PICARD GROUPS: REAL AND COMPLEX, ALGEBRAIC AND CONTINOUS

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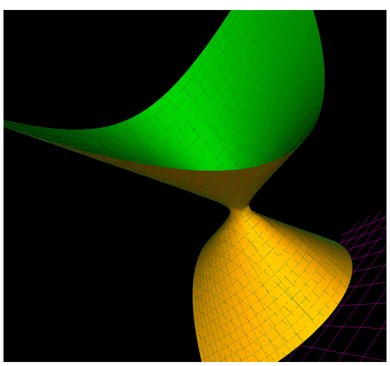
Abstract. An algebraic vector bundle on a smooth variety over $\mathbb{R}$ or $\mathbb{C}$ can also be considered as a continous bundle over the corresponding Hausdorff space. The aim of this note is to consider an elementary example showing the difference this makes.

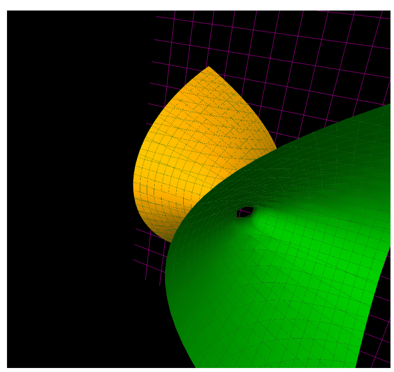
1. Introduction

Let k be a field and consider the set

$$X := \{(x, y, z) \in k^3 \mid xz - y - y^2 = 0\}.$$

Below you can view a piece of X for $k = \mathbb{R}$ from two angles:





Essentially every branch of mathematics would consider this surface as a prototypical example of its theory, it can be made in the obvious way into a measure space, topological space, smooth manifold and by very definition real affine variety. So there is a choice to make, and when it comes to the study of certain invariants such as K-theory, Picard groups, cohomology and so on, these will heavily depend on the choice you make. We are going to demonstrate this by considering a certain line bundle on X in the various categories and will see that in some settings it is trivial but in others not.

2. Topological Picard groups

From now on we only consider $k = \mathbb{R}$ and $k = \mathbb{C}$. The topologist's version of the story is governed by characteristic classes and homological arguments. For $k = \mathbb{R}$ our space is contractible to a circle S^1 (just look at the picture). So real line bundles over X correspond to real line bundles over S^1, and they are classified by their first Stiefel-Whitney class which lies in $H^1(X, \mathbb{Z}_2) = H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, so there is up to isomorphism the trivial line bundle $X \times \mathbb{R}$ and one nontrivial one which corresponds to the Möbius bundle over S^1.

If we work over $k = \mathbb{C}$ we can make a change of coordinates

$$x = \frac{1}{2}(ia + b), \quad y = \frac{1}{2}(c - 1), \quad z = \frac{1}{2}(ia - b),$$

then the defining equation of X becomes

$$a^2 + b^2 + c^2 = 1$$

so we are talking about a complexified S^2 here. If we write $a = a_0 + ia_1$ and similarly for b, c, then the defining equation is equivalent to

$$a_0^2 + b_0^2 + c_0^2 = 1 + a_1^2 + b_1^2 + c_1^2, \quad a_0a_1 + b_0b_1 + c_0c_1 = 0.$$

From this one sees that X is homeomorphic to the tangent bundle TS^2 of the two-sphere: the homeomorphism maps $(a, b, c) \in X$ to the tangent vector $(a_1, b_1, c_1) \in T_pS^2$ at the base point

$$p = \frac{1}{\sqrt{1 + a_1^2 + b_1^2 + c_1^2}}(a_0, b_0, c_0) \in S^2.$$

In particular, X is contractible to S^2, and if we now study complex line bundles over X then they are classified by their first Chern class and from $H^2(X, \mathbb{Z}) = H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ we see that over $\mathbb{C}$ there are infinitely many nonisomorphic line bundles.

3. Algebraic Picard groups

There is no difference to the above when we consider X as a smooth manifold and classify smooth bundles, but purely algebraically the story changes. Let us abbreivate $A := k[X]$ for the coordinate ring of the affine variety X. We shall mean by this the ring of polynomial functions on X and hence the quotient of $k[x, y, z]$ by the ideal generated by $xz - y - y^2$. For $k = \mathbb{C}$ Hilbert's Nullstellensatz implies that a polynomial function which has no zero on X is an invertible element of A, but for $k = \mathbb{R}$ this is not true. Some authors cure this by defining $\mathbb{R}[X]$ as the ring of formal fractions $\frac{f}{g}$ of polynomial functions, where g has no zero on X. We will denote this localisation of A by $\bar{A}$.

Let now M be an invertible A-module, that is, one for which there exists an A-module N such that $M \otimes_A N \simeq A$ as an A-module. Note that A is as a finitely generated algebra over a field Noetherian by Hilbert's basis theorem. This implies (see [1, 2]) for example the folloing:

(1) $N \simeq M^* := \mathrm{Hom}_A(M, A)$.
(2) M is finitely generated projective.
(3) Every localisation $M_{\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{Spec} A$, is isomorphic to $A_{\mathfrak{p}}$.
(4) $\Lambda_A^n M = 0$ for $n > 1$.

What this means is that M is the module of sections of an algebraic line bundle over X which is locally trivial even in the Zariski topology. We can view such a bundle also as a topological bundle. One way to describe this is this: put $B := C(X, k)$, the ring of k-valued continuous functions on the space X with its Hausdorff topology. This is a ring extension of A because polynomials are also continous with respect to the Hausdorff topology. It thus makes sense to define $B \otimes_A M$, and if M is finitely generated projective over A, then this module will be finitely generated projective over B and hence corresponds by the Serre-Swan theorem to a topological vector bundle.

As a concrete example, let M be the A-module with generators X, Y, Z and the relations

$$yX = xY, \quad zX = (1+y)Y, \quad yY = xZ, \quad zY = (1+y)Z.$$

The inverse module can be given in terms of generators $\hat{X}, \hat{Y}, \hat{Z}$ and the relations

$$x\hat{Y} = (1+y)\hat{X}, \quad y\hat{Y} = z\hat{X}, \quad x\hat{Z} = (1+y)\hat{Y}, \quad y\hat{Z} = z\hat{Y}.$$

The isomorphism

$$\theta : M \otimes_A N \to A$$

is then given by

$$\begin{array}{llllll} X \otimes_A \hat{X} & \mapsto x^2, & X \otimes_A \hat{Y} & \mapsto x(1+y), & X \otimes_A \hat{Z} & \mapsto (1+y)^2, \\ Y \otimes_A \hat{X} & \mapsto xy, & Y \otimes_A \hat{Y} & \mapsto y(1+y), & Y \otimes_A \hat{Z} & \mapsto (1+y)z, \\ Z \otimes_A \hat{X} & \mapsto y^2, & Z \otimes_A \hat{Y} & \mapsto yz, & Z \otimes_A \hat{Z} & \mapsto z^2. \end{array}$$

It follows by direct computation that this extends to a well-defined module homomorphism θ (one has to prove that a tensor of the form something $\otimes_A$ relation or relation $\otimes_A$ something is mapped to zero). Furthermore, θ is surjective since

$$X \otimes_A \hat{Z} - 2Y \otimes_A \hat{Y} + Z \otimes_A \hat{X} \mapsto 1.$$

To prove injectivity, one has to check that the inverse map

$$\theta^{-1} : A \to M \otimes_A N, \quad a \mapsto aX \otimes_A \hat{Z} - 2aY \otimes_A \hat{Y} + aZ \otimes_A \hat{X}$$

inverts θ (clearly $\theta \circ \theta^{-1} = \mathrm{id}_A$, but one has to check $\theta^{-1} \circ \theta = \mathrm{id}_{M \otimes_A N}$ on all nine generators, which is lengthy but straightforward). This finishes the proof of the invertibility of M.

The above form of θ also provides us with an embedding of M as a direct summand into A^3 which is given by

$$X \mapsto (x^2, x(1+y), (1+y)^2), Y \mapsto (xy, y(1+y), (1+y)z), Z \mapsto (y^2, yz, z^2).$$

From this one can see that

$$\omega := X + Z$$

is for $k = \mathbb{R}$ an element of M which does not vanish in any point of our surface since under the embedding into A^3 it becomes

$$(x^2 + y^2, x(1 + y) + yz, (1 + y)^2 + z^2).$$

And that is now the crucial point: a continous real or complex line bundle on a is trivial if it admits a nowhere vanishing section, the map

$$B \to B \otimes_A M, \quad b \mapsto b \otimes_A \omega$$

is an isomorphism. Hence the topological line bundle over X described by M is for $k = \mathbb{R}$ trivial (it is the tensor square of the nontrivial one).

There is an analogous algebraic statement, but one has to interpret "vanishing nowhere" appropriately: let M be an invertible module over a Noetherian ring A and $\omega \in M$. As mentioned above, the invertibility implies that for all maximal ideals $\mathfrak{m} \subset A$ we have $M_\mathfrak{m} \simeq A_\mathfrak{m}$. If the image $\iota(\omega) = \frac{\omega}{1}$ of ω in $M_\mathfrak{m}$ is not mapped to an element of $\mathfrak{m}A_\mathfrak{m}$ under this isomorphism, then ω does not vanish in the "point" $\mathfrak{m}$, and the map $A_\mathfrak{m} \to M_\mathfrak{m}$, $a \mapsto a\iota(\omega)$ is an isomorphism. And if this is true for all $\mathfrak{m}$ then $A \to M$, $a \mapsto a\omega$ is an isomorphism since a module map is an isomorphism if it is locally so (in the Noetherian world).

But this does not apply to our concrete M over $A = \mathbb{R}[X]$ since $\iota(\omega)$ does vanish for some maximal ideals of A, these simply do not correspond to real points on the variety $X \subset \mathbb{R}^3$. To see that M is indeed not free one can use characteristic classes living in algebraic de Rham cohomology, see our joint article [3] with N. Kowalzig where we used this example to construct a Lie-Rinehart algebra whose universal enveloping algebra is not a Hopf algebroid.

Over $k = \mathbb{C}$ the element ω vanishes precisely on the subset

$$\{(\pm iy, y, \mp i(1 + y)) \in X \mid y \in \mathbb{C}\},$$

so here ω is by no means an indicator for the triviality of M.

So we see that when we describe a real affinve variety in terms of polynomial functions, then the topology we detect resembles rather that of the Hausdorff space underlying its complexification than that of the variety itself. If we rather would work with $\bar{A}$ and define algebraic line bundles as invertible modules over $\bar{A}$, then this changes and the algebraic geometry of $\bar{A}$ has much more to do with the topology of $X \subset \mathbb{R}^3$, see for example the extensive work on the subject by Bochnak, Kucharz and their coworkers.

References

[1] Nicolas Bourbaki, *Elements of Mathematics. Commutative algebra*, Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass., 1972.

[2] David Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[3] Niels Kowalzig, Ulrich Krähmer, *$A \times_A$-Hopf algebra without antipode*, to appear