LECTURE NOTES - VIII

« FLUID MECHANICS »

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CHAPTER 8

DIMENSIONAL ANALYSIS

8.1 INTRODUCTION

Dimensional analysis is one of the most important mathematical tools in the study of fluid mechanics. It is a mathematical technique, which makes use of the study of dimensions as an aid to the solution of many engineering problems. The main advantage of a dimensional analysis of a problem is that it reduces the number of variables in the problem by combining dimensional variables to form non-dimensional parameters.

By far the simplest and most desirable method in the analysis of any fluid problem is that of direct mathematical solution. But, most problems in fluid mechanics such complex phenomena that direct mathematical solution is limited to a few special cases. Especially for turbulent flow, there are so many variables involved in the differential equation of fluid motion that a direct mathematical solution is simply out of question. In these problems dimensional analysis can be used in obtaining a functional relationship among the various variables involved in terms of non-dimensional parameters.

Dimensional analysis has been found useful in both analytical and experimental work in the study of fluid mechanics. Some of the uses are listed:

1) Checking the dimensional homogeneity of any equation of fluid motion.
2) Deriving fluid mechanics equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter.
3) Planning tests and presenting experimental results in a systematic manner.
4) Analyzing complex flow phenomena by use of scale models (model similitude).

8.2 DIMENSIONS AND DIMENSIONAL HOMOGENEITY

Scientific reasoning in fluid mechanics is based quantitatively on concepts of such physical phenomena as length, time, velocity, acceleration, force, mass, momentum, energy, viscosity, and many other arbitrarily chosen entities, to each of which a unit of measurement has been assigned. For the purpose of obtaining a numerical solution, we adopt for computation the quantities in SI or MKS units. In a more general sense, however, it is desirable to adopt a consistent dimensional system composed of the smallest number of dimensions in terms of which all the physical entities may be expressed. The fundamental dimensions of mechanics are length [L], time [T], mass [M], and force [F], related by Newton's second law of motion, F = ma.

Dimensionally, the law may also be written as,

$$[F]=\left[\frac{ML}{T^2}\right] \qquad \text{or} \qquad \left[\frac{FT^2}{ML}\right]=1 \tag{8.1}$$

Which indicates that when three of the dimensions are known, the fourth may be expressed in the terms of the other three. Hence three independent dimensions are sufficient for any physical phenomenon encountered in Newtonian mechanics. They are usually chosen as either [MLT] (mass, length, time) or [FLT] (force, length, time). For example, the specific mass (ρ) may be expressed either as $[M/L^3]$or as $[FT^2/L^4]$, and a fluid pressure (p), which is commonly expressed as force per unit area $[F/L^2]$ may also be expressed as $[ML/T^2]$ using the (mass, length, time) system. A summary of some of the entities frequently used in fluid mechanics together with their dimensions in both systems is given in Table 8.1.

TABLE 8.1
ENTITIES COMMONLY USED IN FLUID MECHANICS
AND THEIR DIMENSIONS

Entity	**MLT System**	**FLT System**
Length (L)	L	L
Area (A)	L^2	L^2
Volume (V)	L^3	L^3
Time (t)	T	T
Velocity (v)	LT^{-1}	LT^{-1}
Acceleration (a)	LT^{-2}	LT^{-2}
Force (F) and weight (W)	MLT^{-2}	F
Specific weight (γ)	$ML^{-2}T^{-2}$	FL^{-3}
Mass (m)	M	$FL^{-1}T^{-2}$
Specific mass (ρ)	ML^{-3}	$FL^{-4}T^2$
Pressure (p) and stress (τ)	$ML^{-1}T^{-2}$	FL^{-2}
Energy (E) and work	ML^2T^{-2}	FL
Momentum (mv)	MLT^{-1}	FT
Power (P)	ML^2T^{-3}	FLT^{-1}
Dynamic viscosity (μ)	$ML^{-1}T^{-1}$	$FL^{-2}T$
Kinematic viscosity (υ)	L^2T^{-1}	L^2T^{-1}

With the selection of three independent dimensions –either [MLT] or [FLT]- it is possible to express all physical entities of fluid mechanics. An equation which expresses the physical phenomena of fluid motion must be both algebraically correct and dimensionally homogenous. A dimensionally homogenous equation has the unique characteristic of being independent of units chosen for measurement.

Equ. (8.1) demonstrates that a dimensionally homogenous equation may be transformed to a non-dimensional form because of the mutual dependence of fundamental dimensions. Although it is always possible to reduce dimensionally homogenous equation to a non-dimensional form, the main difficulty in a complicated flow problem is in setting up the correct equation of motion. Therefore, a special mathematical method called *dimensional analysis* is required to determine the functional relationship among all the variables involved in any complex phenomenon, in terms of non-dimensional parameters.

8.3 DIMENSIONAL ANALYSIS

The fact that a complete physical equation must be dimensionally homogenous and is, therefore, reducible to a functional equation among non-dimensional parameters forms the basis for the theory of dimensional analysis.

8.3.1 Statement of Assumptions

The procedure of dimensional analysis makes use of the following assumptions:

1) It is possible to select m independent fundamental units (in mechanics, m=3, i.e., length, time, mass or force).
2) There exist n quantities involved in a phenomenon whose dimensional formulae may be expressed in terms of m fundamental units.
3) The dimensional quantity A_0 can be related to the independent dimensional quantities A_1, A_2, ..., A_{n-1} by,

$$A_0 = F\left(A_1, A_2, \ldots\ldots, A_{n-1}\right) = KA_1^{y_1} A_2^{y_2} \ldots\ldots A_{n-1}^{y_{n-1}} \tag{8.2}$$

   Where K is a non-dimensional constant, and y_1, y_2,.....,y_{n-1} are integer components.
4) Equ. (8.2) is independent of the type of units chosen and is dimensionally homogenous, i.e., the quantities occurring on both sides of the equation must have the same dimension.

EXAMPLE 8.1: Consider the problem of a freely falling body near the surface of the earth. If x, w, g, and t represent the distance measured from the initial height, the weight of the body, the gravitational acceleration, and time, respectively, find a relation for x as a function of w, g, and t.

SOLUTION: Using the fundamental units of force F, length L, and time T, we note that the four physical quantities, A_0=x, A_1=w, A_2=g, and A_3=t, involve three fundamental units; hence, m=3 and n=4 in assumptions (1) and (2). By assumption (3) we assume a relation of the form:

$$x = F(w, g, t) = Kw^{y_1} g^{y_2} t^{y_3} \qquad \text{(a)}$$

Where K is an arbitrary non-dimensional constant.

Let [·] denote “dimensions of a quantity”. Then the relation above can be written (using assumption (4)) as,

$$[x] = [w]^{y_1} [g]^{y_2} [t]^{y_3}$$

or

$$F^0 L^1 T^0 = (F)^{y_1} \left(LT^{-2}\right)^{y_2} (T)^{y_3} = F^{y_1} L^{y_2} T^{-2y_2 + y_3}$$

Equating like exponents, we obtain

$$F: 0 = y_1$$
$$L: 1 = y_2$$
$$T: 0 = -2y_2 + y_3 \quad \text{or} \quad y_3 = 2y_2 = 2$$

Therefore, Equ. (a) becomes

$$x = Kw^0 g^1 t^2$$

or

$$x = Kgt^2$$

According to the elementary mechanics we have $x=gt^2/2$. The constant K in this case is ½, which cannot be obtained from dimensional analysis.

EXAMPLE 8.2: Consider the problem of computing the drag force on a body moving through a fluid. Let D, ρ, μ, l, and V be drag force, specific mass of the fluid, dynamic viscosity of the fluid, body reference length, and body velocity, respectively.

SOLUTION: For this problem m=3, n=5, $A_0=D$, $A_1=\rho$, $A_2=\mu$, $A_3=l$, and $A_4=V$. Thus, according to Equ (8.2), we have

$$D = F(\rho, \mu, l, V) = K\rho^{y_1}\mu^{y_2}l^{y_3}V^{y_4} \qquad \text{(a)}$$

or

$$[D] = [\rho]^{y_1}[\mu]^{y_2}[l]^{y_3}[V]^{y_4}$$
$$F^1L^0T^0 = \left(FL^{-4}T^2\right)^{y_1}\left(FL^{-2}T\right)^{y_2}(L)^{y_3}\left(LT^{-1}\right)^{y_4}$$
$$F^1L^0T^0 = F^{y_1+y_2}L^{-4y_1-2y_2+y_3+y_4}T^{2y_1+y_2-y_4}$$

Equating like exponents, we obtain

$$F: 1 = y_1 + y_2$$
$$L: 0 = -4y_1 - 2y_2 + y_3 + y_4$$
$$T: 0 = 2y_1 + y_2 - y_4$$

In this case we have three equations and four unknowns. Hence, we can only solve for three of the unknowns in terms of the fourth unknown (a one-parameter family of solutions exists). For example, solving for y_1, y_3 and y_4 in terms of y_2, one obtains

$$y_1 = 1 - y_2$$
$$y_3 = 2 - y_2$$
$$y_4 = 2 - y_2$$

The required solution is

$$D = K\rho^{1-y_2}\mu^{y_2} l^{2-y_2} V^{2-y_2}$$

or

$$D = (2K)\left(\frac{\rho V l}{\mu}\right)^{-y_2}\left(\frac{\rho V^2}{2}\right) l^2$$

If the Reynolds number is denoted by Re=ρVl/μ, dynamic pressure by $q=\rho V^2/2$, and area by $A=l^2$, we have

$$D = \frac{2K}{(\mathrm{Re})^{y_2}} qA = C_D qA$$

where

$$C_D = \frac{2K}{(\mathrm{Re})^{y_2}}$$

Theoretical considerations show that for laminar flow

$$2K = 1.328 \qquad \text{and} \qquad y_2 = \frac{1}{2}$$

8.3.2 Buckingham-π (Pi) Theorem

It is seen from the preceding examples that m fundamental units and n physical quantities lead to a system of m linear algebraic equations with n unknowns of the form

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \ldots\ldots + a_{1n}y_n &= b_1 \\ a_{21}y_1 + a_{22}y_2 + \ldots\ldots + a_{2n}y_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}y_1 + a_{m2}y_2 + \ldots\ldots + a_{mn}y_n &= b_m \end{aligned} \qquad (8.3)$$

or, in matrix form,

$$Ay = b \qquad (8.4)$$

where

$$A = \begin{bmatrix} a_{11}a_{12}\ldots\ldots a_{1n} \\ a_{12}a_{22}\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}a_{m2}\ldots\ldots a_{mn} \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ y_n \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

A is referred to as the coefficient matrix of order m×n, and y and b are of order n×1 and m×1 respectively.

The matrix A in Equ. (8.4) is rectangular and the largest determinant that can be formed will have the order n or m, whichever is smaller. If any matrix C has at least one determinant of order r, which is different from zero, and nonzero determinant of order greater than r, then the matrix is said to be of rank r, i.e.,

$$R(C) = r \qquad (8.5)$$

In order to determine the condition for the solution of the linear system of Equ. (8.3) it is convenient to define the rank of the augmented matrix B. The matrix B is defined as

$$B = A^b = \begin{bmatrix} a_{11}a_{12}.......a_{1n}b_1 \\ a_{21}a_{22}.......a_{2n}b_2 \\ .... \\ a_{m1}a_{m2}.......a_{mn}b_m \end{bmatrix} \qquad (8.6)$$

For the solution of the linear system in Equ. (8.3), three possible cases arise:

1) R (A)<R (B): No solution exists,
2) R(A)=R(B)= r = n: A unique solution exits,
3) R (A)=R (B)=r<n: An infinite number of solutions with (n-r) arbitrary unknowns exist.

Example 8.2 falls in case (3) where

$$R(A) = R(B) = 3 < n = 4 \quad \text{and} \quad (n-r) = (4-3) = 1$$

an arbitrary unknown exits.

The mathematical reasoning above leads to the following Pi theorem due to Buckingham.

Let n quantities A_1, A_2,…..,A_n be involved in a phenomenon, and their dimensional formulae be described by (m<n) fundamental units. Let the rank of the augmented matrix B be R (B)= r ≤ m. Then the relation

$$F_1(A_1, A_2, ..., A_n) = 0 \qquad (8.7)$$

is equivalent to the relation

$$F_2(\pi_1, \pi_2, ..., \pi_{n-r}) = 0 \qquad (8.8)$$

Where π_1, π_2,…..,π_{n-r} are dimensionless power products of A_1, A_2,…..,A_n taken r+1 at a time.

Thus, the Pi theorem allows one to take n quantities and find the minimum number of non-dimensionless parameter, π_1, π_2,….., π_{n-r} associated with these n quantities.

8.3.2.1 Determination of Minimum Number of π Terms

In order to apply the Buckingham π Theorem to a given physical problem the following procedure should be used:

Step 1. Given n quantities involving m fundamental units, set up the augmented matrix B by constructing a table with the quantities on the horizontal axis and the fundamental units on the vertical axis. Under each quantity list a column of numbers, which represent the powers of each fundamental units that makes up its dimensions. For example,

	ρ	p	d	Q
F	1	1	0	1
L	-4	-2	1	3
T	2	0	0	-1

Where ρ, p, d, and Q are the specific mass, pressure, diameter, and discharge, respectively. The resulting array of numbers represents the augmented matrix B in Buckingham's π Theorem, i.e.,

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -4 & -2 & 1 & 3 \\ 2 & 0 & 0 & -1 \end{bmatrix}$$

The matrix B is sometimes referred to as dimensional matrix.

Step 2. Having constructed matrix B, find its rank. From step1, since

$$\begin{vmatrix} 1 & 1 & 0 \\ -4 & -2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step3. Having determined the number of π dimensionless (n-r) terms, following rules are used to combine the variables to form π terms.

a) From the independent variables select certain variables to use as repeating variables, which will appear in more than π term. The repeating variables should contain all the dimensions used in the problem and be quantities, which are likely to have substantial effect on the dependent variable.
b) Combine the repeating variables with remaining variables to form the required number of independent dimensionless π terms.
c) The dependent variable should appear in one group only.
d) A variable that is expected to have a minor influence should appear in one group only.

Define π_1 as a power product of r of the n quantities raised to arbitrary integer exponents and any one of the excluded (n-r) quantities, i.e.,

$$\pi_1 = A_1^{y_{11}} A_2^{y_{12}} \ldots\ldots A_r^{y_{1r}} A_{r+1}$$

Step 4. Define π_2, π_3,....., π_{n-r} as power products of the same r quantities used in step 3 raised to arbitrary integer exponents but a different excluded quantity for each π term, i.e.,

$$\pi_2 = A_1^{y_{21}} A_2^{y_{22}} \ldots\ldots\ldots\ldots A_r^{y_{2r}} A_{r+2}$$
$$\pi_3 = A_1^{y_{31}} A_2^{y_{32}} \ldots\ldots\ldots\ldots A_r^{y_{3r}} A_{r+3}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\pi_{n-r} = A_1^{y_{n-r,1}} A_2^{y_{n-r,2}} \ldots\ldots A_r^{y_{n-r,r}} A_n$$

Step 5. Carry out dimensional analysis on each π term to evaluate the exponents.

EXAMPLE 8.3: Rework Example 8.1 using the π theorem.

SOLUTION:

Step 1. With F, L, and T as the fundamental units, the dimensional matrix of the quantities w, g, t and x is,

	W	g	t	x
F	1	0	0	0
L	0	1	0	1
T	0	-2	1	0

Where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

Step 2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Arbitrarily select x, w, and g as the r = m= 3 base quantities. The number n-r of independent dimensional products that can be formed by the four quantities is therefore 1, i.e.,

$$\pi_1 = x^{y_{11}} w^{y_{12}} g^{y_{13}} t$$

Step 4. Dimensional analysis gives,

$$[\pi_1] = [x]^{y_{11}} [w]^{y_{12}} [g]^{y_{13}} [t]$$

or

$$F^0 L^0 T^0 = (L)^{y_{11}} (F)^{y_{12}} (LT^{-2})^{y_{13}} (T)$$

Which results in

$$y_{11} = -\frac{1}{2},\ y_{12} = 0, \text{ and } y_{13} = \frac{1}{2}$$

Hence,

$$\pi_1 = \sqrt{\frac{gt^2}{x}}$$

EXAMPLE 8.4: Rework Example 8.2 using the π theorem.

SOLUTION:

Step 1. With F, L and T as the fundamental units, the dimensional matrix of the quantities D, ρ, μ, l, and V is.

	D	ρ	μ	l	V
F	1	1	1	0	0
L	0	-4	-2	1	1
T	0	2	1	0	-1

Where

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 1 \\ 0 & 2 & 1 & 0 & -1 \end{bmatrix}$$

Step2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Select l, V, and ρ as the r=3 base quantities. By the π-Theorem (n-r),(5-3)=2 π terms exist.

$$\pi_1 = l^{y_{11}} V^{y_{12}} \rho^{y_{13}} D$$
$$\pi_2 = l^{y_{21}} V^{y_{22}} \rho^{y_{23}} \mu$$

Step 4. Dimensional analysis gives

y_{11}=-2, y_{12}=-2, y_{13}= -1

y_{21}=-1, y_{22}=-1, y_{23}=-1

Hence,

$$\pi_1 = \frac{D}{\rho V^2 l^2}$$
$$\pi_2 = \frac{\mu}{\rho V l}$$

8.4 THE USE OF DIMENSIONLESS π-TERMS IN EXPERIMENTAL INVESTIGATIONS

Dimensional analysis can be of assistance in experimental investigation by reducing the number of variables in the problem. The result of the analysis is to replace an unknown relation between n variables by a relationship between a smaller numbers, n-r, of dimensionless π-terms. Any reduction in the number of variables greatly reduces the labor of experimental investigation. For instance, a function of one variable can be plotted as a single curve constructed from a relatively small number of experimental observations, or the results can be represented as a single table, which might require just one page.

A function of two variables will require a chart consisting of a family of curves, one for each value of the second variable, or, alternatively the information can be presented in the form of a book of tables. A function of three variables will require a set of charts or a shelf-full of books of tables.

As the number of variables increases, the number of observations to be taken grows so rapidly that the situation soon becomes impossible. Any reduction in the number of variables is extremely important.

Considering, as an example, the resistance to flow through pipes, the shear stress or resistance R per unit area at the pipe wall when fluid of specific mass ρ and dynamic viscosity μ flows in a smooth pipe can be assumed to depend on the velocity of flow V and the pipe diameter D. Selecting a number of different fluids, we could obtain a set of curves relating frictional resistance (measured as $R/\rho V^2$) to velocity, as shown in Fig. 8.1.

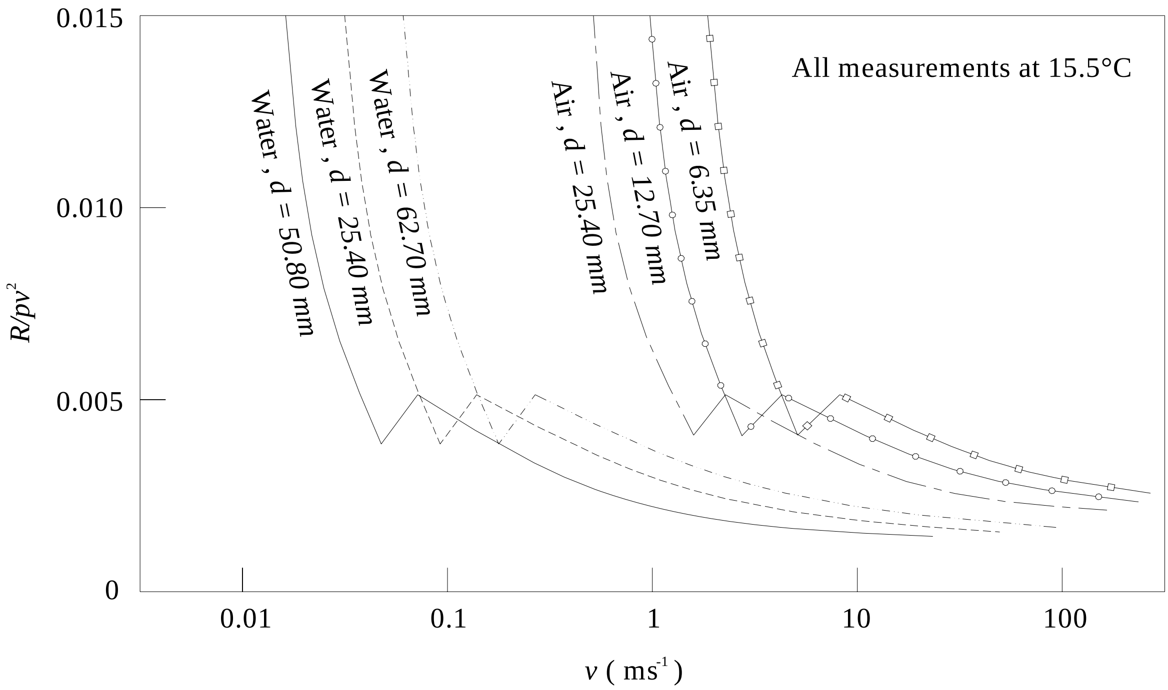

Fig. 8.1

Such a set of curves would be of limited value both for use and for obtaining a proper understanding of the problem. However, it can be shown by dimensional analysis that the relationship can be reduced to the form

$$\frac{R}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right) = \phi(\mathrm{Re})$$

or, using the Darcy resistance coefficient $f = 2R/\rho V^2$,

$$f = \phi\left(\frac{\rho VD}{\mu}\right) = \phi(\mathrm{Re})$$

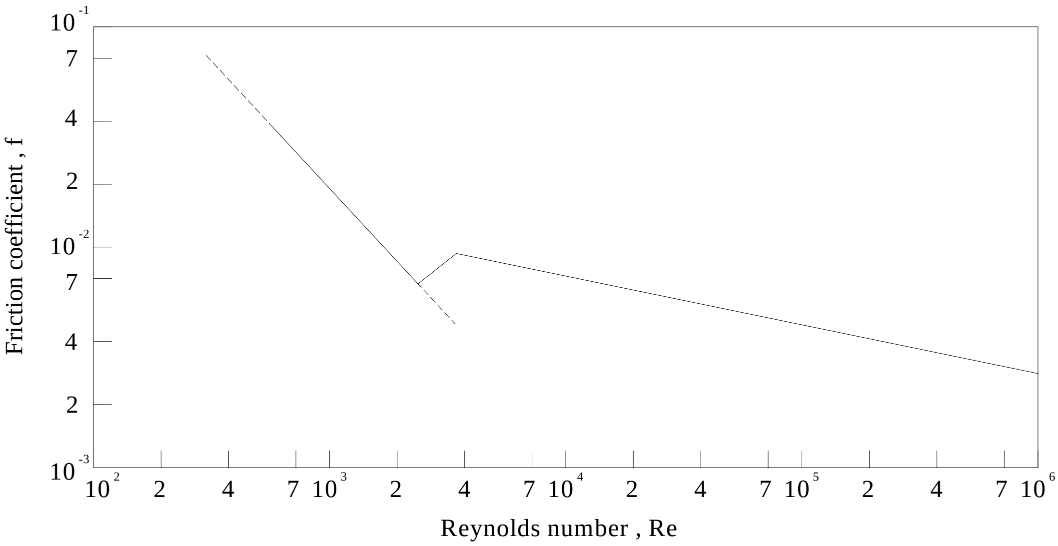

Fig. 8.2

If the experimental points in Fig. 8.1 are used to construct a new graph of Log (f) against Log (Re) the separate sets of experimental data combine to give a single curve as shown in Fig. 8.2. For low values of Reynolds number, when flow is laminar, the slope of this graph is (-1) and $f = 16/Re$, while for turbulent flow at higher values of Reynolds number, $f = 0.08(Re)^{-1/4}$.