Student Seminar on Analytic Number Theory: Schedule

Brad Rodgers and Nicolas Robles

Fall 2014

- **Sept. 18 - No class**
- **Sept. 25 - Some fundamentals of number theory (Nicolas)**

  This lecture covered Bezout's identity and the Euclidean algorithm and Euclid's proof of the infinitude of primes.

  Possible references

  - Apostol, "Introduction to Analytic Number Theory," Chapter 1.
- **Oct. 2 - Some more fundamentals, and arithmetic functions (Nicolas)**

  This lecture proved the fundemental theorem of arithmetic and gave another proof of the infinitude of primes, then defined some elementary arithmetic functions and proved some of their properties.

  Possible references

  - Apostol, "Introduction to Analytic Number Theory," Chapter 1 & 2.
- **Oct. 9 - Modular arithmetic (Milan)** [Assistant: Brad]

  This lecture introduced modular arithmetic, and proved the Fermat-Euler Theorem, Wilson's Theorem, and the Chinese Remainder Theorem.

  Possible references

  - Apostol, "Introduction to Analytic Number Theory," Chapter 5.
- **Oct. 16 - Quadratic reciprocity (Tania)** [Assistant: Brad]

  This lecture introduced the Legendre symbol $\left(\frac{a}{p}\right)$, *where* p *is an odd prime, and proved formulas for* $\left(\frac{-1}{p}\right)$ *and* $\left(\frac{2}{p}\right)$ *and then showed that for* p *and* q *odd primes,*

  $$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

  Possible references

  - Hardy and Wright, "An Introduction to the Theory of Numbers." Chapter 6.
  - http://www.math.harvard.edu/ chaoli/tutorial2012/Lecture2.pdf
- **Oct. 23 - Chebyshev bounds and some complex analysis (Brad/Nicolas)**

  This lecture will prove the bound due to Chebyshev, that for $\pi(x)$ *the number of primes less than* x, *there exist positive constants* c_1, c_2 *such that*

  $$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}.$$

  We will also (separately) review some complex analysis.

  Possible references

  - Iwaniec and Kowalski, "Analytic Number Theory." Chapter 2.
  - Shilov, "Elementary Real and Complex Analysis." Chapters 10 & 11.

- **Oct. 30 - Dirichlet characters (Benjamin)** [Assistant: Nicolas]

  This lecture should define, for all natural numbers m, the group $\hat{G}$ of Dirichlet characters (mod m), and prove the orthogonality relations

$$\frac{1}{\phi(m)}\sum_{\chi\in\hat{G}}\chi(a)\overline{\chi(b)} = \begin{cases}1 & \text{if } a\equiv b\ (\text{mod } m) \text{ and} (a,m)=(b,m)=1\\ 0 & \text{otherwise}\end{cases}$$

$$\frac{1}{\phi(m)}\sum_{a=0}^{m-1}\chi(a)\overline{\psi(a)} = \begin{cases}1 & \text{if } \chi=\psi\\ 0 & \text{otherwise.}\end{cases}$$

  If there is extra time, the lecture should cover some analytic applications of Dirichlet characters, in particular showing that for a non-trivial character χ, the sum

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

  converges for $\Re s > 0$.

  Possible references:

  - Ireland and Rosen, "A Classical Introduction to Modern Number Theory." Chapter 16.3. (And 16.4 for the analytic part.)
  - Montgomery and Vaughan, "Multiplicative Number Theory I." Chapter 4.
  - Apostol, "Introduction to Analytic Number Theory." Chapter 6.

- **Nov. 6 - Introduction to the Riemann zeta function (Roland)** [Assistant: Nicolas]

  Introduction to the Riemann zeta-function as a function of a complex variable $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Derivation of the Euler product and its connection with the fundamental theorem of arithmetic. The analytic continuation and functional equation of $\zeta(s)$ via e.g. Fourier analysis and the Poisson summation. Definition of $\xi(s)$, first results on the zeros on the critical strip and Riemann hypothesis.

  Possible references

  - D. R. Heath-Brown, "Prime Number Theory and the Riemann zeta-function" in "Recent Perspectives in Random Matrix Theory and Number Theory".
  - E. C. Titchmarsh and D. R. Heath-Brown, "The theory of the Riemann zeta-function" Chapters 1 and 2.
  - Apostol, "Introduction to Analytic Number Theory." Chapters 11 and 12 (although derivation of functional equation is made via residue theorem).

- **Nov. 6 - General Dirichlet series (Adrienne)** [Assistant: Nicolas]

  Half-plane of absolute convergence. Multiplication of Dirichlet series. Examples of arithmetic functions are generating Dirichlet series. Analytic properties and the method of contour integration with Perron's formula for Dirichlet series. Examples of sums of arithmetical functions, e.g. $\sum_{n\le x}\phi(n)$, $\sum_{n\le x}\mu(n)$ and combinations of arithmetic functions.

  Possible references

  - Apostol, "Introduction to Analytic Number Theory." Chapter 11
  - Murty, "Problems in Analytic Number Theory." Chapters 4 and some of 7.

- **Nov. 13 - Dirichlet's theorem (Andrea)** [Assistant: Brad or Nicolas]

  Using material from the Oct. 30 lecture, prove Dirichlet's theorem: that if $(a, m) = 1$, there are infinitely many primes of the form $mk + a$, where k is a natural number. A key role is played by the proof that $L(1, \chi) \neq 0$ for all non-trivial χ. This may be proved a number of ways, but a recommended proof is due to de la Vallee Pousson, found in Ireland and Rosen. Another elementary proof considers separately the cases that χ takes on complex values or only the values ± 1, and is given in Rademacher or Apostol.

  Possible references

  - Ireland and Rosen, "A Classical Introduction to Modern Number Theory." Chapter 16
  - Rademacher, "Lectures on Elementary Number Theory." Chapter 14.

– Apostol, "Introduction to Analytic Number Theory." Chapter 6.

- **Nov. 13/20 (20 minutes each day) - Dirichlet's hyperbola trick (Cagla)** [Assistant: Brad or Nicolas]

  This lecture should prove, where $d(n)$ is the number of divisors of an integer n, that

  $$\sum_{n\le x} d(n) = n\log n + (2\gamma - 1)n + O(\sqrt{n}),$$

  where γ is the Euler-Mascheroni constant

  $$\gamma := \lim_{n\to\infty}\Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\Big) - \log n.$$

  Possible references

  – Hardy and Wright, "An Introduction to the Theory of Numbers." Chapter 18.2.

  – Apostol, "Introduction to Analytic Number Theory." Chapter 3.5.

- **Nov. 20 - Prime number theorem (Paulo)** [Assistant: Nicolas]

  Review of the von Mangoldt and Chebyshev function $\psi(x)$. Connection between $\pi(x)$ and $\psi(x)$. Explicit formula for $\psi(x)$ in terms of the non-trivial zeros of the Riemann zeta-function. Proof of $\zeta(1+it)\neq 0$ and its connection to the prime number theorem. If time allows, connection between error term and zero-free region.

  Possible references

  – Apostol, "Introduction to Analytic Number Theory." Chapter 13 for $\zeta(1+it)\neq 0$.

  – Murty, "Problems in Analytic Number Theory." Chapter 4.

  – Zagier, "Newman's Short Proof of the Prime Number Theorem."

- **Nov. 27 - Twin primes and Brun's constant (Alessandro/Gianluca)** [Assistant: Brad]

  Let $\pi_2(x) := \#\{p \le x : \ p \text{ and } p+2 \text{ are both prime.}\}$. This presentation should demonstrate that

  $$\pi_2(x) = O\Big(x\Big(\frac{\log\log x}{\log x}\Big)^2\Big),$$

  and use this to demonstrate that the sum of the reciprocals of all primes within 2 of another prime,

  $$B := \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$$

  converges. (In fact, $\pi_2(x) = O(x/\log^2 x)$, and the presenters may prove this stronger statement if they'd like.)

  Possible references

  – Rademacher "Lectures on Elementary Number Theory." Chapter 15.

  – Montgomery and Vaughan "Multiplicative Number Theory I." Chapter 3.

- **Dec. 4 - Elliptic curves (Violetta/Dario)** [Assistant: Brad]

  This lecture should define elliptic curves and state and prove the group law for them over the rationals. In addition if there is time the lecture might state the Mordell-Weil Theorem, their $(\bmod p)$ theory and the corresponding definition of the L-function $L(s,E)$, and finally the Birch and Swinnerton-Dyer Conjecture. (There is substantial freedom to cover different material in this lecture. I will put forward a more precise outline with time if desired.)

  Possible references

  – http://www.math.brown.edu/ jhs/Presentations/WyomingEllipticCurve.pdf

  – Hardy and Wright, "An Introduction to the Theory of Numbers" 6th Ed. Chapter 25.

  – Silverman and Tate, "Rational Points on Elliptic Curves."

- **Dec. 11 - Erdős-Kac theorem (Robert)** [Assistant: Brad]

  Let $\omega(n)$ be the number of distinct prime divisors of an integer n. It is relatively easy to show that

  $$\frac{1}{x}\sum_{n\leq x}\omega(n) = \log\log x,$$

  so that $\omega(n)$ oscillates around the value $\log\log n$. This lecture should prove the Erdős-Kac theorem, that

  $$\lim_{x\to\infty}\#\left\{n\leq x : \frac{\omega(n)-\log\log x}{\sqrt{\log\log x}}\in[\alpha,\beta]\right\} = \int_\alpha^\beta e^{t^2/2}\frac{dt}{\sqrt{2\pi}}.$$

  (The lecture should also explain – in a way intelligible to people who have not studied probability – the theorem that a normal random variable is determined by its moments, but does not need to prove it.)

  Possible references

  - Granville and Soundararajan, "Sieving and the Erdős-Kac Theorem."
  - http://www.cs.toronto.edu/ yuvalf/CLT.pdf

- **Dec. 18 - Zeros on the critical line: Hardy's theorem (Francesca/Eduardo)** [Assistant: Nicolas]

  Review of the Hadamard product formula. The logarithmic derivative of $\zeta(s)$ and its connection to the non-trivial zeros ρ. The Riemann-von Mangoldt formula for $N(T)$, i.e. counting zeros of $\zeta(s)$ in the critical strip up to height T. The integral concerning $\Xi(t)$ and the Jacobi function $\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$ for $x>0$. The theorems of Fekete and Fejer. Hardy's argument and contradiction to prove that the number of zeros on the critical line is infite.

  Possible references

  - Ivic, "The theory of the Riemann zeta-function with applications ", Chapter 1.
  - Miller and Takloo-Bighash, "An invitation to modern number theory", see §3.2.4 on pages 62 to 64.