

# **Part 1**

**Basic principles  
of fluid mechanics  
and physical  
thermodynamics.**

## Chapter 2. Introduction to Fluid Mechanics

<b>2.1 INTRODUCTION</b> .....	<b>1</b>
2.1.1 The concept of a fluid.....	1
2.1.2 Volume flow, Mass flow and the Continuity Equation.....	3
<b>2.2 FLUID PRESSURE</b> .....	<b>3</b>
2.2.1 The cause of fluid pressure.....	3
2.2.2 Pressure head.....	4
2.2.3 Atmospheric pressure and gauge pressure.....	5
2.2.4. Measurement of air pressure.....	5
2.2.4.1. Barometers.....	5
2.2.4.2. Differential pressure instruments .....	6
<b>2.3 FLUIDS IN MOTION</b> .....	<b>8</b>
2.3.1. Bernoulli's equation for ideal fluids .....	8
<i>Kinetic energy</i> .....	8
<i>Potential energy</i> .....	9
<i>Flow work</i> .....	9
2.3.2. Static, total and velocity pressures. ....	11
2.3.3. Viscosity.....	12
2.3.4. Laminar and turbulent flow. Reynolds Number.....	14
2.3.5. Frictional losses in laminar flow, Poiseuille's Equation.....	16
2.3.6. Frictional losses in turbulent flow.....	22
2.3.6.1. The Chézy-Darcy Equation .....	22
2.3.6.2. The coefficient of friction, $f$ .....	25
2.3.6.3. Equations describing $f$ - Re relationships.....	27
<i>Laminar Flow</i> .....	27
<i>Smooth pipe turbulent curve</i> .....	28
<i>Rough pipes</i> .....	29
<b>Bibliography</b> .....	<b>31</b>

### 2.1 INTRODUCTION

#### 2.1.1 The concept of a fluid

A fluid is a substance in which the constituent molecules are free to move relative to each other. Conversely, in a solid, the relative positions of molecules remain essentially fixed under non-destructive conditions of temperature and pressure. While these definitions classify matter into fluids and solids, the fluids sub-divide further into liquid and gases.

Molecules of any substance exhibit at least two types of forces; an attractive force that diminishes with the square of the distance between molecules, and a force of repulsion that becomes strong when molecules come very close together. In solids, the force of attraction is so dominant that the molecules remain essentially fixed in position while the resisting force of repulsion prevents them from collapsing into each other. However, if heat is supplied to the solid, the energy is absorbed internally causing the molecules to vibrate with increasing amplitude. If that vibration becomes sufficiently violent, then the bonds of attraction will be broken. Molecules will then be free to move in relation to each other - the solid melts to become a liquid.

When two moving molecules in a fluid converge on each other, actual collision is averted (at normal temperatures and velocities) because of the strong force of repulsion at short distances. The molecules behave as near perfectly elastic spheres, rebounding from each other or from the walls of the vessel. Nevertheless, in a liquid, the molecules remain sufficiently close together that the force of attraction maintains some coherence within the substance. Water poured into a vessel will assume the shape of that vessel but may not fill it. There will be a distinct interface (surface) between the water and the air or vapour above it. The mutual attraction between the water molecules is greater than that between a water molecule and molecules of the adjacent gas. Hence, the water remains in the vessel except for a few exceptional molecules that momentarily gain sufficient kinetic energy to escape through the interface (slow evaporation).

However, if heat continues to be supplied to the liquid then that energy is absorbed as an increase in the velocity of the molecules. The rising temperature of the liquid is, in fact, a measure of the **internal kinetic energy** of the molecules. At some critical temperature, depending upon the applied pressure, the velocity of the molecules becomes so great that the forces of attraction are no longer sufficient to hold those molecules together as a discrete liquid. They separate to much greater distances apart, form bubbles of vapour and burst through the surface to mix with the air or other gases above. This is, of course, the common phenomenon of boiling or rapid evaporation. The liquid is converted into gas.

The molecules of a gas are identical to those of the liquid from which it evaporated. However, those molecules are now so far apart, and moving with such high velocity, that the forces of attraction are relatively small. The fluid can no longer maintain the coherence of a liquid. A gas will expand to fill any closed vessel within which it is contained.

The molecular spacing gives rise to distinct differences between the properties of liquids and gases. Three of these are, first, that the volume of gas with its large intermolecular spacing will be much greater than the same mass of liquid from which it evaporated. Hence, the density of gases (mass/volume) is much lower than that of liquids. Second, if pressure is applied to a liquid, then the strong forces of repulsion at small intermolecular distances offer such a high resistance that the volume of the liquid changes very little. For practical purposes most liquids (but not all) may be regarded as incompressible. On the other hand, the far greater distances between molecules in a gas allow the molecules to be more easily pushed closer together when subjected to compression. Gases, then, are compressible fluids.

A third difference is that when liquids of differing densities are mixed in a vessel, they will separate out into discrete layers by gravitational settlement with the densest liquid at the bottom. This is not true of gases. In this case, layering of the gases will take place only while the constituent gases remain unmixed (for example, see Methane Layering, Section 12.4.2). If, however, the gases become mixed into a homogenous blend, then the relatively high molecular velocities and large intermolecular distances prevent the gases from separating out by gravitational settlement. The internal molecular energy provides an effective continuous mixing process.

Subsurface ventilation engineers need to be aware of the properties of both liquids and gases. In this chapter, we shall confine ourselves to incompressible fluids. Why is this useful when we are well aware that a ventilation system is concerned primarily with air, a mixture of gases and, therefore, compressible? The answer is that in a majority of mines and other subsurface facilities, the ranges of temperature and pressure are such that the variation in air density is fairly limited. Airflow measurements in mines are normally made to within 5 per cent accuracy. A 5 per cent change in air density occurs by moving through a vertical elevation of some 500 metres in the gravitational field at the surface of the earth. Hence, the assumption of incompressible flow with its simpler analytical relationships gives acceptable accuracy in most cases. For the deeper and (usually) hotter facilities, the effects of pressure and temperature on air density should be taken into account through thermodynamic analyses if a good standard of accuracy is to be attained. The principles of physical steady-flow thermodynamics are introduced in Chapter 3.

### 2.1.2 Volume flow, Mass flow and the Continuity Equation

Most measurements of airflow in ventilation systems are based on the volume of air ( $\text{m}^3$ ) that passes through a given cross section of a duct or airway in unit time (1 second). The units of volume flow,  $Q$ , are, therefore,  $\text{m}^3/\text{s}$ . However, for accurate analyses when density variations are to be taken into account, it is preferable to work in terms of mass flow - that is, the mass of air (kg) passing through the cross section in 1 second. The units of mass flow,  $M$ , are then kg/s.

The relationship between volume flow and mass flow follows directly from the definition of density,  $\rho$ ,

$$\rho = \frac{\text{mass}}{\text{volume}} \quad \frac{\text{kg}}{\text{m}^3} \quad (2.1)$$

and

$$\rho = \frac{\text{mass flow}}{\text{volume flow}} = \frac{M}{Q} \quad \frac{\text{kg s}}{\text{s m}^3}$$

$$\text{giving } M = Q \rho \text{ kg/s} \quad (2.2)$$

In any continuous duct or airway, the mass flows passing through all cross sections along its length are equal, provided that the system is at steady state and there are no inflows or outflows of air or other gases between the two ends. If these conditions are met then

$$M = Q \rho = \text{constant} \quad \text{kg/s} \quad (2.3)$$

This is the simplest form of the Continuity Equation. It can, however, be written in other ways. A common method of measuring volume flow is to determine the mean velocity of air,  $u$ , over a given cross section, then multiply by the area of that cross-section,  $A$ , (Chapter 6):

$$Q = u A \quad \frac{\text{m}}{\text{s}} \text{ m}^2 \text{ or } \frac{\text{m}^3}{\text{s}} \quad (2.4)$$

Then the continuity equation becomes

$$M = \rho u A = \text{constant kg/s} \quad (2.5)$$

As indicated in the preceding subsection, we can achieve acceptable accuracy in most situations within ventilation systems by assuming a constant density. The continuity equation then simplifies back to

$$Q = u A = \text{constant} \quad \text{m}^3/\text{s} \quad (2.6)$$

This shows that for steady-state and constant density airflow in a continuous airway, the velocity of the air varies inversely with cross sectional area.

## 2.2 FLUID PRESSURE

### 2.2.1 The cause of fluid pressure

Section 2.1.1 described the dynamic behaviour of molecules in a liquid or gas. When a molecule rebounds from any confining boundary, a force equal to the rate of change of momentum of that molecule is exerted upon the boundary. If the area of the solid/fluid boundary is large compared to the average distance between molecular collisions then the statistical effect will be to give a uniform force distributed over that boundary. This is the case in most situations of importance in subsurface ventilation engineering.

Two further consequences arise from the bombardment of a very large number of molecules on a surface, each molecule behaving essentially as a perfectly elastic sphere. First, the force exerted by a static fluid will always be normal to the surface. We shall discover later that the situation is rather different when the dynamic forces of a moving fluid stream are considered (Section 2.3). Secondly, at any point within a static fluid, the pressure is the same in all directions. Hence, static pressure is a scalar rather than a vector quantity.

Pressure is sometimes carelessly confused with force or thrust. The quantitative definition of pressure,  $P$ , is clear and simple

$$P = \frac{\text{Force}}{\text{Area}} \quad \frac{\text{N}}{\text{m}^2} \quad (2.7)$$

In the SI system of units, force is measured in Newtons (N) and area in square metres. The resulting unit of pressure, the  $\text{N/m}^2$ , is usually called a Pascal (Pa) after the French philosopher, **Blaise Pascal (1623-1662)**.

### 2.2.2 Pressure head

If a liquid of density  $\rho$  is poured into a vertical tube of cross-sectional area,  $A$ , until the level reaches a height  $h$ , the volume of liquid is

$$\text{volume} = h A \quad \text{m}^3$$

Then from the definition of density (mass/volume), the mass of the liquid is

$$\begin{aligned} \text{mass} &= \text{volume} \times \text{density} \\ &= h A \rho \quad \text{kg} \end{aligned}$$

The weight of the liquid will exert a force,  $F$ , on the base of the tube equal to mass  $\times$  gravitational acceleration ( $g$ )

$$F = h A \rho g \quad \text{N}$$

But as pressure = force/area, the pressure on the base of the tube is

$$P = \frac{F}{A} = \rho g h \quad \frac{\text{N}}{\text{m}^2} \quad \text{or Pa} \quad (2.8)$$

Hence, if the density of the liquid is known, and assuming a constant value for  $g$ , then the pressure may be quoted in terms of  $h$ , the head of liquid. This concept is used in liquid type manometers (Section 2.2.4) which, although in declining use, are likely to be retained for many purposes owing to their simplicity.

Equation (2.8) can also be used for air and other gases. In this case, it should be remembered that the density will vary with height. A mean value may be used with little loss in accuracy for most mine shafts. However, here again, it is recommended that the more precise methodologies of thermodynamics be employed for elevation differences of more than 500 m.

### 2.2.3 Atmospheric pressure and gauge pressure

The blanket of air that shrouds the earth extends to approximately 40 km above the surface. At that height, its pressure and density tend towards zero. As we descend towards the earth, the number of molecules per unit volume increases, compressed by the weight of the air above. Hence, the pressure of the atmosphere also increases. However, the pressure at any point in the lower atmosphere is influenced not only by the column of air above it but also by the action of convection, wind currents and variations in temperature and water vapour content. Atmospheric pressure near the surface, therefore, varies with both place and time. At the surface of the earth, atmospheric pressure is of the order of 100 000 Pa. For practical reference this is often translated into 100 kPa although the basic SI units should always be used in calculations. Older units used in meteorology for atmospheric pressure are the bar ( $10^5$  Pa) and the millibar (100 Pa).

For comparative purposes, reference is often made to standard atmospheric pressure. This is the pressure that will support a 0.760 m column of mercury having a density of  $13.5951 \times 10^3 \text{ kg/m}^3$  in a standard earth gravitational field of  $9.8066 \text{ m/s}^2$ .

Then from equation (2.8)

$$\begin{aligned} \text{One Standard Atmosphere} &= \rho \times g \times h \\ &= 13.5951 \times 10^3 \times 9.8066 \times 0.760 \\ &= 101.324 \times 10^3 \quad \text{Pa} \\ \text{or} \quad &101.324 \quad \text{kPa.} \end{aligned}$$

The measurement of variations in atmospheric pressure is important during ventilation surveys (Chapter 6), for psychrometric measurements (Chapter 14), and also for predicting the emission of stored gases into a subsurface ventilation system (Chapter 12). However, for many purposes, it is necessary to measure *differences* in pressure. One common example is the difference between the pressure within a system such as a duct and the exterior atmosphere pressure. This is referred to as *gauge pressure*.

$$\text{Absolute pressure} = \text{Atmospheric pressure} + \text{gauge pressure} \quad (2.9)$$

If the pressure within the system is below that of the local ambient atmospheric pressure then the negative gauge pressure is often termed the *suction pressure* or *vacuum* and the sign ignored.

Care should be taken when using equation 2.9 as the gauge pressure may be positive or negative. However, the absolute pressure is *always* positive. Although many quoted measurements are pressure differences, it is the absolute pressures that are used in thermodynamic calculations. We must not forget to convert when necessary.

### 2.2.4. Measurement of air pressure.

#### 2.2.4.1. Barometers

Equation (2.8) showed that the pressure at the bottom of a column of liquid is equal to the product of the head (height) of the liquid, its density and the local value of gravitational acceleration. This principle was employed by **Evangelista Torricelli (1608-1647)**, the Italian who invented the mercury barometer in 1643.. Torricelli poured mercury into a glass tube, about one metre in length, closed at one end, and upturned the tube so that the open end dipped into a bowl of mercury. The level in the tube would then fall until the column of mercury,  $h$ , produced a pressure at the base that just balanced the atmospheric pressure acting on the open surface of mercury in the bowl.

The atmospheric pressure could then be calculated as (see equation (2.8) )

$$P = \rho g h \quad \text{Pa}$$

where, in this case,  $\rho$  is the density of mercury.

Modern versions of the Torricelli instrument are still used as standards against which other types of barometer may be calibrated. Barometric (atmospheric) pressures are commonly quoted in millimetres (or inches) of mercury. However, for precise work, equation (2.8) should be employed using the density of mercury corresponding to its current temperature. Accurate mercury barometers have a thermometer attached to the stem of the instrument for this purpose and a sliding micrometer to assist in reading the precise height of the column. Furthermore, and again for accurate work, the local value of gravitational acceleration should be ascertained as this depends upon latitude and altitude. The space above the mercury in the barometer will not be a perfect vacuum as it contains mercury vapour. However, this exerts a pressure of less than 0.00016 kPa at 20 °C and is quite negligible compared with the surface atmospheric pressure of near 100 kPa. This, coupled with the fact that the high density of mercury produces a barometer of reasonable length, explains why mercury rather than any other liquid is used. A water barometer would need to be about 10.5m in height.

Owing to their fragility and slowness in reacting to temperature changes, mercury barometers are unsuitable for underground surveys . An aneroid barometer consists of a closed vessel which has been evacuated to a near perfect vacuum. One or more elements of the vessel are flexible. These may take the form of a flexing diaphragm, or the vessel itself may be shaped as a helical or spiral spring. The near zero pressure within the vessel remains constant. However, as the surrounding atmospheric pressure varies, the appropriate element of the vessel will flex. The movement may be transmitted mechanically, magnetically or electrically to an indicator and/or recorder.

Low cost aneroid barometers may be purchased for domestic or sporting use. Most altimeters are, in fact, aneroid barometers calibrated in metres (or feet) head of air. For the high accuracy required in ventilation surveys (Chapter 6) precision aneroid barometers are available.

Another principle that can be employed in pressure transducers, including barometers, is the piezoelectric property of quartz. The natural frequency of a quartz beam varies with the applied pressure. As electrical frequency can be measured with great precision, this allows the pressure to be determined with good accuracy.

#### 2.2.4.2. Differential pressure instruments

Differences in air pressure that need to be measured frequently in subsurface ventilation engineering rarely exceed 7 or 8 kPa and are often of the order of only a few Pascals. The traditional instrument for such low pressure differences is the manometer. This relies upon the displacement of liquid to produce a column, or head, that balances the differential pressure being measured. The most rudimentary manometer is the simple glass U tube containing water, mercury or other liquid. A pressure difference applied across the ends of the tube causes the liquid levels in the two limbs to be displaced in opposite directions. A scale is used to measure the vertical distance between the levels and equation (2.8) used to calculate the required pressure differential. Owing to the past widespread use of water manometers, the millimetre (or inch) of water column came to be used commonly as a measure of small pressure differentials, much as a head of mercury has been used for atmospheric pressures. However, it suffers from the same disadvantages in that it is not a primary unit but depends upon the liquid density and local gravitational acceleration.

When a liquid other than water is used, the linear scale may be increased or decreased, dependent upon the density of the liquid, so that it still reads directly in head of water. A pressure head in one

fluid can be converted to a head in any other fluid provided that the ratio of the two densities is known.

$$p = \rho_1 g h_1 = \rho_2 g h_2 \quad \text{Pa}$$

$$\text{or} \quad h_2 = \frac{\rho_1}{\rho_2} h_1 \quad \text{m} \quad (2.10)$$

For high precision, the temperature of the liquid in a manometer should be obtained and the corresponding density determined. Equation (2.10) is then used to correct the reading,  $h_1$  where  $\rho_1$  is the actual liquid density and  $\rho_2$  is the density at which the scale is calibrated.

Many variations of the manometer have been produced. Inclining one limb of the U tube shortens its practicable range but gives greater accuracy of reading. Careful levelling of inclined manometers is required and they are no longer used in subsurface pressure surveys. Some models have one limb of the U tube enlarged into a water reservoir. The liquid level in the reservoir changes only slightly compared with the balancing narrow tube. In the direct lift manometer, the reservoir is connected by flexible tubing to a short sight-glass of variable inclination which may be raised or lowered against a graduated scale. This manipulation enables the meniscus to be adjusted to a fixed mark on the sight-glass. Hence the level in the reservoir remains unchanged. The addition of a micrometer scale gives this instrument both a good range and high accuracy.

One of the problems in some water manometers is a misformed meniscus, particularly if the inclination of the tube is less than 5 degrees from the horizontal. This difficulty may be overcome by employing a light oil, or other liquid that has good wetting properties on glass. Alternatively, the two limbs may be made large enough in diameter to give horizontal liquid surfaces whose position can be sensed electronically or by touch probes adjusted through micrometers.

U tube manometers, or water gauges as they are commonly known, may feature as part of the permanent instrumentation of main and booster fans. Provided that the connections are kept firm and clean, there is little that can go wrong with these devices. Compact and portable inclined gauges are available for rapid readings of pressure differences across doors and stoppings in underground ventilation systems. However, in modern pressure surveying (Chapter 6) manometers have been replaced by the diaphragm gauge. This instrument consists essentially of a flexible diaphragm, across which is applied the differential pressure. The strain induced in the diaphragm is sensed electrically, mechanically or by magnetic means and transmitted to a visual indicator or recorder.

In addition to its portability and rapid reaction, the diaphragm gauge has many advantages for the subsurface ventilation engineer. First, it reflects directly a true pressure (force/area) rather than indirectly through a liquid medium. Secondly, it reacts relatively quickly to changes in temperature and does not require precise levelling. Thirdly, diaphragm gauges can be manufactured over a wide variety of ranges. A ventilation survey team may typically carry gauges ranging from 0 - 100 Pa to 0 - 5 kPa (or to encompass the value of the highest fan pressure in the system). One disadvantage of the diaphragm gauge is that its calibration may change with time and usage. Re-calibration against a laboratory precision manometer is recommended prior to an important survey.

Other appliances are used occasionally for differential pressures in subsurface pressure surveys. Piezoelectric instruments are likely to increase in popularity. The aerostat principle eliminates the need for tubing between the two measurement points and leads to a type of differential barometer. In this instrument, a closed and rigid air vessel is maintained at a constant temperature and is connected to the outside atmospheres via a manometer or diaphragm gauge. As the inside of the vessel remains at near constant pressure, any variations in atmospheric pressure cause a reaction on the manometer or gauge. Instruments based on this principle require independent calibration as slight movements of the diaphragm or liquid in the manometer result in the inside pressure not remaining truly constant.



## 2.3 FLUIDS IN MOTION

### 2.3.1. Bernoulli's equation for ideal fluids

As a fluid stream passes through a pipe, duct or other continuous opening, there will, in general, be changes in its velocity, elevation and pressure. In order to follow such changes it is useful to identify the differing forms of energy contained within a given mass of the fluid. For the time being, we will consider that the fluid is ideal; that is, it has no viscosity and proceeds along the pipe with no shear forces and no frictional losses. Secondly, we will ignore any thermal effects and consider mechanical energy only.

Suppose we have a mass,  $m$ , of fluid moving at velocity,  $u$ , at an elevation,  $Z$ , and a barometric pressure  $P$ . There are three forms of mechanical energy that we need to consider. In each case, we shall quantify the relevant term by assessing how much work we would have to do in order to raise that energy quantity from zero to its actual value in the pipe, duct or airway.

#### *Kinetic energy*

If we commence with the mass,  $m$ , at rest and accelerate it to velocity  $u$  in  $t$  seconds by applying a constant force  $F$ , then the acceleration will be uniform and the mean velocity is

$$\frac{0 + u}{2} = \frac{u}{2} \quad \frac{m}{s}$$

Then

distance travelled = mean velocity x time

$$= \frac{u}{2} t \quad m$$

Furthermore, the acceleration is defined as

$$\frac{\text{increase in velocity}}{\text{time}} = \frac{u}{t} \quad m / s^2$$

The force is given by

$$F = \text{mass} \times \text{acceleration}$$

$$= m \frac{u}{t} \quad N$$

and the work done to accelerate from rest to velocity  $u$  is

$$WD = \text{force} \times \text{distance} \quad Nm$$

$$= m \frac{u}{t} \times \frac{u}{2} t$$

$$= m \frac{u^2}{2} \quad Nm \text{ or } J \quad (2.11)$$

The kinetic energy of the mass  $m$  is, therefore,  $m u^2/2$  Joules.

*Potential energy*

Any base elevation may be used as the datum for potential energy. In most circumstances of underground ventilation engineering, it is differences in elevation that are important. If our mass  $m$  is located on the base datum then it will have a potential energy of zero relative to that datum. We then exert an upward force,  $F$ , sufficient to counteract the effect of gravity.

$$F = \text{mass} \times \text{acceleration}$$

$$= m g \quad \text{N}$$

where  $g$  is the gravitational acceleration.

In moving upward to the final elevation of  $Z$  metres above the datum, the work done is

$$\text{WD} = \text{Force} \times \text{distance}$$

$$= m g Z \quad \text{Joules} \quad (2.12)$$

This gives the potential energy of the mass at elevation  $Z$ .

*Flow work*

Suppose we have a horizontal pipe, open at both ends and of cross sectional area  $A$  as shown in Figure 2.1. We wish to insert a plug of fluid, volume  $v$  and mass  $m$  into the pipe. However, even in the absence of friction, there is a resistance due to the pressure of the fluid,  $P$ , that already exists in the pipe. Hence, we must exert a force,  $F$ , on the plug of fluid to overcome that resisting pressure. Our intent is to find the work done on the plug of fluid in order to move it a distance  $s$  into the pipe.

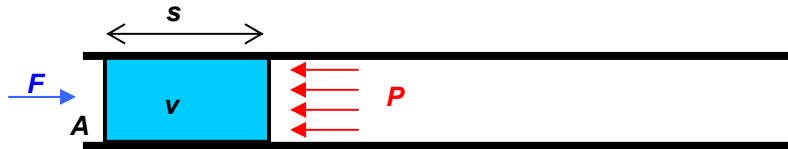


Figure 2.1 Flow work done on a fluid entering a pipe

The force,  $F$ , must balance the pressure,  $P$ , which is distributed over the area,  $A$ .

$$F = P A \quad \text{N}$$

$$\text{Work done} = \text{force} \times \text{distance}$$

$$= P A s \quad \text{J or Joules}$$

However, the product  $As$  is the swept volume  $v$ , giving

$$\text{WD} = P v$$

Now, by definition, the density is

$$\rho = \frac{m}{v} \quad \frac{\text{kg}}{\text{m}^3}$$

or

$$v = \frac{m}{\rho}$$

Hence, the work done in moving the plug of fluid into the pipe is

$$WD = \frac{Pm}{\rho} \quad \text{J} \quad (2.13)$$

or  $P/\rho$  Joules per kilogram.

As fluid continues to be inserted into the pipe to produce a continuous flow, then each individual plug must have this amount of work done on it. That energy is retained within the fluid stream and is known as the **flow work**. The appearance of pressure,  $P$ , within the expression for flow work has resulted in the term sometimes being labelled "pressure energy". This is very misleading as flow work is entirely different to the "elastic energy" stored when a closed vessel of fluid is compressed. Some authorities also object to the term "flow work" and have suggested "convected energy" or, simply, the " $Pv$  work". Note that in Figure 2.1 the pipe is open at both ends. Hence the pressure,  $P$ , inside the pipe does not change with time (the fluid is not compressed) when plugs of fluid continue to be inserted in a frictionless manner. When the fluid exits the system, it will carry kinetic and potential energy, and the corresponding flow work with it.

Now we are in a position to quantify the total mechanical energy of our mass of fluid,  $m$ . From expressions (2.11, 2.12 and 2.13)

$$\begin{aligned} \text{total mechanical energy} &= \text{kinetic energy} + \text{potential energy} + \text{flow work} \\ &= \frac{mu^2}{2} + mZg + m\frac{P}{\rho} \quad \text{J} \quad (2.14) \end{aligned}$$

If no mechanical energy is added to or subtracted from the fluid during its traverse through the pipe, duct or airway, and in the absence of frictional effects, the total mechanical energy must remain constant throughout the airway. Then equation (2.14) becomes

$$m \left\{ \frac{u^2}{2} + Zg + \frac{P}{\rho} \right\} = \text{constant} \quad \text{J} \quad (2.15)$$

Another way of expressing this equation is to consider two stations, 1 and 2 along the pipe, duct or airway. Then

$$m \left\{ \frac{u_1^2}{2} + Z_1g + \frac{P_1}{\rho_1} \right\} = m \left\{ \frac{u_2^2}{2} + Z_2g + \frac{P_2}{\rho_2} \right\}$$

Now as we are still considering the fluid to be incompressible (constant density),

$$\rho_1 = \rho_2 = \rho \quad (\text{say})$$

giving

$$\boxed{\frac{u_1^2 - u_2^2}{2} + (Z_1 - Z_2)g + \frac{P_1 - P_2}{\rho} = 0} \quad \frac{\text{J}}{\text{kg}} \quad (2.16)$$

Note that dividing by  $m$  on both sides has changed the units of each term from J to J/kg. Furthermore, if we multiplied throughout by  $\rho$  then each term would take the units of pressure. Bernoulli's equation has, traditionally, been expressed in this form for incompressible flow.

Equation (2.16) is of fundamental importance in the study of fluid flow. It was first derived by **Daniel Bernoulli (1700-1782)**, a Swiss mathematician, and is known throughout the world by his name.

As fluid flows along any closed system, Bernoulli's equation allows us to track the inter-relationships between the variables. Velocity  $u$ , elevation  $Z$ , and pressure  $P$  may all vary, but their combination as expressed in Bernoulli's equation remains true. It must be remembered, however, that it has been derived here on the assumptions of ideal (frictionless) conditions, constant density and steady-state flow. We shall see later how the equation must be amended for the real flow of compressible fluids.

### 2.3.2. Static, total and velocity pressures.

Consider the level duct shown on Figure 2.2. Three gauge pressures are measured. To facilitate visualization, the pressures are indicated as liquid heads on U tube manometers. However, the analysis will be conducted in terms of true pressure ( $\text{N/m}^2$ ) rather than head of fluid.

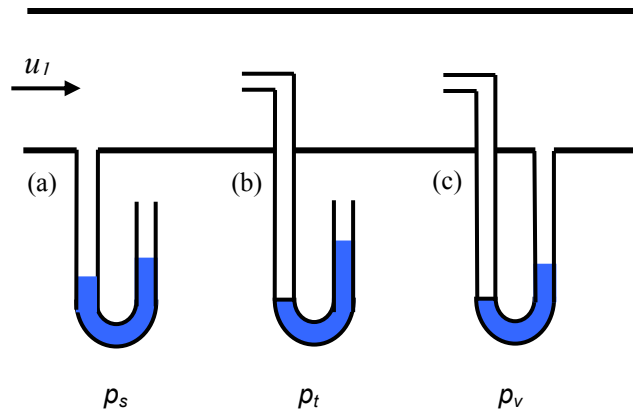


Figure 2.2 (a) static, (b) total and (c) velocity pressures

In position (a), one limb of the U tube is connected perpendicular through the wall of the duct. Any drilling burrs on the inside have been smoothed out so that the pressure indicated is not influenced by the local kinetic energy of the air. The other limb of the manometer is open to the ambient atmosphere. The gauge pressure indicated is known as the static pressure,  $p_s$ .

In position (b) the left tube has been extended into the duct and its open end turned so that it faces directly into the fluid stream. As the fluid impacts against the open end of the tube, it is brought to rest and the loss of its kinetic energy results in a local increase in pressure. The pressure within the tube then reflects the sum of the static pressure and the kinetic effect. Hence the manometer indicates a higher reading than in position (a). The corresponding pressure,  $p_t$ , is termed the total pressure. The increase in pressure caused by the kinetic energy can be quantified by using Bernoulli's equation (2.16). In this case  $Z_1 = Z_2$ , and  $u_2 = 0$ . Then

$$\frac{P_2 - P_1}{\rho} = \frac{u_1^2}{2}$$

The local increase in pressure caused by bringing the fluid to rest is then

$$p_v = P_2 - P_1 = \rho \frac{u_1^2}{2} \quad \text{Pa}$$

This is known as the velocity pressure and can be measured directly by connecting the manometer as shown in position (c). The left connecting tube of the manometer is at gauge pressure  $p_t$  and the right tube at gauge pressure  $p_s$ . It follows that

$$p_v = p_t - p_s$$

$$\text{or } p_t = p_s + p_v \quad \text{Pa} \quad (2.18)$$

In applying this equation, care should be taken with regard to sign as the static pressure,  $p_s$ , will be negative if the barometric pressure inside the duct is less than that of the outside atmosphere.

If measurements are actually made using a liquid in glass manometer as shown on Figure 2.2 then the reading registered on the instrument is influenced by the head of fluid in the manometer tubes above the liquid level. If the manometer liquid has a density  $\rho_1$ , and the superincumbent fluid in both tubes has a density  $\rho_d$ , then the head indicated by the manometer,  $h$ , should be converted to true pressure by the equation

$$p = (\rho_1 - \rho_d)gh \quad \text{Pa} \quad (2.19)$$

Reflecting back on equation (2.8) shows that this is the usual equation relating fluid head and pressure with the density replaced by the difference in the two fluid densities. In ventilation engineering, the superincumbent fluid is air, having a very low density compared with liquids. Hence, the  $\rho_d$  term in equation (2.19) is usually neglected. However, if the duct or pipe contains a liquid rather than a gas then the full form of equation (2.19) should be employed.

A further situation arises when the fluid in the duct has a density,  $\rho_d$ , that is significantly different to that of the air (or other fluid),  $\rho_a$ , which exists above the liquid in the right hand tube of the manometer in Fig. 2.2(a). Then

$$p = (\rho_1 - \rho_d)gh - (\rho_d - \rho_a)gh_2 \quad \text{Pa} \quad (2.20)$$

where  $h_2$  is the vertical distance between the liquid level in the right side of the manometer and the connection into the duct.

Equations (2.19) and (2.20) can be derived by considering a pressure balance on the two sides of the U tube above the lower of the two liquid levels.

### 2.3.3. Viscosity

Bernoulli's equation was derived in Section 2.3.1. on the assumption of an ideal fluid; i.e. that flow could take place without frictional resistance. In subsurface ventilation engineering almost all of the work input by fans (or other ventilating devices) is utilized against frictional effects within the airways. Hence, we must find a way of amending Bernoulli's equation for the frictional flow of real fluids.

The starting point in an examination of '**frictional flow**' is the concept of viscosity. Consider two parallel sheets of fluid a very small distance,  $dy$ , apart but moving at different velocities  $u$  and  $u + du$  (Figure 2.3). An equal but opposite force,  $F$ , will act upon each layer, the higher velocity sheet tending to pull its slower neighbour along and, conversely, the slower sheet tending to act as a brake on the higher velocity layer.

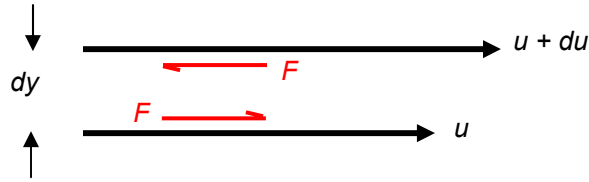


Figure 2.3 Viscosity causes equal but opposite forces to be exerted on adjacent laminae of fluid.

If the area of each of the two sheets in near contact is  $A$ , then the shear stress is defined as  $\tau$  (Greek 'tau') where

$$\tau = \frac{F}{A} \quad \frac{\text{N}}{\text{m}^2} \quad (2.21)$$

Among his many accomplishments, **Isaac Newton (1642-1727)** proposed that for parallel motion of streamlines in a moving fluid, the shear stress transmitted across the fluid in a direction perpendicular to the flow is proportional to the rate of change of velocity,  $du/dy$  (velocity gradient)

$$\tau = \frac{F}{A} = \mu \frac{du}{dy} \quad \frac{\text{N}}{\text{m}^2} \quad (2.22)$$

where the constant of proportionality,  $\mu$ , is known as the coefficient of dynamic viscosity (usually referred to simply as dynamic viscosity). The dynamic viscosity of a fluid varies with its temperature. For air, it may be determined from

$$\mu_{air} = (17.0 + 0.045 t) \times 10^{-6} \quad \frac{\text{Ns}}{\text{m}^2}$$

and for water

$$\mu_{water} = \left( \frac{64.72}{t + 31.766} - 0.2455 \right) \times 10^{-3} \quad \frac{\text{Ns}}{\text{m}^2}$$

where  $t$  = temperature ( $^{\circ}\text{C}$ ) in the range 0 - 60  $^{\circ}\text{C}$

The units of viscosity are derived by transposing equation (2.22)

$$\mu = \tau \frac{dy}{du} \quad \frac{\text{N}}{\text{m}^2} \frac{\text{m}}{\text{m}} \frac{\text{s}}{\text{m}} \quad \text{or} \quad \frac{\text{Ns}}{\text{m}^2}$$

A term which commonly occurs in fluid mechanics is the ratio of dynamic viscosity to fluid density. This is called the *kinematic viscosity*,  $\nu$  (Greek 'nu')

$$\nu = \frac{\mu}{\rho} \quad \frac{\text{Ns}}{\text{m}^2} \frac{\text{m}^3}{\text{kg}} \quad \text{or} \quad \text{Nm} \frac{\text{s}}{\text{kg}}$$

As  $1 \text{ N} = 1 \text{ kg} \times 1 \text{ m/s}^2$ , these units become

$$\text{kg} \frac{\text{m}}{\text{s}^2} \frac{\text{m}^3}{\text{kg}} = \frac{\text{m}^2}{\text{s}}$$

It is the transmission of shear stress that produces frictional resistance to motion in a fluid stream. Indeed, a definition of an 'ideal fluid' is one that has zero viscosity. Following from our earlier discussion on the molecular behaviour of fluids (Section 2.1.1.), there would appear to be at least two effects that produce the phenomenon of viscosity. One is the attractive forces that exist between molecules - particularly those of liquids. This will result in the movement of some molecules tending to drag others along, and for the slower molecules to inhibit motion of faster neighbours. The second effect may be visualized by glancing again at Figure 2.3. If molecules from the faster moving layer stray sideways into the slower layer then the inertia that they carry will impart kinetic energy to that layer. Conversely, migration of molecules from the slower to the faster layer will tend to retard its motion.

In liquids, the molecular attraction effect is dominant. Heating a liquid increases the internal kinetic energy of the molecules and also increases the average inter-molecular spacing. Hence, as the attractive forces diminish with distance, the viscosity of a liquid decreases with respect to temperature. In a gas, the molecular attractive force is negligible. The viscosity of gases is much less than that of liquids and is caused by the molecular inertia effect. In this case, the increased velocity of molecules caused by heating will tend to enhance their ability to transmit inertia across streamlines and, hence, we may expect the viscosity of gases to increase with respect to temperature. This is, in fact, the situation observed in practice.

In both of these explanations of viscosity, the effect works between consecutive layers equally well in both directions. Hence, dynamic equilibrium is achieved with both the higher and lower velocity layers maintaining their net energy levels. Unfortunately, no real process is perfect in fluid mechanics. Some of the useful mechanical energy will be transformed into the much less useful heat energy. In a level duct, pipe or airway, the loss of mechanical energy is reflected in an observable drop in pressure. This is often termed the '**frictional pressure drop**'

Recalling that Bernoulli's equation was derived for mechanical energy terms only in Section 2.3.1, it follows that for the flow of real fluids, the equation must take account of the frictional loss of mechanical energy. We may rewrite equation (2.16) as

$$\frac{u_1^2}{2} + Z_1 g + \frac{P_1}{\rho} = \frac{u_2^2}{2} + Z_2 g + \frac{P_2}{\rho} + F_{12} \quad \frac{\text{J}}{\text{kg}} \quad (2.23)$$

where  $F_{12}$  = energy converted from the mechanical form to heat (J/kg).

The problem now turns to one of quantifying the frictional term  $F_{12}$ . For that, we must first examine the nature of fluid flow.

#### 2.3.4. Laminar and turbulent flow. Reynolds Number

In our everyday world, we can observe many examples of the fact that there are two basic kinds of fluid flow. A stream of oil poured out of a can flows smoothly and in a controlled manner while water, poured out at the same rate, would break up into cascading rivulets and droplets. This example seems to suggest that the type of flow depends upon the fluid. However, a light flow of water falling from a circular outlet has a steady and controlled appearance, but if the flowrate is increased the stream will assume a much more chaotic form. The type of flow seems to depend upon the flowrate as well as the type of fluid.

Throughout the nineteenth century, it was realized that these two types of flow existed. The German engineer **G.H.L. Hagen (1797-1884)** found that the type of flow depended upon the velocity and viscosity of the fluid. However, it was not until the 1880's that Professor **Osborne Reynolds** of Manchester University in England established a means of characterizing the type of flow regime

through a combination of experiments and logical reasoning. Reynolds' laboratory tests consisted of injecting a filament of colored dye into the bell mouth of a horizontal glass tube that was submerged in still water within a large glass-walled tank. The other end of the tube passed through the end of the tank to a valve which was used to control the velocity of water within the tube. At low flow rates, the filament of dye formed an unbroken line in the tube without mixing with the water. At higher flow rates the filament of dye began to waver. As the velocity in the tube continued to be increased the wavering filament suddenly broke up to mix almost completely with the water.

In the initial type of flow, the water appeared to move smoothly along streamlines, layers or laminae, parallel to the axis of the tube. We call this **laminar flow**. Appropriately, we refer to the completely mixing type of behavior as **turbulent flow**. Reynolds' experiments had, in fact, identified a third regime - the wavering filament indicated a transitional region between fully laminar and fully turbulent flow. Another observation made by Reynolds was that the break-up of the filament always occurred, not at the entrance, but about thirty diameters along the tube.

The essential difference between laminar and turbulent flow is that in the former, movement across streamlines is limited to the molecular scale, as described in Section 2.3.3. However, in turbulent flow, swirling packets of fluid move sideways in small turbulent eddies. These should not be confused with the larger and more predictable oscillations that can occur with respect to time and position such as the vortex action caused by fans, pumps or obstructions in the airflow. The turbulent eddies appear random in the complexity of their motion. However, as with all "random" phenomena, the term is used generically to describe a process that is too complex to be characterized by current mathematical knowledge. Computer simulation packages using techniques known generically as **computational fluid dynamics (CFD)** have produced powerful means of analysis and predictive models of turbulent flow. At the present time, however, many practical calculations involving turbulent flow still depend upon empirical factors.

The flow of air in the vast majority of 'ventilated' places underground is turbulent in nature. However, the sluggish movement of air or other fluids in zones behind stoppings or through fragmented strata may be laminar. It is, therefore, important that the subsurface ventilation engineer be familiar with both types of flow. Returning to Osborne Reynolds, he found that the development of full turbulence depended not only upon velocity, but also upon the diameter of the tube. He reasoned that if we were to compare the flow regimes between differing geometrical configurations and for various fluids we must have some combination of geometric and fluid properties that quantified the degree of similitude between any two systems. Reynolds was also familiar with the concepts of "inertial (kinetic) force",  $\rho u^2/2$  (Newtons per square metre of cross section) and "viscous force",  $\tau = \mu du/dy$  (Newtons per square metre of shear surface). Reynolds argued that the dimensionless ratio of "inertial forces" to "viscous forces" would provide a basis of comparing fluid systems

$$\frac{\text{inertial force}}{\text{viscous force}} = \rho \frac{u^2}{2} \frac{1}{\mu} \frac{dy}{du} \quad (2.24)$$

Now, for similitude to exist, all steady state velocities,  $u$ , or differences in velocity between locations,  $du$ , within a given system are proportional to each other. Furthermore, all lengths are proportional to any chosen characteristic length,  $L$ . Hence, in equation (2.24) we can replace  $du$  by  $u$ , and  $dy$  by  $L$ . The constant, 2, can also be dropped as we are simply looking for a combination of variables that characterize the system. That combination now becomes

$$\rho u^2 \frac{1}{\mu} \frac{L}{u}$$

or

$$\frac{\rho u L}{\mu} = \text{Re} \quad (2.25)$$



As equation (2.24) is dimensionless then so, also, must this latter expression be dimensionless. This can easily be confirmed by writing down the units of the component variables. The result we have reached here is of fundamental importance to the study of fluid flow. The dimensionless group  $\rho u L / \mu$  is known universally as Reynolds Number,  $Re$ . In subsurface ventilation engineering, the characteristic length is normally taken to be the hydraulic mean diameter of an airway,  $d$ , and the characteristic velocity is usually the mean velocity of the airflow. Then

$$Re = \frac{\rho u d}{\mu}$$

At Reynolds Numbers of less than 2 000 in fluid flow systems, viscous forces prevail and the flow will be laminar. The Reynolds Number over which fully developed turbulence exists is less well defined. The onset of turbulence will occur at Reynolds Numbers of 2 500 to 3 000 assisted by any vibration, roughness of the walls of the pipe or any momentary perturbation in the flow.

### Example

A ventilation shaft of diameter 5m passes an airflow of 200 m<sup>3</sup>/s at a mean density of 1.2 kg/m<sup>3</sup> and an average temperature of 18 °C. Determine the Reynolds Number for the shaft.

### Solution

For air at 18 °C

$$\begin{aligned} \mu &= (17.0 + 0.045 \times 18) \times 10^{-6} \\ &= 17.81 \times 10^{-6} \text{ Ns/m}^2 \end{aligned}$$

$$\text{Air velocity, } u = \frac{Q}{A} = \frac{200}{\pi 5^2 / 4} = 10.186 \text{ m/s}$$

$$Re = \frac{\rho u d}{\mu} = \frac{1.2 \times 10.186 \times 5}{17.81 \times 10^{-6}} = 3.432 \times 10^6$$

This Reynolds Number indicates that the flow will be turbulent.

### 2.3.5. Frictional losses in laminar flow, Poiseuille's Equation.

Now that we have a little background on the characteristics of laminar and turbulent flow, we can return to Bernoulli's equation corrected for friction (equation (2.23)) and attempt to find expressions for the work done against friction,  $F_{12}$ . First, let us deal with the case of laminar flow.

Consider a pipe of radius  $R$  as shown in Figure 2.4. As the flow is laminar, we can imagine concentric cylinders of fluid telescoping along the pipe with zero velocity at the walls and maximum velocity in the center. Two of these cylinders of length  $L$  and radii  $r$  and  $r + dr$  are shown. The velocities of the cylinders are  $u$  and  $u - du$  respectively.

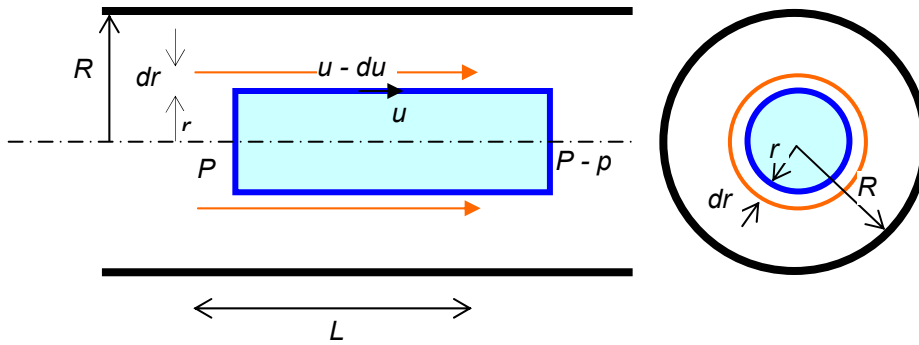


Figure 2.4 Viscous drag opposes the motive effect of applied pressure difference

The force propagating the inner cylinder forward is produced by the pressure difference across its two ends,  $p$ , multiplied by its cross sectional area,  $\pi r^2$ . This force is resisted by the viscous drag of the outer cylinder,  $\tau$ , acting on the 'contact' area  $2\pi rL$ . As these forces must be equal at steady state conditions,

$$2\pi r \tau L = \pi r^2 p$$

However,  $\tau = -\mu \frac{du}{dr}$  (equation (2.22) with a negative  $du$ )

giving

$$-\mu \frac{du}{dr} = \frac{r p}{2 L}$$

$$\text{or } du = -\frac{p r}{L 2\mu} dr \quad \frac{\text{m}}{\text{s}} \quad (2.26)$$

For a constant diameter tube, the pressure gradient along the tube  $p/L$  is constant. So, also, is  $\mu$  for the Newtonian fluids that we are considering. (A Newtonian fluid is defined as one in which viscosity is independent of velocity). Equation (2.26) can, therefore, be integrated to give

$$u = -\frac{p}{L} \frac{1}{2\mu} \frac{r^2}{2} + C \quad (2.27)$$

At the wall of the tube,  $r = R$  and  $u = 0$ . This gives the constant of integration to be

$$C = \frac{p R^2}{L 4\mu}$$

Substituting back into equation (2.27) gives

$$u = \frac{1}{4\mu} \frac{p}{L} (R^2 - r^2) \quad \frac{\text{m}}{\text{s}} \quad (2.28)$$

Equation (2.28) is a general equation for the velocity of the fluid at any radius and shows that the velocity profile across the tube is parabolic (Figure 2.5). Along the centre line of the tube,  $r = 0$  and the velocity reaches a maximum of

$$u_{\max} = \frac{1}{4\mu} \frac{p}{L} R^2 \quad \frac{\text{m}}{\text{s}} \quad (2.29)$$

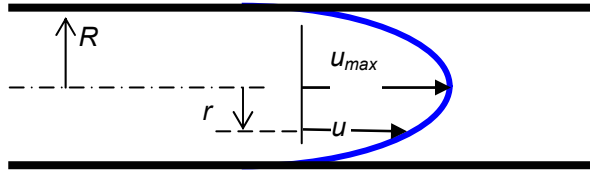


Figure 2.5 The velocity profile for laminar flow is parabolic

The velocity terms in the Bernoulli equation are mean velocities across the relevant cross-sections. It is, therefore, preferable that the work done against viscous friction should also be expressed in terms of a mean velocity,  $u_m$ . We must be careful how we define *mean velocity* in this context. Our convention is to determine it as

$$u_m = \frac{Q}{A} \quad \frac{\text{m}}{\text{s}} \quad (2.30)$$

where  $Q$  = volume airflow ( $\text{m}^3/\text{s}$ ) and  $A$  = cross sectional area ( $\text{m}^2$ )

We could define another mean velocity by integrating the parabolic equation (2.28) with respect to  $r$  and dividing the result by  $R$ . However, this would not take account of the fact that the volume of fluid in each concentric shell of thickness  $dr$  increases with radius. In order to determine the true mean velocity, consider the elemental flow  $dQ$  through the annulus of cross sectional area  $2\pi r dr$  at radius  $r$  and having a velocity of  $u$  (Figure 2.4)

$$dQ = u 2\pi r dr$$

Substituting for  $u$  from equation (2.28) gives

$$dQ = \frac{2\pi}{4\mu} \frac{p}{L} (R^2 - r^2) r dr$$

$$Q = \frac{2\pi}{4\mu} \frac{p}{L} \int_0^R (R^2 r - r^3) dr$$

Integrating gives

$$Q = \frac{\pi R^4}{8\mu} \frac{p}{L} \quad \frac{\text{m}}{\text{s}} \quad (2.31)$$

This is known as the **Poiseuille Equation** or, sometimes, the Hagen-Poiseuille Equation. **J.L.M. Poiseuille (1799-1869)** was a French physician who studied the flow of blood in capillary tubes.

For engineering use, where the dimensions of a given pipe and the viscosity of fluid are known, Poiseuille's equation may be written as a pressure drop - quantity relationship.

$$p = \frac{8\mu L}{\pi R^4} Q$$

or

$$p = R_L Q \quad \text{Pa} \quad (2.32)$$

where  $R_L = \frac{8\mu L}{\pi R^4}$   $\frac{\text{Ns}}{\text{m}^5}$  and is known as the laminar resistance of the pipe.

Equation (2.32) shows clearly that in laminar flow the frictional pressure drop is proportional to the volume flow for any given pipe and fluid. Combining equations (2.30) and (2.31) gives the required mean velocity

$$u_m = \frac{\pi R^4}{8\mu} \frac{p}{L} \frac{1}{\pi R^2} = \frac{R^2 p}{8\mu L} \quad \frac{\text{m}}{\text{s}} \quad (2.33)$$

or

$$p = \frac{8\mu u_m L}{R^2} \quad \text{Pa} \quad (2.34)$$

This latter form gives another expression for the frictional pressure drop in laminar flow.

To see how we can use this equation in practice, let us return the frictional form of Bernoulli's equation

$$\frac{u_1^2 - u_2^2}{2} + (Z_1 - Z_2)g + \frac{(P_1 - P_2)}{\rho} = F_{12} \quad \frac{\text{J}}{\text{kg}} \quad (\text{see equation (2.23)})$$

Now for incompressible flow along a level pipe of constant cross-sectional area,

$$Z_1 = Z_2 \quad \text{and} \quad u_1 = u_2 = u_m$$

then

$$\frac{(P_1 - P_2)}{\rho} = F_{12} \quad \frac{\text{J}}{\text{kg}} \quad (2.35)$$

However,  $(P_1 - P_2)$  is the same pressure difference as  $p$  in equation (2.34).

Hence the work done against friction is

$$F_{12} = \frac{8\mu u_m L}{\rho R^2} \quad \frac{\text{J}}{\text{kg}} \quad (2.36)$$

Bernoulli's equation for incompressible laminar frictional flow now becomes

$$\frac{u_1^2 - u_2^2}{2} + (Z_1 - Z_2)g + \frac{(P_1 - P_2)}{\rho} = \frac{8\mu u_m L}{\rho R^2} \quad \frac{\text{J}}{\text{kg}} \quad (2.37)$$

If the pipe is of constant cross sectional area, then  $u_1 = u_2 = u_m$  and the kinetic energy term disappears. On the other hand, if the cross-sectional area and, hence, the velocity varies along the pipe then  $u_m$  may be established as a weighted mean. For large changes in cross-sectional area, the full length of pipe may be subdivided into increments for analysis.

#### Example.

A pipe of diameter 2 cm rises through a vertical distance of 5m over the total pipe length of 2 000 m. Water of mean temperature 15°C flows up the tube to exit at atmospheric pressure of 100 kPa. If the required flowrate is 1.6 litres per minute, find the resistance of the pipe, the work done against friction and the head of water that must be applied at the pipe entrance.

#### Solution.

It is often the case that measurements made in engineering are not in SI units. We must be careful to make the necessary conversions before commencing any calculations.

Flowrate  $Q = 1.6$  litres/min

$$= \frac{1.6}{1000 \times 60} = 2.667 \times 10^{-5} \quad \frac{\text{m}^3}{\text{s}}$$

Cross sectional area of pipe  $A = \pi d^2 / 4 = \pi \times (0.02^2) / 4 = 3.142 \times 10^{-4} \text{ m}^2$

$$\text{Mean velocity, } u = \frac{Q}{A} = \frac{2.667 \times 10^{-5}}{3.142 \times 10^{-4}} = 0.08488 \quad \text{m/s}$$

(We have dropped the subscript  $m$ . For simplicity, the term  $u$  from this point on will refer to the mean velocity defined as  $Q/A$ )

Viscosity of water at 15 °C (from Section 2.3.3.)

$$\mu = \left( \frac{64.72}{15 + 31.766} - 0.2455 \right) \times 10^{-3} = 1.138 \times 10^{-3} \quad \frac{\text{Ns}}{\text{m}^2}$$

Before we can begin to assess frictional effects we must check whether the flow is laminar or turbulent. We do this by calculating the Reynolds Number

$$\text{Re} = \frac{\rho u d}{\mu}$$

where  $\rho$  = density of water (taken as 1 000 kg/m<sup>3</sup>)

$$\text{Re} = \frac{1000 \times 0.08488 \times 0.02}{1.138 \times 10^{-3}} = 1491 \quad (\text{dimensionless})$$

As Re is below 2 000, the flow is laminar and we should use the equations based on viscous friction.

Laminar resistance of pipe (from equation (2.32))

$$R_L = \frac{8 \mu L}{\pi R^4} = \frac{8 \times 1.1384 \times 10^{-3} \times 2000}{\pi \times (0.01)^4} = 580 \times 10^6 \quad \frac{\text{Ns}}{\text{m}^5}$$

Frictional pressure drop in the pipe (equation (2.32))

$$p = R_L Q = 580 \times 10^{-6} \times 2.667 \times 10^{-5} = 15\,461 \quad \text{Pa}$$

Work done against friction (equation (2.36))

$$F_{12} = \frac{8 \mu u L}{\rho R^2} = \frac{8 \times 1.1384 \times 10^{-3} \times 0.084\,88 \times 2000}{1000 \times (0.01)^2} = 15.461 \quad \frac{\text{J}}{\text{kg}}$$

This is the amount of mechanical energy transformed to heat in Joules per kilogram of water. Note the similarity between the statements for frictional pressure drop,  $p$ , and work done against friction,  $F_{12}$ . We have illustrated, by this example, a relationship between  $p$  and  $F_{12}$  that will be of particular significance in comprehending the behaviour of airflows in ventilation systems, namely

$$\frac{p}{\rho} = F_{12}$$

In fact, having calculated  $p$  as 15 461 Pa, the value of  $F_{12}$  may be quickly evaluated as

$$\frac{15\,461}{1000} = 15.461 \quad \frac{\text{J}}{\text{kg}}$$

To find the pressure at the pipe inlet we may use Bernoulli's equation corrected for frictional effects

$$\frac{u_1^2 - u_2^2}{2} + (Z_1 - Z_2)g + \frac{P_1 - P_2}{\rho} = F_{12} \quad \frac{\text{J}}{\text{kg}} \quad (\text{see equation (2.23)})$$

In this example

$$u_1 = u_2$$

$$Z_1 - Z_2 = -5 \text{ m}$$

and  $P_2 = 100 \text{ kPa} = 100\,000 \text{ Pa}$

$$\text{giving } F_{12} = -5 \times 9.81 + \frac{P_1 - 100\,000}{1000} = 15.461 \quad \frac{\text{J}}{\text{kg}}$$

This yields the absolute pressure at the pipe entry as

$$P_1 = 164.5 \times 10^3 \quad \text{Pa}$$

or  $164.5 \text{ kPa}$

If the atmospheric pressure at the location of the bottom of the pipe is also 100 kPa, then the gauge pressure,  $p_g$ , within the pipe at that same location

$$p_g = 164.5 - 100 = 64.5 \text{ kPa}$$

This can be converted into a head of water,  $h_1$ , from equation (2.8)

$$p_g = \rho g h_1$$

$$h_1 = \frac{64.5 \times 10^{-3}}{1000 \times 9.81} = 6.576 \quad \text{m of water}$$

Thus, a header tank with a water surface maintained 6.576 m above the pipe entrance will produce the required flow of 1.6 litres/minute along the pipe.

The experienced engineer would have determined this result quickly and directly after calculating the frictional pressure drop to be 15 461 Pa. The frictional head loss

$$h = \frac{p}{\rho g} = \frac{15\,461}{1000 \times 9.81} = 1.576 \quad \text{m of water}$$

The head of water at the pipe entrance must overcome the frictional head loss as well as the vertical lift of 5 m. (An intuitive use of Bernoulli's equation). Then

$$h_1 = 5 + 1.576 = 6.576 \quad \text{m of water}$$

### 2.3.6. Frictional losses in turbulent flow

The previous section showed that the parallel streamlines of laminar flow and Newton's perception of viscosity enabled us to produce quantitative relationships through purely analytical means. Unfortunately, the highly convoluted streamlines of turbulent flow, caused by the interactions between both localized and propagating eddies have so far proved resistive to completely analytical techniques. Numerical methods using the memory capacities and speeds of supercomputers allow the flow to be simulated as a large number of small packets of fluids, each one influencing the behaviour of those around it. These mathematical models, using numerical techniques known collectively as **computational fluid dynamics (CFD)**, may be used to simulate turbulent flow in given geometrical systems, or to produce statistical trends. However, the majority of engineering applications involving turbulent flow still rely on a combination of analysis and empirical factors. The construction of physical models for observation in wind tunnels or other fluid flow test facilities remains a common means of predicting the behaviour and effects of turbulent flow.

#### 2.3.6.1. The Chézy-Darcy Equation

The discipline of hydraulics was studied by philosophers of the ancient civilizations. However, the beginnings of our present treatment of fluid flow owe much to the hydraulic engineers of eighteenth and nineteenth century France. During his reign, **Napoleon Bonaparte** encouraged the research and development necessary for the construction of water distribution and drainage systems in Paris.

**Antoine de Chézy (1719-1798)** carried out a series of experiments on the river Seine and on canals in about 1769. He found that the mean velocity of water in open ducts was proportional to the square root of the channel gradient, cross sectional area of flow and inverse of the wetted perimeter.

$$u \propto \sqrt{\frac{A}{per} \frac{h}{L}}$$

where  $h$  = vertical distance dropped by the channel in a length  $L$  ( $h/L$  = hydraulic gradient)

$per$  = wetted perimeter (m)

and  $\propto$  means 'proportional to'

Inserting a constant of proportionality,  $c$ , gives

$$u = c \sqrt{\frac{A}{per} \frac{h}{L}} \quad \frac{\text{m}}{\text{s}} \quad (2.38)$$

where  $c$  is known as the Chézy coefficient.

Equation (2.38) has become known as Chézy's equation for channel flow. Subsequent analysis shed further light on the significance of the Chézy coefficient. When a fluid flows along a channel, a mean shear stress  $\tau$  is set up at the fluid/solid boundaries. The drag on the channel walls is then

$$\tau \text{ per } L$$

where  $per$  is the "wetted" perimeter

This must equal the pressure force causing the fluid to move,  $pA$ , where  $p$  is the difference in pressure along length  $L$ .

$$\tau \text{ per } L = A p \quad \text{N} \quad (2.39)$$

(A similar equation was used in Section 2.3.5. for a circular pipe).

But  $p = \rho g h$  Pa (equation (2.8))

$$\text{giving } \tau = \frac{A}{per} \rho g \frac{h}{L} \quad \frac{\text{N}}{\text{m}^2} \quad (2.40)$$

If the flow is fully turbulent, the shear stress or skin friction drag,  $\tau$ , exerted on the channel walls is also proportional to the inertial (kinetic) energy of the flow expressed in Joules per cubic metre.

$$\tau \propto \rho \frac{u^2}{2} \quad \frac{\text{J}}{\text{m}^3} = \frac{\text{Nm}}{\text{m}^3} \text{ or } \frac{\text{N}}{\text{m}^2}$$

$$\text{or } \tau = f \rho \frac{u^2}{2} \quad \frac{\text{N}}{\text{m}^2} \quad (2.41)$$

where  $f$  is a dimensionless coefficient which, for fully developed turbulence, depends only upon the roughness of the channel walls.

Equating (2.40) and (2.41) gives

$$f \frac{u^2}{2} = \frac{A}{per} g \frac{h}{L}$$

$$\text{or } u = \sqrt{\frac{2g}{f}} \sqrt{\frac{A}{per} \frac{h}{L}} \quad \frac{\text{m}}{\text{s}} \quad (2.42)$$



Comparing this with equation (2.38) shows that Chézy's coefficient,  $c$ , is related to the roughness of the channel.

$$c = \sqrt{\frac{2g}{f}} \quad \frac{\text{m}^{\frac{1}{2}}}{\text{s}} \quad (2.43)$$

The development of flow relationships was continued by **Henri Darcy (1803-1858)**, another French engineer, who was interested in the turbulent flow of water in pipes. He adapted Chézy's work to the case of circular pipes and ducts running full. Then  $A = \pi d^2 / 4$ ,  $per = \pi d$  and the fall in elevation of Chézy's channel became the head loss,  $h$  (metres of fluid) along the pipe length  $L$ . Equation (2.42) now becomes

$$u^2 = \frac{2g}{f} \frac{\pi d^2}{4} \frac{1}{\pi d} \frac{h}{L}$$

or  $h = \frac{4fLu^2}{2gd}$  metres of fluid (2.44)

This is the well known **Chézy-Darcy equation**, sometimes also known simply as the Darcy equation or the Darcy-Weisbach equation. The head loss,  $h$ , can be converted to a frictional pressure drop,  $p$ , by the now familiar relationship,  $p = \rho g h$  to give

$$p = \frac{4fL}{d} \frac{\rho u^2}{2} \quad \text{Pa} \quad (2.45)$$

or a frictional work term

$$F_{12} = \frac{p}{\rho} = \frac{4fL}{d} \frac{u^2}{2} \quad \frac{\text{J}}{\text{kg}} \quad (2.46)$$

The Bernoulli equation for frictional and turbulent flow becomes

$$\frac{u_1^2 - u_2^2}{2} + (Z_1 - Z_2)g + \frac{(P_1 - P_2)}{\rho} = \frac{4fL}{d} \frac{u^2}{2} \quad \frac{\text{J}}{\text{kg}} \quad (2.47)$$

where  $u$  is the mean velocity.

The most common form of the Chézy-Darcy equation is that given as (2.44). Leaving the constant 2 uncanceled provides a reminder that the pressure loss due to friction is a function of kinetic energy  $u^2/2$ . However, some authorities have combined the 4 and the  $f$  into a different coefficient of friction  $\lambda$  ( $= 4f$ ) while others, presumably disliking Greek letters, then replaced the symbol  $\lambda$  by (would you believe it?),  $f$ . We now have a confused situation in the literature of fluid mechanics where  $f$  may mean the original Chézy-Darcy coefficient of friction, or four times that value. When reading the literature, care should be taken to confirm the nomenclature used by the relevant author. Throughout this book,  $f$  is used to mean the original Chézy-Darcy coefficient as used in equation (2.44).

In order to generalize our results to ducts or airways of non-circular cross section, we may define a **hydraulic radius** as

$$\begin{aligned} r_h &= \frac{A}{per} && \text{m} && (2.48) \\ &= \frac{\pi d^2}{4\pi d} = \frac{d}{4} \end{aligned}$$

Reference to the "**hydraulic mean diameter**" denotes  $4A/per$ . This device works well for turbulent flow but must not be applied to laminar flow where the resistance to flow is caused by viscous action throughout the body of the fluid rather than concentrated around the perimeter of the walls.

Substituting for  $d$  in equation (2.45) gives

$$p = fL \frac{per}{A} \frac{\rho u^2}{2} \quad \text{Pa} \quad (2.49)$$

This can also be expressed as a relationship between frictional pressure drop,  $p$ , and volume flow,  $Q$ . Replacing  $u$  by  $Q/A$  in equation (2.49) gives

$$p = \frac{fL}{2} \frac{per}{A^3} \rho Q^2 \quad \text{Pa}$$

$$\text{or} \quad p = R_t \rho Q^2 \quad \text{Pa} \quad (2.50)$$

$$\text{where} \quad R_t = \frac{fL}{2} \frac{per}{A^3} \quad \text{m}^{-4} \quad (2.51)$$

This is known as the rational turbulent resistance of the pipe, duct or airway and is a function only of the geometry and roughness of the opening.

### 2.3.6.2. The coefficient of friction, $f$ .

It is usually the case that a significant advance in research opens up new avenues of investigation and produces a flurry of further activity. So it was following the work of Osborne Reynolds. During the first decade of this century, fluid flow through pipes was investigated in great detail by engineers such as **Thomas E. Stanton (1865-1931)** and **J.R. Pannel** in the United Kingdom, and **Ludwig Prandtl (1875-1953)** in Germany. A major cause for concern was the coefficient of friction,  $f$ .

There were two problems. First, how could one predict the value of  $f$  for any given pipe without actually constructing the pipe and conducting a pressure-flow test on it. Secondly, it was found that  $f$  was not a true constant but varied with Reynolds Number for very smooth pipes and, particularly, at low values of Reynolds Number. The latter is not too surprising as  $f$  was introduced initially as a constant of proportionality between shear stress at the walls and inertial force of the fluid (equation (2.41)) for fully developed turbulence. At the lower Reynolds Numbers we may enter the transitional or even laminar regimes.

Figure 2.6 illustrates the type of results that were obtained. A very smooth pipe exhibited a continually decreasing value of  $f$ . This is labelled as the turbulent smooth pipe curve. However, for rougher pipes, the values of  $f$  broke away from the smooth pipe curve at some point and, after a transitional region, settled down to a constant value, independent of Reynolds Number. This phenomenon was quantified empirically through a series of classical experiments conducted in Germany by **Johann Nikuradse (1894-1979)**, a former student of Prandtl. Nikuradse took a number of smooth pipes of diameter 2.5, 5 and 10 cm, and coated the inside walls uniformly with grains of

graded sand. The roughness of each tube was then defined as  $e/d$  where  $e$  was the diameter of the sand grains and  $d$  the diameter of the tube. The advantages of dimensionless numbers had been well learned from Reynolds. The corresponding  $f$  -  $Re$  relationships are illustrated on Figure 2.6.

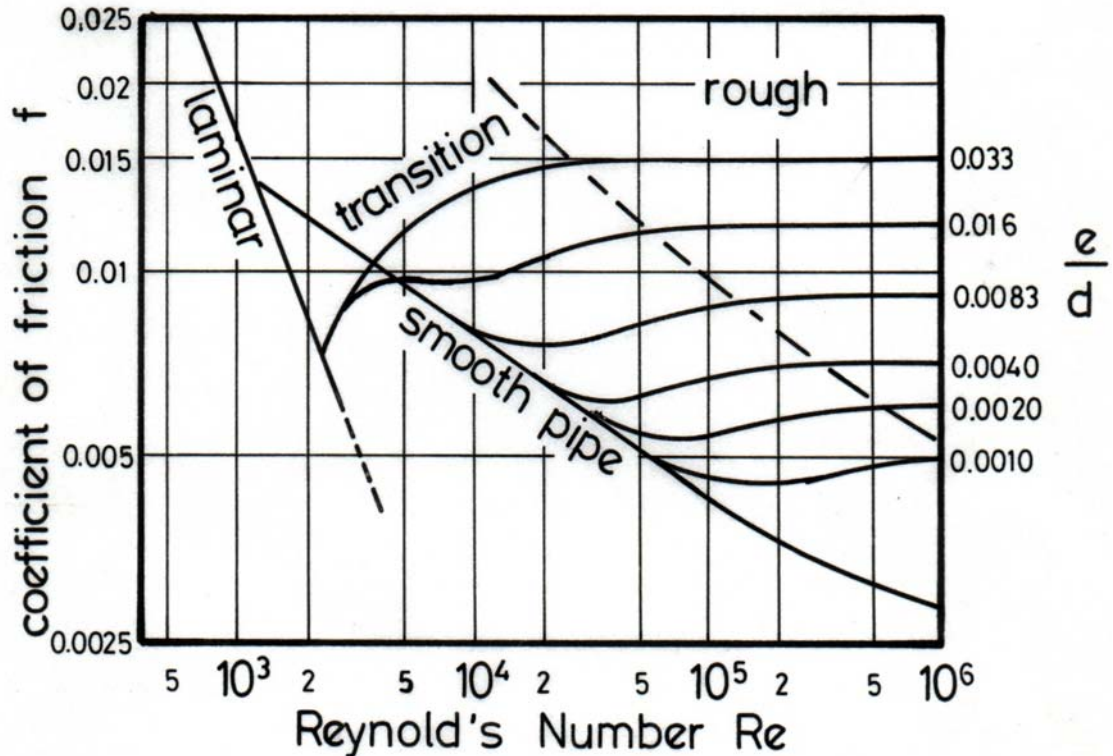


Figure 2.6 variation of  $f$  with respect to  $Re$  as found by Nikuradse

The investigators of the time were then faced with an intriguing question. How could a pipe of given roughness and passing a turbulent flow be "smooth" (i.e. follow the smooth pipe curve) at certain Reynolds Numbers but become "rough" (constant  $f$ ) at higher Reynolds Numbers? The answer lies in our initial concept of turbulence - the formation and maintenance of small, interacting and propagating eddies within the fluid stream. These necessitate the existence of cross velocities with vector components perpendicular to the longitudinal axis of the tube. At the walls there can be no cross velocities except on a molecular scale. Hence, there must be a thin layer close to each wall through which the velocity increases from zero (actually at the wall) to some finite velocity sufficiently far away from the wall for an eddy to exist. Within that thin layer the streamlines remain parallel to each other and to the wall, i.e. laminar flow.

Although this **laminar sublayer** is very thin, it has a marked effect on the behaviour of the total flow in the pipe. All real surfaces (even polished ones) have some degree of roughness. If the peaks of the roughness, or asperities, do not protrude through the laminar sublayer then the surface may be described as "hydraulically smooth" and the wall resistance is limited to that caused by viscous shear within the fluid. On the other hand, if the asperities protrude well beyond the laminar sublayer then the disturbance to flow that they produce will cause additional eddies to be formed, consuming mechanical energy and resulting in a higher resistance to flow. Furthermore, as the velocity and,

hence, the Reynolds Number increases, the thickness of the laminar sublayer decreases. Any given pipe will then be hydraulically smooth if the asperities are submerged within the laminar sublayer and hydraulically rough if the asperities project beyond the laminar sublayer. Between the two conditions there will be a transition zone where some, but not all, of the asperities protrude through the laminar sublayer. The hypothesis of the existence of a laminar sublayer explains the behaviour of the curves in Figure 2.6. The recognition and early study of boundary layers owe a great deal to the work of Ludwig Prandtl and the students who started their careers under his guidance.

Nikuradse's work marked a significant step forward in that it promised a means of predicting the coefficient of friction and, hence, the resistance of any given pipe passing turbulent flow. However, there continued to be difficulties. In real pipes, ducts or underground airways, the wall asperities are not all of the same size, nor are they uniformly dispersed. In particular, mine airways show great variation in their roughness. Concrete lining in ventilation shafts may have a uniform  $e/d$  value as low as 0.001. On the other hand, where shaft tubing or regularly spaced airway supports are used, the turbulent wakes on the downstream side of the supports create a dependence of airway resistance on their distance apart. Furthermore, the immediate wall roughness may be superimposed upon larger scale sinuosity of the airways and, perhaps, the existence of cross-cuts or other junctions. The larger scale vortices produced by these macro effects may be more energy demanding than the smaller eddies of normal turbulent flow and, hence, produce a much higher value of  $f$ . Many airways also have wall roughnesses that exhibit a directional bias, produced by the mechanized or drill and blast methods of driving the airway, or the natural cleavage of the rock.

For all of these reasons, there may be a significant divergence between Nikuradse's curves and results obtained in practice, particularly in the transitional zone. Further experiments and analytical investigations were carried out in the late 1930's by **C.F. Colebrook** in England. The equations that were developed were somewhat awkward to use. However, the concept of "equivalent sand grain roughness" was further developed by the American engineer **Lewis F. Moody** in 1944. The ensuing chart, shown on Figure 2.7, is known as the Moody diagram and is now widely employed by practicing engineers to determine coefficients of friction.

### 2.3.6.3. Equations describing $f$ - Re relationships

The literature is replete with relationships that have been derived through combinations of analysis and empiricism to describe the behavior of the coefficient of friction,  $f$ , with respect to Reynolds' Number on the Moody Chart. No attempt is made here at a comprehensive discussion of the merits and demerits of the various relationships. Rather, a simple summary is given of those equations that have been found to be most useful in ventilation engineering.

#### *Laminar Flow*

The straight line that describes laminar flow on the log-log plot of Figure 2.7 is included in the Moody Chart for completeness. However, Poiseuille's equation (2.31) can be used directly to establish frictional pressure losses for laminar flow without using the chart. The corresponding  $f$ -Re relationship is easily established. Combining equations (2.34) and (2.45) gives

$$p = \frac{8\mu uL}{R^2} = \frac{4fL}{d} \frac{\rho u^2}{2} \quad \text{Pa}$$

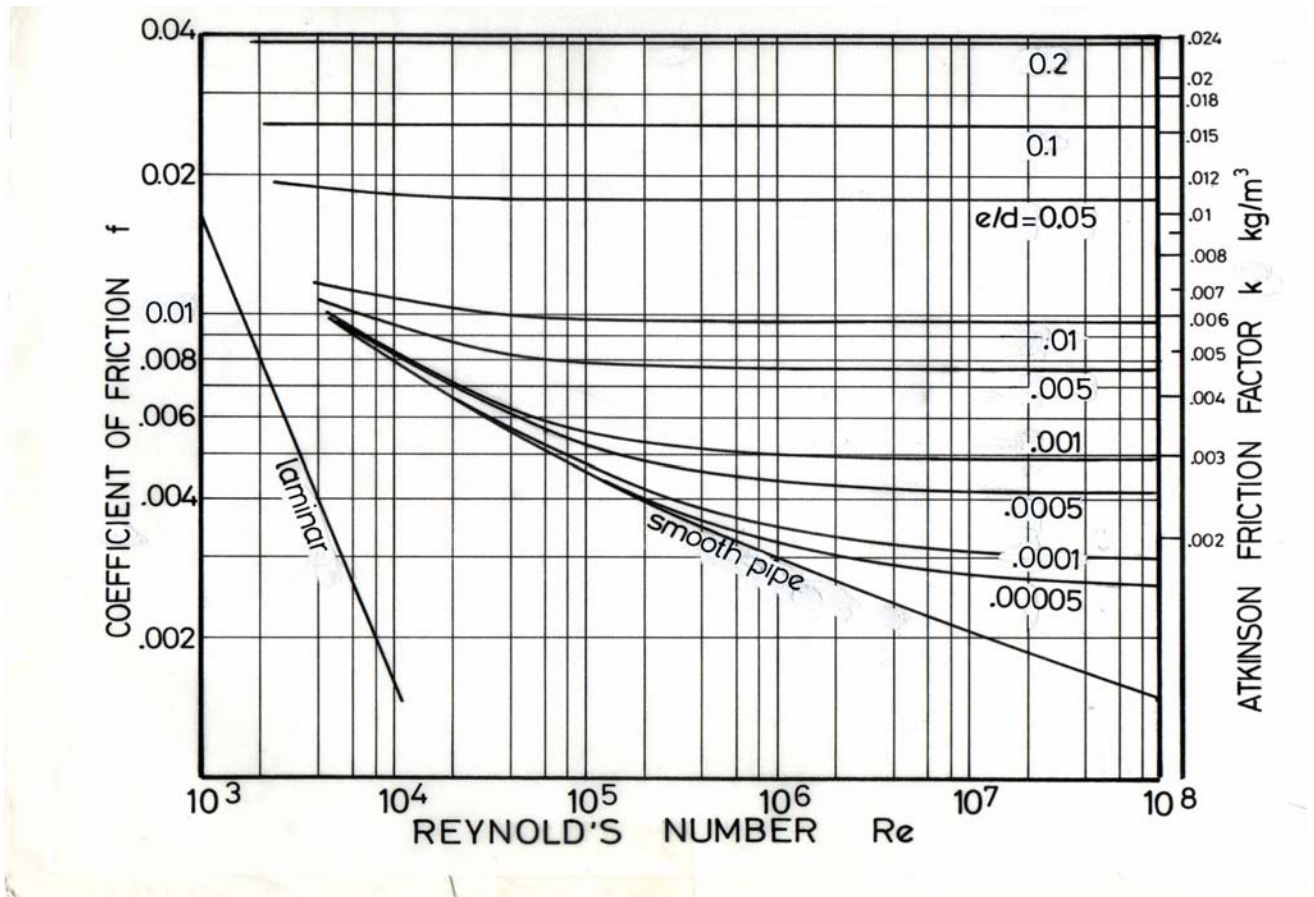


Figure 2.7 Type of chart developed by Moody.

Substituting  $R = d/2$  gives

$$f = 16 \frac{\mu}{\rho u d}$$

or  $f = \frac{16}{Re}$  dimensionless (2.52)

*Smooth pipe turbulent curve*

Perhaps the most widely accepted equation for the smooth pipe turbulent curve is that produced by both Nikuradse and the Hungarian engineer **Theodore Von Kármán (1881-1963)**.

$$\frac{1}{\sqrt{f}} = 4 \log_{10}(Re \sqrt{f}) - 0.4$$

This suffers from the disadvantage that  $f$  appears on both sides of the equation. **Paul R.H. Blasius (1873-1970)**, one of Prandtl's earlier students, suggested the approximation for Reynolds Numbers in the range 3 000 to  $10^5$ .

$$f = \frac{0.0791}{\text{Re}^{0.25}} \quad (2.54)$$

while a better fit to the smooth pipe curve for Reynolds Numbers between 20 000 to  $10^7$  is given as

$$f = \frac{0.046}{\text{Re}^{0.2}}$$

### Rough pipes

When fully developed rough pipe turbulence has been established, the viscous forces are negligible compared with inertial forces. The latter are proportional to the shear stress at the walls (equation (2.41)). Hence, in this condition  $f$  becomes independent of Reynolds Number and varies only with  $e/d$ . A useful equation for this situation was suggested by Von Kármán.

$$f = \frac{1}{4(2\log_{10}(d/e) + 1.14)^2} \quad (2.55)$$

The most general of the  $f$  - Re relationships in common use is the **Colebrook White** equation. This has been expressed in a variety of ways, including

$$\frac{1}{\sqrt{4f}} = 1.74 - 2\log_{10}\left(2\frac{e}{d} + \frac{18.7}{\text{Re}\sqrt{4f}}\right) \quad (2.56)$$

and

$$\frac{1}{\sqrt{f}} = -4\log_{10}\left(\frac{e/d}{3.7} + \frac{1.255}{\text{Re}\sqrt{f}}\right) \quad (2.57)$$

Here again,  $f$ , appears on both sides making these equations awkward to use in practice. It was, in fact, this difficulty that led Moody into devising his chart.

The advantage of the Colebrook White equation is that it is applicable to both rough and smooth pipe flow and for the transitional region as well as fully developed turbulence. For hydraulically smooth pipes,  $e/d = 0$ , and the Colebrook White equation simplifies to the Nikuradse relationship of equation (2.53). On the other hand, for high Reynolds Numbers, the term involving Re in equation (2.57) may be ignored. The equation then simplifies to

$$f = \left[4\log_{10}\left(\frac{e/d}{3.7}\right)\right]^{-2} \quad (2.58)$$

This gives the same results as Von Kármán's rough pipe equation (2.55) for fully developed turbulence.

### Example

A vertical shaft is 400 m deep, 5 m diameter and has wall roughenings of height 5 mm. An airflow of  $150 \text{ m}^3/\text{s}$  passes at a mean density of  $1.2 \text{ kg}/\text{m}^3$ . Taking the viscosity of the air to be  $17.9 \times 10^{-6} \text{ N}\cdot\text{s}/\text{m}^2$  and ignoring changes in kinetic energy, determine:

- (i) the coefficient of friction,  $f$
- (ii) the turbulent resistance,  $R_t$  ( $\text{m}^{-4}$ )
- (iii) the frictional pressure drop  $p$  (Pa)
- (iv) the work done against friction,  $F_{12}$  (J/kg)
- (v) the barometric pressure at the shaft bottom if the shaft top pressure is 100 kPa.

**Solution**

For a 400 m deep shaft, we can assume incompressible flow (Section 2.1.1.)

$$\text{Cross-sectional area, } A = \frac{\pi \times 5^2}{4} = 19.635 \text{ m}^2$$

$$\text{Perimeter, } per = 5\pi = 15.708 \text{ m}$$

$$\text{Air velocity, } u = \frac{Q}{A} = \frac{150}{19.635} = 7.639 \text{ m/s}$$

In order to determine the regime of flow, we must first find the Reynolds Number

$$Re = \frac{\rho u d}{\mu} = \frac{1.2 \times 7.639 \times 5}{17.9 \times 10^{-6}} = 2.561 \times 10^6$$

(i) *Coefficient of friction,  $f$ :*

At this value of  $Re$ , the flow is fully turbulent (Section 2.3.4.). We may then use the Moody Chart to find the coefficient of friction,  $f$ . However, for this we need the equivalent roughness

$$\frac{e}{d} = \frac{5 \times 10^{-3}}{5} = 0.001$$

Hence at  $e/d = 0.001$  and  $Re = 2.561 \times 10^6$  on Figure 2.7 we can estimate  $f = 0.0049$ . (Iterating equation (2.57) gives  $f = 0.00494$ . As the friction coefficient is near constant at this Reynolds Number, we could use equation (2.55) to give  $f = 0.00490$  or equation (2.58) which gives  $f = 0.00491$ ).

(ii) *Turbulent resistance,  $R_t$ :* (equation (2.51))

$$R_t = \frac{f L per}{2A^3} = \frac{0.0049 \times 400 \times 15.708}{2(19.635)^3} = 0.002036 \text{ m}^{-4}$$

(iii) *Frictional pressure drop,  $p$ :* (equation (2.50))

$$p = R_t \rho Q^2 = 0.002036 \times 1.2 \times (150)^2 = 54.91 \text{ Pa}$$

(iv) *Work done against friction,  $F_{12}$ :* (equation (2.46))

$$F_{12} = \frac{p}{\rho} = \frac{54.91}{1.2} = 45.76 \frac{\text{J}}{\text{kg}}$$

(v) *Barometric pressure at shaft bottom,  $P_2$ :* This is obtained from Bernoulli's equation (2.47) with no change in kinetic energy.

$$(Z_1 - Z_2)g + \frac{P_1 - P_2}{\rho} = F_{12}$$

$$\begin{aligned} \text{giving } P_2 &= (Z_1 - Z_2)g\rho - F_{12} \rho + P_1 \\ &= (400 \times 9.81 \times 1.2) - 54.91 + 100\,000 \\ P_2 &= 104\,654 \text{ Pa} \quad \text{or} \quad 104.654 \text{ kPa} \end{aligned}$$

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