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Fractional Fourier Transform Algorithms for Interference Suppression: Rotational Parameter and Mean Square Error Comparison

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***Abstract*— Minimum mean-square error (MMSE) filtering in the Fractional Fourier Transform (FrFT) domain, or MMSE-FrFT, performs better interference suppression (IS) over the fast Fourier Transform (FFT) when the signal-of-interest (SOI) or interference is non-stationary. This technique estimates the optimum FrFT rotational parameter, 'a', as that which gives the MMSE between a signal and its estimate. However, MMSE filtering requires computational covariance matrix inversion. Furthermore, few samples must be used to form the covariance matrix in non-stationary environments, where statistics rapidly change, but MMSE techniques usually require many samples. Hence, MMSE-FrFT filtering results in errors. In this paper, we describe three recently developed algorithms for FrFT domain IS that each estimate 'a' differently: (1) using domain decomposition (DD), (2) using the relation of the FrFT to the Wigner Distribution (WD), and (3) using the correlations subtraction architecture of the multistage Wiener filter (CSA-MWF). The former two algorithms then use MMSE to perform the filtering, whereas the last uses the CSA-MWF. We compare the proposed algorithm to the MMSE-FrFT algorithm by simulation, and we show that all three new algorithms outperform MMSE-FrFT by several orders of magnitude, using just N = 4 samples per block. The CSA-MWF is the most robust in low E_b/N_0, e.g. $E_b/N_0 < 5$ dB. At high E_b/N_0, both the DD and WD algorithms slightly outperform CSA-MWF at low carrier-to interference ratios (CIRs), and all three algorithms perform equally at high CIRs.**

***Keywords*—Domain Decomposition, Fractional Fourier Transform, MMSE, Multistage Wiener Filter, Wigner Distribution.**

I. Introduction

The Fractional Fourier Transform (FrFT) has numerous applications in fields such as signal and image processing, optics, and quantum mechanics ([7] and [8]). It is a powerful tool for separating a signal-of-interest (SOI) from interference and/or noise when the statistics of either are non-stationary, as is often the case [10].

The FrFT translates the received signal to an axis in the time-frequency plane, known as the Wigner Distribution (WD), where the SOI and interference may be separable, when they are not separable in the time domain or in the frequency domain, with a fast Fourier Transform (FFT).

The FrFT of a function f(x) of order a is defined as [10]

$$\mathbf{F}^a[f(x)] = \int_{-\infty}^{\infty} B_a(x,x')f(x')dx', \tag{1}$$

Where the kernel $B_a(x,x')$ is defined as

$$B_a(x,x') = \frac{e^{i(\pi\hat{\phi}/4-\phi/2)}}{|sin\phi|^{1/2}} \times e^{i\pi(x^2cot\phi-2xx'csc\phi+x'^2cot\phi)}, \tag{2}$$

$\phi = a\pi/2$, and $\hat{\phi} = sgn[sin(\phi)]$. This applies to the range $0 < |\phi| < \pi$, or $0 < |a| < 2$. In discrete time, the FrFT of an N × 1 vector **x** is

$$\mathbf{X}_a = \mathbf{F}^a\mathbf{x}, \tag{3}$$

Where $\mathbf{F}^a$ is an N × N matrix with elements ([2] and [10])

$$\mathbf{F}^a[m,n] = \sum_{k=0,k\neq(N-1+(N)_2)}^{N} u_k[m]e^{-j\frac{\pi}{2}ka}u_k[n], \tag{4}$$

$u_k[m]$ and $u_k[n]$ are eigenvectors of the matrix [2]

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix}, \tag{5}$$

and

$$C_n = 2cos(\frac{2\pi}{N}n) - 4. \tag{6}$$

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An MMSE-FrFT solution for estimating an SOI in the presence of unknown, non-stationary interference and noise in the FrFT domain is presented in [13]. When the environment is non-stationary, it is necessary to perform this estimation with very few samples, i.e. before the statistics of the received signal change. MMSE-based algorithms, however, require a large number of samples in practice, which produces errors and hence greatly limits performance in non-stationary cases [11]. In this paper, we summarize three recently developed algorithms using domain decomposition (DD), the Wigner Distribution (WD), and the reduced rank correlations subtraction architecture of the multistage Wiener filter (CSA-MWF) that improve performance over the MMSE-FrFT solution and operate with few samples.

An outline of the paper is as follows: Section II describes the adaptive filtering problem, now in the FrFT domain. Section III presents the full rank MMSE-FrFT solution proposed in [13]. Sections IV - VI describe the three new algorithms, termed DD-FrFT, WD-FrFT, and MWF-FrFT, respectively; MMSE-FrFT has been previously shown to outperform FFT methods, so we do not discuss the FFT, which fails here [14]. Section VII shows simulation results to compare all four algorithms. We compare the values of ‘a’ as well as the mean-square error (MSE) estimates. Finally, conclusions and remarks on future work are given in Section VIII.

II. Problem Formulation

Without loss of generality, we ignore the carrier and model the SOI as a baseband binary phase shift keying (BPSK) signal whose elements are in (−1,+1), denoted in vector form as the N × 1 vector **x**(i). The number of bits per block is denoted N_1, we upsample each bit by a factor of SPB (samples per bit), giving N = N_1SPB samples per block. The SOI is corrupted by a non-stationary interferer $\mathbf{x}_I$(i), which we describe in Section VII and by an additive white Gaussian noise (AWGN) signal **n**(i). Here, index i denotes the i^{th} block, where i = 1, 2, …, M, and M is the total number of blocks that we process. The received signal **y**(i) is then

$$\mathbf{y}(i) = \mathbf{x}(i) + \mathbf{x}_I(i) + \mathbf{n}(i). \tag{7}$$

We obtain an estimate of the transmitted signal **x**(i), denoted $\hat{\mathbf{x}}$(i), by first transforming the received signal to the FrFT domain, applying an adaptive filter, and taking the inverse FrFT. This is written as [13]

$$\hat{\mathbf{x}}(i) = \mathbf{F}^{-a}\mathbf{G}\mathbf{F}^{a}\mathbf{y}(i), \tag{8}$$

Where $\mathbf{F}^a$ and $\mathbf{F}^{-a}$ are the N × N FrFT and inverse FrFT matrices of order ‘a’, respectively, and

$$\mathbf{g} = diag(\mathbf{G}) = (g_0, g_1, g_{N-1}) \tag{9}$$

is an N × 1 set of optimum filter coefficients to be found that gives the MMSE between the desired signal **x**(i) and its estimate $\hat{\mathbf{x}}$ (i). That is, we minimize

$$J(\mathbf{g}) = \frac{1}{M}\sum_{i=1}^{M}||\mathbf{F}^{-a}\mathbf{G}\mathbf{F}^{a}\mathbf{y}(i) - \mathbf{x}(i)||^2, \tag{10}$$

The notation diag(**G**) = (g_0,g_1,…, g_{N-1}) is a shorthand way of denoting a matrix **G** whose diagonal elements are the scalar coefficients g_0, g_1, ..., and g_{N-1}, with all other elements set to zero.

III. Full Rank MMSE Solution

The most well-known solution that minimizes the cost function in Eq. (10), denoted here as $\mathbf{g}_0$, is obtained by setting the partial derivative of the cost function to zero [13]. That is, we solve for $\mathbf{g}_0$ such that

$$\frac{\partial J(\mathbf{g})}{\partial \mathbf{g}}|_{\mathbf{g}=\mathbf{g}_0} = 0. \tag{11}$$

This is the MMSE-FrFT solution in [13], given by

$$\mathbf{g}_{0,MMSE-FrFT} = \frac{1}{2}\mathbf{Q}^{-1}\mathbf{b}, \tag{12}$$

Where

$$\mathbf{Q} = \sum_{i=1}^{M}\mathbf{Q}(i), \tag{13}$$

$$\mathbf{b} = \sum_{i=1}^{M}\mathbf{b}(i), \tag{14}$$

$$\mathbf{Q}(i) = (\mathbf{F}^{-a}\mathbf{Z}(i))^H(\mathbf{F}^{-a}\mathbf{Z}(i)), \tag{15}$$

$$\mathbf{z}(i) = [z_0(i)\ z_1(i)\ ...\ z_{N-1}(i)]^T = diag(\mathbf{Z}(i)) = \mathbf{F}^a\mathbf{y}(i), \tag{16}$$

and

$$\mathbf{b}(i) = (-2Re(\mathbf{x}(i)^H\mathbf{F}^{-a}\mathbf{Z}(i)))^T. \tag{17}$$

Note that this solution requires a known, training sequence **x**(i) and a search to find the best 'a'. The three new solutions, presented next, will also require the same.

IV. Domain Decomposition Method

We represent the kernel of our interference vector $\mathbf{x}_I(i)$ as an N×N matrix $\mathbf{X}_{I,N}$ [17]. Note that we could also assume this is a non-stationary channel, replacing $\mathbf{X}_{I,N}$ with $\mathbf{H}_N$ [17]. Let $\mathbf{x}_I$ be a complex, time-varying L × 1 vector, written as

$$\mathbf{x}_I = [x_{I,1}\ x_{I,2}\ ...\ x_{I,L}]^T, \tag{18}$$

So that in matrix form, we can write $\mathbf{X}_{I,N}$ as

$$\mathbf{X}_{I,N} = \begin{bmatrix} x_{I,1} & 0 & \dots & \dots & 0 \\ x_{I,2} & x_{I,1} & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ x_{I,L} & x_{I,L-1} & \dots & \dots & \vdots \\ 0 & x_{I,L} & \dots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & x_{I,L} & x_{I,L-1} \\ 0 & 0 & \dots & 0 & x_{I,L} \end{bmatrix}. \tag{19}$$

Note that if L = N, we can truncate $\mathbf{X}_{I,N}$ to be an N × N matrix. For reference, the singular value decomposition (SVD) of $\mathbf{X}_{I,N}$ is [17]

$$\mathbf{X}_{I,N} = \mathbf{U}_N \Sigma_N \mathbf{V}_N^H, \tag{20}$$

Where $\mathbf{U}_N$ and $\mathbf{V}_N$ are unitary N × N matrices whose columns are the eigenvectors of $\mathbf{X}_{I,N}\mathbf{X}^H_{I,N}$, and Σ_N is an N×N diagonal matrix whose elements are the positive square roots of the eigenvalues of $\mathbf{X}_{I,N}\mathbf{X}^H_{I,N}$. Here, $(\cdot)^H$ denotes Hermitian (complex conjugate) transpose of the matrix (·).

The Domain Decomposition (DD) of $\mathbf{X}_{I,N}$ uses the FrFT matrix $\mathbf{F}^a$ and is defined as [17]

$$\mathbf{X}_{I,N} = \sum_{k=1}^{K} \mathbf{F}^{-a_k} \mathbf{\Lambda}_k (\mathbf{F}^{-a_k})^H, \tag{21}$$

Where the Λ_k's are matrices whose diagonal elements contain weighting coefficients similar to the weights Σ_N of Eq. (20).

Fig. 1 shows the WD of the desired signal **x**(i) and a potential corrupting interferer $\mathbf{x}_I(i)$.

The optimum FrFT axis t_{ak} where the interference can best be filtered out, corresponds to the axis where the projection of the WD of $\mathbf{x}_I(i)$ is maximum. The WD of a continuous time signal x(t) can be written as

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi j\tau f}d\tau. \tag{22}$$

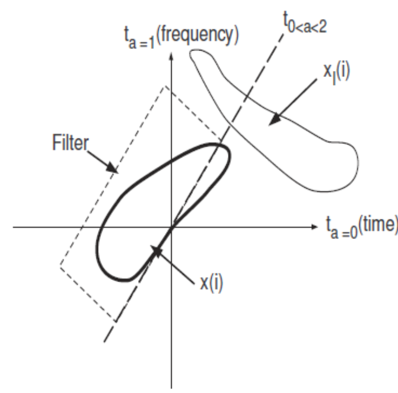

Fig. 1. Wigner Distribution of Signal x(i) and Interferer $x_I(i)$; Optimum rotation axis t_{ak}

We can now summarize the DD-FrFT algorithm in [15]: It is well-known that the projection of the WD of a signal, specifically the interfering signal $\mathbf{x}_I(t)$, onto an axis t_{ak} gives the energy in the FrFT, $|\mathbf{X}_{I,ak}(t)|^2$ (see e.g. [7] or [8]). Letting $\alpha_k = a_k\pi/2$, this is written as

$$|\mathbf{X}_{I,\alpha_k}(t)|^2 = \int_{-\infty}^{\infty} W_{x_I}(tcos\alpha_k - fsin\alpha_k, tsin\alpha_k + fcos\alpha_k)df, \tag{23}$$

So this is the quantity to be maximized. Now, examining Eq. (21), we observe that the amount of energy contained in $\mathbf{X}_{I,N}$ in the domain given by a_k is given by the coefficient Λ_k. Based upon the above, we choose the a_k which maximizes Λ_k in Eq. (21) and hence the energy of the interferer. We then rotate to this domain to best filter the interferer out. Thus, the problem is to determine the maximum Λ_k in Eq. (21). The solution is a maximization problem, written as [15]

$$\underset{\substack{a_k \\ k=1,2,\dots,K}}{arg\ max}\ \mathbf{\Lambda}_k = max(diag([\mathbf{F}^{-a_k}(\mathbf{F}^{-a_k})^H]^{-1}\mathbf{X}_{I,N})), \tag{24}$$

So that the value of a_k which produces the largest Λ_k is the optimum FrFT rotational parameter. We then apply the MMSE solution to filter out the interference, using Eqs. (12) and (8), but with the new 'a', denoting the new solution as $\mathbf{g}_{0,DD-FrFT}$.

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V. WIGNER DISTRIBUTION METHOD

The WD technique presented in [16] estimates the optimum value of 'a' without using the received signal **y**(i), because this signal contains the SOI and interference together, without needing a large number of samples N, and without matrix inversions [16]. Referring back to Fig. 1, note that we can choose the FrFT rotational axis as that for which the desired signal and interference do not overlap, or in the practical case, overlap as little as possible. To avoid computing the WD of the SOI and interferer (i.e. SOI turned off) separately, which is difficult in practice, this method recognizes that we can more easily compute the WD along each axis t_a by computing the energy of the FrFT along that axis [16]. Hence, this algorithm computes the energy of the FrFT of both the SOI and interference, computes their product, sums the values over the new time-frequency axis t_a defined by the rotational parameter 'a', and selects as the optimum 'a' the value for which the result is minimum. The algorithm is summarized as follows [16]: For 0 < a < 2, compute

$$|\mathbf{X}_a(i)|^2 = |\mathbf{F}^a \mathbf{x}(i)|^2, \tag{25}$$

$$|\mathbf{X}_{\mathbf{I}a}(i)|^2 = |\mathbf{F}^a \mathbf{x}_I(i)|^2, \tag{26}$$

and

$$\Re_{\mathbf{X}\mathbf{X}_I}(a) \equiv \sum_{i=1}^{N} |\mathbf{X}_a(i)|^2 |\mathbf{X}_{I\,a}(i)|^2. \tag{27}$$

Choose the value of 'a' for which $\Re_{XXI}(a)$ in Eq. (27) is minimum. Recall that $\mathbf{F}^a$ was defined in Eq. (4). Again, note that the interferer could be replaced with a non-stationary channel, and the algorithm could still be applied. It is also important to mention that computing $|\mathbf{X}_{Ia}(i)|^2$ in Eq. (26) above requires calculation of the FrFT of the interference. Since the algorithm operates with very few samples, we can compute this in gaps where the SOI is off, e.g. using empty sub-carriers, as in OFDM, etc. [16]. Once we have the best 'a', we again compute the filter coefficients from Eq. (12), now denoted $\mathbf{g}_{0,WD-FrFT}$ and apply Eq. (8) to obtain the bit estimates.

VI. REDUCED RANK MWF SOLUTION

The full rank MMSE solution in Eq. (12) can be implemented efficiently and with better performance using the correlations subtraction architecture of the multistage Wiener filter (CSAMWF). The MWF was first introduced in [3] − [6] and the efficient CSA implementation of the MWF was first presented in [12].

The MWF-FrFT algorithm was introduced in [14]. Referring to [14], we initialize the CSA-MWF with

$$d_0(i) = (\mathbf{F}^a \mathbf{x}(i))^T, \tag{28}$$

and

$$\mathbf{x}_0(i) = \mathbf{Z}(i). \tag{29}$$

Recall that **Z**(i) was defined in Eq. (16). Using the recursion equations in Table I, the CSA-MWF computes the D scalar weights w_j, j = 1, 2,..., D and vectors h_j, from which we form the optimum filter

$$\mathbf{g}_{0,MWF-FrFT} = w_1\mathbf{h}_1 - w_1 w_2 \mathbf{h}_2 + ... - (-1)^D w_1 w_2 ... w_D \mathbf{h}_D. \tag{30}$$

Rank reduction is achieved because we set D < N. Again, we search over all 'a' to find the best one, computing the filter coefficients from Eqs. (28) - (30) and Table I. Hence, this is the only algorithm that applies the MWF filter versus the MMSE filter.

VII. SIMULATIONS

We present simulation examples to compare the performance of the four techniques discussed in this paper: MMSE-FrFT, DD-FrFT, WD-FrFT, and MWF-FrFT, showing the robustness of the latter three methods, with the MWF-FrFT showing the most promise when the E_b/N_0 is very low. After computing the best rotational parameter 'a' for each algorithm, we compute the filter coefficients using Eq. (12) for the first three algorithms and Eq. (30) for the fourth and then apply them to Eq. (8) to compute the bit estimates. These are compared to the true bit to determine if an error occurs, so we can compute the MSE, given by the cost function in Eq. (10). In the first example we assume the desired BPSK signal is corrupted by a non-stationary chirp interferer signal given in vector form by (see Example 3 in [7])

TABLE I
RECURSION EQUATIONS FOR THE CSA-MWF

Initialization: $d_0(i)$ and $\mathbf{x}_0(i)$
Forward Recursion: For $j = 1, 2, ..., D$: $\mathbf{h}_j = \frac{\sum_\Omega \{d_{j-1}^*(i)\mathbf{x}_{j-1}(i)\}}{\lVert \sum_\Omega \{d_{j-1}^*(i)\mathbf{x}_{j-1}(i)\}\rVert}$ $d_j(i) = \mathbf{h}_j^H \mathbf{x}_{j-1}(i)$ $\mathbf{x}_j(i) = \mathbf{x}_{j-1}(i) - \mathbf{h}_j d_j(i)$
Backward Recursion: $\epsilon_D(i) = d_D(i)$ For $j = D, D-1, ..., 1$: $w_j = \frac{\sum_\Omega \{d_{j-1}^*(i)\epsilon_j(i)\}}{\sum_\Omega \{\lvert\epsilon_j(i)\rvert^2\}}$ $\epsilon_{j-1}(i) = d_{j-1}(i) - w_j^*\epsilon_j(i)$

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$$\mathrm{x}_I = e^{-j1.73\pi((0:N-1)/f_s)^2}, \quad (31)$$

Where f_s is an arbitrary sampling rate and we have dropped the block index i for convenience. We set $E_b/N_0 = -5$ dB by scaling the amplitude of the AWGN. We vary the CIR, which is the ratio of the desired signal power to the chirp interferer power, from −10 to 10 dB. Here, due to the non-stationarity of the interference, we choose a very small block size for best performance, so we let $N_1 = 2$, SPB = 2, and therefore N = 4. We let the rank of the MWF-FrFT algorithm be D = 1, since there is only a single desired signal and we have a training sequence. The performance is not very sensitive to rank, so we choose D = 1 for faster computation. Since the sample size (N = 4) is so small, the choice of rank is $1 \leq D \leq 4$. We search over $0 < a < 2$ using a step size of $\Delta a = 0.01$. We run M = 10,000 trials to obtain good statistical averages at each CIR and plot both the best 'a' and the MSE as a function of CIR in Fig. 2. The subsequent examples use E_b/N_0 = 0,5,10, and 15 dB, respectively, and are shown in Figs. 3 to 6.

In Fig. 2, the MSE for the MMSE-FrFT method is much greater than one, so it goes off the scale of the plot and is not seen; this occurs to a lesser extent in Figs. 3 and 4 also. From all the figures, we see that all three new algorithms (DD-FrFT, WD-FrFT, and MWF-FrFT) are robust over a large range of CIR and produce a much lower MSE between the signal and its estimate than the conventional MMSE-FrFT algorithm, by several orders of magnitude. When E_b/N_0 is high, the DD and WD algorithms are better than the MWF, at low CIR, but the MWF performs well too; at high CIR, all three are the same. When E_b/N_0 is low, however, the MWF is best. Also, we observe that the best value of 'a' is different for all four algorithms - it is typically less than 0.05 for the MWF-FrFT, it is about 0.25 for the DD-FrFT and WD-FrFT for this interfering signal, and it varies significantly for MMSE-FrFT (confirming that obtaining a good estimate of the bits is difficult for MMSE-FrFT in the non-stationary environment). Since all four algorithms estimate the best 'a' using different criteria, differences in the outcome are expected.

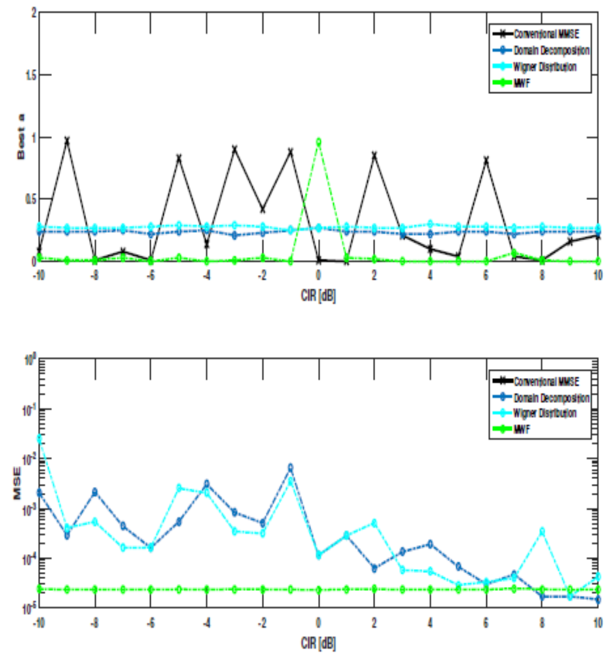

Fig. 2. CIR [dB] vs. BER; BPSK SOI + Chirp Interferer; $E_b/N_0 = -5$ dB

VIII. Conclusion

In this paper, we study three new algorithms for estimating the optimum rotational parameter 'a' in the FrFT domain for suppressing interference. The three algorithms are based on domain decomposition of the signal of interest, the relationship of the FrFT to the Wigner Distribution, and the reduced rank MWF. All three greatly outperform conventional MMSE filtering in the FrFT domain and work over a range of CIR and E_b/N_0. At low E_b/N_0, MWF is best, but for high E_b/N_0 the three algorithms perform comparably. Future work is to study how the DD-FrFT and WD-FrFT algorithms perform if the filtering is done with the CSA-MWF after the best 'a' is computed. This is expected to improve performance even further.

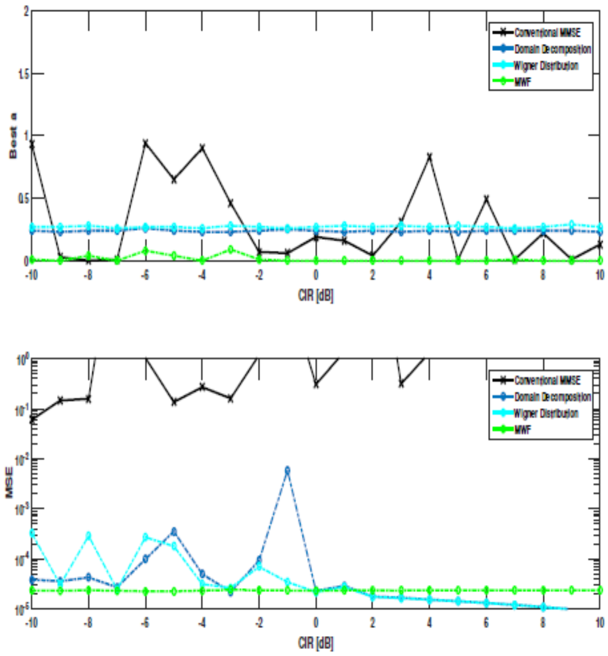

Fig. 3. CIR [dB] vs. BER; BPSK SOI + Chirp Interferer; E_b/N_0 = 0 dB

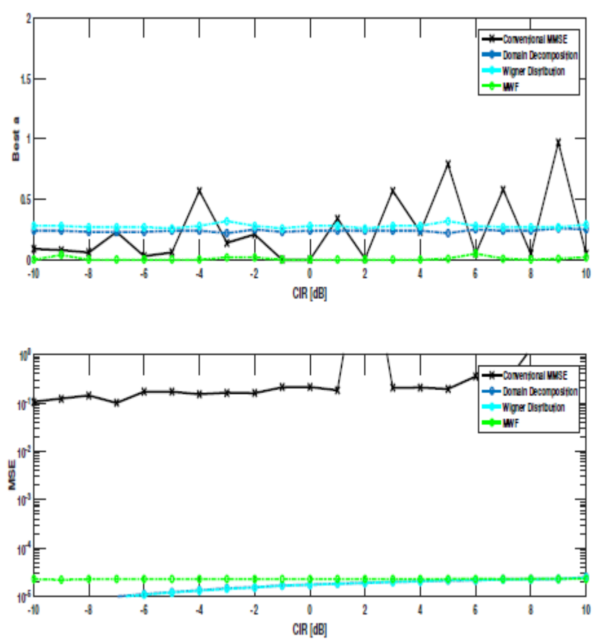

Fig. 4. CIR [dB] vs. BER; BPSK SOI + Chirp Interferer; E_b/N_0 = 5 dB

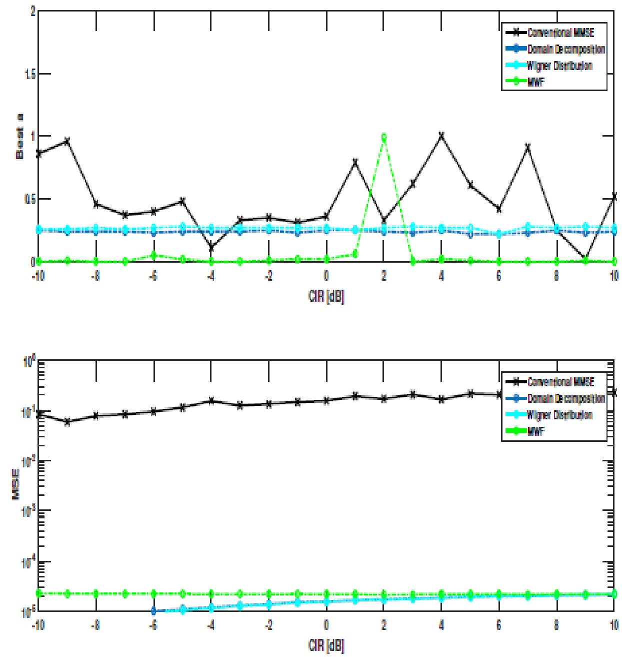

Fig. 5. CIR [dB] vs. BER; BPSK SOI + Chirp Interferer; E_b/N_0 = 10 dB

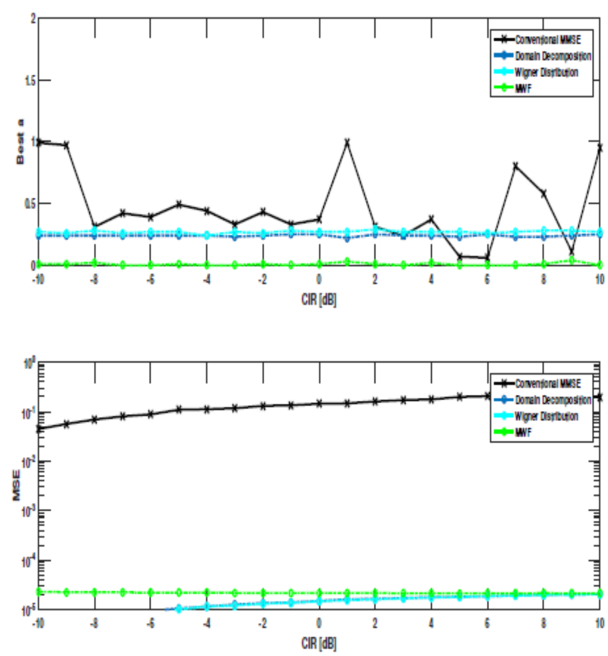

Fig. 6. CIR [dB] vs. BER; BPSK SOI + Chirp Interferer; E_b/N_0 = 15 dB

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