac{\partial^2 F(x,s)}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq n}.$$

We associate the following μ holomorphic functions $h_0(t), \cdots, h_\tau(t) \in \mathcal{O}_S$ to the function $h(x,t)$,

$$h(x,t) \equiv \sum_{\ell=0}^{\tau} h_\ell(t) e_\ell(x) \ \ mod(d_x F(x,s)). \tag{4.8}$$

Further by means of (4.7) we introduce the matrix

$$B^H(t) := \sum_{\ell=0}^{\tau} \eta^{\ell}(t) \tau^{\ell}(t), \tag{4.9}$$

where

$$\begin{pmatrix} \eta^0(t) \\ \vdots \\ \eta^\tau(t) \end{pmatrix} = T(t) \cdot \begin{pmatrix} h_0(t) \\ \vdots \\ h_\tau(t) \end{pmatrix}.$$

We consider the matrix $B^{HF}(s) = (\cdot)_{0\le a,b\le\tau}$ whose $(a,b)-$element is defined by the trace of the following expression on the Milnor ring Q_F,

$$h(x,t)F(x,s)e_a(x)e_b(x) \equiv (\sum_{\ell=0}^{\tau} h_\ell(t)e_\ell(x))(\sum_{c=0}^{\tau}\sigma_a^c(s)\sum_{m=0}^{\tau}\tau_{c,b}^m(t)e_m(x)) \tag{4.10}$$

$$\equiv \sum_{\ell=0}^{\tau}\sum_{c=0}^{\tau}\sum_{m=0}^{\tau} h_\ell(t)\sigma_a^c(s)\tau_{c,b}^m(t)e_\ell(x)e_m(x)$$

$$\equiv \sum_{\ell=0}^{\tau}\sum_{c=0}^{\tau}\sum_{m=0}^{\tau} h_\ell(t)\sigma_a^c(s)\tau_{c,b}^m(t)\sum_{r=0}^{\tau}\tau_{\ell,m}^r(t)e_r(x) \quad mod(d_xF(x,s)).$$

If we take the trace of this, we get

$$\sum_{c=0}^{\tau}\sigma_a^c(s)\sum_{m=0}^{\tau}\sum_{\ell=0}^{\tau}h_\ell(s)\tau_{c,b}^m(t)(\sum_{r=0}^{\tau}\tau_{\ell,m}^r(t)\zeta_r(t)).$$

After (4.8) and (4.9) this matrix has the following expression,

$$B^{HF}(s) = \Sigma(s)\cdot B^H(t). \tag{4.11}$$

We consider the following closures of semi-algebraic sets,

$$W_{=0} := \{x \in \tilde{X}\cap \mathbf{R}^n; F(x,s) = 0\},$$

$$W_{\ge 0} := \{x \in \tilde{X}\cap \mathbf{R}^n; F(x,s) \ge 0\}, W_{\le 0} := \{x \in \tilde{X}\cap \mathbf{R}^n; F(x,s) \le 0\}.$$

Theorem 4.1 *The following expression of the Euler characteristics for W_* holds,*

$$\chi(W_{\ge 0}) - \chi(W_{=0}) = \frac{sign(B^H(t)) + sign(B^{HF}(s))}{2}.$$

$$\chi(W_{\le 0}) - \chi(W_{=0}) = (-1)^n\frac{sign(B^H(t)) - sign(B^{HF}(s))}{2}.$$

Proof. After Szafraniec [5], or simply applying Morse theory to the real fibres of $F(x,s)$, we have the following equalities,

$$\sum_{x\in\text{critical points of} F(x,s)} (sgn\ h(x,t))$$

$$= sign\langle tr(h(x,t)e_i(x)\cdot e_j(x)\cdot)\rangle_{1\le i,j\le n} = \sum_{x\in\text{critical points of} F(x,s)} (-1)^{\lambda(x)}.$$

Here we denoted by $tr(h(x,t)e_i(x)\cdot e_j(x)\cdot)$ the trace of a matrix defined by the multiplication by the element $h(x,t)e_i(x)\cdot e_j(x)$ considered $mod(d_xF(x,s))$ for the basis $e_i(x), 1\leq i\leq \mu$.

$$\sum_{x\in \text{critical points of} F(x,s)} (sgn\ h(x,t))(sgn\ F(x,s))$$
$$= sign\langle tr(h(x,t)F(x,s)e_i(x)\cdot e_j(x)\cdot)\rangle_{1\leq i,j\leq n}$$
$$= \sum_{x\in \text{critical points of} F(x,s)} (-1)^{\lambda(x)}(sgn\ F(x,s)).$$

We denoted by $tr(h(x,t)F(x,s)e_i(x)\cdot e_j(x)\cdot)$ the trace of a matrix defined by the multiplication by the element $h(x,t)F(x,s)e_i(x)\cdot e_j(x)$ considered $mod(d_xF(x,s))$ for the basis $e_i(x), 1\leq i\leq \mu$. The exponent $\lambda(x)$ is the Morse index of the function $F(x,s)$ at x and $sgn\ h(x,t) = (-1)^{\lambda(x)}$. ■

5 Topology of real complete intersections

Let us reconsider the situation (3.1) for the deformation of the CI,

$$\vec{F}(x,u,t) = \begin{pmatrix} F_1(x,t)-u \\ F_2(x,t) \\ \vdots \\ F_k(x,t) \end{pmatrix}, \tag{5.1}$$

with $s=(u,t)\in S$. Define the ideal $I_{C_0}(t)\subset \mathcal{O}_{\tilde{X}\times S}$ generated by $k\times k$ minors of the matrix $(\frac{\partial \vec{F}(x,0,t)}{\partial x_1},\cdots,\frac{\partial \vec{F}(x,0,t)}{\partial x_n})$.

We have the following isomorphisms

$$\Phi = \frac{\mathcal{O}_X}{\mathcal{O}_X\langle f_1(x)-u, f_2(x),\cdots,f_k(x)\rangle + I_{C_0}(0)}$$

$$\cong \frac{\mathcal{O}_{\tilde{X}}}{\mathcal{O}_{\tilde{X}}\langle f_2(x),\cdots,f_k(x)\rangle + I_{C_0}(0)}, \tag{5.2}$$

where $I_{C_0}(0)$ is the corresponding ideal in $\mathcal{O}_{\tilde{X}}$. The dimension of this space is equal to μ introduced in Proposition 1. As for this number we remember that it can be expressed by means of the Milnor number of the singularity $X_1 := \{x\in\tilde{X}; f_2(x) = \cdots = f_k(x)=0\}$ and the Milnor number of the function f_1 restricted on X_1 i.e. that of the singularity $\tilde{X}_0 := \{x\in\tilde{X}; f_1(x)=f_2(x)=\cdots=f_k(x)=0\}$,

$$\mu = \mu(X_1)+\mu(\tilde{X}_0).$$

This formula is known under the name of Lê-Greuel formula [18, 17].

Let us denote by $\phi_i(x) \in \Phi, 1 \leq i \leq \mu$ a basis of Φ.

Proposition 6 *We have the following free $\mathcal{O}_S$ module of rank μ,*

$$\Phi(s) = \frac{\mathcal{O}_{\tilde{X}\times S}}{\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)}.$$

Proof. We reproduce the argument by [4], Lemma A 1. First of all we see that the module $\Phi(s)$ is a finitely generated $\mathcal{O}_S$ module. This can be shown by a combination of the Weierstraß-Malgrange preparation theorem and the fact that for each fixed $s \in S$ the space

$$\frac{\mathcal{O}_{\tilde{X}}}{\mathcal{O}_{\tilde{X}}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)}, \tag{5.3}$$

is a finite dimensional ($\leq \mu$) $\mathbf{C}$ vector space (see [5]).

The above space (5.3) is isomorphic to the direct sum of $\mathbf{C}$ vector spaces,

$$\bigoplus_{\{x';(x',s)\in Cr(\vec{F})\}} \frac{\mathcal{O}_{\tilde{X},x'}}{\mathcal{O}_{\tilde{X},x'}\langle F_2(x,t), \cdots, F_k(x,t)\rangle_{x'} + I_{C_0}(t)_{x'}}.$$

Since this direct sum has dimension μ = the multiplicity of the critical point $(x,u) = 0$ of the height function on X_0, as mentioned at the very beginning of the paper, it follows that $\{\phi_i(x)\}_{0\leq i\leq\tau}$ form in fact a $\mathbf{C}$ basis of (5.3). Now we see that they form in fact $\Phi(s)$ freely. If not, there exist holomorphic functions $\{a_i(s)\}_{0\leq i\leq\tau}$ such that $\sum_{i=0}^{\tau} a_i(s)\phi_i(x) = 0$ in $\Phi(s)$. It would contradict the fact that for each fixed s, $\{\phi_i(x)\}_{0\leq i\leq\tau}$ are linearly independent in (5.3). ■

Let us consider the multiplication table

$$(F_1(x,t) - u)\phi_i(x) \tag{5.4}$$

$$\equiv \sum_{\ell=0}^{\tau} \rho_i^{\ell}(s)\phi_\ell(x) \ mod(\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)).$$

Thus the matrix

$$P(s) := (\rho_i^{\ell}(s))_{0\leq i,\ell\leq\tau} = (\tilde{\rho}_i^{\ell}(t) - u\cdot\delta_{i,\ell})_{0\leq i,\ell\leq\tau}, \tag{5.5}$$

is defined. In analogy with (3.3), we define another multiplication table

$$\phi_i(x)\phi_j(x) \equiv \sum_{\ell=0}^{\tau} w_{i,j}^{\ell}(t)\phi_\ell(x) \ \ mod(\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)). \tag{5.6}$$

We will denote by $W^c(t)$ the matrix $(w^c_{\ell,b}(t))_{0\leq \ell,b\leq \tau}$. Hence,

$$(F_1(x,t)-u)\phi_a(x)\phi_b(x) \equiv \sum_{\ell=0}^{\tau} \rho_a^\ell(s)\phi_\ell(x)\phi_b(x) \tag{5.7}$$

$$\equiv \sum_{\ell=0}^{\tau} \rho_a^\ell(s) \sum_{c=0}^{\tau} w^c_{\ell,b}(t)\phi_c(x) \;\; mod(\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t),\cdots,F_k(x,t)\rangle + I_{C_0}(t)). \tag{5.7}$$

$$\zeta_c(t) := tr(\phi_c(x)\cdot) = \sum_{\ell=0}^{\tau} w^\ell_{c,\ell}(t). \tag{5.8}$$

Thus

$$tr((F_1(x,t)-u)\phi_a(x)\phi_b(x)\cdot) = \sum_{\ell=0}^{\tau} \rho_a^\ell(s) \sum_{c=0}^{\tau} w^c_{\ell,b}(t)\zeta_c(t). \tag{5.9}$$

We introduce the notation,

$$T(t) = \sum_{c=0}^{\tau} \zeta_c(t) W^c(t). \tag{5.10}$$

¿From here on we assume that $|s|$ is small enough and denote by $\tilde{X} = \{x \in \mathbf{C}^n; |x| \leq \delta\}$ a closed ball such that all critical points of $F_1(x,t)-u$ on $F_2(x,t) = \cdots = F_k(x,t) = 0$ are located inside $\tilde{X}$.

In combining the results of [5], theorem 2.1, Theorem 2.4, Theorem 3.1, with our above arguments we get the following.

Theorem 5.1 *1. The discriminant set of the deformation of projection X_t is given by the matrix* (5.5),

$$D = \{s \in S; det(P(s)) = 0\}. \tag{5.11}$$

2. { number of positive critical points of $F_1(x,t)-u$ on $F_2(x,t) = \cdots = F_k(x,t) = 0$, $x \in \tilde{X} \cap \mathbf{R}^n$} - { number of negative critical points of $F_1(x,t)-u$ on $F_2(x,t) = \cdots = F_k(x,t) = 0$, $x \in \tilde{X} \cap \mathbf{R}^n$ }

$$= sign(P(s)\cdot T(t)).$$

In opposition to the case $k = 1$, we cannot write down a simple formula for Euler characteristic of closures of semi-algebraic sets,

$$W_* = \{x \in \tilde{X} \cap \mathbf{R}^n; F_1(x,t) - u * 0, F_2(x,t) = \cdots = F_k(x,t) = 0\},$$

with $* = \geq, \leq, =$. As a matter of fact, it is quite easy to establish an analogous theorem to [5] Theorem 3.3 on $\chi(W_{\geq 0}) \pm \chi(W_{\leq 0})$ by the aid of matrices introduced above. We leave this task as an exercise in view of complicated form of the analogy to the Hessian.

The bifurcation set B_{F_1} is defined as $B_{F_1} := \{t \in T;$ number of critical points of $F_1(x,t) - u$ on $F_2(x,t) = \cdots = F_k(x,t) = 0$ is strictly less than $\mu\} \setminus B_M$. Here B_M denotes the Maxwell set of $F_1(x,t) - u$, namely $B_M := \{t \in T;$ two critical values of $F_1(x,t) - u$ on $F_2(x,t) = \cdots = F_k(x,t) = 0$ coincides $\}$.

Proposition 7 *The bifurcation set has the following expression*

$$B_{F_1} = \{t \in T; det\, T(t) = 0\}. \tag{5.12}$$

Proof. We consider the critical set

$$C_0(t) := \{x \in \tilde{X}; dF_1(x,t) \wedge dF_2(x,t) \wedge \cdots \wedge dF_k(x,t) = 0,$$
$$F_2(x,t) = \cdots = F_k(x,t) = 0\}.$$

Here we remark that the critical set $C_0(t)$ has codimension n in $\tilde{X}$ for a fixed generic value t and it is a set of points. After [5] Corollary 2.5, the rank of $T(t)$ is equal to the number of points $\{p \in C_0(t)\}$. Therefore $T(t)$ degenerates if and only if $|C_0(t)| < \mu$ which means our statement. ■

Regretfully, to the moment we cannot state how to deduce the basis of $Der_S(log\ D)$ from the matrix $P(s)$. Consequently we cannot establish the relationship between the Gauss-Manin system and the topology of the real algebraic sets. This fact is due to the situation mentioned in the Remark 1 below.

To remedy the situation, we state a proposition on the multiplication table and the coefficients to the Gauss-Manin system.

Let us consider the multiplication between ϕ_i and $\vec{v}_j$ by the following way,

$$\frac{\partial(\phi_i(x) h_{j,p}(x,s))}{\partial x_p} \equiv \tag{5.13}$$

$$\equiv \sum_{r=0}^{\tau} R^r_{i,j}(s) \phi_r(x) mod(\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)).$$

Here $\check{\vec{v}}_j = \sum_{p=1}^{n} h_{j,p}(x,s) \frac{\partial}{\partial x_p}$ denotes the vector field that has been defined in Lemma 4.

Proposition 8 *The Gauss-Manin system for the period integrals $I^{(-\mathbf{1})}_{\phi_i}(s)$ introduced in the Theorem 2.3 is expressed by means of multiplication tables* (5.6) *and* (5.13) *as follows,*

$$\vec{v}_j(I^{(-\mathbf{1})}_{\phi_i}(s)) = \sum_{\ell=1}^{\mu} \left((tr\ M^0)\cdot w^{\ell}_{i,j}(s) + R^{\ell}_{i,j}(s)\right) I^{(-\mathbf{1})}_{\phi_\ell}(s) \ \ 1 \leq j, q \leq \mu.$$

Here tr M^0 stands for the trace of the non-degenerate matrix M^0 defined in Lemma 3.

Proof. First of all we remark the following chain of equalities,

$$\vec{v}_j\left(\int_{t(\gamma)} \phi_i(x)\vec{F}^{-\mathbf{1}}dx\right) = \int_{t(\gamma)} \phi_i(x)\left(\sum_{\ell=1}^{k}\sigma^{\ell}_j(s)\frac{\partial}{\partial s_\ell}\vec{F}^{-\mathbf{1}}\right)dx$$

$$= \sum_{q=1}^{k}\int_{t(\gamma)} \phi_i(x)F_q^{-1}\vec{F}^{-\mathbf{1}}(\sum_{\ell=1}^{k}\sigma^{\ell}_j(s)\frac{\partial F_q(x,s)}{\partial s_\ell})dx.$$

Here we remember Lemmata 3, 4 and see that the above expression equals to

$$\int_{t(\gamma)} \phi_i(x)\phi_j(x)\left(\sum_{q=1}^{k}F_q^{-1}(\sum_{\ell=1}^{k}C^{\ell}_qF_\ell(x,s)+m_q(x,\vec{F}(x,s)))\right)\vec{F}^{-\mathbf{1}}dx$$

$$-\int_{t(\gamma)} \phi_i(x)\sum_{q=1}^{k}(-1)^{q-1}dF_q\wedge i_{\check{\vec{v}}_j}(dx)F_q^{-1}\vec{F}^{-\mathbf{1}}.$$

As the terms with C'^{ℓ}_q, $\ell \neq q$ (resp. terms with $m_q(x,\vec{F}(x,s)) \in \mathcal{O}_{\tilde{X}}\otimes m_S^2$) vanish because of the lack of residues along $F_\ell(x,s)=0$ (resp. some other $F_r(x,s)=0$), the last expression in its turn equals to

$$\int_{t(\gamma)} \phi_i(x)\phi_j(x)(\sum_{q=1}^{k}C^q_qF_q(x,s))F_q^{-1}\vec{F}^{-\mathbf{1}}dx + \int_{t(\gamma)} d(\phi_i(x)i_{\check{\vec{v}}_j}(dx))F^{-\mathbf{1}}$$

$$= (\sum_{q=1}^{k}C^q_q)\int_{t(\gamma)} \phi_i(x)\phi_j(x)\vec{F}^{-\mathbf{1}}dx + \sum_{\ell=1}^{k}\int_{t(\gamma)} R^{\ell}_{i,j}(s)\phi_\ell(x)\vec{F}^{-\mathbf{1}}dx$$

$$= \sum_{\ell=1}^{\mu}\left((tr\ M^0)\cdot w^{\ell}_{i,j}(t) + R^{\ell}_{i,j}(s)\right) I^{(-\mathbf{1})}_{\phi_\ell}(s).$$

■

Remark 1 The rank of $\mathbf{C}-$module of Leray coboundaries $t(\gamma) \in H_n(\tilde{X} \setminus \cup_{q=1}^{k}\{x \in \tilde{X}; F_q(x,s) = 0\})$ is equal to $\mu(\tilde{X}_0)$: the Milnor number of the singularity $\tilde{X}_0$ due to the tube operation isomorphism t : defined in Lemma 2. In view of the Lê-Greuel formula mentioned in connection with (5.2), the dimension μ of the space Φ is bigger than $\mu(\tilde{X}_0)$ as it represents the sum of the ranks of $(n-k)-$dimensional cycles and $(n-k+1)-$dimensional cycles. Thus we have no exact duality between the integrands and the integration cycles. This means that the Gauss-Manin system of the above Proposition 8 is defined only for the Riemann period matrix of size $\mu \times \mu(\tilde{X}_0)$.

To get the the Gauss-Manin system defined for the Riemann period matrix of size $\mu(\tilde{X}_0) \times \mu(\tilde{X}_0)$, one need to consider the multiplication table on the Brieskorn-Greuel lattice

$$\mathcal{H}'' := \frac{\Omega^n_{\tilde{X}}}{dF_1(x,s) \wedge \cdots \wedge dF_k(x,s) \wedge d\Omega^{n-k-1}_{\tilde{X}} + \langle F_1(x,s), \cdots, F_k(x,s)\rangle \Omega^n_{\tilde{X}}},$$

that is known to be a $\mathcal{O}_S$ free module of rank $\mu(\tilde{X}_0)$. This procedure can be done in an analogous way to that in Proposition 8. For the case of quasihomogeneous i.c.i.s., the concrete calculus of the the Gauss-Manin system is done by means of Brieskorn-Greuel lattice in [19].

6 Examples

1. Let us consider the simplest example of the Pham-Brieskorn singularity,

$$F(x_1, x_2) = x_1^3 + x_2^3 + u + bx_1x_2 + cx_1 + dx_2,$$

with deformation parameters $s = (u, t) = (u, b, c, d)$. We calculate the data (4.4), (2.4), (4.10), (4.11) as follows.

$$\tau^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3\,d & 0 & 1/9\,bc \\ 0 & 0 & -1/3\,c & 1/9\,bd \\ 0 & 1/9\,bc & 1/9\,bd & 1/9\,dc \end{bmatrix}$$

$$\tau^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1/9\,b^2 \\ 0 & 0 & -1/3\,b & -1/3\,c \\ 0 & 1/9\,b^2 & -1/3\,c & 1/9\,bd \end{bmatrix}$$

$$\tau^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1/3\,b & 0 & -1/3\,d \\ 1 & 0 & 0 & 1/9\,b^2 \\ 0 & -1/3\,d & 1/9\,b^2 & 1/9\,bc \end{bmatrix}$$

$$\tau^4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1/9\,b^2 \end{bmatrix}$$

$$\Sigma(s)$$

$$= \begin{bmatrix} 3\,u & 2\,d & 2\,c & b \\ -2/3\,d^2 + 1/9\,b^2c & 3\,u + 1/9\,b^3 & -bd & 2\,c \\ -2/3\,c^2 + 1/9\,b^2d & -bc & 3\,u + 1/9\,b^3 & 2\,d \\ 5/9\,bcd & -2/3\,c^2 + 1/3\,b^2d & -2/3\,d^2 + 1/3\,b^2c & 3\,u + 1/9\,b^3 \end{bmatrix}$$

$$B^H = \begin{bmatrix} 8\,b^2 & 16\,bc & 16\,bd & b^4 + 16\,dc \\ 16\,bc & -8\,b^2d & b^4 + 16\,dc & 8/3\,b^3c - 16/3\,bd^2 \\ 16\,bd & b^4 + 16\,dc & -8\,b^2c & 8/3\,b^3d - 16/3\,bc^2 \\ b^4 + 16\,dc & 8/3\,b^3c - 16/3\,bd^2 & 8/3\,b^3d - 16/3\,bc^2 & \frac{56}{9}\,b^2dc + 1/9\,b^6) \end{bmatrix}$$

$$B^{HF}(s) = \Sigma(s) \cdot B^H = \begin{bmatrix} b_{1.1} & b_{1.2} & b_{1.3} & b_{1.4} \\ b_{2.1} & b_{2.2} & b_{2.3} & b_{2.4} \\ b_{3.1} & b_{3.2} & b_{3.3} & b_{3.4} \\ b_{4.1} & b_{4.2} & b_{4.3} & b_{4.4} \end{bmatrix},$$

where

$$b_{1.1} = 24\,ub^2 + 80\,bcd + b^5, b_{1.2} = b_{2.1} = -\frac{64}{3}\,d^2b^2 + 14/3\,b^4c + 48\,ubc + 32\,dc^2,$$

$$b_{1.3} = b_{3.1} = -\frac{64}{3}\,c^2b^2 + 14/3\,b^4d + 48\,ubd + 32\,d^2c,$$

$$b_{1.4} = b_{4.1} = \frac{152}{9}\,b^3dc - \frac{32}{3}\,bc^3 - \frac{32}{3}\,bd^3 + 3\,ub^4 + 48\,udc + 1/9\,b^7,$$

$$b_{2.2} = -\frac{112}{3}\,bcd^2 + \frac{64}{9}\,b^3c^2 - 24\,b^2du - \frac{17}{9}\,b^5d,$$

$$b_{2.3} = b_{3.2} = \frac{152}{9}\, b^3dc - \frac{32}{3}\, bc^3 - \frac{32}{3}\, bd^3 + 3\, ub^4 + 48\, udc + 1/9\, b^7,$$

$$b_{2.4} = b_{4.2} = b_{3.4} = b_{4.3}$$

$$= -\frac{106}{27}\, c^2b^4 - \frac{32}{3}\, c^3d + \frac{17}{27}\, b^6d + \frac{176}{9}\, b^2cd^2 + 8\, ub^3d - 16\, ubc^2,$$

$$b_{3.3} = -\frac{112}{3}\, bdc^2 + \frac{64}{9}\, b^3d^2 - \frac{17}{9}\, b^5c - 24\, b^2cu$$

$$b_{4.4} = \frac{245}{81}\, b^5cd + 16\, bc^2d^2 - \frac{32}{9}\, c^3b^3 - \frac{32}{9} b^3d^3 + \frac{56}{3}\, ub^2dc + 1/3\, ub^6 + \frac{1}{81}\, b^9.$$

After Theorem 4.1 the signature of this matrix gives us the Euler characteristic of real algebraic sets defined by $F(x, s) \geq, \leq, = 0$.

We calculate the determinants of these matrices.

$$det(B^H) = 1/9\, \left(256\, b^2d^3 + 768\, d^2c^2 + 96\, b^4dc - b^8 + 256\, c^3b^2\right)^2,$$

$$det(\Sigma(s)) =$$

$$8/3\, b^2c^4d - \frac{1}{243}\, b^8cd + 8/3\, d^4cb^2 + \frac{23}{27}\, b^4d^2c^2 + 32\, ubc^2d^2 - \frac{11}{9}\, ub^5cd - 30\, u^2b^2dc$$

$$-\frac{1}{243}\, b^6d^3 - \frac{1}{243}\, b^6c^3 - \frac{32}{9}\, d^3c^3 + 24\, u^2d^3 + 1/3\, u^2b^6 + 9\, u^3b^3 + \frac{1}{243}\, ub^9$$

$$-\frac{20}{9}\, uc^3b^3 - \frac{20}{9}\, ub^3d^3 + 24\, c^3u^2 + 81\, u^4 + \frac{16}{9}\, d^6 + \frac{16}{9}\, c^6$$

The discriminant of the polynomial $det(\Sigma)(s)$ with respect to the variable u is calculated as follows,

$$Dscrim(det(\Sigma), u)$$

$$= 27\, (d-c)^2(d^2 + dc + c^2)^2(256\, b^2d^3 + 768\, d^2c^2 + 96\, b^4dc - b^8 + 256\, c^3b^2)^3.$$

These results combined with the Proposition 7 calculate the Maxwell set,

$$M = \{s \in \mathbf{C}^3; (d-c)^2 \left(d^2 + dc + c^2\right)^2 = 0\}.$$

Example 2. The versal deformation of the singularity E_6.
We consider the following deformation,

$$F(x, y, t) + u = x^3 + y^4 + gxy^2 + dy^2 + cxy + by + ax + u.$$

with $t = (a, b, c, d, g)$. As $F(x, y, 0)$ is a quasihomogeneous polynomial in (x, y), we attribute to the deformation parameters $(u, t) \in S$ corresponding quasihomogeneous weights. This means that there is a $\mathbf{C}^*$ action on the space of deformation parameters S. This allows us to consider $\tilde{X} = \mathbf{C}^2$, $S = \mathbf{C}^6$ in the arguments of §4.

Thus we deal with the global parameter values $t \in \mathbf{C}^5$. Essentially all the informations on the multiplication table (4.3) are contained in the following equivalence relations,

$x^2 \equiv -1/3\, gy^2 - 1/3\, cy - 1/3\, a \;\; mod(d_x F(x,y,t), d_y F(x,y,t))$

$y^3 \equiv (-1/2\, gy - 1/4\, c)x - 1/2\, dy - 1/4\, b$

$x^2 y \equiv (1/12\, gc + 1/6\, g^2 y)x + 1/12\, gb - 1/3\, cy^2 + (1/6\, dg - 1/3\, a)y$

$x^2 y^2 \equiv (1/6\, g^2 y^2 + 1/4\, gcy + 1/12\, c^2)x + (1/6\, dg - 1/3\, a)y^2 + (1/12\, gb + 1/6\, cd)y + 1/12\, cb$

$xy^3 \equiv ((-1/2\, d - 1/12\, g^3)y - 1/24\, g^2 c - 1/4\, b)x + 1/4\, gcy^2 + (1/6\, ga + 1/12\, c^2 - 1/12\, g^2 d)y - 1/24\, g^2 b + 1/12\, ca$

$x^2 y^3 \equiv (1/4\, gcy^2 + (1/3\, ga - 1/3\, g^2 d + 1/12\, c^2 - 1/36\, g^5)y - \frac{1}{72}\, g^4 c + 1/6\, ca - 1/12\, g^2 b - 1/12\, dgc)x + (1/12\, gb + 1/6\, cd + 1/12\, g^3 c)y^2 + (-1/36\, g^4 d + 1/18\, g^3 a + 1/3\, ad + 1/36\, g^2 c^2 - 1/6\, d^2 g + 1/12\, cb)y + 1/36\, g^2 ca - 1/12\, dgb - \frac{1}{72}\, g^4 b + 1/6\, ab$

$xy^4 \equiv ((-1/12\, g^3 - 1/2\, d)y^2 + (-1/4\, b - 1/6\, g^2 c)y - 1/16\, gc^2)x + (1/12\, c^2 - 1/12\, g^2 d + 1/6\, ga)y^2 + (-1/24\, g^2 b - 1/8\, dgc + 1/12\, ca)y - 1/16\, gcb$

$$
\begin{gathered}
x^2 y^4 \equiv ((1/12\, c^2 + 1/3\, ga - 1/36\, g^5 - 1/3\, g^2 d)y^2 + \\
(-\frac{11}{144}\, g^4 c - 1/8\, g^2 b + 1/6\, ca - \frac{7}{24}\, dgc)y \\
-1/12\, gcb - 1/32\, g^3 c^2 - 1/24\, c^2 d)x \\
+(1/3\, ad - 1/6\, d^2 g + 1/18\, g^3 a - 1/36\, g^4 d + 1/12\, cb + \frac{13}{144}\, g^2 c^2)y^2 \\
+(-1/12\, cd^2 - 1/16\, g^3 cd + 1/6\, ab - 1/8\, dgb + 1/48\, gc^3 + \frac{5}{72}\, g^2 ca - \frac{1}{72}\, g^4 b)y \\
-1/48\, gb^2 - 1/24\, cdb + 1/48\, gc^2 a - 1/32\, g^3 cb.
\end{gathered}
$$

We can write down these results in the form of matrices (4.4) and the polynomials $\zeta_k(t)$, $1 \le k \le 6$, (4.6),

$$
\tau^1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{a}{3} & 0 & \frac{bg}{12} & 0 & \frac{bc}{12} \\
0 & 0 & 0 & 0 & -\frac{b}{4} & \frac{ac}{12} - \frac{bg^2}{24} \\
0 & \frac{bg}{12} & 0 & \frac{bc}{12} & \frac{ac}{12} - \frac{bg^2}{24} & p_1 \\
0 & 0 & -\frac{b}{4} & \frac{ac}{12} - \frac{bg^2}{24} & 0 & -\frac{bcg}{16} \\
0 & \frac{bc}{12} & \frac{ac}{12} - \frac{bg^2}{24} & p_1 & -\frac{bcg}{16} & -\frac{bcd}{24} - \frac{b^2 g}{48} + \frac{ac^2 g}{48} - \frac{bcg^3}{48}
\end{bmatrix}
$$

where $p_1 = \frac{ab}{12} - \frac{bdg}{24} + \frac{acg^2}{72} - \frac{bg^4}{144}$,

$$\zeta_1(t) = 6.$$

$$\tau^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{c\,g}{12} & 0 & \frac{c^2}{12} \\ 0 & 0 & 0 & 0 & -\frac{c}{4} & -\frac{b}{4} - \frac{c\,g^2}{24} \\ 0 & \frac{c\,g}{12} & 0 & \frac{c^2}{12} & -\frac{b}{4} - \frac{c\,g^2}{24} & p_2 \\ 0 & 0 & -\frac{c}{4} & -\frac{b}{4} - \frac{c\,g^2}{24} & 0 & -\frac{c^2\,g}{16} \\ 0 & \frac{c^2}{12} & -\frac{b}{4} - \frac{c\,g^2}{24} & p_2 & -\frac{c^2\,g}{16} & -\frac{c^2\,d}{24} - \frac{b\,c\,g}{12} - \frac{c^2\,g^3}{48} \end{bmatrix},$$

where $p_2 = \frac{a\,c}{12} - \frac{c\,d\,g}{24} - \frac{b\,g^2}{24} - \frac{c\,g^4}{144}$,

$$\zeta_2 = \frac{g^2}{3}.$$

$$\tau^3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{c}{3} & 0 & -\frac{a}{3} + \frac{d\,g}{6} & 0 & \frac{c\,d}{6} + \frac{b\,g}{12} \\ 1 & 0 & 0 & 0 & -\frac{d}{2} & r_1 \\ 0 & -\frac{a}{3} + \frac{d\,g}{6} & 0 & \frac{c\,d}{6} + \frac{b\,g}{12} & r_1 & r_3 \\ 0 & 0 & -\frac{d}{2} & r_1 & -\frac{b}{4} & r_2 \\ 0 & \frac{c\,d}{6} + \frac{b\,g}{12} & r_1 & r_3 & r_2 & r_4 \end{bmatrix},$$

where

$$r_1 = \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{12},$$

$$r_2 = \frac{a\,c}{12} - \frac{c\,d\,g}{8} - \frac{b\,g^2}{24},$$

$$r_3 = \frac{b\,c}{12} + \frac{a\,d}{6} - \frac{d^2\,g}{12} + \frac{c^2\,g^2}{72} + \frac{a\,g^3}{36} - \frac{d\,g^4}{72},$$

$$r_4 = \frac{a\,b}{12} - \frac{c\,d^2}{12} + \frac{c^3\,g}{48} - \frac{b\,d\,g}{12} + \frac{a\,c\,g^2}{18} - \frac{c\,d\,g^3}{24} - \frac{b\,g^4}{144}.$$

$$\zeta_3(t) = 0.$$

$$\tau^4$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{g^2}{6} & 0 & \frac{c\,g}{4} \\ 0 & 1 & 0 & 0 & -\frac{g}{2} & -\frac{d}{2} - \frac{g^3}{12} \\ 1 & \frac{g^2}{6} & 0 & \frac{c\,g}{4} & -\frac{d}{2} - \frac{g^3}{12} & \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{6} - \frac{g^5}{72} \\ 0 & 0 & -\frac{g}{2} & -\frac{d}{2} - \frac{g^3}{12} & -\frac{c}{4} & -\frac{b}{4} - \frac{c\,g^2}{6} \\ 0 & \frac{c\,g}{4} & -\frac{d}{2} - \frac{g^3}{12} & \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{6} - \frac{g^5}{72} & -\frac{b}{4} - \frac{c\,g^2}{6} & \frac{a\,c}{12} - \frac{c\,d\,g}{4} - \frac{b\,g^2}{12} - \frac{7\,c\,g^4}{144} \end{bmatrix}.$$

$$\zeta_4(t) = \frac{5\,c\,g}{6}.$$

$$\tau^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{g}{3} & 0 & -\frac{c}{3} & 0 & -\frac{a}{3} + \frac{d\,g}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{c\,g}{4} \\ 0 & -\frac{c}{3} & 0 & -\frac{a}{3} + \frac{d\,g}{6} & \frac{c\,g}{4} & \frac{c\,d}{6} + \frac{b\,g}{12} + \frac{c\,g^3}{24} \\ 1 & 0 & 0 & \frac{c\,g}{4} & -\frac{d}{2} & \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{12} \\ 0 & -\frac{a}{3} + \frac{d\,g}{6} & \frac{c\,g}{4} & \frac{c\,d}{6} + \frac{b\,g}{12} + \frac{c\,g^3}{24} & \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{12} & q_5 \end{bmatrix},$$

where

$$q_5 = \frac{b\,c}{12} + \frac{a\,d}{6} - \frac{d^2\,g}{12} + \frac{11\,c^2\,g^2}{144} + \frac{a\,g^3}{36} - \frac{d\,g^4}{72}.$$

$$\zeta_5(t) = -2\,d - \frac{g^3}{6}.$$

$$\tau^6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{g^2}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{g^2}{6} & 0 & \frac{c\,g}{4} \\ 0 & 1 & 0 & 0 & -\frac{g}{2} & -\frac{d}{2} - \frac{g^3}{12} \\ 1 & \frac{g^2}{6} & 0 & \frac{c\,g}{4} & -\frac{d}{2} - \frac{g^3}{12} & \frac{c^2}{12} + \frac{a\,g}{6} - \frac{d\,g^2}{6} - \frac{g^5}{72} \end{bmatrix}$$

$$\zeta_6(t) = \frac{5\,c^2}{12} + \frac{2\,a\,g}{3} - \frac{d\,g^2}{2} - \frac{g^5}{36}.$$

Finally we get the matrix (4.5) as follows.

$$T(t) = 6\tau^1(t) + \zeta_3(t)\tau^3(t) + \zeta_4(t)\tau^4(t) + \zeta_5(t)\tau^5(t) + \zeta_6(t)\tau^6(t) =$$

$$= \begin{bmatrix} 6 & \frac{g^2}{3} & 0 & \frac{5\,c\,g}{6} & -2\,d - \frac{g^3}{6} & T_{1,6} \\ \frac{g^2}{3} & -2\,a + \frac{2\,d\,g}{3} + \frac{g^4}{18} & \frac{5\,c\,g}{6} & \frac{2\,c\,d}{3} + \frac{b\,g}{2} + \frac{2\,c\,g^3}{9} & T_{2,5} & T_{2,6} \\ 0 & \frac{5\,c\,g}{6} & -2\,d - \frac{g^3}{6} & T_{3,4} & -\frac{3\,b}{2} - \frac{c\,g^2}{2} & T_{3,6} \\ \frac{5\,c\,g}{6} & \frac{2\,c\,d}{3} + \frac{b\,g}{2} + \frac{2\,c\,g^3}{9} & T_{4,3} & T_{4,4} & T_{4,5} & T_{4,6} \\ -2\,d - \frac{g^3}{6} & T_{5,2} & -\frac{3\,b}{2} - \frac{c\,g^2}{2} & T_{5,4} & T_{5,5} & T_{5,6} \\ T_{6,1} & T_{6,2} & T_{6,3} & T_{6,4} & T_{6,5} & T_{6,6} \end{bmatrix},$$

where

$$\begin{gathered}
T_{1,6} = T_{2,5} = T_{3,4} = T_{4,3} = T_{5,2} = T_{6,1} = \tfrac{5\,c^2}{12} + \tfrac{2\,a\,g}{3} - \tfrac{d\,g^2}{2} - \tfrac{g^5}{36}, \\
T_{2,6} = T_{6,2} = \tfrac{b\,c}{2} + \tfrac{2\,a\,d}{3} - \tfrac{d^2\,g}{3} + \tfrac{11\,c^2\,g^2}{36} + \tfrac{a\,g^3}{6} - \tfrac{d\,g^4}{9} - \tfrac{g^7}{216}, \\
T_{4,4} = \tfrac{b\,c}{2} + \tfrac{2\,a\,d}{3} - \tfrac{d^2\,g}{3} + \tfrac{11\,c^2\,g^2}{36} + \tfrac{a\,g^3}{6} - \tfrac{d\,g^4}{9} - \tfrac{g^7}{216}. \\
T_{4,5} = T_{5,4} = \tfrac{a\,c}{2} - \tfrac{11\,c\,d\,g}{12} - \tfrac{b\,g^2}{3} - \tfrac{c\,g^4}{8} \\
T_{3,6} = T_{6,3} = \tfrac{a\,c}{2} - \tfrac{11\,c\,d\,g}{12} - \tfrac{b\,g^2}{3} - \tfrac{c\,g^4}{8} \\
T_{5,5} = d^2 - \tfrac{5\,c^2\,g}{12} - \tfrac{a\,g^2}{3} + \tfrac{d\,g^3}{3} + \tfrac{g^6}{72} \\
T_{4,6} = T_{6,4} = \tfrac{a\,b}{2} - \tfrac{c\,d^2}{3} + \tfrac{25\,c^3\,g}{144} - \tfrac{5\,b\,d\,g}{12} + \tfrac{5\,a\,c\,g^2}{12} - \tfrac{7\,c\,d\,g^3}{18} - \tfrac{5\,b\,g^4}{72} - \tfrac{c\,g^6}{36}, \\
T_{5,6} = T_{6,5} = -\tfrac{3\,c^2\,d}{8} - \tfrac{7\,b\,c\,g}{12} - \tfrac{2\,a\,d\,g}{3} + \tfrac{5\,d^2\,g^2}{12} - \tfrac{5\,c^2\,g^3}{24} - \tfrac{a\,g^4}{12} + \tfrac{5\,d\,g^5}{72} + \tfrac{g^8}{432}, \\
T_{6,6} = \tfrac{5\,c^4}{144} - \tfrac{5\,b\,c\,d}{12} - \tfrac{a\,d^2}{3} - \tfrac{b^2\,g}{8} + \tfrac{23\,a\,c^2\,g}{72} + \tfrac{d^3\,g}{6} + \\
+ \tfrac{a^2\,g^2}{9} - \tfrac{35\,c^2\,d\,g^2}{72} - \tfrac{17\,b\,c\,g^3}{72} - \tfrac{5\,a\,d\,g^3}{18} + \tfrac{d^2\,g^4}{8} - \tfrac{59\,c^2\,g^5}{864} - \tfrac{a\,g^6}{54} + \tfrac{d\,g^7}{72} + \tfrac{g^{10}}{2592}.
\end{gathered}$$

It is a conceptually easy exercise to calculate further $B^H(s)$ and $B^{HF}(s)$ to establish correspondence between parameter value $s = (a, b, c, d, g, u)$ and the Euler characteristic of a semi-algebraic set defined by $F(x, y, t) + u$.

For instance, for the values

$$-0.6 \leq a \leq 1, (b, c, d, g, u) = (-0.4, 0.1, 0.1, -0.1, -10),$$

we calculate with computer (Mathematica computation achieved by Galina Filipuk) $\chi(W_{\geq 0}) = \chi(W_{\leq 0}) = 0$, while for the values

$$-1 \leq a \leq -0.8, (b, c, d, g, u) = (-0.4, 0.1, 0.1, -0.1, -10),$$

we have $\chi(W_{\geq 0}) = 1, \chi(W_{\leq 0}) = -1$.

For the values

$$-1 \leq a \leq -0.8, (b, c, d, g, u) = (-0.4, 0.1, 0.1, -0.1, 8.5),$$

we have $\chi(W_{\geq 0}) = -1, \chi(W_{\leq 0}) = 1$, and

$$-0.6 \leq a \leq 1, (b, c, d, g, u) = (-0.4, 0.1, 0.1, -0.1, 8.5),$$

we have $\chi(W_{\geq 0}) = 0, \chi(W_{\leq 0}) = 0$.

It is worthy noticing that the first two cases (resp. last two cases) give us examples of topologically different isotopy types of the real curve for the same sign combination of coefficients $(-, -, +, +, -, -)$ (resp. $(-, -, +, +, -, +)$). These examples show the cases that Viro's patchworking method could not distinguish.

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