Novi Sad J. Math.
Vol. 45, No. 1, 2015, 31-44

SOME APPLICATIONS OF LAPLACE TRANSFORMS IN ANALYTIC NUMBER THEORY

Aleksandar Ivić[1]

Dedicated to Professor Stanković on the occasion of his 90^{th} *birthday.*

Abstract. In this overview paper, presented at the meeting DANS14, Novi Sad, July 3-7, 2014, we give some applications of Laplace transforms to analytic number theory. These include the classical circle and divisor problem, moments of $|\zeta(\frac{1}{2}+it)|$, and a discussion of two functional equations connected to a work of Prof. Bogoljub Stanković.

AMS Mathematics Subject Classification (2010): 44A10, 39B22, 11N37, 11M06

Key words and phrases: Laplace transforms; circle problem; divisor problem; Riemann zeta-function

1. Introduction

1.1. Integral transforms

Integral transforms play an important rôle in Analytic number theory, the part of Number theory where problems of a number-theoretic nature are solved by the use of various methods from Analysis. The most common integral transforms that are used are: *Mellin transforms* (Robert Hjalmar Mellin, 1854-1933), *Laplace transforms* (Pierre-Simon, marquis de Laplace, 1749-1827) and *Fourier transforms* (Joseph Fourier, 1768-1830). Crudely speaking, suppose that one has an integral transform

$$F(s) := \int_{\mathcal{I}} f(t)K_1(s,t)\,\mathrm{d}t, \tag{1.1}$$

where $\mathcal{I}$ is an interval, and $K_1(s,t)$ is a suitable kernel function. If a problem involving the initial function $f(t)$ can be solved by means of the transforms $F(s)$, then by the inverse transform

$$f(t) := \int_{\mathcal{J}} F(s)K_2(s,t)\,\mathrm{d}s \tag{1.2}$$

one can obtain information about $f(t)$ itself. Here $\mathcal{J}$ is a suitable contour in $\mathbb{C}$ and $K_2(s,t)$ is another kernel function. Naturally the passage from (1.1) to (1.2) and back requires knowledge about the kernels, the convergence (existence) of the integrals that are involved, etc.

[1]Serbian Academy of Science and Arts, Knez Mihailova 35, 11000 Beograd, Serbia, e-mail: aleksandar.ivic@rgf.bg.ac.rs, aivic_2000@yahoo.com

1.2. Mellin transforms

If $f(x)x^{\sigma-1} \in L(0,\infty)$ and $f(x)$ is of bounded variation in every finite x-interval, then

$$F(s) := \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x \qquad (s = \sigma + it,\ \sigma, t \in \mathbb{R}) \tag{1.3}$$

is the Mellin transform of $f(x)$. The notation of a complex variable $s = \sigma + it$ in Number Theory originates with B. Riemann (1826-1866). If (1.3) holds, then under suitable conditions

$$\tfrac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} F(s)x^{-s}\,\mathrm{d}s. \tag{1.4}$$

The relation (1.4) is the Mellin inversion formula. Note that if the function $f(x)$ is continuous, then the left-hand side is simply $f(x)$. As an example, take $f(x) = e^{-x}$ ($\Re s > 0$). Then $F(s) = \Gamma(s)$, the familiar gamma-function of Euler. Hence (1.4) becomes

$$e^{-z} = \int_{c-i\infty}^{c+i\infty} \Gamma(s)z^{-s}\,\mathrm{d}s \qquad (c > 0, \Re z > 0). \tag{1.5}$$

Henceforth we shall use the notation $\int_{c-i\infty}^{c+i\infty}\cdots$ to denote

$$\lim_{T\to\infty}\int_{c-iT}^{c+iT}\cdots\,.$$

The Mellin transform is, because of the presence of n^{-s}, particularly useful in dealing with Dirichlet series

$$F(s) = \sum_{n=1}^\infty f(n)n^{-s} \qquad (\Re s > \sigma_0 > 0).$$

A prototype of such series is the the Riemann zeta-function (p denotes primes)

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \prod_p \left(1 - p^{-s}\right)^{-1} \quad (\Re s > 1). \tag{1.6}$$

For the values s such that $\Re s \leqslant 1$ (both the series and the product in (1.6) clearly diverge for $s = 1$) $\zeta(s)$ is defined by analytic continuation (see e.g., [12] and [13] for an account on $\zeta(s)$).

1.3. Laplace transforms

The Laplace transform of $f(x)$ (under suitable conditions on $f(x)$) is

$$\mathcal{L}\{f(x)\} \equiv F(s) := \int_{0-}^{\infty} e^{-sx} f(x)\,\mathrm{d}x \qquad (\Re s > 0).$$

Then $\mathcal{L}^{-1}\{F(s)\} = f(x)$ is the inverse Laplace transform. It is unique if e.g., $f(x)$ is continuous. The inverse Laplace transform can be represented by a complex inversion integral (so-called Bromwich's inversion integral). This transform is

$$f(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{sx} F(s)\,\mathrm{d}s \qquad (x > 0)$$

and $f(x) = 0$ for $x < 0$. Here γ is a real number such that the contour of integration lies in the region of convergence of $F(s)$. The Laplace transforms are practical in view of the fast decay factor e^{-sx}, which usually ensures good convergence. For an account on Laplace transforms, we refer the reader to G. Doetsch's classic works [5] and [6].

1.4. Fourier transforms

Mellin and Laplace transforms are special cases (by a change of variable) of Fourier transforms (see A.M. Sedletskii [31]). This is

$$\hat{f}(\alpha) := \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,\mathrm{d}x \qquad (\alpha \in \mathbb{R})$$

if $f(x) \in L_1(-\infty,\infty)$. Some sources define $\hat{f}(\alpha)$ with the exponential $e^{-i\alpha x}$ or $e^{2\pi i\alpha x}$.

The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x}\,\mathrm{d}\alpha.$$

This holds for almost all real x if

$$f(x) \in L_1(-\infty,\infty), \qquad \hat{f}(x) \in L_1(-\infty,\infty).$$

The purpose of this paper is to give first an overview of applications of Laplace transforms to some solutions of two functional equations, connected to a work of Prof. B. Stanković, whose 90th birthday is celebrated this year. Then we shall move to the application of Laplace transforms to some classical problems of Analytic Number Theory. These include the circle problem, the divisor problem, the Voronoï summation formula for the divisor function, and moments of $|\zeta(\frac{1}{2} + it)|$.

2. Solving two functional equations

2.1. Prof. Stanković's work

Walter K. Hayman (1926–) in 1967 [7] asked: Is it true that the equation

$$\int_0^\infty \frac{F(wt)}{F(t)}\,\mathrm{d}t = \frac{1}{1-w} \tag{2.1}$$

has the unique solution $F(t) = ce^t$ (with positive coefficients in its power series expansion) among the entire functions?

This is a special case of the functional equation

$$\int_0^\infty \frac{F(wt,w)}{F(t,w)}\,\mathrm{d}t = G(w). \tag{2.2}$$

Both (2.1) and (2.2) were investigated in 1973 by Prof. B. Stanković [32] and the speaker [11] (independently), and some partial solutions were obtained.

In 1972, during the seminar on Functional Analysis in Novi Sad[1], Prof. Stanković noted that J. Vincze (Budapest) drew his attention to this problem, which was originally proposed by him and A. Rényi (1919-1970) (after whom the Mathematical Institute of the Hungarian Academy of Sciences was later named). The author was a member of this Seminar from 1971-1976, and the work that was subsequently done represents his first research work.

Prof. B. Stanković proved [32] several results concerning this problem. Two of them are the following

Theorem 1. *If the integral in* (2.1) *exists for a function* $F(t)$ *when* $w_0 \leqslant w < 1$ *or* $1 < w \leqslant w_0'$, *then there does not exist a function* $f(t)$ *continuous in a neighborhood* $\mathcal{V}$ *of* $t_0 > 0, f(t_0) \neq 0$, *and such that*

$$F(t) = (t-t_0)^k f(t) \qquad (t \in \mathcal{V}, k \in \mathbb{R}, k > 1 \vee k < -1).$$

Theorem 2. *Let* $e^t f(t)$ *be a solution of (1) for* $G(w) = 1/(1-w), w' \leqslant w < 1$. *If* $f(t)$ *is a monotonic function of constant sign, then* $f(t)$ *is a constant.*

2.2. The author's work

Under the substitution $F(t) = e^t f(t,w)$ the equation (2.2) becomes

$$\int_0^\infty e^{-(1-w)t}\frac{f(wt,w)}{f(t,w)}\,\mathrm{d}t = G(w),$$

[01] The seminar on Functional Analysis was founded by Prof. B. Stanković in 1964 and is held at the Department of Mathematics at the University of Novi Sad continuously for 50 years, which is a record that has hardly been attained anywhere. The seminar was very influential in the scientific formation of many generations of mathematicians.

which gives, in the special case when $G(w) = 1/(1-w)$,

$$\int_0^\infty e^{-(1-w)t} \left\{ \frac{f(wt,w)}{f(t,w)} - 1 \right\} \mathrm{d}t = 0.$$

Therefore we obtain, with

$$1 - w = s,\ G(1-s) = g(s),\ H(s,t) = \frac{f(t-st, 1-s)}{f(t, 1-s)},$$

$$\int_0^\infty e^{-st} H(s,t)\,\mathrm{d}t = g(s). \tag{2.3}$$

The representation (2.2) allows, by the use of the inverse Laplace transform, to find some families of solutions.

Theorem 3. *A solution of the functional equation*

$$\int_0^\infty \frac{F(wt,w)}{F(t,w)}\,\mathrm{d}t \ = \ \frac{h(w)}{1-w}$$

is given by

$$F(t,w) \ = \ t^c \exp\left(\frac{w^c}{h(w)} t\right) \qquad (0 < w < 1,\ h(w) > 0),$$

and is also valid for $w > 1$ *if* $h(w) < 0$ *, where* c *is a constant.*

3. Laplace transform in the circle and divisor problem

3.1. Introduction

The circle and divisor problems represent two classical problems of Analytic Number Theory. Much work was done on them, and the reader is referred e.g., to [12], [9].

The (Gauss) circle problem is the estimation of

$$P(x) \ := \ {\sum_{n\leqslant x}}' r(n) - \pi x + 1,$$

where $x > 0$, ${\sum_{n\leqslant x}}'$ means that the last term in the sum is to be halved if x is an integer and $r(n) = \sum_{n=a^2+b^2} 1$ denotes the number of representations of n $(\in \mathbb{N})$ as a sum of two integer squares.

The (Dirichlet) divisor problem is the estimation of

$$\Delta(x) \ := \ {\sum_{n\leqslant x}}' d(n) - x(\log x + 2\gamma - 1) - \tfrac{1}{4},$$

where $x > 0$, $d(n) = \sum_{\delta|n} 1$ is the number of divisors of n, and $\gamma = -\Gamma'(1) = 0.5772157\ldots$ is Euler's constant.

One of the main problems is to determine the value of the constants α, β such that

$$\alpha = \inf\Big\{\ a \mid P(x) \ll x^a\ \Big\}, \quad \beta = \inf\Big\{\ b \mid \Delta(x) \ll x^b\ \Big\}.$$

It is known that

$$1/4 \leqslant \alpha \leqslant 131/416 = 0,314903\ldots, \quad 1/4 \leqslant \beta \leqslant 131/416 = 0,314903\ldots.$$

It is also generally conjectured that $\alpha = \beta = 1/4$, but this conjecture seems very difficult to prove.

3.2. Mean square results

The author in two papers [16], [17] investigated the Laplace transform of $P^2(x)$ and $\Delta^2(x)$. The key tools are the explicit formulas of G.H. Hardy (1916) [8]:

$$P(x) = x^{1/2}\sum_{n=1}^{\infty} r(n)n^{-1/2}J_1(2\pi\sqrt{xn}), \tag{3.1}$$

and G.F. Voronoï (1904) [34]:

$$\Delta(x) = -\frac{2\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\Big\{K_1(4\pi\sqrt{xn}\,) + \frac{\pi}{2}Y_1(4\pi\sqrt{xn}\,)\Big\}.$$

Here J_1, K_1, Y_1 are the familiar Bessel functions (see e.g., Lebedev's monograph [26] and Watson's classic [35] for definitions and properties of Bessel functions), and the above series for $P(x)$ and $\Delta(x)$ are boundedly, but not absolutely convergent. We have

$$J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{p+2k}}{k!\Gamma(p+k+1)} \qquad (p \in \mathbb{R}, z \in \mathbb{C}),$$

and more complicated power series expansions for K_1, Y_1.

All Bessel functions have asymptotic expansions involving sines, cosines and negative powers of z, to any degree of accuracy.

Remark 1. Note that the case of the Laplace transform of $P(x)$ and $\Delta(x)$ is much easier. Namely, using (3.1) and

$$\int_0^{\infty} e^{-sx}x^{\nu/2}J_\nu(2\sqrt{ax}\,)\,\mathrm{d}x = e^{-a/s}a^{\nu/2}s^{\nu-1} \qquad (\Re s > 0,\ \Re\nu > -1)$$

one obtains immediately

$$\int_0^{\infty} e^{-sx}P(x)\,\mathrm{d}x = \pi s^{-2}\sum_{n=1}^{\infty} r(n)e^{-\pi^2 n/s} \qquad (\Re s > 0),$$

and an analogous formula holds for $\Delta(x)$. We have the following results.

Theorem 4. *For any given* $\varepsilon > 0$

$$\int_0^\infty P^2(x)e^{-x/T}\,\mathrm{d}x = \frac{1}{4}\left(\frac{T}{\pi}\right)^{3/2}\sum_{n=1}^\infty r^2(n)n^{-3/2} - T + O_\varepsilon(T^{2/3+\varepsilon}).$$

Theorem 5. *For any given* $\varepsilon > 0$

$$\int_0^\infty \Delta^2(x)e^{-x/T}\,\mathrm{d}x = \frac{1}{8}\left(\frac{T}{\pi}\right)^{3/2}\sum_{n=1}^\infty d^2(n)n^{-3/2} + TP_2(\log T) + O_\varepsilon(T^{2/3+\varepsilon}).$$

Remark 2. The standard notation $f(x) = O_\varepsilon(g(x))$ means that $|f(x)| \leqslant Cg(x)$ for $C = C(\varepsilon) > 0, g(x) > 0$ and $x \geqslant x_0(\varepsilon)$.

Remark 3. In Theorem 5, $P_2(x) = a_0x^2 + a_1x + a_2$ ($a_0 > 0$) with effectively computable a_0, a_1, a_2. Moreover, the series over n in both formulas are both absolutely convergent, since $r(n) = O_\varepsilon(n^\varepsilon), d(n) = O_\varepsilon(n^\varepsilon)$.

Remark 4. Mean square formulas for $P(x)$ and $\Delta(x)$ are a classical problem in Analytic Number Theory. The best known results are due to W.G. Nowak [29] and Lau–Tsang [25], respectively. They are

$$\int_0^T P^2(x)\,\mathrm{d}x \;=\; AT^{3/2} + O(T\log^{3/2}T\log\log T) \qquad (A > 0)$$

and

$$\int_0^T \Delta^2(x)\,\mathrm{d}x \;=\; BT^{3/2} + O(T\log^3 T\log\log T) \qquad (B > 0).$$

There is an analogy with the formulas of Theorem 4 and Theorem 5, but due primarily to the presence of the factor $e^{-x/T}$, the formulas of Theorem 4 and Theorem 5 are much more precise. For an account on $\Delta(x)$ and the mean square of $|\zeta(\frac{1}{2}+it)|$, see K.-M. Tsang [33].

The key formula for the proof of Theorem 4 is the Laplace transform

$$\begin{aligned}&\int_0^\infty e^{-st}tJ_1(a\sqrt{t}\,)J_1(b\sqrt{t}\,)\,\mathrm{d}t\\ &= \exp\left(-\frac{a^2+b^2}{4s}\right)(4s^3)^{-1}\left\{2abI_0\Big(\frac{ab}{2s}\Big) - (a^2+b^2)I_1\Big(\frac{ab}{2s}\Big)\right\}.\end{aligned}$$

This is valid for $\Re s > 0; a, , b \in \mathbb{R}$. The asymptotic expansion for the Bessel function $I_\nu(x)$, for fixed $|x| \geqslant 1$, is (first three terms only)

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left\{1 - \frac{4\nu^2-1}{8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + O\left(\frac{1}{|x|^3}\right)\right\}.$$

Namely when we square (3.1), we are led to the integrals of the above type with $s = 1/T, a = \sqrt{2\pi m}, b = \sqrt{2\pi n}$; $m, n \in \mathbb{N}$.

The arithmetic part of the proof of Theorem 4 and Theorem 5 is based on the formulas of F. Chamizo [3] and Y. Motohashi [27], respectively. These are

$$\sum_{n\leqslant x} r(n)r(n+h) = \frac{(-1)^h 8x}{h}\sum_{d|h}(-1)^d d + O_\varepsilon(x^{2/3+\varepsilon}),$$

$$\sum_{n\leqslant x} d(n)d(n+h) = x\sum_{i=0}^{2}(\log x)^i\sum_{j=0}^{2} c_{ij}\sum_{d|h}\frac{(\log d)^j}{d} + O_\varepsilon(x^{2/3+\varepsilon}).$$

The point is that h, called "the shift parameter", is not necessarily fixed, but may vary with x. These formulas hold uniformly for $1 \leqslant h \leqslant x^{1/2}$, where the c_{ij}'s are absolute constants, and $c_{22} = c_{21} = 0, c_{20} = 6\pi^{-2}$. The exponents $2/3+\varepsilon$ in the above formulas account essentially for the same exponents in Theorem 4 and Theorem 5.

4. The Voronoï summation formula via Laplace transforms

The author [15] proved the classical Voronoï formula

$$\Delta(x) = -\frac{2\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left\{K_1(4\pi\sqrt{xn}\,) + \frac{\pi}{2}Y_1(4\pi\sqrt{xn}\,)\right\} \tag{4.1}$$

by the use of Laplace transforms. Although there are several proofs of this important result in the literature (see e.g., Chapter 3 of [12]), this was the first proof of it by means of Laplace transforms. We shall present now a sketch of the proof. Denote the right-hand side of (4.1) by $f(x)$. It can be shown that

$$\mathcal{L}[\Delta(x)] = \mathcal{L}[f(x)] \qquad (x > 0). \tag{4.2}$$

Suppose that $x_0 \notin \mathbb{N}$. Then both $\Delta(x)$ and $f(x)$ are continuous at $x = x_0$. Hence by the uniqueness theorem for Laplace transforms it follows that (4.1) holds for $x = x_0$. But if $x \in \mathbb{N}$, then the validity of (4.1) follows from the validity of (4.2) when $x \notin \mathbb{N}$, as shown e.g., by M. Jutila [21]. Thus it suffices to show that (4.2) holds when $x \notin \mathbb{N}$. To see this note that, for $\Re s > 0$,

$$
\begin{aligned}
\mathcal{L}[\Delta(x)] &= \int_0^\infty \Biggl(\sum_{n\leqslant x}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4}\Biggr) e^{-sx}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty d(n) \int_n^\infty e^{-sx}\,\mathrm{d}x + \frac{\log s - \gamma}{s^2} - \frac{1}{4s} \\
&= \frac{1}{s}\sum_{n=1}^\infty d(n) e^{-sn} + \frac{\log s - \gamma}{s^2} - \frac{1}{4s} \\
&= \frac{1}{2\pi i s}\int_{(2)} \zeta^2(w)\Gamma(w)s^{-w}\,\mathrm{d}w + \frac{\log s - \gamma}{s^2} - \frac{1}{4s} \\
&= \frac{1}{2\pi i s}\int_{(1/2)} \zeta^2(w)\Gamma(w)s^{-w}\,\mathrm{d}w - \frac{1}{4s}.
\end{aligned}
$$

Here we used the well-known Mellin integral (1.5) and the series representation

$$
\zeta^2(s) = \sum_{k=1}^\infty k^{-s} \sum_{\ell=1}^\infty \ell^{-s} = \sum_{k\ell=n,n\geqslant 1}^\infty n^{-s} = \sum_{n=1}^\infty d(n) n^{-s} \qquad (\Re s > 1).
$$

Change of summation and integration was justified by absolute convergence, and in the last step the residue theorem was used together with

$$
\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \dots\,, \quad \Gamma(s) = 1 - \gamma(s-1) + \dots \qquad (s \to 1).
$$

The crucial step is to show that

$$
\begin{aligned}
&\left(\frac{x}{n}\right)^{1/2} \left(K_1(4\pi\sqrt{xn}\,) + \frac{\pi}{2} Y_1(4\pi\sqrt{xn}\,)\right) \\
&= \frac{1}{4\pi^2 n i}\int_{(1)} \Gamma(w)\Gamma(w-1)\cos^2\left(\frac{\pi w}{2}\right)\left(2\pi\sqrt{xn}\right)^{2-2w}\,\mathrm{d}w.
\end{aligned}
$$

For this we need properties of Bessel functions and the functional equation for $\zeta(s)$, proved by first B. Riemann [30] in 1859, namely

$$
\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \quad (s\in\mathbb{C}),
$$

$$
\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{it+i\pi/4}\left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \geqslant t_0 > 0).
$$

With these ingredients the proof of (4.2) is completed.

5. Laplace transforms of moments of $|\zeta(\frac{1}{2}+it)|$

5.1. Introduction

Let

$$(5.1)\qquad L_k(s) \;:=\; \int_0^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}\, e^{-sx}\,\mathrm{d}x \qquad (k\in\mathbb{N},\ \Re s>0)$$

denote that Laplace transform of $|\zeta(\frac{1}{2}+ix)|^{2k}$. Also let

$$(5.2)\qquad I_k(T) \;:=\; \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \qquad (k\in\mathbb{N}).$$

The investigations of the moments $I_k(T)$ is one of the central themes in the theory of $\zeta(s)$ (see e.g., the monographs [12] and [13]). One trivially has

$$I_k(T) \leqslant e\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} e^{-t/T}\,\mathrm{d}t = eL_k\left(\frac{1}{T}\right).$$

Therefore any nontrivial bound of the form

$$(5.3)\qquad L_k(\sigma) \;\ll_\varepsilon\; \left(\frac{1}{\sigma}\right)^{c_k+\varepsilon} \qquad (\sigma\to 0+,\ c_k\geqslant 1)$$

gives (with $\sigma = 1/T$) the bound

$$(5.4)\qquad I_k(T) \;\ll_\varepsilon\; T^{c_k+\varepsilon}.$$

Conversely, if (5.4) holds, we obtain (5.3) from the identity

$$L_k\left(\frac{1}{T}\right) \;=\; \frac{1}{T}\int_0^\infty I_k(t)\mathrm{e}^{-t/T}\,\mathrm{d}t.$$

For a more detailed discussion on $L_k(s)$ and $I_k(T)$, see the author's work [18].

5.2. The mean square of $|\zeta(\frac{1}{2}+it)|$

A classical result of H. Kober [24] from 1936 says that, as $\sigma\to 0+$,

$$L_1(2\sigma) = \frac{\gamma-\log(4\pi\sigma)}{2\sin\sigma} + \sum_{n=0}^{N} c_n\sigma^n + O_N(\sigma^{N+1})$$

for any given integer $N\geqslant 1$, where the c_n's are effectively computable constants.

For complex values of s the function $L_1(s)$ was studied by F.V. Atkinson [1] in 1941. More recently M. Jutila [23] refined Atkinson's method and proved

Theorem 6. *One has*

$$L_1(s) = -ie^{\frac12 is}\left(\log(2\pi) - \gamma + (\frac{\pi}{2} - s)i\right) +$$

$$+2\pi e^{-\frac12 is}\sum_{n=1}^{\infty} d(n)\exp(-2\pi i n e^{-is}) + \lambda_1(s)$$

in the strip $0 < \Re s < \pi$*, where the function* $\lambda_1(s)$ *is holomorphic in the strip* $|\Re s| < \pi$*. Moreover, in any strip* $|\Re s| \leqslant \theta$ *with* $0 < \theta < \pi$*, we have*

$$\lambda_1(s) \ = \ O_\varepsilon\left((|s|+1)^{-1}\right).$$

In 1997 M. Jutila [22] gave a discussion on the application of Laplace transforms to the evaluation of sums of coefficients of certain Dirichlet series. His work showed how a powerful tool Laplace transforms can be in Analytic number theory in a general setting.

Remark 5. For $I_1(T)$ one has

$$I_1(T) = \int_0^T |\zeta(\tfrac12 + it)|^2\,\mathrm{d}t = T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right) + E(T), \tag{5.5}$$

say, where γ is Euler's constant and $E(T)$ is the error term in the asymptotic formula (5.5). For an account on $E(T)$ see e.g., [12] and [13]. If

$$\rho \ := \ \inf\Big\{\ r \mid E(x) \ll x^r\ \Big\},$$

then it is known (see N. Watt [36]) that $1/4 \leqslant \rho \leqslant \frac{131}{416} = 0.314903\ldots$. We note that Kober's formula for $L_1(s)$ is different (and in some ways more precise) than the formula (5.5) for $I_1(T)$.

6. The Laplace transform of $|\zeta(\frac12 + it)|^4$

6.1. The explicit formula

Atkinson [2] obtained the asymptotic formula, as $\sigma \to 0+$,

$$L_2(\sigma) = \frac{1}{\sigma}\left(A\log^4\frac{1}{\sigma} + B\log^3\frac{1}{\sigma} + C\log^2\frac{1}{\sigma} + D\log\frac{1}{\sigma} + E\right) + \lambda_2(\sigma), \tag{6.1}$$

where

$$A = \frac{1}{2\pi^2},\ B = \frac{1}{\pi^2}\left(2\log(2\pi) - 6\gamma + \frac{24\zeta'(2)}{\pi^2}\right),$$

and

$$\lambda_2(\sigma) \ \ll_\varepsilon \ \left(\frac{1}{\sigma}\right)^{\frac{13}{14}+\varepsilon},$$

and indicated how the exponent 13/14 can be replaced by 8/9. The author in [17] sharpened this result and proved, by means of spectral theory of the non-Euclidean Laplacian, the following result:

Theorem 7. *Let* $0 \leqslant \phi < \frac{\pi}{2}$ *be given. Then for* $0 < |s| \leqslant 1$ *and* $|\arg s| \leqslant \phi$ *we have*

$$
\begin{aligned}
L_2(s) = {} & \frac{1}{s}\left(A\log^4\frac{1}{s} + B\log^3\frac{1}{s} + C\log^2\frac{1}{s} + D\log\frac{1}{s} + E\right) \\
& + s^{-\frac{1}{2}}\left\{\sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac{1}{2})\Big(s^{-i\kappa_j}R(\kappa_j)\Gamma(\tfrac{1}{2}+i\kappa_j) + s^{i\kappa_j}R(-\kappa_j)\Gamma(\tfrac{1}{2}-i\kappa_j)\Big)\right\} \\
& + G_2(s).
\end{aligned}
$$

Remark 6. We have

$$
R(y) \;:=\; \sqrt{\frac{\pi}{2}}\Big(2^{-iy}\frac{\Gamma(\frac{1}{4}-\frac{i}{2}y)}{\Gamma(\frac{1}{4}+\frac{i}{2}y)}\Big)^3\Gamma(2iy)\cosh(\pi y)
$$

and in the above region $G_2(s)$ is a regular function satisfying ($C > 0$ is a suitable constant)

$$
G_2(s) \ll \frac{1}{\sqrt{|s|}}\exp\left\{-\frac{C\log(|s|^{-1}+20)}{(\log\log(|s|^{-1}+20))^{2/3}(\log\log\log(|s|^{-1}+20))^{1/3}}\right\},
$$

where $f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$. Here

$$
\Big\{\lambda_j = \kappa_j^2 + \frac{1}{4}\Big\} \cup \{0\} \qquad (j = 1, 2, \dots)
$$

denotes the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$–automorphic forms, and

$$
\alpha_j = |\rho_j(1)|^2(\cosh\pi\kappa_j)^{-1},
$$

where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue λ_j to which the Hecke series $H_j(s)$ is attached. We note that

$$
\sum_{\kappa_j \leqslant K}\alpha_j H_j^3(\tfrac{1}{2}) \ll K^2\log^C K \qquad (C > 0).
$$

See Y. Motohashi [28] for a detailed account on the spectral theory of the non-Euclidean Laplacian and its applications to the moments of $\zeta(s)$.

Remark 7. For $I_2(T)$ (see (5.2)) one has

$$
I_2(T) = (a_0\log^4 T + a_1\log^3 T + a_2\log^2 T + a_3\log T + a_4)T + E_2(T). \tag{6.2}
$$

We have $a_0 = 1/(2\pi^2)$, proved already by A.E. Ingham [10]. For other coefficients in (6.2) see e.g., the author's paper [14] and J.B. Conrey [4], who independently obtained the coefficients in a different form. We also have $A = a_0 = 1/(2\pi^2)$ in Theorem 7, but it is easy to see that, for B in (6.1), $B \neq a_1$ etc. Theorem 3 of [14] provides the exact connection between the two sets of coefficients. Since the series in Theorem 7 is absolutely convergent, it is seen that

$$
L_2(1/T) = T\left(A\log^4 T + B\log^3 T + C\log^2 T + D\log T + E\right) + O(\sqrt{T}). \tag{6.3}
$$

Remark 8. For the results on $E_2(T)$ see [19] and [20]. For example, one has $E_2(T) \ll T^{2/3} \log^C T$. This is weaker than the O-term in (6.2).

References

[1] Atkinson, F.V., The mean value of the zeta-function on the critical line, Quart. J. Math. Oxford 10 (1939), 122–128.

[2] F.V. Atkinson, The mean value of the zeta-function on the critical line, Proc. London Math. Soc. 47 (1941), 174–200.

[3] Chamizo, F., Correlated sums of $r(n)$, J. Math. Soc. Japan 51 (1999), 237–252.

[4] Conrey, J.B., A note on the fourth power moment of the Riemann zeta-function. Analytic number theory, Vol. 1 (Allerton Park, IL, 1995), 225-230, Progr. Math., 138, Birkhäuser Boston, MA 1996.

[5] Doetsch, G., Theorie und Anwendung der Laplace-Transformation. (German) Dover Publication, N. Y. 1943.

[6] Doetsch, G., Handbuch der Laplace-Transformation. Band I. Theorie der Laplace-Transformation. (German) Verlag Birkhäuser, Basel 1950.

[7] Hayman, W.K., Research problems in function theory, Athlone Press, University of London, London 1967.

[8] Hardy, G.H., The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc. London Math. Soc. (2) 15 (1916), 192–213.

[9] Huxley, M.N., Ivić, A., Subconvexity for the Riemann zeta-function and the divisor problem, Bulletin CXXXIV de l'Académie Serbe des Sciences et des Arts - 2007, Classe des Sciences mathématiques et naturelles, Sciences mathématiques 32, 13–32.

[10] Ingham, A.E., Mean-value theorems in the theory of the Riemann zeta-function, Proc. London Math. Soc. (2) 27 (1926), 273–300.

[11] Ivić, A., On a certain integral equation, Matematički Vesnik (Belgrade) 10(25) (1973), 259–262.

[12] Ivić, A., The Riemann-zeta function, John Wiley & Sons, New York 1985.

[13] Ivić, A., The mean values of the Riemann zeta-function, Tata Institute of Fundamental Research, Lecture Notes 82, Bombay 1991.

[14] Ivić, A., On the fourth moment of the Riemann zeta-function, Publs. Inst. Math. (Belgrade) 57(71) (1995), 101–110.

[15] Ivić, A., The Voronoï identity via the Laplace transform, The Ramanujan Journal 2 (1998), 39–45.

[16] Ivić, A., The Laplace transform of the square in the circle and divisor problems, Studia Scient. Math. Hungarica 32 (1996), 181–205 and ibid. 37 (2001), 391–399.

[17] Ivić, A., The Laplace transform of the fourth moment of of the zeta-function, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 11 (2000), 41–48.

[18] Ivić, A., The Laplace and Mellin transforms of powers of the Riemann zeta-function, International Journal of Mathematics and Analysis 1(2) (2006), 113–140.

[19] Ivić A. and Motohashi Y., The mean square of the error term for the fourth moment of the zeta-function, Proc. London Math. Soc. (3) 66 (1994), 309–329.

[20] Ivić A. and Motohashi Y., The fourth moment of the Riemann zeta-function, J. Number Theory 51 (1995), 16–45.

[21] Jutila, M., A method in the theory of exponential sums, LNs 80, Tata Institute of Fundamental Research, Bombay 1987.

[22] Jutila, M., Mean values of Dirichlet series via Laplace transforms. Analytic number theory (Kyoto, 1996), 169-207, London Math. Soc. Lecture Note Ser., 247, Cambridge Univ. Press, Cambridge, 1997.

[23] Jutila, M., The Mellin transform of the square of Riemann's zeta-function, Periodica Math. Hung. 42 (2001), 179–190.

[24] Kober, H., Eine Mittelwertformel der Riemannschen Zetafunktion, Compositio Math. 3 (1936), 174–189.

[25] Lau Y. K., Tsang K. M., On the mean square formula of the error term in the Dirichlet divisor problem. Mathematical Proceedings of the Cambridge Philosophical Society 146 (2009), 277–287.

[26] Lebedev, N.N., Special functions and their applications, Dover Publications, Inc., New York 1972.

[27] Motohashi, Y., The binary additive divisor problem, Ann. Sci. École Normale Supérieure (4) 27 (1994), 529–572.

[28] Motohashi, Y., Spectral theory of the Riemann zeta-function, Cambridge University Press, Cambridge 1997.

[29] Nowak, W. G., Lattice points in a circle: an improved mean-square asymptotics, Acta Arith. 113 (2004), 259–272.

[30] Riemann, B., Über die Anzahl der Primzahlen unter einer gegebener Größe, Monatsber. Akad. Berlin (1859), 671–680.

[31] Sedletskii, A.M., Fourier Transforms and Approximations, Gordon and Breach Science Publishers, Amsterdam 2000.

[32] Stanković, B., On two integral equations, Mathematical Structures – Computational Mathematics – Mathematical Modelling, Papers dedicated to Professor L. Iliev's 60th birthday, Sofia, 1975, 439–445.

[33] Tsang, K.-M., Recent progress on the Dirichlet divisor problem and the mean square of the Riemann zeta-function. Sci. China Math. 53 (2010), 2561–2572.

[34] G.F. Voronoï, Sur une fonction transcendante et ses applications à la sommation de quelques séries, Ann. École Normale (3) 21(1904), 459–533.

[35] Watson, G.N., A Treatise on the Theory of Bessel Functions, Second Edition (1995), Cambridge University Press Cambridge.

[36] Watt, N., A note on the mean square of $|\zeta(\frac{1}{2}+it)|$, J. London Math. Soc. 82(2) (2010), 279–294.

Received by the editors August 28, 2014