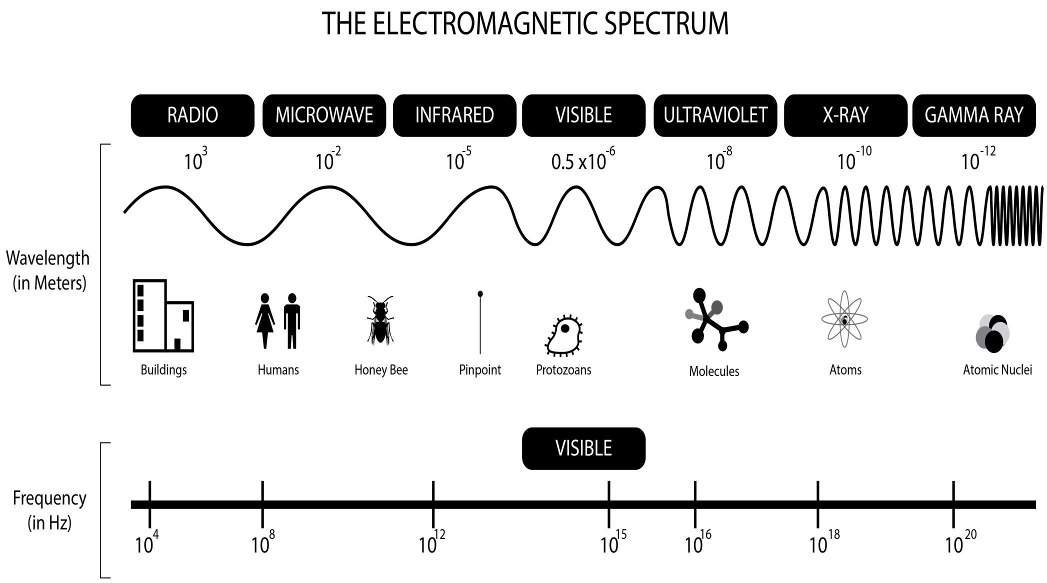
CARDIFF UNIVERSITY

School of Physics and Astronomy

MATHEMATICAL FORMULAE AND PHYSICAL CONSTANTS



Contents

1 ELEMENTARY ALGEBRA AND TRIGNOMETRY

1.1 Logarithms and exponentials

$$\ln x = \log_e x = \int_1^x \frac{dt}{t}, \quad x > 0, \quad e = 2.718281828\ldots$$
$$\log_a x = (\log_b x)(\log_a b)$$
$$\log_a b = \frac{1}{\log_b a}$$
$$a^x = \exp(x \ln a)$$

1.2 Trigonometric functions

$$\sec\theta = 1/\cos\theta \qquad \operatorname{cosec}\theta = 1/\sin\theta \qquad \cot\theta = 1/\tan\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\sin^2\theta + \cos^2\theta = \sec^2\theta - \tan^2\theta = \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

1.3 Compound formulae: sines, cosines and tangents

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\ \tan B}$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = -\cos(A+B) + \cos(A-B) \quad \text{note minus sign of first term}$$
$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad \text{(note minus signs)}$$

1.4 Double-angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

1.5 "Tan of half-angle" formulae

If $t = \tan\theta/2$, then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

1.6 Triangle sine and cosine formulae

If in a triangle A, B and C are the angles opposite sides of lengths a, b and c respectively,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

1.7 Hyperbolic functions

$$\cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}) \qquad \sinh\theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta})$$
$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$
$$\coth\theta = \frac{\cosh\theta}{\sinh\theta} = \frac{1}{\tanh\theta} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}$$
$$\mathrm{sech}\,\theta = \frac{1}{\cosh\theta} \qquad \mathrm{cosech}\,\theta = \frac{1}{\sinh\theta}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$
$$\mathrm{sech}^2\theta + \tanh^2\theta = 1$$
$$\coth^2\theta - \mathrm{cosech}^2\theta = 1$$

1.8 Stirling's approximation

$$\ln(n!) \approx n\ln n - n \qquad \text{for} \quad n \gg 1$$

An even closer approximation is

$$\ln n! \approx n\ln n - n + \tfrac{1}{2}\ln(2\pi n)$$

2 SERIES FORMULAE

2.1 Sums of progressions to n terms

(i) Arithmetic Progression (A.P.):

$$\sum_{m=0}^{n-1}(a+md) = a+(a+d)+(a+2d)+...+(a+(n-1)d)$$
$$= (n/2)\,[2a+(n-1)d] = (n/2)(\text{first term} + \text{last term})$$

(ii) Geometric Progression (G.P.):

$$S_n = \sum_{m=0}^{n-1}(ar^m) = a+ar+ar^2+.......+ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

For an infinite number of terms, if $|r|<1$

$$S_\infty = \frac{a}{1-r}$$

2.2 Binomial series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+....+\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+...$$

(Note that $0! = 1$).
If n is a positive integer, the series terminates.
Otherwise, the series converges so long as $|x|<1$.

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$$

2.3 Taylor's Theorem

(i) Single Variable:

The value of a function $f(x)$ given the value of the function and its relevant derivatives at $x=a$, is

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}(x-a)^n f^{(n)}(a) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$

If $a=0$, this series expansion is often called a Maclaurin Series.

(ii) Two Variables:

$$f(x,y) = f(x_0,y_0)+\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}\Delta x^2+2\frac{\partial^2 f}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 f}{\partial y^2}\Delta y^2\right]+\ \cdots$$

where $\Delta x = x-x_0, \Delta y = y-y_0$
and all the derivatives are evaluated at (x_0,y_0).

2.4 Power series in algebra and trigonometry

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x| < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x| < 1.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1.$$

3 DERIVATIVES AND INTEGRALS

3.1 Derivatives

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

Product rule:
Given $f(x) = u(x)v(x)$ then

$$\frac{df}{dx} = u\,\frac{dv}{dx} + \frac{du}{dx}\,v$$

Chain rule:
Given $u(x)$ and $f(u)$, then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

3.2 Partial differentiation

The total differential df of a function $f(x, y)$ is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

The chain rule for partial differentiation.
If $f(x, y)$ and x and y are functions of another variable, so that $x(u)$ and $y(u)$, then

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

3.3 Indefinite integrals

The constant of integration is omitted. Where the logarithm of a quantity is given, that quantity is taken as positive. a is a positive constant.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad \text{or} \quad -\cos^{-1}\frac{x}{a} \quad \text{(principal value)}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad \text{(principal value)}$$

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{2a}\ln\frac{a+x}{a-x} = \frac{1}{a}\tanh^{-1}\frac{x}{a} & (\text{if } |x| < a) \\ \frac{1}{2a}\ln\frac{x+a}{x-a} = \frac{1}{a}\coth^{-1}\frac{x}{a} & (\text{ if } |x| > a) \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} \quad \text{or} \quad \ln(x + \sqrt{a^2 + x^2})$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad \text{or} \quad \ln(x + \sqrt{x^2 - a^2})$$

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a} \quad \text{(principal value)}$$

$$\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2\ln(x + \sqrt{x^2 \pm a^2})$$

3.4 Indefinite integrals involving sines, cosines and exponentials

$$\int \tan x \; dx = -\ln(\cos x) = \ln(\sec x)$$

$$\int \cot x dx = \ln(\sin x)$$

$$\int \sec x \; dx = \ln(\sec x + \tan x) = \ln\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right) = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)$$

$$\int \operatorname{cosec} x \; dx = \ln(\operatorname{cosec} x - \cot x) = \ln\left(\tan\frac{x}{2}\right) = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\int \sin^{-1}\frac{x}{a} \; dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}$$

$$\int \cos^{-1}\frac{x}{a} \; dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

$$\int a^x \; dx = \frac{a^x}{\ln a}$$

$$\int x^n e^{-ax} \; dx = -e^{-ax}\left(\frac{x^n}{a} + \frac{nx^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} + \cdots + \frac{n!x}{a^n} + \frac{n!}{a^{n+1}}\right) \; (n \text{ a non-negative integer})$$

$$\int e^{ax}\sin bx \; dx = e^{ax}\frac{a\sin bx - b\cos bx}{a^2 + b^2}$$

$$\int e^{ax}\cos bx \; dx = e^{ax}\frac{a\cos bx + b\sin bx}{a^2 + b^2}$$

$$\int x\sin ax \; dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

$$\int \ln x \; dx = x\ln x - x$$

$$\int \sinh x \; dx = \cosh x \qquad \int \cosh x \; dx = \sinh x$$

$$\int \tanh x \; dx = \ln(\cosh x)$$

3.5 Integration by parts

If u and v are functions of x,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

3.6 Definite integrals involving sines and cosines

If m and n are positive integers

$$\int_0^{\pi} \sin mx \sin nx \; dx = \frac{\pi}{2}\delta_{mn} \qquad \int_0^{\pi} \cos mx \cos nx \; dx = \frac{\pi}{2}\delta_{mn}$$

where

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

is the Kronecker delta.

$$\int_{-\pi/2}^{\pi/2} \sin mx \cos nx \; dx = 0$$

3.7 Definite integrals involving exponentials

$$\int_0^\infty x\, e^{-\alpha x}\, dx = \frac{1}{\alpha^2} \qquad \int_0^\infty x^2\, e^{-\alpha x}\, dx = \frac{2}{\alpha^3}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \qquad \int_{-\infty}^\infty e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x e^{-\alpha x^2}\, dx = \frac{1}{2\alpha} \qquad \int_{-\infty}^\infty x e^{-\alpha x^2}\, dx = 0$$

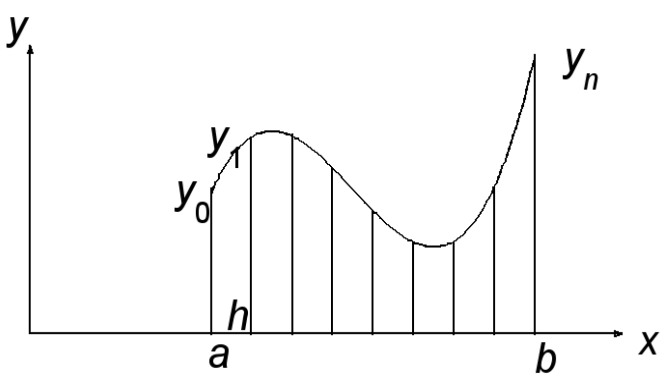
$$\int_0^\infty x^2 e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \qquad \int_{-\infty}^\infty x^2 e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_0^\infty x^3 e^{-\alpha x^2}\, dx = \frac{1}{2\alpha^2} \qquad \int_{-\infty}^\infty x^3 e^{-\alpha x^2}\, dx = 0$$

$$\int_0^\infty x^4 e^{-\alpha x^2}\, dx = \frac{3}{8}\sqrt{\frac{\pi}{\alpha^5}} \qquad \int_{-\infty}^\infty x^4 e^{-\alpha x^2}\, dx = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

$$\int_0^y e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y) \qquad \int_0^y e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,\mathrm{erf}(\sqrt{\alpha}\, y)$$

3.8 Numerical integration



The interval between a and b is divided into equal intervals h.
y has values $y_0,\ y_1,\ y_2\ \cdots\ y_n$.

3.8.1 Trapezoidal rule

$$\int_a^b y dx = h\left(\frac{y_0}{2} + y_1 + y_2 + ... + \frac{y_n}{2}\right)$$

3.8.2 Simpson rule

If there is an odd number of y-values (an even number of intervals),

$$\int_a^b y dx = \frac{h}{3}\left\{y_0 + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n\right\}.$$

3.9 Newton-Raphson Method for Finding the Root of an Equation

The root is found by successive approximations.
If the equation is $f(x) = 0$ and x_j is the jth approximation of the root

$$x_j = x_{j-1} - \frac{f(x_{j-1})}{f'(x_{j-1})} \text{ where } f' = \frac{df}{dx}$$

4 COMPLEX NUMBERS

4.1 Definitions etc.

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Complex conjugate of z is $z^* = x - iy = re^{-i\theta}$

Modulus or amplitude of z is $|z| = \sqrt{x^2 + y^2} = r = \sqrt{zz^*}$

Argument of z is $\arg z = \tan^{-1}\dfrac{y}{x} = \theta$

Real part of z is $\mathrm{Re}(z) = x = r\cos\theta = \dfrac{z + z^*}{2}$

Imaginary part of z is $\mathrm{Im}(z) = y = r\sin\theta = \dfrac{z - z^*}{2i}$

4.2 De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

4.3 Formulae involving $e^{i\theta}$ etc.

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$i\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = \frac{e^{2i\theta} - 1}{e^{2i\theta} + 1} = \frac{1 - e^{-2i\theta}}{1 + e^{-2i\theta}}$$

5 SPECIAL FUNCTIONS

5.1 Spherical harmonics

A general equation which gives the 'right' phase factors (as used in quantum mechanics) is

$$Y_l^m = \left\{ \frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!} \right\}^{1/2} \frac{1}{2^l l!} e^{im\phi} (-\sin\theta)^m \left\{ \frac{d}{d(\cos\theta)} \right\}^{l+m} (\cos^2\theta - 1)^l$$

which can also be expressed

$$Y_l^m(\theta, \phi) = P_l^m(\cos\theta) \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

where $P_l^m(\cos\theta)$ is a normalised associated Legendre polynomial.

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(2\cos^2\theta - \sin^2\theta\right)$$

$$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta e^{\pm i\phi}$$

$$Y_2^{\pm 2} = \mp \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}$$

5.2 The gamma function

This is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt = \int_0^1 \left(\ln \frac{1}{t} \right)^{n-1} dt$$

where $n > 0$ (n can be an integer or a non-integer)

$$\Gamma(n+1) = n\Gamma(n)$$

If n is an integer ≥ 0, $\Gamma(n+1) = n!$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ for } n \text{ a non-integer}$$

5.3 Bessel functions

$$J_n(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{\Gamma(\lambda+1)\Gamma(\lambda+n+1)} \left(\frac{x}{2}\right)^{n+2\lambda}$$

$$\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x) \qquad \frac{d}{dx}\left\{x^{n} J_n(x)\right\} = x^{n} J_{n-1}(x)$$

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(ix\cos\phi\right)\, d\phi \qquad zJ_1(z) = \frac{1}{2\pi}\int_0^{z} xJ_0(x)dx$$

6 DETERMINANTS AND MATRICES

6.1 Definition of a determinant

$$|A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} & ... & A_{1n} \\ A_{21} & A_{22} & A_{23} & ... & A_{2n} \\ A_{31} & A_{32} & A_{33} & ... & A_{3n} \\ ... & ... & ... & ... & ... \\ A_{n1} & A_{n2} & A_{n3} & ... & A_{nn} \end{vmatrix}$$
$$= \sum_j (-1)^{k+j} A_{kj} M_{kj} = \sum_i (-1)^{k+i} A_{ik} M_{ik}$$

where M_{ij} is the minor of A_{ij} in A, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column passing through A_{ij}. The number $(-1)^{i+j} M_{ij}$ is called the cofactor of A_{ij}. By repeating this process the determinant of A can be found.

Properties of Determinants

- $|A|$ is unaltered if rows and columns are interchanged.
- $|A|$ is unaltered if any row (or constant any row) is added to or subtracted from another row.
- $|A|$ is unaltered if any column (or constant any column) is added to or subtracted from another column.
- $|A| = 0$ if any row or column is zero.
- $|A| = 0$ if the matrix has two identical rows or columns.
- If all the elements of any two rows, or any two columns, are interchanged, $|A|$ changes sign.
- If all the elements of any row or column are multiplied by a constant λ, $|A|$ is multiplied by λ.
- $|AB| = |A|\ |B|$ the determinant of the product is the product of the determinants.
- If a $n \times n$ matrix is nultiplied by a scalar a, then its determinant is increaseed by factor a^n.

6.2 Consistency of n simultaneous equations with n variables and no constants.

If the equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + ... + A_{1n}x_n &= 0 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + ... + A_{2n}x_n &= 0 \\ &\cdots \\ A_{n1}x_1 + A_{n2}x_2 + A_{n3}x_3 + ... + A_{nn}x_n &= 0 \end{aligned}$$

are consistent, then

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & ... & A_{1n} \\ A_{21} & A_{22} & A_{23} & ... & A_{2n} \\ A_{31} & A_{32} & A_{33} & ... & A_{3n} \\ ... & ... & ... & ... & ... \\ A_{n1} & A_{n2} & A_{n3} & ... & A_{nn} \end{vmatrix} = 0$$

6.3 Solutions of n simultaneous equations with n variables and with constants.

The equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_{2+}A_{13}x_3 + ... + A_{1n}x_n + C_1 = 0 \\ A_{21}x_1 + A_{22}x_{2+}A_{23}x_3 + ... + A_{2n}x_n + C_2 = 0 \\ .... \\ A_{n1}x_1 + A_{n2}x_{2+}A_{n3}x_3 + ... + A_{nn}x_n + C_n = 0 \end{aligned}$$

have a solution

$$\frac{x_1}{\begin{vmatrix} A_{12} & A_{13} & ... & C_1 \\ A_{22} & A_{23} & ... & C_2 \\ ... & ... & ..... & \\ A_{n2} & A_{n3} & ... & C_n \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} A_{11} & A_{13} & ... & C_1 \\ A_{21} & A_{23} & ... & C_2 \\ ... & .... & ... & .... \\ A_{n1} & A_{n3} & ... & C_n \end{vmatrix}} = ... = \frac{(-1)^n}{\begin{vmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ ... & ... & ... & ... \\ A_{n1} & A_{n2} & ... & A_{nn} \end{vmatrix}}$$

6.4 Matrices: basic equations

Linear equations like:

$$\begin{aligned} y_1 = A_{11}x_1 + A_{12}x_{2+}A_{13}x_3 + ... + A_{1n}x_n \\ y_2 = A_{21}x_1 + A_{22}x_{2+}A_{23}x_3 + ... + A_{2n}x_n \\ .... \\ y_n = A_{n1}x_1 + A_{n2}x_{2+}A_{n3}x_3 + ... + A_{nn}x_n \end{aligned}$$

can be expressed in matrix form as:

$$\begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & ... & A_{1n} \\ A_{21} & A_{22} & A_{23} & ... & A_{2n} \\ A_{31} & A_{32} & A_{33} & ... & A_{3n} \\ ... & ... & ... & ... & ... \\ A_{n1} & A_{n2} & A_{n3} & ... & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

x and y are column vectors, A is a n by n matrix.
Usually the matrices to be considered are square.

6.5 Rules of matrix algebra

Given matrices A and B,

$$(A+B)_{ij} = A_{ij} + B_{ij}$$
$$(\lambda A)_{ij} = \lambda A_{ij}$$
$$(AB)_{ij} = \sum_l A_{il} B_{lj}$$

You must remember that matrix algebra is not commutative in general; in other words we generally have:

$$AB \neq BA.$$

6.6 Trace of a square matrix

$$\mathrm{Tr}(A) = \sum_i A_{ii}$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

6.7 Transpose of matrix

The transpose of a matrix A is written as A^T and is obtained by interchanging rows and columns.

$$A^T_{ij} = A_{ji}$$

The complex conjugate transpose of a matrix A is denoted by $A^\dagger$.
It is also called the Hermitian conjugate.

$$A^\dagger_{ij} = A^*_{ji}$$

6.8 Inverse of a matrix

A^{-1} is the inverse of the matrix A if $AA^{-1} = A^{-1}A = I$ where I is the unit matrix.

An explicit expression for A^{-1} is:

$$(A^{-1})_{ij} = \frac{(-1)^{j+i} M_{ji}}{|A|}$$

where $(-1)^{i+j} M_{ij}$ is called the cofactor of A_{ij}.

$$(ABC..X)^{-1} = X^{-1}...B^{-1}A^{-1}$$

6.9 Special matrices

If a square matrix is equal to its transpose, ie, $A = A^T$, it is said to be *symmetric*. If $A = -A^T$, it is anti-symmetric. Any real, square matrix can be written as the sum of a symmetric and an anti-symmetric matrix.

An orthogonal matrix is one such that $A^T = A^{-1}$, ie, its inverse is its transpose. This implies that A is non-singular and as $A^T A = I$, its determinant is ± 1.

A Hermitian matrix satisfies the relation $A = A^\dagger$. Any complex n by n matrix can be written as a sum of a Hermitian and an anti-Hermitian matrix.

Unitary matrices have the special property that $A^\dagger = A^{-1}$. Finally, normal matrices are ones that commute with their Hermitian conjugates.

6.10 Eigenvalues and eigenvectors of a square matrix

For a square $n \times n$ matrix A there are n eigenvalues λ with associated eigenvalues x which satisfy:

$$Ax = \lambda x$$

x is a vector, which when operated on by A is simply scaled. The eigenvalues are determined by finding the non-trivial solutions of

$$|A - \lambda I| = 0.$$

The left-hand side is a polynomial of order n, so this equation – the characteristic equation – has n roots giving the n eigenvalues (which are not necessarily distinct).

6.11 Similarity transform

The operation on a matrix A to produce a matrix $B = Q^{-1}AQ$ is called a similarity transformation. Under a similarity transform,

$$\mathrm{Tr}B = \mathrm{Tr}A$$
$$|B| = |A|$$

6.12 Diagonalisation of a matrix A with different eigenvalues

If Q is a matrix whose columns are the eigenvectors of a matrix A, then $Q^{-1}AQ$ is diagonal and has elements which are the eigenvalues of A.

6.13 Representation of a rotation by a matrix R

A real orthogonal 3×3 matrix R with determinant $= 1$ represents a rotation in 3-dimensional space.

The angle of implied rotation θ is given by $\mathrm{Tr}R = 1 + 2\cos\theta$.
The axis of implied rotation is a column vector u which is the solution of $Ru = u$.

7 VECTORS

Throughout, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to Ox, Oy and Oz respectively.

7.1 Definition of the scalar (or dot) product of two vectors

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ (with $0 \le \theta \le \pi$) where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

7.2 Properties of the scalar product

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1.$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0.$

If $\mathbf{a}\cdot\mathbf{b} = 0$, and the moduli of $\mathbf{a}$ and $\mathbf{b}$ are non-zero, then $\mathbf{a}$ is perpendicular (orthogonal) to $\mathbf{b}$.

If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$,

$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z$

$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2$

$\mathbf{a} = (\mathbf{a}\cdot\mathbf{i})\mathbf{i} + (\mathbf{a}\cdot\mathbf{j})\mathbf{j} + (\mathbf{a}\cdot\mathbf{k})\mathbf{k}$

7.3 Definition of the vector (or cross) product of two vectors

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ (with $0 \le \theta \le \pi$) where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ and such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ form a right-handed system.

7.4 Properties of the vector product

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$

$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta$ is the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, having an angle θ between the adjacent sides.

$\mathbf{a} \times \mathbf{b} = 0$ and the moduli of $\mathbf{a}$ and $\mathbf{b}$ are both non-zero, then $\mathbf{a}$ and $\mathbf{b}$ are parallel or anti-parallel.

7.5 Scalar triple product

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ which also equals $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}]$.

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

The volume of a parallelopiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

7.6 Vector triple product

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(Note that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, ie the vector product is not associative.)

7.7 The del operator ∇

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

7.8 The gradient of a scalar function $\phi(x, y, z)$

$$\text{grad}\ \phi = \nabla\phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}.$$

$\nabla\phi$ gives the magnitude and direction of the maximum (spatial) rate of change of ϕ.

7.9 The divergence of a vector function $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$

$$\text{div}\ \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{div}\ \mathbf{F} = \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{F}}{\partial z}$$

7.10 The curl of a vector function $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

$$\text{curl } \mathbf{F} = \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

7.11 Compound operations

$$\text{div grad } \phi = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

(∇^2 is the Laplacian).

$$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = [\nabla\ \nabla\ \mathbf{F}] = 0.$$

$$\text{curl grad } \phi = \nabla \times (\nabla \phi) = 0$$

$$\text{curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} = \text{grad div} \mathbf{F} - \nabla^2 \mathbf{F}$$

where

$$\nabla^2 \mathbf{F} = \frac{\partial^2 \mathbf{F}}{\partial x^2} + \frac{\partial^2 \mathbf{F}}{\partial y^2} + \frac{\partial^2 \mathbf{F}}{\partial z^2}$$

These equations can 'deduced' by regarding ∇ as a vector

7.12 Operations on sums and products

$\nabla(\phi + \psi) = \nabla \phi + \nabla \psi$

$\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$

$\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$

$\nabla(\phi\ \psi) = \phi\ \nabla \psi + \psi\ \nabla \phi$

$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} + (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{b})$

$\nabla \cdot (\phi\ \mathbf{a}) = \phi\ \nabla \cdot \mathbf{a} + (\nabla \phi)\ \mathbf{a}$

$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$

$\nabla \times (\phi\ \mathbf{a}) = \phi\ \nabla \times \mathbf{a} + (\nabla \phi) \times \mathbf{a}$

$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\ (\nabla \cdot \mathbf{b}) - \mathbf{b}\ (\nabla \cdot \mathbf{a})$

7.13 Gauss's (divergence) theorem

Let V be a region, completely bounded by a closed surface S with outward drawn unit normal $\mathbf{n}$. Then, for a well-behaved vector function $\mathbf{F}(x, y, z)$

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

where $d\mathbf{S} = \hat{\mathbf{n}} dS$ and dS is an element of the surface.

7.14 Stokes's theorem

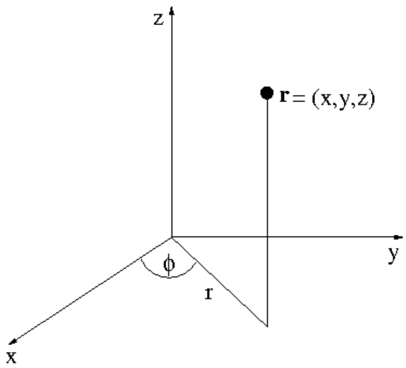
Let S be a surface with unit normal $\mathbf{n}$, bounded by a closed curve C. Then, for a "well-behaved" vector function $\mathbf{F}(x, y, z)$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where $d\mathbf{r} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$ and $d\mathbf{S} = \hat{\mathbf{n}} dS$.

8 CYLINDRICAL AND SPHERICAL POLAR COORDINATES

8.1 Cylindrical coordinates

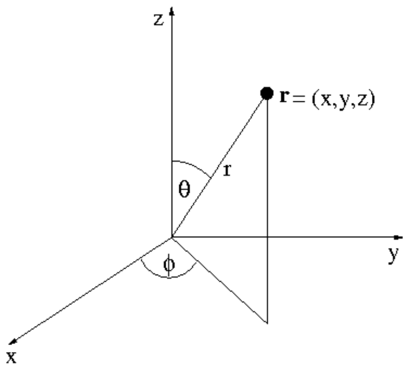


$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$
where $r \geq 0, \quad 0 \leq \phi \leq 2\pi, \quad -\infty \leq z \leq \infty$

The inverse relations are $r = \sqrt{(x^2+y^2)}, \quad \phi = \tan^{-1}(y/x), \quad z = z$

Note: The polar coordinates in two dimensions are the same as those for the cylindrical systems with $z = 0$.

8.2 Spherical polar coordinates



$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$
where $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$

The inverse relations are $r = \sqrt{(x^2+y^2+z^2)}, \quad \phi = \tan^{-1}(y/x), \quad \theta = \cos^{-1}(z/r)$

8.3 ∇^2 in cylindrical polar coordinates $(r, \ \phi, \ z)$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

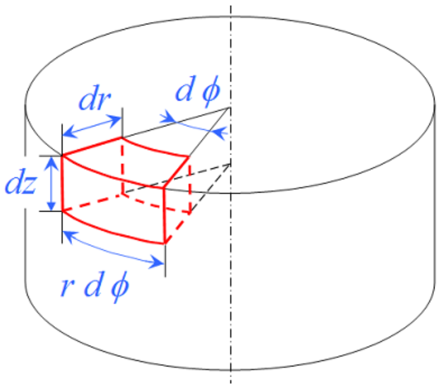
8.4 ∇^2 in spherical polar coordinates $(r,\ \theta,\ \phi)$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

8.5 Line area and volume elements

The line element $ds = |d\mathbf{r}|$ and volume element dV in cylindrical and spherical polar coordinates are

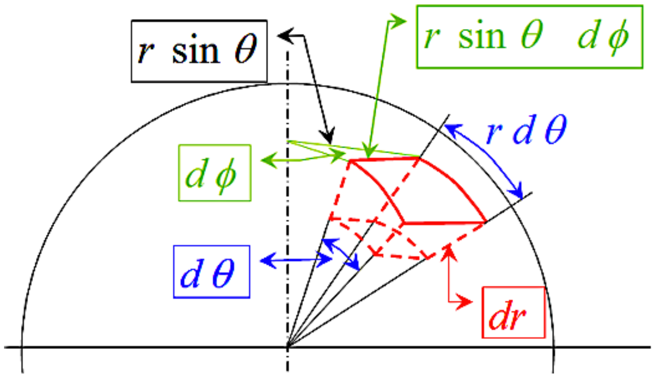
8.5.1 Cylindrical



line element: $ds = \sqrt{(dr)^2 + r^2(d\phi)^2 + (dz)^2}$

volume element: $dV = r\ dr\ d\phi\ dz$

8.5.2 Spherical



line element: $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2}$

volume element: $dV = r^2\sin\theta\ dr\ d\theta\ d\phi$

9 FOURIER SERIES AND TRANSFORMS

A function $f(t)$ which is periodic in t with period T satisfies $f(t+T) = f(t)$.
It can be expanded in an infinite series of exponentials or of sines and cosines.

9.1 Fourier Series

(a) Complex expansion

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{-i\omega_n t},$$
$$\text{where } \omega_n = \frac{2\pi n}{T} \quad (n = 0, \pm 1, \pm 2.....\infty)$$
$$\text{and } F_n = \frac{1}{T}\int_T e^{i\omega_n t} f(t)\ dt$$

Here, the integral is taken over one complete period
(e.g. from 0 to T or from $-T/2$ to $T/2$).
Note the orthogonality relation

$$\frac{1}{T}\int_T e^{-i\omega_n t} e^{i\omega_m t} dt = \delta_{nm}$$

where δ_{nm} is the Kronecker delta.

(b) Real expansion
By separating the above result into real and imaginary parts, for real $f(t)$,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$
$$\text{where } a_n = \frac{2}{T}\int_T f(t)\cos\omega_n t\ dt$$
$$b_n = \frac{2}{T}\int_T f(t)\sin\omega_n t\ dt$$
$$\text{and } a_0 = \frac{1}{T}\int_T f(t)\ dt$$

9.2 Fourier transforms

By letting $T \to \infty$ and replacing sums by integrals, one finds that (suitably restricted) functions $f(t)$ can be expressed as a 'superposition' of exponential functions.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$
$$\text{where } F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

The functions $f(t)$ and $F(\omega)$ are 'Fourier mates', and the results can be viewed as a consequence of the fact that

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega' t}\ dt = 2\pi\delta(\omega - \omega')$$

9.3 Shift theorems in Fourier transforms

(a) If $f(t)$ is replaced by $f(t-a)$ (ie. a translation in time by a),

$$F(\omega) \text{ is replaced by } F(\omega)e^{i\omega a}$$

(b) If $f(t)$ is multiplied by $e^{i\omega' t}$

$$F(\omega) \text{ is `translated' into } F(\omega+\omega')$$

9.4 Convolutions

If $f(t)$ and $g(t)$ are two functions, their convolution (with respect to t) $h(t)$, is defined by

$$\begin{aligned} h(t) = f(t) * g(t) &= \int_{-\infty}^{\infty} f(u)g(t-u)du \\ &= \int_{-\infty}^{\infty} f(t-u)g(u)du \end{aligned}$$

The Fourier transform of $h(t)$ is $H(\omega) = F(\omega)G(\omega)$, where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$.

Similarly, $H(\omega) = F(\omega) * G(\omega)$ is the Fourier transform of $h(t) = f(t)g(t)$

9.5 Some common Fourier mates

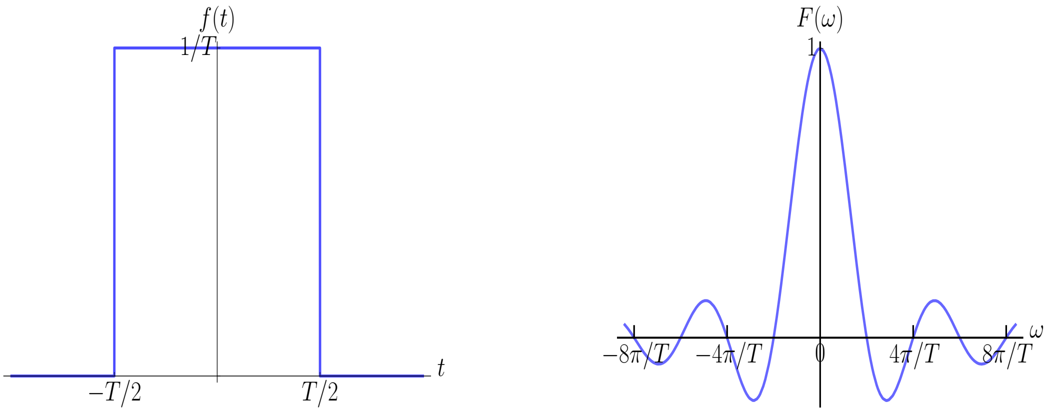
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

$$f(t) = e^{-i\omega_0 t} \qquad F(\omega) = 2\pi\delta(\omega-\omega_0)$$

$$f(t) = \sin\omega_0 t \qquad F(\omega) = \frac{\pi}{i}\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

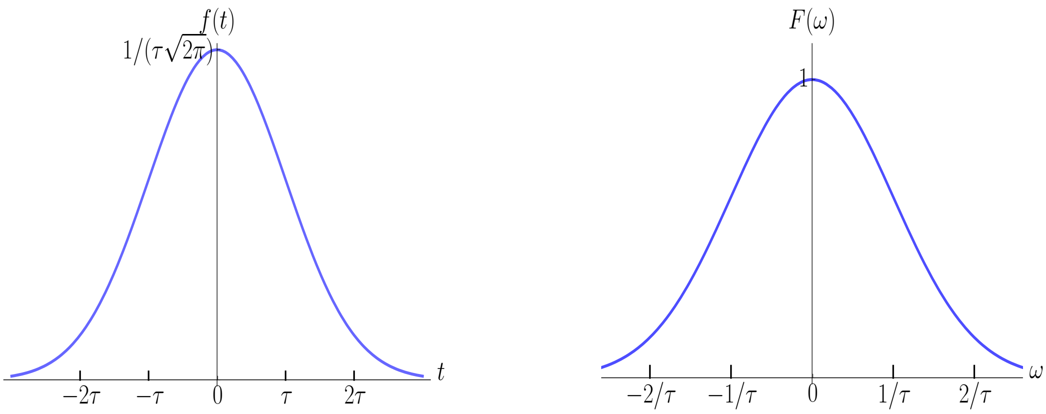
$$f(t) = \cos\omega_0 t \qquad F(\omega) = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

$$f(t) = \delta(t-t_0) \qquad F(\omega) = e^{i\omega t_0}$$



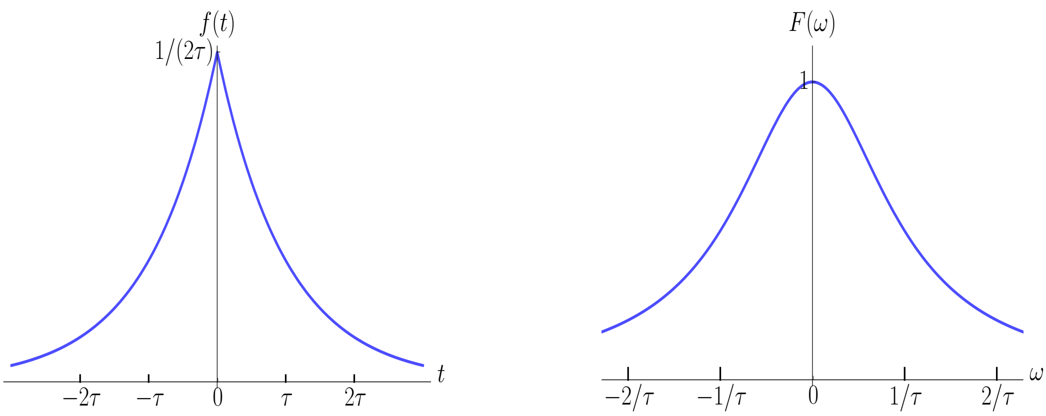
(a) $f(t)\begin{cases} = \frac{1}{T} & (|t| < T/2) \\ = 0 & (|t| \geq T/2) \end{cases}$ (b) $F(\omega) = \frac{\sin(\omega T/2)}{\omega T/2}$

Figure 9.1: The slit function



(a) $f(t) = \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{t^2}{2\tau^2}\right)$ (b) $F(\omega) = \exp\left(-\omega^2\tau^2/2\right)$

Figure 9.2: The Gaussian function



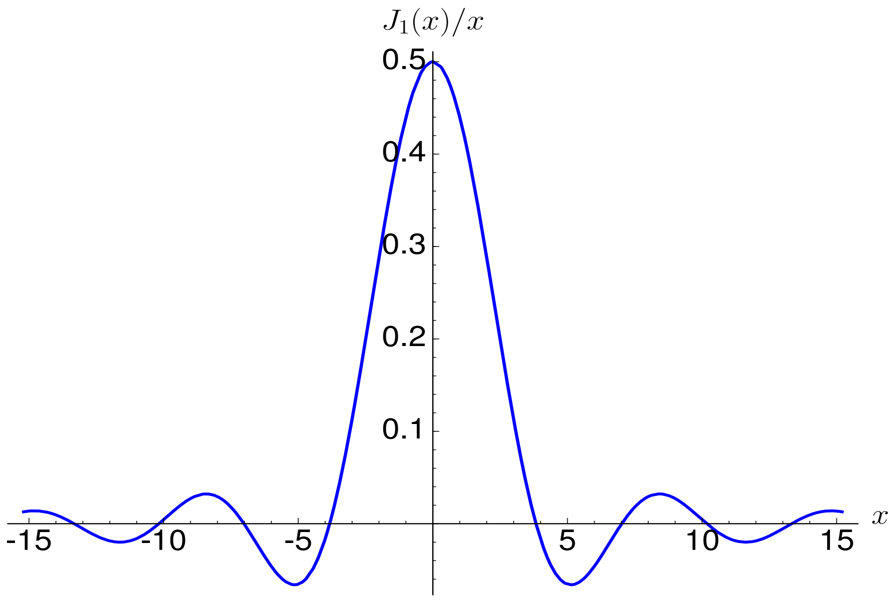
(a) $f(t) = \frac{1}{2\tau} \exp\left(-\frac{|t|}{\tau}\right)$ (b) $F(\nu) = 1/(1 + \omega^2\tau^2)$

Figure 9.3: The exponential function

9.6 Diffraction at a circular aperture

The integral of $e^{2\pi i S r \cos\phi}$ over the area of a circle is

$$\int_{\phi=0}^{2\pi}\int_{r=0}^{a} e^{2\pi i S r \cos\phi} r dr d\phi = \frac{aJ_1(2\pi Sa)}{S}$$



$$\begin{aligned}\frac{J_1(x)}{x} &= \text{ when } |x| = 1.22\pi(= 3.3833), 2.233\pi(= 7.016), 3.238\pi(= 10.174), ...\\ &= \text{ max. when } |x| = 0, 2.679\pi(= 8.417), ...\\ &= \text{ min. when } |x| = 1.635\pi(= 5.136), 3.699\pi(= 11.620), ...\end{aligned}$$

10 LAPLACE TRANSFORMS

10.1 Definition and table of transforms

The Laplace transform $F(s)$ of $f(t)$ is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Function $f(t)$	Laplace transform $F(x)$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$e^{at}f(t)$	$F(s-a)$
$f(t) = \begin{cases} (t-a) & t > a \\ 0 & t < a \end{cases}$	$e^{-as}F(s)$
$\frac{df(t)}{dt}$	$sF(s) - f(0)$
$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0) - \frac{df}{dt}(0)$
$\int_0^t f(u)du$	$\frac{F(s)}{s}$
$\int_0^t \frac{(t-u)^{n-1}}{(n-1)!}f(u)du$	$\frac{F(s)}{s^n}$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$
$t^n f(t) \quad (n = 0, 1, 2, 3, \text{ etc})$	$(-1)^n \frac{d^sF}{ds^n}(s)$
$t^{-1}f(t)$	$\int_s^\infty F(u)du$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\delta(t)$	1
$\delta(t-T)$	e^{-sT}

11 PROBABILITY, STATISTICS AND DATA INTERPRETATION

11.1 Mean and variance

(a) Discretely distributed random variables (variates)

For a variate x which can take on the N values, x_i ($i = 1,\ .... N$) with respective probabilities f_i,

$$\sum_{i=1}^{n} f_i = 1$$

$$\text{Mean of } x \text{ is } \overline{x} = \sum_{i=1}^{n} f_i x_i$$

$$\text{Variance of } x \text{ is } \sigma^2 = \text{Var}(x) = \overline{(x-\overline{x})^2} = \overline{x^2} - \overline{x}^2 = \sum_{i=1}^{n} f_i x_i^2 - \overline{x}^2$$

where σ is the standard deviation.

(b) Continuously distributed variates

For a continuously distributed variate x, with probability density function $f(x)$, normalised as

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\overline{x} = \int_{-\infty}^{\infty} x f(x)dx$$

$$\text{Var}(x) = \int_{-\infty}^{\infty} (x-\overline{x})^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \overline{x}^2 = \overline{x^2} - \overline{x}^2$$

(c) Scale factor and change of origin

If $y = k(x-a)$, where k and a are constants, then

$$\overline{y} = k(\overline{x} - a) \text{ and}$$
$$\text{Var}(y) = k^2 \text{Var}(x)$$

11.2 Binomial distribution

In n identical independent trials with probability, p, of success (and $q = 1 - p$ of failure) at each trial, the probability of exactly r successes is

$$^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

Mean number of successes, $\overline{r} = np$.
Variance of number of successes $\mathrm{Var}(r) = npq$.
Variance of proportion successes $= \mathrm{Var}\left(\frac{r}{n}\right) = \frac{pq}{n}$.

11.3 Poisson distribution

For a non-negative integer variate x (ie. $x = 0, 1, 2, ...r, ...$)

Probability that $x = r$ is

$$P_r = \frac{\mu^r e^{-\mu}}{r!}$$

where μ is a constant.

$$\overline{x} = \mu$$
$$\mathrm{Var}(x) = \mu$$
$$\text{If } \mu \gg 1, \quad P_r \to \frac{1}{\sqrt{2\pi\mu}} \exp\left(\frac{-(r-\mu)^2}{2\mu}\right)$$

11.4 Normal (Gaussian) distribution

If the continuous variate x is distributed normally with mean μ and standard deviation σ, then its probability density function $f(x)$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

The standard normal variate $X = (x - \mu)/\sigma$ has mean zero, variance unity and a probability density function $\phi(X)$ given by

$$\phi(X) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X^2}{2}\right)$$

$$\text{Probability } \; (-\infty \leq X \leq u) = \int_{-\infty}^{u} \phi(X) dX$$

$$\text{Error function } \; \mathrm{erf}\, u = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-t^2) dt$$

In particular, $\mathrm{erf}(\infty) = 1$.

11.5 Statistics

Suppose n statistically independent measurements, x_1, x_2, x_3, ..., x_i, ... x_n are made of a certain quantity which are 'samplings' of a variate x with a variance σ^2. The sample variance is

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

(where μ is the mean of the x_i).

$$\text{Mean of the sample variance } S^2 = \left(\frac{n-1}{n}\right)\sigma^2$$

$$\text{The standard error of the mean } s = \frac{S}{\sqrt{n}}$$

11.6 Data interpretation: least squares fitting of a straight line

The best straight line $y = ax + b$ through n points $(x_i,\ y_i)$ (where $i = 1,\ 2,\ \ldots\ n$) has for the best estimate of slope and intercept

$$a = \frac{n\sum xy - \sum x \sum y}{\Delta}\ , \qquad b = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}$$

$$\text{where } \Delta = n\sum x^2 - \left(\sum x\right)^2$$

The standard errors are

$$S(a) = \frac{n\sigma(y)}{\sqrt{(n-2)\Delta}}\ , \qquad S(b) = \sigma(y)\sqrt{\frac{n\sum x^2}{(n-2)\Delta}}$$

$$\text{where } n^2\sigma^2(y) = n\sum y^2 - \left(\sum y\right)^2 - \frac{(n\sum xy - \sum x \sum y)^2}{\Delta.}$$

In all of the above,

$$\sum A = \sum_{i=1}^{n} A_i$$

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12 SOME PHYSICS FORMULAE

12.1 Newton's laws and conservation of energy and momentum

The frictional force $f = \mu F_N$ where F_N is the normal force.

The centripetal force is $mv^2/r = m\omega^2 r$.

The work done by a force: $\int F dx$ or force $\times$ dist for a constant force.

The mechanical energy $= K + U$ is conserved.

Conservation of momentum: $(\sum_i m_i v_i)_{init} = (\sum_i m_i v_i)_{final}$.

For rocket motion: $v_f - v_i = v_{rel} \ln{(m_i/m_f)}$.

12.2 Rotational motion and angular momentum

Angular speed $\omega = v/r$.

The rotational inertia is $I = \sum m_i r_i^2$.
For mass M rotating about an axis distance R away, $I = MR^2$.

Newton's second (angular) law is net torque, $\tau_{net} = I\alpha$ and $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

For a rolling ball, $K = K_{rot} + K_{trans} = 0.5I\omega^2 + 0.5mv_{com}^2$.
For a wheel (radius R) rolling smoothly: $v_{com} = \omega R$.

Angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$.

Angular momentum $L = I\omega$ is conserved.

12.3 Gravitation and Planetary motion

Gravitational force: $\mathbf{F} = GmM\mathbf{r}/r^3$

Gravitational law in differential form: $\nabla \cdot \mathbf{g} = -4\pi G\rho$

Gravitational potential energy is $U = -GMm/r$.

Escape speed : $v = \sqrt{2GM/R}$.

Kepler's second law: $\dot{A} = L/2M =$ constant.

Kepler's third law: $T^2 = (4\pi^2/GM)r^3$.

12.4 Oscillations - Simple harmonic motion, Springs

Spring restoring force: $F = -kx$.

Displacement : $x = x_m \cos{(\omega t + \phi)}$, where $\omega^2 = k/m$.

Period $T = 2\pi\sqrt{m/k} = 2\pi/\omega$

Energy: $K = m\dot{x}^2/2$, $U = kx^2/2$.

12.5 Thermodynamics, gases and fluids

Change in heat energy is $\Delta Q = mc\Delta T$.

Heat of transformation $\Delta Q = Lm$.

Ideal gas equation of state: $pV = nRT$.

1st law of thermodynamics : $dE_{\text{int}} = dQ - dW$.
Also $\Delta E_{\text{int}} = \Delta E_{\text{int,f}} - \Delta E_{\text{int,i}} = Q - W$.
For cyclical processes: $\Delta E_{\text{int}} = 0, \quad Q = W$.

Work done: $W = \int dW = \int pdV$
For an isothermal process $W = nRT \ln V_f/V_i$.

root mean square velocity is $v_{rms} = \sqrt{(3RT/M)}$ where M is the molecular mass.

Maxwell-Boltzmann distribution

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp -(\frac{mv^2}{2k_B T})$$

where v is the velocity and m the mass of the each particle.

Bernoulli's equation for the flow of an ideal fluid

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = \text{constant}$$

12.6 Waves

Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

The speed of the wave $v = f\lambda$ where λ is the wavelength and f is the frequency.
The angular frequency $\omega = 2\pi f$.

Energy of one photon: $E = hf = hc/\lambda$

Photoelectric effect equation: $eV_0 = hf - \phi$.
ϕ is the workfunction of the surface and V_0 is the applied voltage.

Speed of electromagnetic waves: $c = 1/\sqrt{\epsilon_0 \mu_0}$

Index of refraction: $n = c/v$

Snell's law of refraction between media a and b: $n_a \sin\theta_a = n_b \sin\theta_b$

Constructive interference: $d \sin\theta = m\lambda$

Destructive interference: $d \sin\theta = (m + 1/2)\lambda$

Transverse wave in a string of tension T and mass/length μ: $v = \sqrt{T/\mu}$

Longitudinal wave in a fluid of density ρ and bulk modulus B: $v = \sqrt{B/\rho}$

12.7 Electricity and Magnetism

Coulomb's Law

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

Electric field

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Potential difference

$$V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{r}$$

12.8 Maxwell's equations

	Integral form	Differential form
Gauss' law for electricity	$\oint\!\!\oint \mathbf{E}\cdot d\mathbf{S} = \frac{\sum q_i}{\epsilon_0} = \frac{Q}{\epsilon_0}$	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss' law for magnetism	$\oint\!\!\oint \mathbf{B}\cdot d\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$
Faraday's law	$\oint \mathbf{E}\cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
Ampere-Maxwell law	$\oint \mathbf{B}\cdot d\mathbf{r} = \frac{1}{c^2}\frac{d\Phi_E}{dt} + \mu_0 I$	$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{j}$

12.9 Special Relativity

Lorentz contraction : $L = L_0/\gamma$ where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$.

time dilation : $\Delta t = \gamma\Delta t_0$.

Lorentz transformation eqns:
$x' = \gamma(x - vt)$, $t' = \gamma(t - vx/c^2)$, $y' = y$ and $z' = z$.

Relativistic momentum $p = \gamma mv$.

Relativistic energy $E = mc^2 + K = \gamma mc^2$.

Relativistic energy equation: $E^2 = (pc)^2 + (mc^2)^2$.

12.10 Photons, atoms and quantum mechanics

Photons: $E = hf$, $p = h/\lambda$.

Photoelectric equation: $hf = K_{max} + \Phi$, where Φ is the work function.

Compton scattering: $\Delta\lambda = h(1 - \cos\phi)/mc$.

Heisenberg uncertainty principle: $\Delta p_x \Delta x \geq \hbar/2$

A particle with momentum p has de Broglie wavelength: $\lambda = h/p$

The Schrödinger equation

$$\text{One-dimension:} \quad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\text{Three dimension:} \quad -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$\text{Hyrogen atom:} \quad -\frac{\hbar^2}{2m}\nabla^2 u(\mathbf{r}) - \frac{e^2}{4\pi\epsilon_0 r}u(\mathbf{r}) = Eu(\mathbf{r})$$

The energy levels of a particle (mass m) in an infinite square well of width L are given by

$$E_n = \frac{h^2}{8mL^2}n^2.$$

The electron energy levels in the hydrogen atom are:

$$E_n = -\frac{13.6}{n^2} \quad eV.$$

The probability of finding a particle, described by a wavefunction $\psi(x)$, between positions $x = a$ and $x = b$ is $P = \int_a^b |\psi(x)|^2 \, dx$.

The wavelength of radiation absorbed/emitted by an electron going from energy level E_i to E_f is

$$\frac{1}{\lambda} = R_\infty \left[\frac{1}{n_i^2} - \frac{1}{n_f^2}\right]$$

where R_∞ is the Rydberg constant.

The transmission coefficient for a particle of mass m tunnelling across a barrier of height V and width L is

$$T = e^{-2bL} \quad \text{where} \quad b = \sqrt{\frac{8\pi^2 m(V - E)}{\hbar^2}}$$

Fermi-Dirac distribution: $f(E) = \left[\exp\{(E - \mu)/k_B T\} + 1\right]^{-1}$

Bose-Einstein distribution: $f(E) = \left[\exp\{(E - \mu)/k_B T\} - 1\right]^{-1}$

12.11 Nuclear Physics

Rutherford scattering: For α-particle of kinetic energy K, the distance of closest approach to a gold nucleus is

$$d = \frac{q_\alpha q_{Au}}{4\pi\epsilon_0 K}$$

Mass excess: $\Delta = M - A$.

Binding energy: $\Delta E_{be} = \sum mc^2 - Mc^2$.
BE per nucleon: $\Delta E_{ben} = \Delta E_{be}/A$.

Radioactive decay:

$$R = -\frac{dN}{dt} = \lambda N \qquad \rightarrow \qquad N(t) = N_0 \exp(-\lambda t)$$

Half-life: $T_{1/2} = \ln 2/\lambda$.

α-decay:

$${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}X' + {}^{4}_{2}He$$

β-decay: $p \rightarrow n + e^+ + \nu$ and $n \rightarrow p + e^- + \bar{\nu}$.

13 PHYSICAL CONSTANTS AND CONVERSIONS

13.1 Physical constants

speed of light in vacuum	$c = 3.00 \times 10^8$m s^{-1}	$= 3.00 \times 10^{10}$cm s^{-1}
elementary charge	$e = 1.6 \times 10^{-19}$C	
(elementary charge)^2 (e in esu not Coulombs)	$e^2 = 2.31 \times 10^{-28}$J m	$= 2.31 \times 10^{-19}$erg cm
Planck constant	$h = 6.63 \times 10^{-34}$J s	$= 6.63 \times 10^{-27}$erg cm
	$h/2\pi = 1.055 \times 10^{-34}$J s	$= 1.055 \times 10^{-27}$erg cm
unified atomic mass constant	$m_u = 1.66 \times 10^{-27}$kg	$= 931$ MeV/c^2
mass of proton	$m_p = 1.67 \times 10^{-27}$kg	$= 1.67 \times 10^{-24}$g
mass of electron	$m_e = 9.11 \times 10^{-31}$kg	$= 9.11 \times 10^{-28}$g
ratio of proton to electron mass	$m_p/m_e = 1836$	
Bohr radius	$a_0 = 5.29 \times 10^{-11}$m	
Rydberg constant	$R_\infty = 1.097 \times 10^7$m $^{-1}$	
Rydberg energy of hydrogen	$R_H = 13.6$ eV	
Bohr magneton	$\mu_B = 9.27 \times 10^{-24}$J T $^{-1}$	
Fine structure constant	$\alpha = 1/137.0$	
permeabililty of a vacuum	$\mu_0 = 4\pi \times 10^{-7}$H m $^{-1}$	
permittivity of a vacuum	$\epsilon_0 = 8.85 \times 10^{-12}$F m $^{-1}$	
Avogadro constant	$N_A = 6.02 \times 10^{23}$mol^{-1}	
Faraday constant	$F = 9.65 \times 10^4$C mol^{-1}	
Boltzmann constant	$k_B = 1.38 \times 10^{-23}$J K^{-1}	$= 1.38 \times 10^{-16}$erg K^{-1}

gas constant	$R = 8.31$ J $K^{-1}mol^{-1}$	
Stefan-Boltzmann constant	$\sigma_{SB} = 5.67 \times 10^{-8}$J $s^{-1}m^{-2}K^{-4}$	$= 5.67 \times 10^{-5}$erg $s^{-1}cm^{-2}K^{-4}$
Gravitational constant	$G = 6.67 \times 10^{-11}m^3kg^{-1}s^{-2}$	$= 6.67 \times 10^{-8}cm^3g^{-1}s^{-2}$
acceleration of free fall	$g = 9.81$m s^{-2}	
radiant energy density const	$a = 7.56 \times 10^{-16}$J $m^{-3}K^{-4}$	$= 7.56 \times 10^{-15}$erg $cm^{-3}K^{-4}$

13.2 Astronomical constants

Mass associated with one hydrogen nucleus for cosmic composition	$m = 2.38 \times 10^{-24}$g	$= 2.38 \times 10^{-27}$kg
Solar mass	$M_\odot = 1.99 \times 10^{33}$g	$= 1.99 \times 10^{30}$kg
Solar radius	$R_\odot = 6.96 \times 10^{10}$cm	$= 6.96 \times 10^{8}$m
Earth mass	$M_\oplus = 6.0 \times 10^{27}$g	$= 6.0 \times 10^{24}$kg
Earth radius	$R_\oplus = 6.4 \times 10^{8}$cm	$= 6.4 \times 10^{6}$m
Solar luminosity	$L_\odot = 3.83 \times 10^{33}$erg s^{-1}	$= 3.83 \times 10^{26}$J s^{-1}
Astronomical unit	$AU = 1.50 \times 10^{13}$cm	$= 1.50 \times 10^{11}$m
Parsec	$pc = 3.09 \times 10^{18}$cm	$= 3.09 \times 10^{16}$m

13.3 Conversions

1 km	$= 10^3$ m	$= 10^5$ cm
1Å (angström unit)	$= 10^{-10}$ m	$= 10^{-8}$ cm
1 year	$= 3.16 \times 10^7$ s	
1 eV	$= 1.6 \times 10^{-19}$ J	

Celsius temperature = thermodynamic temperature - 273.15