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TABLES OF SOME INDEFINITE INTEGRALS OF BESSEL FUNCTIONS

Integrals of the type

$$\int x J_0^2(x)\,dx \qquad \text{or} \qquad \int x J_0(ax) J_0(bx)\,dx$$

are well-known.

Most of the following integrals are not found in the widely used tables of Gradstein/Ryshik, Bateman/Erdélyi, Abramowitz/ Stegun, Prudnikov/Brychkov/Marichev or Jahnke/Emde/Lösch. The goal of this table was to get tables for practicians. So the integrals should be expressed by Bessel and Struve functions. Indeed, there occured some exceptions. Generally, integrals of the type $\int x^\mu J_\nu(x)\,dx$ may be written with Lommel functions, see [8], 10 -74, or [3], III .
In many cases reccurence relations define more integrals in a simple way.

Partially the integrals may be found by MAPLE as well. In some cases MAPLE gives results with hypergeometric functions, see also [2], 9.6., or [4].
Some known integrals are included for completeness.

Here $Z_\nu(x)$ denotes some Bessel function or modified Bessel function of the first kind. Partially the functions $Y_\nu(x)$ [sometimes called Neumann's functions or Weber's functions and denoted by $N_\nu(x)$] and the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are also considered. The same holds for the modified Bessel function of the second kind $K_\nu(x)$.

When a formula is continued in the next line, then the last sign '+' or '-' is repeated in the beginning of the new line.

On page 443 the used special functions and defined functions are described.

E - This sign marks formulas, that were incorrect in previous editions. The pages with corrected errors are listed in the errata in the end.

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1. Integrals with one Bessel Function:

See also [10], 2. .

1.1. $x^n Z_\nu(x)$ with integer values of n

1.1.1. Integrals of the type $\int x^{2n} Z_0(x)\, dx$

Let

$$\Phi(x) \;=\; \frac{\pi x}{2}\left[J_1(x)\cdot \mathbf{H}_0(x) - J_0(x)\cdot \mathbf{H}_1(x)\right] \;,$$

where $\mathbf{H}_\nu(x)$ denotes the Struve function, see [1], chapter 11.1.7, 11.1.8 and 12.
And let

$$\Psi(x) \;=\; \frac{\pi x}{2}\left[I_0(x)\cdot \mathbf{L}_1(x) - I_1(x)\cdot \mathbf{L}_0(x)\right]$$

be defined with the modified Struve function $\mathbf{L}_\nu(x)$.
Furthermore, let

$$\Phi_Y(x) \;=\; \frac{\pi x}{2}\left[Y_1(x)\cdot \mathbf{H}_0(x) - Y_0(x)\cdot \mathbf{H}_1(x)\right] \;,$$

$$\Phi_H^{(1)}(x) \;=\; \frac{\pi x}{2}\left[H_1^{(1)}(x)\cdot \mathbf{H}_0(x) - H_0^{(1)}(x)\cdot \mathbf{H}_1(x)\right] \;,$$

$$\Phi_H^{(2)}(x) \;=\; \frac{\pi x}{2}\left[H_1^{(2)}(x)\cdot \mathbf{H}_0(x) - H_0^{(2)}(x)\cdot \mathbf{H}_1(x)\right]$$

and

$$\Psi_K(x) \;=\; \frac{\pi x}{2}\left[K_0(x)\cdot \mathbf{L}_1(x) + K_1(x)\cdot \mathbf{L}_0(x)\right]$$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x),\; p = 1, 2$ and $\Phi_H^{(p)}(x)$.

Well-known integrals:

$$\int J_0(x)\, dx \;=\; xJ_0(x) + \Phi(x) \;=\; \Lambda_0(x)$$

$$\int I_0(x)\, dx \;=\; xI_0(x) + \Psi(x) \;=\; \Lambda_0^*(x)$$

$$\int K_0(x)\, dx \;=\; xK_0(x) + \Psi_K(x)$$

The new-defined function $\Lambda_0(x)$ is discussed in 1.2.11 a) on page 110 and so is $\Lambda_0^*(x)$ on page 112.
See also [1], 11.1 .

$$\int Y_0(x)\, dx \;=\; xY_0(x) + \Phi_Y(x)$$

$$\int H_0^{(p)}(x)\, dx \;=\; xH_0^{(p)}(x) + \Phi_H^{(p)}(x) \;, \quad p = 1,\, 2$$

$$\int x^2\, J_0(x)\, dx \;=\; x^2 J_1(x) - \Phi(x)$$

$$\int x^2\, I_0(x)\, dx \;=\; x^2 I_1(x) + \Psi(x) \qquad *E*$$

$$\int x^2\, K_0(x)\, dx \;=\; -x^2 K_1(x) + \Psi_K(x)$$

$$\int x^4\, J_0(x)\, dx \;=\; (x^4 - 9x^2)J_1(x) + 3x^3 J_0(x) + 9\Phi(x)$$

$$\int x^4\, I_0(x)\, dx \;=\; (x^4 + 9x^2)I_1(x) - 3x^3 I_0(x) + 9\Psi(x)$$

$$\int x^4\, K_0(x)\, dx \;=\; -(x^4 + 9x^2)K_1(x) - 3x^3 K_0(x) + 9\Psi_K(x)$$

$$\int x^6\, J_0(x)\, dx \;=\; (x^6 - 25x^4 + 225x^2)J_1(x) + (5x^5 - 75x^3)J_0(x) - 225\Phi(x)$$

$$\int x^6\, I_0(x)\, dx \;=\; (x^6 + 25x^4 + 225x^2)I_1(x) - (5x^5 + 75x^3)I_0(x) + 225\Psi(x)$$

$$\int x^6\, K_0(x)\, dx \;=\; -(x^6 + 25x^4 + 225x^2)K_1(x) - (5x^5 + 75x^3)K_0(x) + 225\Psi_K(x) \quad \text{and so on.}$$

$$\int x^8\, J_0(x)\, dx \;=\; (x^8 - 49x^6 + 1\,225x^4 - 11\,025x^2)J_1(x) + (7x^7 - 245x^5 + 3\,675x^3)J_0(x) + 11\,025\Phi(x)$$

$$\int x^8\, I_0(x)\, dx \;=\; (x^8 + 49x^6 + 1\,225x^4 + 11\,025x^2)I_1(x) - (7x^7 + 245x^5 + 3\,675x^3)I_0(x) + 11\,025\Psi(x)$$

$$\begin{aligned}\int x^{10}\, J_0(x)\, dx \;&=\; (x^{10} - 81x^8 + 3\,969x^6 - 99\,225x^4 + 893\,025)J_1(x) + \\ &+(9x^9 - 567x^7 + 19\,845x^5 - 297\,675x^3)J_0(x) - 893\,025\Phi(x)\end{aligned}$$

$$\begin{aligned}\int x^{10}\, I_0(x)\, dx \;&=\; (x^{10} + 81x^8 + 3\,969x^6 + 99\,225x^4 + 893\,025)I_1(x) - \\ &-(9x^9 + 567x^7 + 19\,845x^5 + 297\,675x^3)I_0(x) + 893\,025\Psi(x)\end{aligned}$$

$$\begin{aligned}\int x^{12}\, J_0(x)\, dx \;&=\; (11x^{11} - 1\,089x^9 + 68\,607x^7 - 2\,401\,245x^5 + 36\,018\,675x^3)J_0(x) + \\ &+(x^{12} - 121x^{10} + 9\,801x^8 - 480\,249x^6 + 12\,006\,225x^4 - 108\,056\,025x^2)J_1(x) + 108\,056\,025\Phi(x)\end{aligned}$$

$$\begin{aligned}\int x^{12}\, I_0(x)\, dx \;&=\; (x^{12} + 121x^{10} + 9\,801x^8 + 480\,249x^6 + 12\,006\,225x^4 + 108\,056\,025x^2)I_1(x) - \\ &-(11x^{11} + 1\,089x^9 + 68\,607x^7 + 2\,401\,245x^5 + 36\,018\,675x^3)I_0(x) + 108\,056\,025\Psi(x)\end{aligned}$$

Let

$$n!! \;=\; \begin{cases} 2\cdot 4\cdot\ldots\cdot(n-2)\cdot n & , \quad n = 2m \\ 1\cdot 3\cdot 5\cdot\ldots\cdot(n-2)\cdot n & , \quad n = 2m+1 \end{cases}$$

and $n!! = 1$ in the case $n \le 0$.

<u>General formulas:</u>

$$\begin{aligned}\int x^{2n}\, J_0(x)\, dx &= \left(\sum_{k=0}^{n-2}(-1)^k \frac{[(2n-1)!!]^2\, x^{2n-2k-1}}{[(2n-1-2k)!!]\cdot[(2n-3-2k)!!]}\right) J_0(x) + \\ &+\left(\sum_{k=0}^{n-1}(-1)^k \left[\frac{(2n-1)!!}{(2n-1-2k)!!}\right]^2 x^{2n-2k}\right) J_1(x) + (-1)^n\,[(2n-1)!!]^2\,\Phi(x) = \\ &= \left(\sum_{k=0}^{n-2}(-1)^k \frac{[(2n)!]^2\cdot(n-k)!\cdot(n-k-1)!\cdot x^{2n-2k-1}}{2^{2k+1}\cdot(n!)^2\cdot(2n-2k)!\cdot(2n-2-2k)!}\right) J_0(x) + \\ &+\left(\sum_{k=0}^{n-1}(-1)^k \left[\frac{(2n)!\cdot(n-k)!}{2^k\cdot(n!)\cdot(2n-2k)!}\right]^2 x^{2n-2k}\right) J_1(x) + (-1)^n \left[\frac{(2n)!}{2^n\cdot n!}\right]^2 \Phi(x)\end{aligned}$$

and

$$\begin{aligned}\int x^{2n}\, I_0(x)\, dx &= \left(\sum_{k=0}^{n-1}\left[\frac{(2n-1)!!}{(2n-1-2k)!!}\right]^2 x^{2n-2k}\right) I_1(x) - \\ &-\left(\sum_{k=0}^{n-2}\frac{[(2n-1)!!]^2\, x^{2n-2k-1}}{[(2n-1-2k)!!]\cdot[(2n-3-2k)!!]}\right) I_0(x) + [(2n-1)!!]^2\,\Psi(x) =\end{aligned}$$

$$= \left(\sum_{k=0}^{n-1} \left[\frac{(2n)! \cdot (n-k)!}{2^k \cdot (n!) \cdot (2n-2k)!} \right]^2 x^{2n-2k} \right) I_1(x) -$$

$$- \left(\sum_{k=0}^{n-2} \frac{[(2n)!]^2 \cdot (n-k)! \cdot (n-k-1)! \cdot x^{2n-2k-1}}{2^{2k+1} \cdot (n!)^2 \cdot (2n-2k)! \cdot (2n-2-2k)!} \right) I_0(x) + \left[\frac{(2n)!}{2^n \cdot n!} \right]^2 \Psi(x)$$

<u>Recurrence formulas:</u>

$$\int x^{2n+2} J_0(x)\,dx = (2n+1)x^{2n+1} J_0(x) + x^{2n+2} J_1(x) - (2n+1)^2 \int x^{2n} J_0(x)\,dx$$

$$\int x^{2n+2} I_0(x)\,dx = -(2n+1)x^{2n+1} I_0(x) + x^{2n+2} I_1(x) + (2n+1)^2 \int x^{2n} I_0(x)\,dx$$

$$\int x^{2n+2} K_0(x)\,dx = -(2n+1)x^{2n+1} K_0(x) - x^{2n+2} K_1(x) + (2n+1)^2 \int x^{2n} K_0(x)\,dx$$

In the case $n < 0$ the previous formulas give

$$\int \frac{J_0(x)}{x^2}\,dx = J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x)$$

$$\int \frac{I_0(x)}{x^2}\,dx = \frac{x^2-1}{x} I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_0(x)}{x^2}\,dx = \frac{x^2-1}{x} K_0(x) + K_1(x) + \Psi_K(x)$$

$$\int \frac{J_0(x)}{x^4}\,dx = \frac{1}{9} \left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^2-1}{x^2} J_1(x) + \Phi(x) \right] \qquad *E*$$

$$\int \frac{I_0(x)}{x^4}\,dx = \frac{1}{9} \left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^2+1}{x^2} I_1(x) + \Psi(x) \right] \qquad *E*$$

$$\int \frac{K_0(x)}{x^4}\,dx = \frac{1}{9} \left[\frac{x^4-x^2-3}{x^3} K_0(x) + \frac{x^2+1}{x^2} K_1(x) + \Psi_K(x) \right]$$

$$\int \frac{J_0(x)}{x^6}\,dx = \frac{1}{225} \left[\frac{x^4-x^2+9}{x^4} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^6}\,dx = \frac{1}{225} \left[\frac{x^6-x^4-3x^2-45}{x^5} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x) \right]$$

$$\int \frac{K_0(x)}{x^6}\,dx = \frac{1}{225} \left[\frac{x^6-x^4-3x^2-45}{x^5} K_0(x) + \frac{x^4+x^2+9}{x^4} K_1(x) + \Psi_K(x) \right] \quad \text{and so on.}$$

$$\int \frac{J_0(x)}{x^8}\,dx = \frac{1}{11\,025} \left[\frac{x^8+x^6-3x^4+45x^2-1\,575}{x^7} J_0(x) - \frac{x^6-x^4+9x^2-225}{x^6} J_1(x) + \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^8}\,dx = \frac{1}{11\,025} \left[\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7} I_0(x) - \frac{x^6+x^4+9x^2+225}{x^6} I_1(x) + \Psi(x) \right]$$

$$\int \frac{J_0(x)}{x^{10}}\,dx = \frac{1}{893\,025} \left[\frac{x^8-x^6+9x^4-225x^2+11\,025}{x^8} J_1(x) - \right.$$

$$\left. - \frac{x^{10}+x^8-3x^6+45x^4-1\,575x^2+99\,225}{x^9} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^{10}}\,dx = \frac{1}{893\,025} \left[\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9} I_0(x) - \right.$$

$$-\frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8}I_1(x)+\Psi(x)\Bigg]$$

$$\int\frac{J_0(x)}{x^{12}}\,dx = \frac{1}{108\,056\,025}\left[\frac{x^{12}+x^{10}-3x^8+45x^6-1\,575x^4+99\,225x^2-9\,823\,275}{x^{11}}\,J_0(x)-\right.$$

$$\left.-\;\frac{x^{10}-x^8+9x^6-225x^4+11\,025x^2-893\,025}{x^{10}}J_1(x)+\Phi(x)\right]$$

$$\int\frac{I_0(x)}{x^{12}}\,dx = \frac{1}{108\,056\,025}\left[\frac{x^{12}-x^{10}-3x^8-45x^6-1\,575x^4-99\,225x^2-9\,823\,275}{x^{11}}\,I_0(x)-\right.$$

$$\left.-\;\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}I_1(x)+\Psi(x)\right]$$

<u>General formula:</u> With $n!!$ as defined on page 8 holds

$$\int\frac{J_0(x)\,dx}{x^{2n}}=\frac{(-1)^n}{[(2n-1)!!]^2}\left[\left(x+\sum_{k=0}^{n-1}(-1)^k\cdot(2k+1)!!\cdot(2k-1)!!\cdot x^{-2k-1}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-2}(-1)^k\cdot[(2k+1)!!]^2\,x^{-2k-2}\right)J_1(x)+\Phi(x)\right]=$$

$$=\frac{(-1)^n\cdot 2^{2n}\cdot(n!)^2}{(2n)!}\left\{\left(x+\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!\cdot(2k)!}{2^{2k+1}\cdot(k+1)!\cdot k!\cdot x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-2}\frac{(-1)^k}{x^{2k+2}}\left[\frac{(2k+2)!}{2^{k+1}\cdot(k+1)!}\right]^2\right)J_1(x)+\Phi(x)\right\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n}I_0(x)\,dx$ and $\int x^{-2n}K_0(x)\,dx$.

1.1.2. Integrals of the type $\int x^{2n+1} Z_0(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p=1,2.$

$$\int x\,J_0(x)\,dx \;=\; xJ_1(x)$$

$$\int x\,I_0(x)\,dx \;=\; x\,I_1(x)$$

$$\int x\,K_0(x)\,dx \;=\; -x\,K_1(x)$$

$$\int x^3\,J_0(x)\,dx \;=\; x\left[2x\,J_0(x)+(x^2-4)\,J_1(x)\right]$$

$$\int x^3\,I_0(x)\,dx \;=\; x\left[(x^2+4)\,I_1(x)-2x\,I_0(x)\right]$$

$$\int x^3\,K_0(x)\,dx \;=\; -x\left[(x^2+4)\,K_1(x)+2x\,K_0(x)\right]$$

$$\int x^5\,J_0(x)\,dx \;=\; x\left[(4x^3-32x)\,J_0(x)+(x^4-16x^2+64)\,J_1(x)\right]$$

$$\int x^5\,I_0(x)\,dx \;=\; x\left[(x^4+16x^2+64)\,I_1(x)-(4x^3+32x)\,I_0(x)\right]$$

$$\int x^5\,K_0(x)\,dx \;=\; -x\left[(x^4+16x^2+64)\,K_1(x)+(4x^3+32x)\,K_0(x)\right]$$

$$\int x^7\,J_0(x)\,dx \;=\; x\left[(6x^5-144x^3+1\,152x)\,J_0(x)+(x^6-36x^4+576x^2-2\,304)\,J_1(x)\right]$$

$$\int x^7\,I_0(x)\,dx \;=\; x\left[(x^6+36x^4+576x^2+2\,304)\,I_1(x)-(6x^5+144x^3+1\,152x)\,I_0(x)\right]$$

$$\int x^7\,K_0(x)\,dx \;=\; -x\left[(x^6+36x^4+576x^2+2\,304)\,K_1(x)+(6x^5+144x^3+1\,152x)\,K_0(x)\right]$$

$$\int x^9\,J_0(x)\,dx \;=$$
$$=\; x\left[(8x^7-384x^5+9\,216x^3-73\,728x)\,J_0(x)+(x^8-64x^6+2\,304x^4-36\,864x^2+147\,456)\,J_1(x)\right]$$

$$\int x^9\,I_0(x)\,dx \;=$$
$$=\; x\left[(x^8+64x^6+2\,304x^4+36\,864x^2+147\,456)\,I_1(x)-(8x^7+384x^5+9\,216x^3+73\,728x)\,I_0(x)\right]$$

$$\int x^9\,K_0(x)\,dx \;=$$
$$=\; -x\left[(x^8+64x^6+2\,304x^4+36\,864x^2+147\,456)\,K_1(x)+(8x^7+384x^5+9\,216x^3+73\,728x)\,K_0(x)\right]$$

Let

$$\int x^m J_0(x)\,dx = x[P_m(x)J_0(x)+Q_m(x)J_1(x)] \quad \text{and} \quad \int x^m I_0(x)\,dx = x[Q_m^*(x)I_1(x)-P_m^*(x)I_0(x)]\,,$$

$$\int x^m K_0(x)\,dx = -x[Q_m^*(x)K_1(x)+P_m^*(x)K_0(x)]\,,$$

then holds

$P_{11}(x)=10\,x^9-800\,x^7+38400\,x^5-921600\,x^3+7372800\,x$
$Q_{11}(x)=x^{10}-100\,x^8+6400\,x^6-230400\,x^4+3686400\,x^2-14745600$
$P^*_{11}(x)=10\,x^9+800\,x^7+38400\,x^5+921600\,x^3+7372800\,x$ *E*
$Q^*_{11}(x)=x^{10}+100\,x^8+6400\,x^6+230400\,x^4+3686400\,x^2+14745600$

$P_{13}(x) = 12\,x^{11} - 1440\,x^9 + 115200\,x^7 - 5529600\,x^5 + 132710400\,x^3 - 1061683200\,x$
$Q_{13}(x) = x^{12} - 144\,x^{10} + 14400\,x^8 - 921600\,x^6 + 33177600\,x^4 - 530841600\,x^2 + 2123366400$
$P^*_{13}(x) = 12\,x^{11} + 1440\,x^9 + 115200\,x^7 + 5529600\,x^5 + 132710400\,x^3 + 1061683200\,x$ *E*
$Q^*_{13}(x) = x^{12} + 144\,x^{10} + 14400\,x^8 + 921600\,x^6 + 33177600\,x^4 + 530841600\,x^2 + 2123366400$

$P_{15}(x) = 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 - 22579200\,x^7 + 1083801600\,x^5 - 26011238400\,x^3 + 208089907200\,x$
$Q_{15}(x) =$
$x^{14} - 196\,x^{12} + 28224\,x^{10} - 2822400\,x^8 + 180633600\,x^6 - 6502809600\,x^4 + 104044953600\,x^2 - 416179814400$
$P^*_{15}(x) =$
$= 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 + 22579200\,x^7 + 1083801600\,x^5 + 26011238400\,x^3 + 208089907200\,x$ *E*
$Q^*_{15}(x) =$
$x^{14} + 196\,x^{12} + 28224\,x^{10} + 2822400\,x^8 + 180633600\,x^6 + 6502809600\,x^4 + 104044953600\,x^2 + 416179814400$

<u>Recurrence formulas:</u>

$$\int x^{2n+1}\,J_0(x)\,dx \;=\; 2nx^{2n}\,J_0(x) + x^{2n+1}\,J_1(x) - 4n^2 \int x^{2n-1}\,J_0(x)\,dx$$

$$\int x^{2n+1}\,I_0(x)\,dx \;=\; -2nx^{2n}\,I_0(x) + x^{2n+1}\,I_1(x) + 4n^2 \int x^{2n-1}\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,dx \;=\; -2nx^{2n}\,K_0(x) - x^{2n+1}\,K_1(x) + 4n^2 \int x^{2n-1}\,K_0(x)\,dx$$

General formula: With $n!!$ as defined on page 8 holds

$$\int x^{2n+1}\,J_0(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{[(2n)!!]^2\, x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2k-2)!!]}\right) J_0(x) +$$

$$+ \left(\sum_{k=0}^{n} (-1)^k \left[\frac{(2n)!!}{(2n-2k)!!}\right]^2 x^{2n+1-2k}\right) J_1(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{2^{2k+1}\cdot (n!)^2\, x^{2n-2k}}{(n-k)!\cdot(n-k-1)!}\right) J_0(x) + \left(\sum_{k=0}^{n} (-1)^k \left[\frac{2^k\cdot n!}{(n-k)!}\right]^2 x^{2n+1-2k}\right) J_1(x)\;.$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1}\,I_0(x)\,dx$ and $\int x^{2n+1}\,K_0(x)\,dx$.

1.1.3. Integrals of the type $\int x^{-2n-1} \cdot Z_0(x)\,dx$

The basic integral

$$\int \frac{J_0(x)\,dx}{x} \quad \text{can be expressed by} \quad \int_0^x \frac{1-J_0(t)}{t}\,dt \quad \text{or} \quad -\int_x^\infty \frac{J_0(t)\,dt}{t} = Ji_0(x)\ ,$$

see [1], equation 11.1.19 and the following formulas. There are given asymptotic expansions and polynomial approximations as well. Tables of these functions may be found by [1], [11.13] or [11.22]. The function $Ji_0(x)$ is introduced and discussed in [9].
For fast computations of this integrals one should use approximations with Chebyshev polynomials, see [2], tables 9.3 .
I got the information from F. Nouguier, that there is an error in a formula in [9], p. 278.
The true formula is

$$Ji_0(x) - \ln x = \frac{\sin \pi x}{\pi x}(\gamma - \ln 2) + \frac{2x \sin \pi x}{\pi} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^2 - x^2} \left[Ji_0(s) - \ln s \right] .$$

The power series in

$$\int \frac{I_0(x)\,dx}{x} \quad = \quad \ln x \; + \; \sum_{k=1}^{\infty} \frac{1}{2k \cdot (k!)^2} \left(\frac{x}{2} \right)^{2k} \qquad *E*$$

can be used without numerical problems.

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p = 1, 2.$

$$\int \frac{J_0(x)\,dx}{x^3} = -\frac{J_0(x)}{2x^2} + \frac{J_1(x)}{4x} - \frac{1}{4} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^3} = -\frac{I_0(x)}{2x^2} - \frac{I_1(x)}{4x} + \frac{1}{4} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^5} = \left(\frac{1}{32x^2} - \frac{1}{4x^4} \right) J_0(x) + \left(-\frac{1}{64x} + \frac{1}{16x^3} \right) J_1(x) + \frac{1}{64} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^5} = -\left(\frac{1}{32x^2} + \frac{1}{4x^4} \right) I_0(x) - \left(\frac{1}{64x} + \frac{1}{16x^3} \right) I_1(x) + \frac{1}{64} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^7} = \frac{-x^4 + 8\,x^2 - 192}{1152\,x^6} J_0(x) + \frac{x^4 - 4\,x^2 + 64}{2304\,x^5} J_1(x) - \frac{1}{2304} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^7} = -\frac{x^4 + 8\,x^2 + 192}{1152\,x^6} I_0(x) - \frac{x^4 + 4\,x^2 + 64}{2304\,x^5} I_1(x) + \frac{1}{2304} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^9} =$$

$$= \frac{x^6 - 8\,x^4 + 192\,x^2 - 9216}{73728\,x^8} J_0(x) + \frac{-x^6 + 4\,x^4 - 64\,x^2 + 2304}{147456\,x^7} J_1(x) + \frac{1}{147456} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^9} =$$

$$= -\frac{x^6 + 8\,x^4 + 192\,x^2 + 9216}{73728\,x^8} I_0(x) - \frac{x^6 + 4\,x^4 + 64\,x^2 + 2304}{147456\,x^7} I_1(x) + \frac{1}{147456} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^{11}} = \frac{-x^8 + 8\,x^6 - 192\,x^4 + 9216\,x^2 - 737280}{7372800\,x^{10}} J_0(x) +$$

$$+ \frac{x^8 - 4\,x^6 + 64\,x^4 - 2304\,x^2 + 147456}{14745600\,x^9} J_1(x) - \frac{1}{14745600} \int \frac{J_0(x)\,dx}{x}$$

$$*E* \qquad \int \frac{I_0(x)\,dx}{x^{11}} = -\frac{x^8 + 8\,x^6 + 192\,x^4 + 9216\,x^2 + 737280}{7372800\,x^{10}} I_0(x) -$$

$$- \frac{x^8 + 4\,x^6 + 64\,x^4 + 2304\,x^2 + 147456}{14745600\,x^9} I_1(x) + \frac{1}{14745600} \int \frac{I_0(x)\,dx}{x}$$

Descending recurrence formulas:

$$\int x^{-2n-1}\,J_0(x)\,dx = \frac{1}{4n^2}\,\left[x^{-2n+1}J_1(x) - 2nx^{-2n}\,J_0(x) - \int x^{-2n+1}\,J_0(x)\,dx\right]$$

$$\int x^{-2n-1}\,I_0(x)\,dx = \frac{1}{4n^2}\,\left[-x^{-2n+1}I_1(x) - 2nx^{-2n}\,I_0(x) + \int x^{-2n+1}\,I_0(x)\,dx\right]$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_0(x)\,dx}{x^{2n+1}} =$$

$$= \frac{(-1)^n}{[(2n)!!]^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!!\cdot(2k)!!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{[(2k)!!]^2}{x^{2k+1}}\right)J_1(x) + \int\frac{J_0(x)\,dx}{x}\right\} = \quad *E*$$

$$= \frac{(-1)^n}{2^{2n}\cdot(n!)^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k+1}\cdot(k+1)!\cdot k!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k}\cdot(k!)^2}{x^{2k+1}}\right)J_1(x) + \int\frac{J_0(x)\,dx}{x}\right\} \quad *E*$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n-1}\,I_0(x)\,dx$.

1.1.4. Integrals of the type $\int x^{2n} Z_1(x)\, dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)},\ p = 1, 2$.

$$\int J_1(x)\, dx = -J_0(x)$$

$$\int I_1(x)\, dx = I_0(x)$$

$$\int K_1(x)\, dx = -K_0(x)$$

$$\int x^2 J_1(x)\, dx = x\,[2J_1(x) - x\, J_0(x)]$$

$$\int x^2 I_1(x)\, dx = x\,[x\, I_0(x) - 2I_1(x)]$$

$$\int x^2 K_1(x)\, dx = -x\,[x\, K_0(x) + 2K_1(x)]$$

$$\int x^4 J_1(x)\, dx = x\left[(4x^2 - 16)\, J_1(x) - (x^3 - 8x)\, J_0(x)\right]$$

$$\int x^4 I_1(x)\, dx = x\left[(x^3 + 8x)\, I_0(x) - (4x^2 + 16)\, I_1(x)\right]$$

$$\int x^4 K_1(x)\, dx = -x\left[(x^3 + 8x)\, K_0(x) + (4x^2 + 16)\, K_1(x)\right]$$

$$\int x^6 J_1(x)\, dx = x\left[(6x^4 - 96x^2 + 384)\, J_1(x) - (x^5 - 24x^3 + 192x)\, J_0(x)\right]$$

$$\int x^6 I_1(x)\, dx = x\left[(x^5 + 24x^3 + 192x)\, I_0(x) - (6x^4 + 96x^2 + 384)\, I_1(x)\right]$$

$$\int x^6 K_1(x)\, dx = -x\left[(x^5 + 24x^3 + 192x)\, K_0(x) + (6x^4 + 96x^2 + 384)\, K_1(x)\right]$$

$$\int x^8 J_1(x)\, dx =$$
$$= x\left[(8x^6 - 288x^4 + 4\,608x^2 - 18\,432)\, J_1(x) - (x^7 - 48x^5 + 1\,152x^3 - 9\,216x)\, J_0(x)\right]$$

$$\int x^8 I_1(x)\, dx =$$
$$= x\left[(x^7 + 48x^5 + 1\,152x^3 + 9\,216x)\, I_0(x) - (8x^6 + 288x^4 + 4\,608x^2 + 18\,432)\, I_1(x)\right]$$

$$\int x^8 K_1(x)\, dx =$$
$$= -x\left[(x^7 + 48x^5 + 1\,152x^3 + 9\,216x)\, K_0(x) + (8x^6 + 288x^4 + 4\,608x^2 + 18\,432)\, K_1(x)\right]$$

$$\int x^{10} J_1(x)\, dx = x\left[(10x^8 - 640x^6 + 23\,040x^4 - 368\,640x^2 + 1\,474\,560)\, J_1(x) - \right.$$
$$\left. - (x^9 - 80x^7 + 3\,840x^5 - 92\,160x^3 + 737\,280x)\, J_0(x)\right]$$

$$\int x^{10} I_1(x)\, dx = x\left[(x^9 + 80x^7 + 3\,840x^5 + 92\,160x^3 + 737\,280x)\, I_0(x) - \right.$$
$$\left. -(10x^8 + 640x^6 + 23\,040x^4 + 368\,640x^2 + 1\,474\,560)\, I_1(x)\right]$$

$$\int x^{10} K_1(x)\, dx = -x\left[(x^9 + 80x^7 + 3\,840x^5 + 92\,160x^3 + 737\,280x)\, K_0(x) + \right.$$

$$+(10x^8+640x^6+23\,040x^4+368\,640x^2+1\,474\,560)\,K_1(x)\big]$$

Let

$$\int x^m J_1(x)\,dx = x[Q_m(x)J_1(x)-P_m(x)J_0(x)] \quad \text{and} \quad \int x^m I_1(x)\,dx = x[P_m^*(x)I_0(x)-Q_m^*(x)I_1(x)]\ ,$$

$$\int x^m K_1(x)\,dx = -x[P_m^*(x)I_0(x)+Q_m^*(x)I_1(x)]\ ,$$

then holds

$P_{12}(x)=x^{11}-120\,x^9+9600\,x^7-460800\,x^5+11059200\,x^3-88473600\,x$
$Q_{12}(x)=12\,x^{10}-1200\,x^8+76800\,x^6-2764800\,x^4+44236800\,x^2-176947200$
$P_{12}^*(x)=x^{11}+120\,x^9+9600\,x^7+460800\,x^5+11059200\,x^3+88473600\,x$
$Q_{12}^*(x)=12\,x^{10}+1200\,x^8+76800\,x^6+2764800\,x^4+44236800\,x^2+176947200$ *E*

$P_{14}(x)=x^{13}-168\,x^{11}+20160\,x^9-1612800\,x^7+77414400\,x^5-1857945600\,x^3+14863564800\,x$
$Q_{14}(x)=14\,x^{12}-2016\,x^{10}+201600\,x^8-12902400\,x^6+464486400\,x^4-7431782400\,x^2+29727129600$
$P_{14}^*(x)=x^{13}+168\,x^{11}+20160\,x^9+1612800\,x^7+77414400\,x^5+1857945600\,x^3+14863564800\,x$
$Q_{14}^*(x)=14\,x^{12}+2016\,x^{10}+201600\,x^8+12902400\,x^6+464486400\,x^4+7431782400\,x^2+29727129600$

<u>Recurrence formulas:</u>

$$\int x^{2n+2}\,J_1(x)\,dx = -x^{2n+2}\,J_0(x)+(2n+2)x^{2n+1}\,J_1(x)-4n(n+1)\int x^{2n}\,J_1(x)\,dx$$

$$\int x^{2n+2}\,I_1(x)\,dx = x^{2n+2}\,I_0(x)-(2n+2)x^{2n+1}\,I_1(x)+4n(n+1)\int x^{2n}\,I_1(x)\,dx$$

$$\int x^{2n+2}\,K_1(x)\,dx = -x^{2n+2}\,K_0(x)-(2n+2)x^{2n+1}\,K_1(x)+4n(n+1)\int x^{2n}\,K_1(x)\,dx$$

<u>General formula:</u> With $n!!$ as defined on page 8 holds

$$\int x^{2n}\,J_1(x)\,dx = \left(\sum_{k=0}^{n-1}(-1)^k\,\frac{[(2n)!!]\,\cdot[(2n-2)!!]\,\cdot x^{2n-1-2k}}{[(2n-2-2k)!!]^2}\right)J_1(x)-$$

$$-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{(2n)!!\cdot(2n-2)!!\cdot x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2-2k)!!]}\right)J_0(x)=$$

$$=\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k+1}\cdot(n!)\cdot(n-1)!\cdot x^{2n-1-2k}}{[(n-1-k)!]^2}\right)J_1(x)-$$

$$-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k}\cdot n!\cdot(n-1)!!\cdot x^{2n-2k}}{(n-k)!\cdot(n-1-k)!}\right)J_0(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n}\,I_1(x)\,dx$ and $\int x^{2n}\,K_1(x)\,dx$.

1.1.5. Integrals of the type $\int x^{-2n} \cdot Z_1(x)\,dx$

About the integrals

$$\int \frac{J_0(x)\,dx}{x} \qquad \text{and} \qquad \int \frac{I_0(x)\,dx}{x}$$

see 1.1.3, page 13.

In the following formulas $J_0(x)$ may be substituted by $Y_0(x)$ and simultaneously $J_1(x)$ by $Y_1(x)$.

$$\int \frac{J_1(x)\,dx}{x^2} = -\frac{1}{2x} J_1(x) + \frac{1}{2} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^2} = -\frac{1}{2x} I_1(x) + \frac{1}{2} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2} J_0(x) + \frac{x^2-4}{16\,x^3} J_1(x) - \frac{1}{16} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2} I_0(x) - \frac{x^2+4}{16\,x^3} I_1(x) + \frac{1}{16} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^6} =$$

$$= \frac{x^2-8}{192\,x^4} J_0(x) + \frac{-x^4+4\,x^2-64}{384\,x^5} J_1(x) + \frac{1}{384} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^6} = -\frac{x^2+8}{192\,x^4} I_0(x) - \frac{x^4+4\,x^2+64}{384\,x^5} I_1(x) + \frac{1}{384} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^8} =$$

$$= \frac{-x^4+8\,x^2-192}{9216\,x^6} J_0(x) + \frac{x^6-4\,x^4+64\,x^2-2304}{18432\,x^7} J_1(x) - \frac{1}{18432} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^8} = -\frac{x^4+8\,x^2+192}{9216\,x^6} I_0(x) - \frac{x^6+4\,x^4+64\,x^2+2304}{18432\,x^7} I_1(x) + \frac{1}{18432} \int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^{10}} =$$

$$= \frac{x^6-8\,x^4+192\,x^2-9216}{737280\,x^8} J_0(x) + \frac{-x^8+4\,x^6-64\,x^4+2304\,x^2-147456}{1474560\,x^9} J_1(x) +$$

$$+ \frac{1}{1474560} \int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^{10}} =$$

$$= -\frac{x^6+8\,x^4+192\,x^2+9216}{737280\,x^8} I_0(x) - \frac{x^8+4\,x^6+64\,x^4+2304\,x^2+147456}{1474560\,x^9} I_1(x) +$$

$$+ \frac{1}{1474560} \int \frac{I_0(x)\,dx}{x}$$

Recurrence formulas:

$*E*$ $$\int \frac{J_1(x)\,dx}{x^{2n+2}} = -\frac{J_0(x)}{4n(n+1)x^{2n}} - \frac{J_1(x)}{(2n+2)x^{2n+1}} - \frac{1}{4n(n+1)} \int \frac{J_1(x)\,dx}{x^{2n}}$$

$*E*$ $$\int \frac{I_1(x)\,dx}{x^{2n+2}} = -\frac{I_0(x)}{4n(n+1)x^{2n}} - \frac{I_1(x)}{(2n+2)x^{2n+1}} + \frac{1}{4n(n+1)} \int \frac{I_1(x)\,dx}{x^{2n}}$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_1(x)\,dx}{x^{2n}} = \frac{(-1)^{n+1}}{(2n)!!\cdot(2n-2)!!}\cdot$$

$$\cdot\left\{\left(\sum_{k=0}^{n-2}(-1)^k\,\frac{(2k+2)!!\cdot(2k)!!}{x^{2k+2}}\right)J_0(x)-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{[(2k)!!]^2}{x^{2k+1}}\right)J_1(x)+\int\frac{J_0(x)\,dx}{x}\right\}=$$

$$=\frac{(-1)^{n+1}}{2^{2n-1}\cdot n!\cdot(n-1)!}\cdot$$

$$\cdot\left[\left(\sum_{k=0}^{n-2}(-1)^k\,\frac{2^{2k+1}\cdot(k+1)!\cdot k!}{x^{2k+2}}\right)J_0(x)-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k}\cdot(k!)^2}{x^{2k+1}}\right)J_1(x)+\int\frac{J_0(x)\,dx}{x}\right]$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n}\,I_1(x)\,dx$.

1.1.6. Integrals of the type $\int x^{2n+1} Z_1(x)\,dx$

$\Phi(x)$, $\Phi_Y(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in 1.1.1, page 7 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$ and $\Phi_H^{(p)}(x)$.

$$\int x\, J_1(x)\,dx = \Phi(x)$$

$$\int x\, I_1(x)\,dx = -\Psi(x)$$

$$\int x\, K_1(x)\,dx = \Psi_K(x)$$

$$\int x^3\, J_1(x)\,dx = 3x^2\, J_1(x) - x^3\, J_0(x) - 3\Phi(x)$$

$$\int x^3\, I_1(x)\,dx = -3x^2\, I_1(x) + x^3\, I_0(x) - 3\Psi(x)$$

$$\int x^3\, K_1(x)\,dx = -3x^2\, K_1(x) - x^3\, K_0(x) + 3\Psi_K(x)$$

$$\int x^5\, J_1(x)\,dx = (5x^4 - 45x^2)\, J_1(x) - (x^5 - 15x^3)\, J_0(x) + 45\Phi(x)$$

$$\int x^5\, I_1(x)\,dx = -(5x^4 + 45x^2)\, I_1(x) + (x^5 + 15x^3)\, I_0(x) - 45\Psi(x)$$

$$\int x^5\, K_1(x)\,dx = -(5x^4 + 45x^2)\, K_1(x) - (x^5 + 15x^3)\, K_0(x) + 45\Psi_K(x)$$

$$\int x^7\, J_1(x)\,dx = (7x^6 - 175x^4 + 1\,575x^2)\, J_1(x) - (x^7 - 35x^5 + 525x^3)\, J_0(x) - 1\,575\Phi(x) \qquad *E*$$

$$\int x^7\, I_1(x)\,dx = -(7x^6 + 175x^4 + 1\,575x^2)\, I_1(x) + (x^7 + 35x^5 + 525x^3)\, I_0(x) - 1\,575\Psi(x)$$

$$\int x^7\, K_1(x)\,dx = -(7x^6 + 175x^4 + 1\,575x^2)\, K_1(x) - (x^7 + 35x^5 + 525x^3)\, K_0(x) + 1\,575\Psi_K(x)$$

$$\int x^9\, J_1(x)\,dx =$$
$$= (9x^8 - 441x^6 + 11\,025x^4 - 99\,225x^2)\, J_1(x) - (x^9 - 63x^7 + 2\,205x^5 - 33\,075x^3)\, J_0(x) + 99\,225\,\Phi(x)$$

$$\int x^9\, I_1(x)\,dx =$$
$$= -(9x^8 + 441x^6 + 11\,025x^4 + 99\,225x^2)\, I_1(x) + (x^9 + 63x^7 + 2\,205x^5 + 33\,075x^3)\, I_0(x) - 99\,225\,\Psi(x)$$

$$\int x^9\, K_1(x)\,dx =$$
$$= -(9x^8 + 441x^6 + 11\,025x^4 + 99\,225x^2)\, K_1(x) - (x^9 + 63x^7 + 2\,205x^5 + 33\,075x^3)\, K_0(x) + 99\,225\,\Psi(x)$$

General formula: With $n!!$ as defined on page 8 holds

$$\int x^{2n+1}\, J_1(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{(2n+1)!! \cdot (2n-1)!! \cdot x^{2n-2k}}{[(2n-1-2k)!!]^2} \right) J_1(x) -$$
$$- \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{(2n+1)!! \cdot (2n-1)!! \cdot x^{2n+1-2k}}{(2n+1-2k)!! \cdot (2n-1-2k)!!} \right) J_0(x) + (-1)^n \cdot (2n+1)!! \cdot (2n-1)!!\, \Phi(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k \, \frac{(2n+2)! \cdot (2n)! \cdot [(n-k)!]^2 \cdot x^{2n-2k}}{2^{2k+1} \cdot (n+1)! \cdot n! \cdot [(2n-2k)!]^2} \right) J_1(x)$$

$$- \left(\sum_{k=0}^{n-1} (-1)^k \, \frac{(2n+2)! \cdot (2n)! \cdot (n+1-k)! \cdot (n-k)! \cdot x^{2n+1-2k}}{2^{2k} \cdot (n+1)! \cdot n! \cdot (2n+2-2k)! \cdot (2n-2k)!} \right) J_0(x) +$$

$$+ (-1)^n \frac{(2n+2)! \cdot (2n)!}{2^{2n+1} \cdot (n+1)! \cdot n!} \, \Phi(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1} I_1(x)\,dx$ and $\int x^{2n+1} K_1(x)\,dx$.

Recurrence formulas:

$$\int x^{2n+1} J_1(x)\,dx = -x^{2n+1} J_0(x) + (2n+1)x^{2n} J_1(x) - (2n-1)(2n+1) \int x^{2n-1} J_1(x)\,dx$$

$$\int x^{2n+1} I_1(x)\,dx = x^{2n+1} I_0(x) - (2n+1)x^{2n} I_1(x) + (2n-1)(2n+1) \int x^{2n-1} I_1(x)\,dx$$

$$\int x^{2n+1} K_1(x)\,dx = -x^{2n+1} K_0(x) - (2n+1)x^{2n} K_1(x) + (2n-1)(2n+1) \int x^{2n-1} K_1(x)\,dx$$

Descending:

$$\int \frac{J_1(x)\,dx}{x^{2n+1}} = -\frac{J_0(x)}{(4n^2-1)x^{2n-1}} - \frac{J_1(x)}{(2n+1)x^{2n}} - \frac{1}{4n^2-1} \int \frac{J_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{I_1(x)\,dx}{x^{2n+1}} = -\frac{I_0(x)}{(4n^2-1)x^{2n-1}} - \frac{I_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{I_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{K_1(x)\,dx}{x^{2n+1}} = \frac{K_0(x)}{(4n^2-1)x^{2n-1}} - \frac{K_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{K_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{J_1(x)}{x}\,dx = x \cdot J_0(x) - J_1(x) + \Phi(x)$$

$$\int \frac{I_1(x)}{x}\,dx = x \cdot I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_1(x)}{x}\,dx = -x \cdot K_0(x) - K_1(x) - \Psi_K(x)$$

$$\int \frac{J_1(x)}{x^3}\,dx = \frac{1}{3}\left[\frac{x^2-1}{x^2} J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^3}\,dx = \frac{1}{3}\left[-\frac{x^2+1}{x^2} I_1(x) + \frac{x^2-1}{x} I_0(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^3}\,dx = \frac{1}{3}\left[-\frac{x^2+1}{x^2} K_1(x) - \frac{x^2-1}{x} K_0(x) - \Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^5}\,dx = \frac{1}{45}\left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^4-x^2+9}{x^4} J_1(x) + \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^5}\,dx = \frac{1}{45}\left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^5}\,dx = \frac{1}{45}\left[-\frac{x^4-x^2-3}{x^3} K_0(x) - \frac{x^4+x^2+9}{x^4} K_1(x) - \Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^7}\,dx = \frac{1}{1\,575}\left[\frac{x^6-x^4+9x^2-225}{x^6} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^7}\,dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6+x^4+9x^2+225}{x^6}I_1(x)+\frac{x^6-x^4-3x^2-45}{x^5}I_0(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^7}\,dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6+x^4+9x^2+225}{x^6}K_1(x)-\frac{x^6-x^4-3x^2-45}{x^5}K_0(x)-\Psi_k(x)\right]$$

$$\int \frac{J_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[\frac{x^8+x^6-3x^4+45x^2-1\,575}{x^7}J_0(x)-\frac{x^8-x^6+9x^4-225x^2+11\,025}{x^8}J_1(x)+\Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7}I_0(x)-\frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8}I_1(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[-\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7}K_0(x)-\frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8}I_1(x)-\Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[\frac{x^{10}-x^8+9x^6-225x^4+11\,025x^2-893\,025}{x^{10}}J_1(x)-\right.$$

$$\left.-\frac{x^{10}+x^8-3x^6+45x^4-1\,575x^2+99\,225}{x^9}J_0(x)-\Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}I_1(x)+\right.$$

$$\left.+\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9}I_0(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}K_1(x)-\right.$$

$$\left.-\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9}K_0(x)+\Psi_K(x)\right]$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_1(x)\,dx}{x^{2n+1}} = \frac{(-1)^n}{(2n+1)!!\cdot(2n-1)!!}\left\{\left(x+\sum_{k=0}^{n-1}\frac{(-1)^k\cdot(2k+1)!!\cdot(2k-1)!!}{x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-1}(-1)^k\frac{[(2k+1)!!]^2}{x^{2k+2}}\right)J_1(x)+\Phi(x)\right\} =$$

$$=\frac{2^{2n+1}\cdot(n+1)!\cdot n!}{(2n+2)!\cdot(2n)!}\left\{\left(x-\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!\cdot(2k)!}{2^{2k+1}\cdot(k+1)!\cdot k!\cdot x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-1}(-1)^k\frac{[(2k+2)!]^2}{2^{2k+2}\cdot[(k+1)!]^2\cdot x^{2k+2}}\right)J_1(x)+\Phi(x)\right\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n-1}I_1(x)\,dx$ and $\int x^{-2n-1}K_1(x)\,dx$.

1.1.7. Integrals of the type $\int x^n Z_\nu(x)\,dx,\ \nu > 1$:

From the well-known recurrence relations one gets immadiately

$$\int J_{\nu+1}(x)\,dx \;=\; -2J_\nu(x) + \int J_{\nu-1}(x)\,dx \quad \text{and} \quad \int I_{\nu+1}(x)\,dx \;=\; 2I_\nu(x) - \int I_{\nu-1}(x)\,dx\,.$$

With this formulas follows

$$\int_0^x J_{2\nu}(t)\,dt = \Lambda_0(x) - 2\sum_{\kappa=1}^{n} J_{2\kappa-1}(x)\,, \qquad \int_0^x J_{2\nu+1}(t)\,dt = 1 - J_0(x) - 2\sum_{\kappa=1}^{n} J_{2\kappa}(x)$$

$$\int_0^x I_{2\nu}(t)\,dt = (-1)^n\Lambda_0^*(x) + 2\sum_{\kappa=1}^{n}(-1)^{n+\kappa} I_{2\kappa-1}(x)\,, \quad \int_0^x I_{2\nu+1}(t)\,dt = (-1)^n[I_0(x)-1] + 2\sum_{\kappa=1}^{n}(-1)^{n+\kappa} I_{2\kappa}(x)$$

The integrals $\Lambda_0(x)$ and $\Lambda_0^*(x)$ are defined on page 7 and discussed on page 110 and 112.
Holds

$$\int Y_{2\nu}(x)\,dx = xY_0(x) + \Phi_Y(x) - 2\sum_{\kappa=1}^{n} Y_{2\kappa-1}(x)\,, \quad \int Y_{2\nu+1}(x)\,dx = -Y_0(x) - 2\sum_{\kappa=1}^{n} Y_{2\kappa-1}(x)$$

$$\int H_{2\nu}^{(1)}(x)\,dx = xH_0^{(1)}(x) + \Phi_H^{(1)}(x) - 2\sum_{\kappa=1}^{n} H_{2\kappa-1}^{(1)}(x)\,, \quad \int H_{2\nu+1}^{(1)}(x)\,dx = -H_0^{(1)}(x) - 2\sum_{\kappa=1}^{n} H_{2\kappa-1}^{(1)}(x)$$

$$\int H_{2\nu}^{(2)}(x)\,dx = xH_0^{(2)}(x) + \Phi_H^{(2)}(x) - 2\sum_{\kappa=1}^{n} H_{2\kappa-1}^{(2)}(x)\,, \quad \int H_{2\nu+1}^{(2)}(x)\,dx = -H_0^{(2)}(x) - 2\sum_{\kappa=1}^{n} H_{2\kappa-1}^{(2)}(x)$$

$$\int K_{2\nu}(x)\,dx = (-1)^n\left\{xK_0(x) + \frac{\pi x}{2}\left[K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)\right]\right\} + 2\sum_{\kappa=1}^{n}(-1)^{n+\kappa}K_{2\kappa-1}(x)\,,$$

$$\int K_{2\nu+1}(x)\,dx = (-1)^{n+1}K_0(x) + 2\sum_{\kappa=1}^{n}(-1)^{n+\kappa+1}K_{2\kappa}(x)$$

About the functions $\Phi_Y(x),\ \Phi_H^{(1)}(x),\ \Phi_H^{(2)}(x)$ see page 7.
Further on, holds

$$\int_0^x t\,J_{2\nu+1}(t)\,dt = (2\nu+1)\Lambda_0(x) - x\left[J_0(x) + 2\sum_{\kappa=1}^{\nu} J_{2\kappa}(x)\right] - 4\sum_{\kappa=0}^{\nu-1}(\nu-\kappa)J_{2\kappa+1}(x)$$

$$\int_0^x t\,J_{2\nu}(t)\,dt = -x\left[J_1(x) + 2\sum_{\kappa=1}^{\nu-1} J_{2\kappa+1}(x)\right] + 2\nu[1-J_0(x)] - 4\sum_{\kappa=1}^{\nu-1}(\nu-\kappa)J_{2\kappa}(x)$$

$$\int_0^x t\,I_{2\nu+1}(t)\,dt = (-1)^{\nu+1}\left[(2\nu+1)\Lambda_0^*(x) - xI_0(x) - 2x\sum_{\kappa=1}^{\nu}(-1)^\kappa I_{2\kappa}(x) - 4\sum_{\kappa=0}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa+1}(x)\right]$$

$$\int_0^x t\,I_{2\nu}(t)\,dt = (-1)^{\nu+1}\left[xI_1(x) + 2x\sum_{\kappa=1}^{\nu-1}(-1)^\kappa I_{2\kappa+1}(x) + 2\nu[1-I_0(x)] - 4\sum_{\kappa=1}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa}(x)\right]$$

Some of the previous sums may cause numerical problems, if x is located near 0. For instance, the sum

$$\int_0^x t\,I_6(t)\,dt = xJ_1(x) - 2xJ_3(x) + 2xJ_5(x) + 6 - 6J_0(x) + 8J_2(x) - 4J_4(x)$$

gives with $x = 0.3$

$$0.045\,508\,152\,001 - 0.000\,339\,402\,714 + 0.000\,000\,381\,114 + 6 - 6.135\,761\,276\,110 + 0.090\,676\,901\,288 -$$

$$-0.000\,084\,755\,400 = 6.136\,185\,434\,403 - 6.136\,185\,434\,224 = 0.000\,000\,000\,179\,,$$

which means the loss of 10 decimal digits.
For that reason the value of such integrals should be computed by the power series or other formulas. See also the following remark.

In the following the integrals are expressed by $Z_0(x)$ and $Z_1(x)$.
Integrals with $-2 \le n \le 4$ are written explicitely: at first $n = 0,\ 1,\ 2,\ 3,\ 4$, after them $n = -1,\ -2$. In the other cases the functions $\mathcal{P}_\nu^{(n)}(x)$, $\mathcal{Q}_\nu^{(n)}(x)$ and the coefficients $\mathcal{R}_\nu^{(n)}$, $\mathcal{S}_\nu^{(n)}$ describe the integral

$$\int x^n \cdot J_\nu(x)\,dx = \mathcal{P}_\nu^{(n)}(x)\,J_0(x) + \mathcal{Q}_\nu^{(n)}(x)J_1(x) + \mathcal{R}_\nu^{(n)}\,\Lambda_0(x) + \mathcal{S}_\nu^{(n)} \int \frac{J_0(x)\,dx}{x}\,.$$

Furthermore, let

$$\int x^n \cdot I_\nu^*(x)\,dx = \mathcal{P}_\nu^{(n),*}(x)\,I_0(x) + \mathcal{Q}_\nu^{(n),*}(x)I_1(x) + \mathcal{R}_\nu^{(n),*}\,\Lambda_0^*(x) + \mathcal{S}_\nu^{(n),*} \int \frac{I_0(x)\,dx}{x}\,.$$

Concerning $\int x^{-1} \cdot Z_0(x)\,dx$ see 1.1.3., page 13.

Simple recurrence formula:

$$\begin{aligned}\int x^n \cdot J_{\nu+1}(x)\,dx &= 2\nu \int x^{n-1} \cdot J_\nu(x)\,dx - \int x^n \cdot J_{\nu-1}(x)\,dx \\ \int x^n \cdot I_{\nu+1}(x)\,dx &= -2\nu \int x^{n-1} \cdot J_\nu(x)\,dx + \int x^n \cdot J_{\nu-1}(x)\,dx\end{aligned}$$

The integrals of $x^n\,Z_0(x)$ and $x^n\,Z_1(x)$ to start this recurrences are already described.

Remark:
Let $F_\nu^{(m)}(x)$ denote the antiderivative of $x^m\,Z_\nu(x)$ as given in the following tables. They do not exist in the point $x = 0$ in the case $\nu + m < 0$. However, even if $\nu + m \ge 0$ the value of $F_\nu^{(m)}(0)$ sometimes turns out to be a limit of the type $\infty - \infty$. For instance, holds

$$\int \frac{J_3(x)\,dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3} = F_3^{(-2)}(x) \quad \text{with} \quad \lim_{x\to 0} F_3^{(-2)}(x) = -\frac{1}{8}\,.$$

With $L_{\nu,m} = \lim_{x\to 0} F_\nu^{(m)}(x)$ for the Bessel functions $J_\nu(x)$ and $L_{\nu,m}^*$ for the modified Bessel functions $I_\nu(x)$ one has the following limits in the tables of integrals (The values $L_{\nu,m} = 0$ are omitted.):

$L_{2,-1} = -1/2,\ L_{2,-1}^* = 1/2$

$L_{3,0} = -1,\ L_{3,-2} = -1/8;\ L_{3,0}^* = -1,\ L_{3,-2}^* = 1/8$

$L_{4,1} = -4,\ L_{4,-1} = -1/4,\ L_{4,-3} = -1/48;\ L_{4,1}^* = 4,\ L_{4,-1}^* = -1/4,\ L_{4,-3}^* = 1/48$

$L_{5,2} = -24,\ L_{5,0} = -1,\ L_{5,-2} = -1/24,\ L_{5,-4} = -1/384;$
$L_{5,2}^* = -24,\ L_{5,0}^* = 1,\ L_{5,-2}^* = -1/24,\ L_{5,-4}^* = 1/384$

$L_{6,3} = -192,\ L_{6,1} = -6,\ L_{6,-1} = -1/6,\ L_{6,-3} = -1/192,\ L_{6,-5} = -1/3840;$
$L_{6,3}^* = 192,\ L_{6,1}^* = -6,\ L_{6,-1}^* = 1/6,\ L_{6,-3}^* = -1/192,\ L_{6,-5}^* = 1/3840$

$L_{7,4} = -1920,\ L_{7,2} = -48,\ L_{7,0} = -1,\ L_{7,-2} = -1/48,\ L_{7,-4} = -1/1920,\ L_{7,-6} = -1/46080;$
$L_{7,4}^* = -1920,\ L_{7,2}^* = 48,\ L_{7,0}^* = -1,\ L_{7,-2}^* = 1/48,\ L_{7,-4}^* = -1/1920,\ L_{7,-6}^* = 1/46080$

$L_{8,5} = -23040,\ L_{8,3} = -480,\ L_{8,1} = -8,\ L_{8,-1} = -1/8,\ L_{8,-3} = -1/480,\ L_{8,-5} = -1/23040;$
$L_{8,5}^* = 23040,\ L_{8,3}^* = -480,\ L_{8,1}^* = 8,\ L_{8,-1}^* = -1/8,\ L_{8,-3}^* = 1/480,\ L_{8,-5}^* = -1/23040$

$L_{9,6} = -322560,\ L_{9,4} = -5760,\ L_{9,2} = -80,\ L_{9,0} = -1,\ L_{9,-2} = -1/80,\ L_{9,-4} = -1/5760,$
$L_{9,-6} = -1/322560;$
$L_{9,6}^* = -322560,\ L_{9,4}^* = 5760,\ L_{9,2}^* = -80,\ L_{9,0}^* = 1,\ L_{9,-2}^* = -1/80,\ L_{9,-4}^* = 1/5760,$
$L_{9,-6}^* = -1/322560$

$L_{10,7} = -5160960,\ L_{10,5} = -80640,\ L_{10,3} = -960,\ L_{10,1} = -10,\ L_{10,-1} = -1/10,\ L_{10,-3} = -1/960,$
$L_{10,-5} = -1/80640;$
$L_{10,7}^* = 5160960,\ L_{10,5}^* = -80640,\ L_{10,3}^* = 960,\ L_{10,1}^* = -10,\ L_{10,-1}^* = 1/10,\ L_{10,-3}^* = -1/960,$
$L_{10,-5}^* = 1/80640$

In the described cases of limits of the type $\infty-\infty$ the numerical computation of $F_\nu^{(m)}(x)$ causes difficulties, if $0 < x << 1$. Then it is preferable to use the power series, which has a fast convergengence for such values of x. With $m+\nu \geq 0$ holds

$$\int_0^x t^m\, J_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{k! \cdot (\nu+k)! \cdot 4^k \cdot (m+\nu+1+2k)}$$

and

$$\int_0^x t^m\, I_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^\infty \frac{x^{2k}}{k! \cdot (\nu+k)! \cdot 4^k \cdot (m+\nu+1+2k)} \,.$$

From this one has

$$F_\nu^{(m)}(x) = L_{\nu,m} + \int_0^x t^m\, J_\nu(t)\, dt \quad \text{and} \quad F_\nu^{*,(m)}(x) = L_{\nu,m}^* + \int_0^x t^m\, I_\nu(t)\, dt \,.$$

For instance,

$$\int_{0.002}^3 \frac{J_3(x)}{x^2}\, dx = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3}\bigg|_{0.002}^3 =$$

$$= (-0.0288946616557703820 - 0.0251154784093286266) - (249999.750000062500 - 249999.875000020834) =$$

$$= -0.0540101400650990086 - (-0.124999958334^*) = 0.070989818269$$

It was a loss of seven decimal digits at $x = 0.002$. This value may be found without problems by the power series:

$$F_3^{(-2)}(0.002) =$$

$$= -\frac{1}{8} + 5\cdot 10^{-7}\,[0.0833333333333333335 - 1.04166666666666667\cdot 10^{-8} + 6.94444444444444445\cdot 10^{-16} - \ldots] =$$

$$= -\frac{1}{8} + 4.16666614583336806 \cdot 10^{-8} = -0.124999958333338542$$

In the previous value, signed by *, the last digit should be 3 instead of 4 and the result had to finish with 8.

The integrals with $I_\nu(x)$ may be computed in the same way.

This method can be used even if $\nu + m < 0$. For instance,

$$\int_{0.002}^3 \frac{J_4(x)}{x^7}\, dx = \int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx + \int_1^3 \frac{J_4(x)}{x^7}\, dx$$

and the second integral is given in the following tables. For the first one holds with the power series of the function $J_4(x)$

$$\int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx =$$

$$= \int_{0.002}^1 \frac{1}{x^7}\left(\frac{1}{384}\, x^4 - \frac{1}{7680}\, x^6 + \frac{1}{368640}\, x^8 - \frac{1}{30965760}\, x^{10} + \frac{1}{3963617280}\, x^{12} - \frac{1}{713451110400}\, x^{14} + \ldots\right) dx =$$

$$= \int_{0.002}^1 \left(\frac{1}{384\, x^3} - \frac{1}{7680\, x} + \frac{1}{368640}\, x - \frac{1}{30965760}\, x^3 + \frac{1}{3963617280}\, x^5 - \frac{1}{713451110400}\, x^7 + \ldots\right) dx =$$

$$= -\frac{1}{768\, x^2} - \frac{\ln x}{7680} + \frac{1}{737280}\, x^2 - \frac{1}{123863040}\, x^4 + \frac{1}{23781703680}\, x^6 - \frac{1}{5707608883200}\, x^8 + \ldots\bigg|_{0.002}^1 =$$

$$= (-0.001302083 - 0.0 + 1.3563 \cdot 10^{-6} - 8.07343 \cdot 10^{-9} + 4.20491 \cdot 10^{-11} - 0.17520 \cdot 10^{-13} + \ldots)$$

$$-(-325.520833333333333 + 0.000809193762815389549 + 5.4253472\cdot 10^{-12} - 1.2917\cdot 10^{-19} + 0.26911\cdot 10^{-27} - \ldots) =$$

$$= -0.00130073502808721678 - (-325.520024139565093) = 325.518723404537006 \,.$$

Here are no differences of nearly the same values.

$\mathbf{Z}_2(\mathbf{x})$:

$$\int J_2(x)\,dx = -2J_1(x) + \Lambda_0(x)$$

$$\int I_2(x)\,dx = 2I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_2(x)\,dx = -2J_0(x) - xJ_1(x)$$

$$\int x\,I_2(x)\,dx = -2I_0(x) + xI_1(x)$$

$$\int x^2\,J_2(x)\,dx = -3xJ_0(x) - x^2 J_1(x) + 3\Lambda_0(x)$$

$$\int x^2\,I_2(x)\,dx = -3xI_0(x) + x^2 I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^3\,J_2(x)\,dx = -4x^2 J_0(x) - \left(x^2-8\right) xJ_1(x)$$

$$\int x^3\,I_2(x)\,dx = -4x^2 I_0(x) + \left(x^2+8\right) xI_1(x)$$

$$\int x^4\,J_2(x)\,dx = -5x\,\left(x^2-3\right) J_0(x) - \left(x^2-15\right) x^2 J_1(x) - 15\Lambda_0(x)$$

$$\int x^4\,I_2(x)\,dx = -5x\,\left(x^2+3\right) I_0(x) + \left(x^2+15\right) x^2 I_1(x) + 15\Lambda_0^*(x)$$

$$\int \frac{J_2(x)\,dx}{x} = -\frac{J_1(x)}{x}$$

$$\int \frac{I_2(x)\,dx}{x} = \frac{I_1(x)}{x}$$

$$\int \frac{J_2(x)\,dx}{x^2} = \frac{1}{3x}\,J_0(x) - \frac{x^2+2}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_2(x)\,dx}{x^2} = -\frac{1}{3x}\,I_0(x) - \frac{x^2-2}{3x^2}\,I_1(x) + \frac{1}{3}\,\Lambda_0^*(x)$$

$$\mathcal{P}_2^{(5)}(x) = -6\,\left(x^2-8\right)x^2\,,\quad \mathcal{Q}_2^{(5)}(x) = -\left(x^4-24\,x^2+96\right)x\,,\quad \mathcal{R}_2^{(5)} = 0\,,\quad \mathcal{S}_2^{(5)} = 0$$

$$\mathcal{P}_2^{(5),*}(x) = -6\,\left(x^2+8\right)x^2\,,\quad \mathcal{Q}_2^{(5),*}(x) = x^5+24\,x^3+96x\,,\quad \mathcal{R}_2^{(5),*} = 0\,,\quad \mathcal{S}_2^{(5),*} = 0$$

$$\mathcal{P}_2^{(6)}(x) = -7\,\left(x^4-15\,x^2+45\right)x\,,\quad \mathcal{Q}_2^{(6)}(x) = -\left(x^4-35\,x^2+315\right)x^2\,,\quad \mathcal{R}_2^{(6)} = 315\,,\quad \mathcal{S}_2^{(6)} = 0$$

$$\mathcal{P}_2^{(6),*}(x) = -7(x^5+15\,x^3+45\,x)\,,\quad \mathcal{Q}_2^{(6),*}(x) = x^6+35\,x^4+315\,x^2\,,\quad \mathcal{R}_2^{(6),*} = 315\,,\quad \mathcal{S}_2^{(6),*} = 0$$

$$\mathcal{P}_2^{(7)}(x) = -8\,\left(x^4-24\,x^2+192\right)x^2\,,\quad \mathcal{Q}_2^{(7)}(x) = -\left(x^6-48\,x^4+768\,x^2-3072\right)x\,,\quad \mathcal{R}_2^{(7)} = 0\,,\quad \mathcal{S}_2^{(7)} = 0$$

$$\mathcal{P}_2^{(7),*}(x) = -(8\,x^4+192\,x^2+1536)\,x^2\,,\quad \mathcal{Q}_2^{(7),*}(x) = x^7+48\,x^5+768\,x^3+3072\,x\,,\quad \mathcal{R}_2^{(7),*} = 0\,,\quad \mathcal{S}_2^{(7),*} = 0$$

$$\mathcal{P}_2^{(8)}(x) = -9\,\left(x^6-35\,x^4+525\,x^2-1575\right)x\,,\quad \mathcal{Q}_2^{(8)}(x) = -\left(x^6-63\,x^4+1575\,x^2-14175\right)x^2\,,$$

$$\mathcal{R}_2^{(8)} = -14175\,,\quad \mathcal{S}_2^{(8)} = 0$$

$$\mathcal{P}_2^{(8),*}(x) = -(9\,x^7+315\,x^5+4725\,x^3+14175\,x)\,,\quad \mathcal{Q}_2^{(8),*}(x) = x^8+63\,x^6+1575\,x^4+14175\,x^2\,,$$

$$\mathcal{R}_2^{(8),*} = 14175\,,\quad \mathcal{S}_2^{(8),*} = 0$$

$$\mathcal{P}_2^{(9)}(x) = -10\,\left(x^6-48\,x^4+1152\,x^2-9216\right)x^2\,,\quad \mathcal{Q}_2^{(9)}(x) = -\left(x^8-80\,x^6+2880\,x^4-46080\,x^2+184320\right)\,,$$

$$\mathcal{R}_2^{(9)} = 0\,,\quad \mathcal{S}_2^{(9)} = 0$$

$$\mathcal{P}_2^{(9),*}(x) = -(10\,x^8+480\,x^6+11520\,x^4+92160\,x^2)\ , \quad \mathcal{Q}_2^{(9),*}(x) = x^9+80\,x^7+2880\,x^5+46080\,x^3+184320\,x\ ,$$

$$\mathcal{R}_2^{(9),*} = 0\ , \quad \mathcal{S}_2^{(9),*} = 0$$

$$\mathcal{P}_2^{(10)}(x) = -11\,\left(x^8 - 63\,x^6 + 2205\,x^4 - 33075\,x^2 + 99225\right)x\ ,$$

$$\mathcal{Q}_2^{(10)}(x) = -\left(x^8 - 99\,x^6 + 4851\,x^4 - 121275\,x^2 + 1091475\right)x^2\ , \quad \mathcal{R}_2^{(10)} = 1091475\ , \quad \mathcal{S}_2^{(10)} = 0$$

$$\mathcal{P}_2^{(10),*}(x) = -(11\,x^9 + 693\,x^7 + 24255\,x^5 + 363825\,x^3 + 1091475\,x)\ ,$$

$$\mathcal{Q}_2^{(10),*}(x) = x^{10} + 99\,x^8 + 4851\,x^6 + 121275\,x^4 + 1091475\,x^2\ , \quad \mathcal{R}_2^{(10),*} = 1091475\ , \quad \mathcal{S}_2^{(10),*} = 0$$

$$\mathcal{P}_2^{(-3)}(x) = \frac{1}{4\,x^2}\ , \quad \mathcal{Q}_2^{(-3)}(x) = -\frac{x^2+4}{8\,x^3}\ , \quad \mathcal{R}_2^{(-3)} = 0\ , \quad \mathcal{S}_2^{(-3)} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-3),*}(x) = -\frac{1}{4x^2}\ , \quad \mathcal{Q}_2^{(-3),*}(x) = -\frac{x^2-4}{8\,x^3}\ , \quad \mathcal{R}_2^{(-3),*} = 0\ , \quad \mathcal{S}_2^{(-3),*} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-4)}(x) = -\frac{x^2-3}{15\,x^3}\ , \quad \mathcal{Q}_2^{(-4)}(x) = \frac{x^4-x^2-6}{15\,x^4}\ , \quad \mathcal{R}_2^{(-4)} = -\frac{1}{15}\ , \quad \mathcal{S}_2^{(-4)} = 0$$

$$\mathcal{P}_2^{(-4),*}(x) = -\frac{x^2+3}{15\,x^3}\ , \quad \mathcal{Q}_2^{(-4),*}(x) = -\frac{x^4+x^2-6}{15\,x^4}\ , \quad \mathcal{R}_2^{(-4),*} = \frac{1}{15}\ , \quad \mathcal{S}_2^{(-4),*} = 0$$

$$\mathcal{P}_2^{(-5)}(x) = -\frac{x^2-8}{48\,x^4}\ , \quad \mathcal{Q}_2^{(-5)}(x) = \frac{x^4-4\,x^2-32}{96\,x^5}\ , \quad \mathcal{R}_2^{(-5)} = 0\ , \quad \mathcal{S}_2^{(-5)} = -\frac{1}{96}$$

$$\mathcal{P}_2^{(-5),*}(x) = -\frac{x^2+8}{48\,x^4}\ , \quad \mathcal{Q}_2^{(-5),*}(x) = -\frac{x^4+4\,x^2-32}{96\,x^5}\ , \quad \mathcal{R}_2^{(-5),*} = 0\ , \quad \mathcal{S}_2^{(-5),*} = \frac{1}{96}$$

$$\mathcal{P}_2^{(-6)}(x) = \frac{x^4-3\,x^2+45}{315\,x^5}\ , \quad \mathcal{Q}_2^{(-6)}(x) = -\frac{x^6-x^4+9\,x^2+90}{315\,x^6}\ , \quad \mathcal{R}_2^{(-6)} = \frac{1}{315}\ , \quad \mathcal{S}_2^{(-6)} = 0$$

$$\mathcal{P}_2^{(-6),*}(x) = -\frac{x^4+3\,x^2+45}{315x^5}\ , \quad \mathcal{Q}_2^{(-6),*}(x) = -\frac{x^6+x^4+9\,x^2-90}{315x^6}\ , \quad \mathcal{R}_2^{(-6),*} = \frac{1}{315}\ , \quad \mathcal{S}_2^{(-6),*} = 0$$

$\underline{\mathbf{Z_3(x)}:}$

$$\int J_3(x)\,dx = J_0(x) - \frac{4}{x}\,J_1(x)$$

$$\int I_3(x)\,dx = I_0(x) - \frac{4}{x}\,I_1(x)$$

$$\int x\,J_3(x)\,dx = xJ_0(x) - 8J_1(x) + 3\Lambda_0(x)$$

$$\int x\,I_3(x)\,dx = xI_0(x) - 8I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^2\,J_3(x)\,dx = (x^2-8)J_0(x) - 6xJ_1(x)$$

$$\int x^2\,I_3(x)\,dx = (x^2+8)I_0(x) - 6xI_1(x)$$

$$\int x^3\,J_3(x)\,dx = \left(x^2-15\right)x\,J_0(x) - 7x^2J_1(x) + 15\Lambda_0(x)$$

$$\int x^3\,I_3(x)\,dx = (x^2+15)x\,I_0(x) - 7x^2\,I_1(x) - 15\Lambda_0^*(x)$$

$$\int x^4\,J_3(x)\,dx = \left(x^2-24\right)x^2J_0(x) - 8\,\left(x^2-6\right)xJ_1(x)$$

$$\int x^4\,I_3(x)\,dx = \left(x^2+24\right)x^2I_0(x) - 8\,\left(x^2+6\right)xI_1(x)$$

$$\int \frac{J_3(x)\,dx}{x} = \frac{4}{3x}\,J_0(x) - \frac{x^2+8}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_3(x)\,dx}{x} = \frac{4}{3x}\,I_0(x) + \frac{x^2-8}{3x^2}\,I_1(x) - \frac{1}{3}\Lambda_0^*(x)$$

$$\int \frac{J_3(x)\,dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3}$$

$$\int \frac{I_3(x)\,dx}{x^2} = \frac{I_0(x)}{x^2} - \frac{2I_1(x)}{x^3}$$

$$\mathcal{P}_3^{(5)}(x) = x^5 - 35\,x^3 + 105\,x\ , \quad \mathcal{Q}_3^{(5)}(x) = -(9\,x^4 - 105\,x^2)\ , \quad \mathcal{R}_3^{(5)} = -105\ , \quad \mathcal{S}_3^{(5)} = 0$$

$$\mathcal{P}_3^{(5),*}(x) = x^5 + 35\,x^3 + 105\,x\ , \quad \mathcal{Q}_3^{(5),*}(x) = -(9\,x^4 + 105\,x^2)\ , \quad \mathcal{R}_3^{(5),*} = -105\ , \quad \mathcal{S}_3^{(5),*} = 0$$

$$\mathcal{P}_3^{(6)}(x) = x^6 - 48\,x^4 + 384\,x^2\ , \quad \mathcal{Q}_3^{(6)}(x) = -(10\,x^5 - 192\,x^3 + 768\,x)\ , \quad \mathcal{R}_3^{(6)} = 0\ , \quad \mathcal{S}_3^{(6)} = 0$$

$$\mathcal{P}_3^{(6),*}(x) = x^6 + 48\,x^4 + 384\,x^2\ , \quad \mathcal{Q}_3^{(6),*}(x) = -(10\,x^5 + 192\,x^3 + 768\,x)\ , \quad \mathcal{R}_3^{(6),*} = 0\ , \quad \mathcal{S}_3^{(6),*} = 0$$

$$\mathcal{P}_3^{(7)}(x) = x^7 - 63\,x^5 + 945\,x^3 - 2835\,x\ , \quad \mathcal{Q}_3^{(7)}(x) = -(11\,x^6 - 315\,x^4 + 2835\,x^2)\ , \quad \mathcal{R}_3^{(7)} = 2835\ , \quad \mathcal{S}_3^{(7)} = 0$$

$$\mathcal{P}_3^{(7),*}(x) = x^7 + 63\,x^5 + 945\,x^3 + 2835\,x\ , \quad \mathcal{Q}_3^{(7),*}(x) = -(11\,x^6 + 315\,x^4 + 2835\,x^2)\ , \quad \mathcal{R}_3^{(7),*} = -2835\ , \quad \mathcal{S}_3^{(7),*} = 0$$

$$\mathcal{P}_3^{(8)}(x) = x^8 - 80\,x^6 + 1920\,x^4 - 15360\,x^2\ , \quad \mathcal{Q}_3^{(8)}(x) = -(12\,x^7 - 480\,x^5 + 7680\,x^3 - 30720\,x)\ ,$$

$$\mathcal{R}_3^{(8)} = 0\ , \quad \mathcal{S}_3^{(8)} = 0$$

$$\mathcal{P}_3^{(8),*}(x) = x^8 + 80\,x^6 + 1920\,x^4 + 15360\,x^2\ , \quad \mathcal{Q}_3^{(8),*}(x) = -(12\,x^7 + 480\,x^5 + 7680\,x^3 + 30720\,x)\ ,$$

$$\mathcal{R}_3^{(8),*} = 0\ , \quad \mathcal{S}_3^{(8),*} = 0$$

$$\mathcal{P}_3^{(9)}(x) = x^9 - 99\,x^7 + 3465\,x^5 - 51975\,x^3 + 155925\,x\ ,$$

$$\mathcal{Q}_3^{(9)}(x) = -(13\,x^8 - 693\,x^6 + 17325\,x^4 - 155925\,x^2)\ , \quad \mathcal{R}_3^{(9)} = -155925\ , \quad \mathcal{S}_3^{(9)} = 0$$

$$\mathcal{P}_3^{(9),*}(x) = x^9 + 99\,x^7 + 3465\,x^5 + 51975\,x^3 + 155925\,x\ ,$$

$$\mathcal{Q}_3^{(9),*}(x) = -(13\,x^8 + 693\,x^6 + 17325\,x^4 + 155925\,x^2)\ , \quad \mathcal{R}_3^{(9),*} = -155925\ , \quad \mathcal{S}_3^{(9),*} = 0$$

$$\mathcal{P}_3^{(10)}(x) = x^{10} - 120\,x^8 + 5760\,x^6 - 138240\,x^4 + 1105920\,x^2\ ,$$

$$\mathcal{Q}_3^{(10)}(x) = -(14\,x^9 - 960\,x^7 + 34560\,x^5 - 552960\,x^3 + 2211840\,x)\ , \quad \mathcal{R}_3^{(10)} = 0\ , \quad \mathcal{S}_3^{(10)} = 0$$

$$\mathcal{P}_3^{(10),*}(x) = x^{10} + 120\,x^8 + 5760\,x^6 + 138240\,x^4 + 1105920\,x^2\ ,$$

$$\mathcal{Q}_3^{(10),*}(x) = -(14\,x^9 + 960\,x^7 + 34560\,x^5 + 552960\,x^3 + 2211840\,x)\ , \quad \mathcal{R}_3^{(10),*} = 0\ , \quad \mathcal{S}_3^{(10),*} = 0$$

$$\mathcal{P}_3^{(-3)}(x) = \frac{x^2+12}{15\,x^3}\ , \quad \mathcal{Q}_3^{(-3)}(x) = -\frac{x^4-x^2+24}{15\,x^4}\ , \quad \mathcal{R}_3^{(-3)} = \frac{1}{15}\ , \quad \mathcal{S}_3^{(-3)} = 0$$

$$\mathcal{P}_3^{(-3),*}(x) = -\frac{x^2-12}{15\,x^3}\ , \quad \mathcal{Q}_3^{(-3),*}(x) = -\frac{x^4+x^2+24}{15\,x^4}\ , \quad \mathcal{R}_3^{(-3),*} = \frac{1}{15}\ , \quad \mathcal{S}_3^{(-3),*} = 0$$

$$\mathcal{P}_3^{(-4)}(x) = \frac{x^2+16}{24\,x^4}\ , \quad \mathcal{Q}_3^{(-4)}(x) = -\frac{x^4-4\,x^2+64}{48\,x^5}\ , \quad \mathcal{R}_3^{(-4)} = 0\ , \quad \mathcal{S}_3^{(-4)} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-4),*}(x) = -\frac{x^2-16}{24\,x^4}\ , \quad \mathcal{Q}_3^{(-4),*}(x) = -\frac{x^4+4\,x^2+64}{48\,x^5}\ , \quad \mathcal{R}_3^{(-4),*} = 0\ , \quad \mathcal{S}_3^{(-4),*} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-5)}(x) = -\frac{x^4-3\,x^2-60}{105\,x^5}\ , \quad \mathcal{Q}_3^{(-5)}(x) = \frac{x^6-x^4+9\,x^2-120}{105\,x^6}\ , \quad \mathcal{R}_3^{(-5)} = -\frac{1}{105}\ , \quad \mathcal{S}_3^{(-5)} = 0$$

$$\mathcal{P}_3^{(-5),*}(x) = -\frac{x^4+3\,x^2-60}{105\,x^5}\ , \quad \mathcal{Q}_3^{(-5),*}(x) = \frac{x^6+x^4+9\,x^2+120}{105\,x^6}\ , \quad \mathcal{R}_3^{(-5),*} = -\frac{1}{105}\ , \quad \mathcal{S}_3^{(-5),*} = 0$$

$$\mathcal{P}_3^{(-6)}(x) = -\frac{x^4-8\,x^2-192}{384\,x^6}\ , \quad \mathcal{Q}_3^{(-6)}(x) = \frac{x^6-4\,x^4+64\,x^2-768}{768\,x^7}\ , \quad \mathcal{R}_3^{(-6)} = 0\ , \quad \mathcal{S}_3^{(-6)} = -\frac{1}{768}$$

$$\mathcal{P}_3^{(-6),*}(x) = -\frac{x^4+8\,x^2-192}{384\,x^6}\ , \quad \mathcal{Q}_3^{(-6),*}(x) = -\frac{x^6+4\,x^4+64\,x^2+768}{768\,x^7}\ , \quad \mathcal{R}_3^{(-6),*} = 0\ , \quad \mathcal{S}_3^{(-6),*} = \frac{1}{768}$$

$\mathbf{Z}_4(\mathbf{x})$:

$$\int J_4(x)\,dx = \frac{8J_0(x)}{x} - \frac{16J_1(x)}{x^2} + \Lambda_0(x)$$

$$\int I_4(x)\,dx = -\frac{8I_0(x)}{x} + \frac{16I_1(x)}{x^2} + \Lambda_0^*(x)$$

$$\int x\,J_4(x)\,dx = 8J_0(x) + \frac{x^2-24}{x}J_1(x)$$

$$\int x\,I_4(x)\,dx = -8I_0(x) + \frac{x^2+24}{x}I_1(x)$$

$$\int x^2\,J_4(x)\,dx = 9xJ_0(x) + (x^2-48)J_1(x) + 15\Lambda_0(x)$$

$$\int x^2\,I_4(x)\,dx = -9xI_0(x) + (x^2+48)I_1(x) - 15\Lambda_0^*(x)$$

$$\int x^3\,J_4(x)\,dx = (10\,x^2-48)J_0(x) + \left(x^2-44\right)xJ_1(x)$$

$$\int x^3\,I_4(x)\,dx = -(10\,x^2+48)I_0(x) + \left(x^2+44\right)xI_1(x)$$

$$\int x^4\,J_4(x)\,dx = \left(11\,x^2-105\right)xJ_0(x) + \left(x^2-57\right)x^2J_1(x) + 105\Lambda_0(x)$$

$$\int x^4\,I_4(x)\,dx = -\left(11\,x^2+105\right)xI_0(x) + \left(x^2+57\right)x^2I_1(x) + 105\Lambda_0^*(x)$$

$$\int \frac{J_4(x)\,dx}{x} = \frac{6J_0(x)}{x^2} + \frac{x^2-12}{x^3}J_1(x)$$

$$\int \frac{I_4(x)\,dx}{x} = -\frac{6J_0(x)}{x^2} + \frac{x^2+12}{x^3}J_1(x)$$

$$\int \frac{J_4(x)\,dx}{x^2} = \frac{x^2+72}{15x^3}J_0(x) - \frac{x^4-16\,x^2+144}{15x^4}J_1(x) + \frac{1}{15}\Lambda_0(x)$$

$$\int \frac{I_4(x)\,dx}{x^2} = \frac{x^2-72}{15x^3}I_0(x) + \frac{x^4+16\,x^2+144}{15x^4}I_1(x) - \frac{1}{15}\Lambda_0^*(x)$$

$$\mathcal{P}_4^{(5)}(x) = 12\,x^4 - 192\,x^2\ , \quad \mathcal{Q}_4^{(5)}(x) = x^5 - 72\,x^3 + 384\,x\ , \quad \mathcal{R}_4^{(5)} = 0\ , \quad \mathcal{S}_4^{(5)} = 0$$

$$\mathcal{P}_4^{(5),*}(x) = -(12\,x^4 + 192\,x^2)\ , \quad \mathcal{Q}_4^{(5),*}(x) = x^5 + 72\,x^3 + 384\,x\ , \quad \mathcal{R}_4^{(5),*} = 0\ , \quad \mathcal{S}_4^{(5),*} = 0$$

$$\mathcal{P}_4^{(6)}(x) = 13\,x^5 - 315\,x^3 + 945\,x\ , \quad \mathcal{Q}_4^{(6)}(x) = x^6 - 89\,x^4 + 945\,x^2\ ,$$

$$\mathcal{R}_4^{(6)} = -945\ , \quad \mathcal{S}_4^{(6)} = 0$$

$$\mathcal{P}_4^{(6),*}(x) = -(13\,x^5 + 315\,x^3 + 945\,x)\ , \quad \mathcal{Q}_4^{(6),*}(x) = x^6 + 89\,x^4 + 945\,x^2\ ,$$

$$\mathcal{R}_4^{(6),*} = 945\ , \quad \mathcal{S}_4^{(6),*} = 0$$

$$\mathcal{P}_4^{(7)}(x) = 14\,x^6 - 480\,x^4 + 3840\,x^2\ , \quad \mathcal{Q}_4^{(7)}(x) = x^7 - 108\,x^5 + 1920\,x^3 - 7680\,x\ ,$$

$$\mathcal{R}_4^{(7)} = 0\ , \quad \mathcal{S}_4^{(7)} = 0$$

$$\mathcal{P}_4^{(7),*}(x) = -(14\,x^6 + 480\,x^4 + 3840\,x^2)\ , \quad \mathcal{Q}_4^{(7),*}(x) = x^7 + 108\,x^5 + 1920\,x^3 + 7680\,x\ ,$$

$$\mathcal{R}_4^{(7),*} = 0\ , \quad \mathcal{S}_4^{(7),*} = 0$$

$$\mathcal{P}_4^{(8)}(x) = 15\,x^7 - 693\,x^5 + 10395\,x^3 - 31185\,x\ , \quad \mathcal{Q}_4^{(8)}(x) = x^8 - 129\,x^6 + 3465\,x^4 - 31185\,x^2\ ,$$

$$\mathcal{R}_4^{(8)} = 31185\ , \quad \mathcal{S}_4^{(8)} = 0$$

$$\mathcal{P}_4^{(8),*}(x) = -(15\,x^7 + 693\,x^5 + 10395\,x^3 + 31185\,x)\ , \quad \mathcal{Q}_4^{(8),*}(x) = x^8 + 129\,x^6 + 3465\,x^4 + 31185\,x^2\ ,$$

$$\mathcal{R}_4^{(8),*} = 31185\ , \quad \mathcal{S}_4^{(8),*} = 0$$

$$\mathcal{P}_4^{(9)}(x) = 16\,x^8 - 960\,x^6 + 23040\,x^4 - 184320\,x^2\ ,$$

$$\mathcal{Q}_4^{(9)}(x) = x^9 - 152\,x^7 + 5760\,x^5 - 92160\,x^3 + 368640\,x\ , \quad \mathcal{R}_4^{(9)} = 0\ , \quad \mathcal{S}_4^{(9)} = 0$$

$$\mathcal{P}_4^{(9),*}(x) = -(16\,x^8 + 960\,x^6 + 23040\,x^4 + 184320\,x^2)\ ,$$

$$\mathcal{Q}_4^{(9),*}(x) = x^9 + 152\,x^7 + 5760\,x^5 + 92160\,x^3 + 368640\,x\ , \quad \mathcal{R}_4^{(9),*} = 0\ , \quad \mathcal{S}_4^{(9),*} = 0$$

$$\mathcal{P}_4^{(10)}(x) = 17\,x^9 - 1287\,x^7 + 45045\,x^5 - 675675\,x^3 + 2027025\,x\ ,$$

$$\mathcal{Q}_4^{(10)}(x) = x^{10} - 177\,x^8 + 9009\,x^6 - 225225\,x^4 + 2027025\,x^2\ ,$$

$$\mathcal{R}_4^{(10)} = -2027025\ , \quad \mathcal{S}_4^{(10)} = 0$$

$$\mathcal{P}_4^{(10),*}(x) = -(17\,x^9 + 1287\,x^7 + 45045\,x^5 + 675675\,x^3 + 2027025\,x)\ ,$$

$$\mathcal{Q}_4^{(10),*}(x) = x^{10} + 177\,x^8 + 9009\,x^6 + 225225\,x^4 + 2027025\,x^2\ ,$$

$$\mathcal{R}_4^{(10),*} = 2027025\ , \quad \mathcal{S}_4^{(10),*} = 0$$

$$\mathcal{P}_4^{(-3)}(x) = \frac{4}{x^4}\ , \quad \mathcal{Q}_4^{(-3)}(x) = \frac{x^2 - 8}{x^5}\ , \quad \mathcal{R}_4^{(-3)} = 0\ , \quad \mathcal{S}_4^{(-3)} = 0$$

$$\mathcal{P}_4^{(-3),*}(x) = -\frac{4}{x^4}\ , \quad \mathcal{Q}_4^{(-3),*}(x) = \frac{x^2 + 8}{x^5}\ , \quad \mathcal{R}_4^{(-3),*} = 0\ , \quad \mathcal{S}_4^{(-3),*} = 0$$

$$\mathcal{P}_4^{(-4)}(x) = \frac{x^4 - 3\,x^2 + 360}{105\,x^5}\ , \quad \mathcal{Q}_4^{(-4)}(x) = -\frac{x^6 - x^4 - 96\,x^2 + 720}{105x^6}\ ,$$

$$\mathcal{R}_4^{(-4)} = \frac{1}{105}\ , \quad \mathcal{S}_4^{(-4)} = 0$$

$$\mathcal{P}_4^{(-4),*}(x) = -\frac{x^4 + 3\,x^2 + 360}{105x^5}\ , \quad \mathcal{Q}_4^{(-4),*}(x) = -\frac{x^6 + x^4 - 96\,x^2 - 720}{105x^6}\ ,$$

$$\mathcal{R}_4^{(-4),*} = \frac{1}{105}\ , \quad \mathcal{S}_4^{(-4),*} = 0$$

$$\mathcal{P}_4^{(-5)}(x) = \frac{x^4 - 8\,x^2 + 576}{192\,x^6}\ , \quad \mathcal{Q}_4^{(-5)}(x) = -\frac{x^6 - 4\,x^4 - 320\,x^2 + 2304}{384\,x^7}\ ,$$

$$\mathcal{R}_4^{(-5)} = 0\ , \quad \mathcal{S}_4^{(-5)} = \frac{1}{384}$$

$$\mathcal{P}_4^{(-5),*}(x) = -\frac{x^4 + 8\,x^2 + 576}{192\,x^6}\ , \quad \mathcal{Q}_4^{(-5),*}(x) = -\frac{x^6 + 4\,x^4 - 320\,x^2 - 2304}{384\,x^7}\ ,$$

$$\mathcal{R}_4^{(-5),*} = 0\ , \quad \mathcal{S}_4^{(-5),*} = \frac{1}{384}$$

$$\mathcal{P}_4^{(-6)}(x) = -\frac{x^6 - 3\,x^4 + 45\,x^2 - 2520}{945\,x^7}\ , \quad \mathcal{Q}_4^{(-6)}(x) = \frac{x^8 - x^6 + 9\,x^4 + 720\,x^2 - 5040}{945\,x^8}\ ,$$

$$\mathcal{R}_4^{(-6)} = -\frac{1}{945}\ , \quad \mathcal{S}_4^{(-6)} = 0$$

$$\mathcal{P}_4^{(-6),*}(x) = -\frac{x^6 + 3\,x^4 + 45\,x^2 + 2520}{945\,x^7}\ , \quad \mathcal{Q}_4^{(-6),*}(x) = \frac{x^8 + x^6 + 9\,x^4 - 720\,x^2 - 5040}{945\,x^8}\ ,$$

$$\mathcal{R}_4^{(-6),*} = \frac{1}{945}\ , \quad \mathcal{S}_4^{(-6),*} = 0$$

$\mathbf{Z}_5(\mathbf{x})$:

$$\int J_5(x)\,dx = -\frac{x^2 - 48}{x^2} J_0(x) + \frac{12\,x^2 - 96}{x^3} J_1(x)$$

$$\int I_5(x)\,dx = \frac{x^2+48}{x^2} I_0(x) - \frac{12\,x^2+96}{x^3} I_1(x)$$

$$\int x\,J_5(x)\,dx = -\frac{x^2-64}{x} J_0(x) + \frac{8\,x^2-128}{x^2} J_1(x) + 5\Lambda_0(x)$$

$$\int x\,I_5(x)\,dx = \frac{x^2+64}{x} I_0(x) - \frac{8\,x^2+128}{x^2} I_1(x) - 5\Lambda_0^*(x)$$

$$\int x^2\,J_5(x)\,dx = -(x^2-72)J_0(x) + \frac{14\,x^2-192}{x} J_1(x)$$

$$\int x^2\,I_5(x)\,dx = (x^2+72)I_0(x) - \frac{14\,x^2+192}{x} I_1(x)$$

$$\int x^3\,J_5(x)\,dx = -(x^3-87\,x)J_0(x) + (15\,x^2-384)J_1(x) + 105\Lambda_0(x)$$

$$\int x^3\,I_5(x)\,dx = (x^3+87\,x)I_0(x) - (15\,x^2+384)I_1(x) + 105\Lambda_0^*(x)$$

$$\int x^4\,J_5(x)\,dx = -(x^4-104\,x^2+384)J_0(x) + (16\,x^3-400\,x)J_1(x)$$

$$\int x^4\,I_5(x)\,dx = (x^4+104\,x^2+384)I_0(x) - (16\,x^3+400\,x)I_1(x)$$

$$\int \frac{J_5(x)\,dx}{x} = -\frac{4\,x^2-192}{5\,x^3} J_0(x) - \frac{x^4-56\,x^2+384}{5\,x^4} J_1(x) + \frac{1}{5}\,\Lambda_0(x)$$

$$\int \frac{I_5(x)\,dx}{x} = \frac{4\,x^2+192}{5\,x^3} I_0(x) - \frac{x^4+56\,x^2+384}{5\,x^4} I_1(x) + \frac{1}{5}\Lambda_0^*(x)$$

$$\int \frac{J_5(x)\,dx}{x^2} = -\frac{x^2-32}{x^4} J_0(x) + \frac{10\,x^2-64}{x^5} J_1(x)$$

$$\int \frac{I_5(x)\,dx}{x^2} = \frac{x^2+32}{x^4} I_0(x) - \frac{10\,x^2+64}{x^5} I_1(x)$$

$$\mathcal{P}_5^{(5)}(x) = -(x^5-123\,x^3+945\,x)\ , \quad \mathcal{Q}_5^{(5)}(x) = 17\,x^4-561\,x^2\ , \quad \mathcal{R}_5^{(5)} = 945\ , \quad \mathcal{S}_5^{(5)} = 0$$

$$\mathcal{P}_5^{(5),*}(x) = x^5+123\,x^3+945\,x\ , \quad \mathcal{Q}_5^{(5),*}(x) = -(17\,x^4+561\,x^2)\ , \quad \mathcal{R}_5^{(5),*} = -945\ , \quad \mathcal{S}_5^{(5),*} = 0$$

$$\mathcal{P}_5^{(6)}(x) = -(x^6-144\,x^4+1920\,x^2)\ , \quad \mathcal{Q}_5^{(6)}(x) = 18\,x^5-768\,x^3+3840\,x\ ,$$

$$\mathcal{R}_5^{(6)} = 0\ , \quad \mathcal{S}_5^{(6)} = 0$$

$$\mathcal{P}_5^{(6),*}(x) = x^6+144\,x^4+1920\,x^2\ , \quad \mathcal{Q}_5^{(6),*}(x) = -(18\,x^5+768\,x^3+3840\,x)\ ,$$

$$\mathcal{R}_5^{(6),*} = 0\ , \quad \mathcal{S}_5^{(6),*} = 0$$

$$\mathcal{P}_5^{(7)}(x) = -(x^7-167\,x^5+3465\,x^3-10395\,x)\ , \quad \mathcal{Q}_5^{(7)}(x) = 19\,x^6-1027\,x^4+10395\,x^2\ ,$$

$$\mathcal{R}_5^{(7)} = -10395\ , \quad \mathcal{S}_5^{(7)} = 0$$

$$\mathcal{P}_5^{(7),*}(x) = x^7+167\,x^5+3465\,x^3+10395\,x\ , \quad \mathcal{Q}_5^{(7),*}(x) = -(19\,x^6+1027\,x^4+10395\,x^2)\ ,$$

$$\mathcal{R}_5^{(7),*} = -10395\ , \quad \mathcal{S}_5^{(7),*} = 0$$

$$\mathcal{P}_5^{(8)}(x) = -(x^8-192\,x^6+5760\,x^4-46080\,x^2)\ , \quad \mathcal{Q}_5^{(8)}(x) = 20\,x^7-1344\,x^5+23040\,x^3-92160\,x\ ,$$

$$\mathcal{R}_5^{(8)} = 0\ , \quad \mathcal{S}_5^{(8)} = 0$$

$$\mathcal{P}_5^{(8),*}(x) = x^8+192\,x^6+5760\,x^4+46080\,x^2\ , \quad \mathcal{Q}_5^{(8),*}(x) = -(20\,x^7+1344\,x^5+23040\,x^3+92160\,x)\ ,$$

$$\mathcal{R}_5^{(8),*} = 0\ , \quad \mathcal{S}_5^{(8),*} = 0$$

$$\mathcal{P}_5^{(9)}(x) = -(x^9-219\,x^7+9009\,x^5-135135\,x^3+405405\,x)\ ,$$

$$\mathcal{Q}_5^{(9)}(x) = 21\,x^8-1725\,x^6+45045\,x^4-405405\,x^2\ , \quad \mathcal{R}_5^{(9)} = 405405\ , \quad \mathcal{S}_5^{(9)} = 0$$

$$\mathcal{P}_5^{(9),*}(x) = x^9 + 219\,x^7 + 9009\,x^5 + 135135\,x^3 + 405405\,x\ ,$$

$$\mathcal{Q}_5^{(9),*}(x) = -(21\,x^8 + 1725\,x^6 + 45045\,x^4 + 405405\,x^2)\ , \quad \mathcal{R}_5^{(9),*} = -405405\ , \quad \mathcal{S}_5^{(9),*} = 0$$

$$\mathcal{P}_5^{(10)}(x) = -(x^{10} - 248\,x^8 + 13440\,x^6 - 322560\,x^4 + 2580480\,x^2)\ ,$$

$$\mathcal{Q}_5^{(10)}(x) = 22\,x^9 - 2176\,x^7 + 80640\,x^5 - 1290240\,x^3 + 5160960\,x\ , \quad \mathcal{R}_5^{(10)} = 0\ , \quad \mathcal{S}_5^{(10)} = 0$$

$$\mathcal{P}_5^{(10),*}(x) = x^{10} + 248\,x^8 + 13440\,x^6 + 322560\,x^4 + 2580480\,x^2\ ,$$

$$\mathcal{Q}_5^{(10),*}(x) = -(22\,x^9 + 2176\,x^7 + 80640\,x^5 + 1290240\,x^3 + 5160960\,x)\ , \quad \mathcal{R}_5^{(10),*} = 0\ , \quad \mathcal{S}_5^{(10),*} = 0$$

$$\mathcal{P}_5^{(-3)}(x) = \frac{x^4 - 108\,x^2 + 2880}{105x^5}\ , \quad \mathcal{Q}_5^{(-3)}(x) = -\frac{x^6 - x^4 - 936\,x^2 + 5760}{105x^6}\ , \quad \mathcal{R}_5^{(-3)} = \frac{1}{105}\ , \quad \mathcal{S}_5^{(-3)} = 0$$

$$\mathcal{P}_5^{(-3),*}(x) = \frac{x^4 + 108\,x^2 + 2880}{105x^5}\ , \quad \mathcal{Q}_5^{(-3),*}(x) = \frac{x^6 + x^4 - 936\,x^2 - 5760}{105x^6}\ , \quad \mathcal{R}_5^{(-3),*} = -\frac{1}{105}\ , \quad \mathcal{S}_5^{(-3),*} = 0$$

$$\mathcal{P}_5^{(-4)}(x) = -\frac{x^2 - 24}{x^6}\ , \quad \mathcal{Q}_5^{(-4)}(x) = \frac{8\,x^2 - 48}{x^7}\ , \quad \mathcal{R}_5^{(-4)} = 0\ , \quad \mathcal{S}_5^{(-4)} = 0$$

$$\mathcal{P}_5^{(-4),*}(x) = \frac{x^2 + 24}{x^6}\ , \quad \mathcal{Q}_5^{(-4),*}(x) = -\frac{8\,x^2 + 48}{x^7}\ , \quad \mathcal{R}_5^{(-4),*} = 0\ , \quad \mathcal{S}_5^{(-4),*} = 0$$

$$\mathcal{P}_5^{(-5)}(x) = \frac{x^6 - 3\,x^4 - 900\,x^2 + 20160}{945\,x^7}\ , \quad \mathcal{Q}_5^{(-5)}(x) = -\frac{x^8 - x^6 + 9\,x^4 - 6840\,x^2 + 40320}{945\,x^8}\ ,$$

$$\mathcal{R}_5^{(-5)} = \frac{1}{945}\ , \quad \mathcal{S}_5^{(-5)} = 0$$

$$\mathcal{P}_5^{(-5),*}(x) = -\frac{x^6 + 3\,x^4 - 900\,x^2 - 20160}{945\,x^7}\ , \quad \mathcal{Q}_5^{(-5),*}(x) = -\frac{x^8 + x^6 + 9\,x^4 + 6840\,x^2 + 40320}{945\,x^8}\ ,$$

$$\mathcal{R}_5^{(-5),*} = \frac{1}{945}\ , \quad \mathcal{S}_5^{(-5),*} = 0$$

$$\mathcal{P}_5^{(-6)}(x) = \frac{x^6 - 8\,x^4 - 1728\,x^2 + 36864}{1920\,x^8}\ , \quad \mathcal{Q}_5^{(-6)}(x) = -\frac{x^8 - 4\,x^6 + 64\,x^4 - 25344\,x^2 + 147456}{3840\,x^9}\ ,$$

$$\mathcal{R}_5^{(-6)} = 0\ , \quad \mathcal{S}_5^{(-6)} = \frac{1}{3840}$$

$$\mathcal{P}_5^{(-6),*}(x) = -\frac{x^6 + 8\,x^4 - 1728\,x^2 - 36864}{1920\,x^8}\ , \quad \mathcal{Q}_5^{(-6),*}(x) = -\frac{x^8 + 4\,x^6 + 64\,x^4 + 25344\,x^2 + 147456}{3840\,x^9}\ ,$$

$$\mathcal{R}_5^{(-6),*} = 0\ , \quad \mathcal{S}_5^{(-6),*} = \frac{1}{3840}$$

$\mathbf{Z_6(x)}$:

$$\int J_6(x)\,dx = -\frac{16\,x^2 - 384}{x^3} J_0(x) - \frac{2\,x^4 - 128\,x^2 + 768}{x^4} J_1(x) + \Lambda_0(x)$$

$$\int I_6(x)\,dx = -\frac{16\,x^2 + 384}{x^3} I_0(x) + \frac{2\,x^4 + 128\,x^2 + 768}{x^4} I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_6(x)\,dx = -\frac{18\,x^2 - 480}{x^2} J_0(x) - \frac{x^4 - 144\,x^2 + 960}{x^3} J_1(x)$$

$$\int x\,I_6(x)\,dx = -\frac{18\,x^2 + 480}{x^2} I_0(x) + \frac{x^4 + 144\,x^2 + 960}{x^3} I_1(x)$$

$$\int x^2\,J_6(x)\,dx = -\frac{19\,x^2 - 640}{x} J_0(x) - \frac{x^4 - 128\,x^2 + 1280}{x^2} J_1(x) + 35\Lambda_0(x)$$

$$\int x^2\,I_6(x)\,dx = -\frac{19\,x^2 + 640}{x} I_0(x) + \frac{x^4 + 128\,x^2 + 1280}{x^2} I_1(x) + 35\Lambda_0^*(x)$$

$$\int x^3\,J_6(x)\,dx = -(20\,x^2 - 768) J_0(x) - \frac{x^4 - 184\,x^2 + 1920}{x} J_1(x)$$

$$\int x^3\, I_6(x)\, dx = -(20\, x^2 + 768) I_0(x) + \frac{x^4 + 184\, x^2 + 1920}{x} I_1(x)$$

$$\int x^4\, J_6(x)\, dx = -(21\, x^3 - 975\, x) J_0(x) - (x^4 - 207\, x^2 + 3840) J_1(x) + 945 \Lambda_0(x)$$

$$\int x^4\, I_6(x)\, dx = -(21\, x^3 + 975\, x) I_0(x) + (x^4 + 207\, x^2 + 3840) I_1(x) - 945 \Lambda_0^*(x)$$

$$\int \frac{J_6(x)\, dx}{x} = -\frac{16\, x^2 - 320}{x^4} J_0(x) - \frac{x^4 - 112\, x^2 + 640}{x^5} J_1(x)$$

$$\int \frac{I_6(x)\, dx}{x} = -\frac{16\, x^2 + 320}{x^4} I_0(x) + \frac{x^4 + 112\, x^2 + 640}{x^5} I_1(x)$$

$$\int \frac{J_6(x)\, dx}{x^2} = \frac{x^4 - 528\, x^2 + 9600}{35\, x^5} J_0(x) - \frac{x^6 + 34\, x^4 - 3456\, x^2 + 19200}{35\, x^6} J_1(x) + \frac{1}{35} \Lambda_0(x)$$

$$\int \frac{I_6(x)\, dx}{x^2} = -\frac{x^4 + 528\, x^2 + 9600}{35\, x^5} I_0(x) - \frac{x^6 - 34\, x^4 - 3456\, x^2 - 19200}{35\, x^6} I_1(x) + \frac{1}{35} \Lambda_0^*(x)$$

$$\mathcal{P}_6^{(5)}(x) = -(22\, x^4 - 1232\, x^2 + 3840)\ , \quad \mathcal{Q}_6^{(5)}(x) = -(x^5 - 232\, x^3 + 4384\, x)\ ,$$

$$\mathcal{R}_6^{(5)} = 0\ , \quad \mathcal{S}_6^{(5)} = 0$$

$$\mathcal{P}_6^{(5),*}(x) = -(22\, x^4 + 1232\, x^2 + 3840)\ , \quad \mathcal{Q}_6^{(5),*}(x) = 22\, x^4 + 1232\, x^2 + 3840\ ,$$

$$\mathcal{R}_6^{(5),*} = 0\ , \quad \mathcal{S}_6^{(5),*} = 0$$

$$\mathcal{P}_6^{(6)}(x) = -(23\, x^5 - 1545\, x^3 + 10395\, x)\ , \quad \mathcal{Q}_6^{(6)}(x) = -(x^6 - 259\, x^4 + 6555\, x^2)\ ,$$

$$\mathcal{R}_6^{(6)} = 10395\ , \quad \mathcal{S}_6^{(6)} = 0$$

$$\mathcal{P}_6^{(6),*}(x) = -(23\, x^5 + 1545\, x^3 + 10395\, x)\ , \quad \mathcal{Q}_6^{(6),*}(x) = x^6 + 259\, x^4 + 6555\, x^2\ ,$$

$$\mathcal{R}_6^{(6),*} = 10395\ , \quad \mathcal{S}_6^{(6),*} = 0$$

$$\mathcal{P}_6^{(7)}(x) = -(24\, x^6 - 1920\, x^4 + 23040\, x^2)\ , \quad \mathcal{Q}_6^{(7)}(x) = -(x^7 - 288\, x^5 + 9600\, x^3 - 46080\, x)\ ,$$

$$\mathcal{R}_6^{(7)} = 0\ , \quad \mathcal{S}_6^{(7)} = 0$$

$$\mathcal{P}_6^{(7),*}(x) = -(24\, x^6 + 1920\, x^4 + 23040\, x^2)\ , \quad \mathcal{Q}_6^{(7),*}(x) = x^7 + 288\, x^5 + 9600\, x^3 + 46080\, x\ ,$$

$$\mathcal{R}_6^{(7),*} = 0\ , \quad \mathcal{S}_6^{(7),*} = 0$$

$$\mathcal{P}_6^{(8)}(x) = -(25\, x^7 - 2363\, x^5 + 45045\, x^3 - 135135\, x)\ , \quad \mathcal{Q}_6^{(8)}(x) = -(x^8 - 319\, x^6 + 13735\, x^4 - 135135\, x^2)\ ,$$

$$\mathcal{R}_6^{(8)} = -135135\ , \quad \mathcal{S}_6^{(8)} = 0$$

$$\mathcal{P}_6^{(8),*}(x) = -(25\, x^7 + 2363\, x^5 + 45045\, x^3 + 135135\, x)\ , \quad \mathcal{Q}_6^{(8),*}(x) = x^8 + 319\, x^6 + 13735\, x^4 + 135135\, x^2\ ,$$

$$\mathcal{R}_6^{(8),*} = 135135\ , \quad \mathcal{S}_6^{(8),*} = 0$$

$$\mathcal{P}_6^{(9)}(x) = -(26\, x^8 - 2880\, x^6 + 80640\, x^4 - 645120\, x^2)\ ,$$

$$\mathcal{Q}_6^{(9)}(x) = -(x^9 - 352\, x^7 + 19200\, x^5 - 322560\, x^3 + 1290240\, x)\ , \quad \mathcal{R}_6^{(9)} = 0\ , \quad \mathcal{S}_6^{(9)} = 0$$

$$\mathcal{P}_6^{(9),*}(x) = -(26\, x^8 + 2880\, x^6 + 80640\, x^4 + 645120\, x^2)\ ,$$

$$\mathcal{Q}_6^{(9),*}(x) = x^9 + 352\, x^7 + 19200\, x^5 + 322560\, x^3 + 1290240\, x\ , \quad \mathcal{R}_6^{(9),*} = 0\ , \quad \mathcal{S}_6^{(9),*} = 0$$

$$\mathcal{P}_6^{(10)}(x) = -(27\, x^9 - 3477\, x^7 + 135135\, x^5 - 2027025\, x^3 + 6081075\, x)\ ,$$

$$\mathcal{Q}_6^{(10)}(x) = -(x^{10} - 387\, x^8 + 26259\, x^6 - 675675\, x^4 + 6081075\, x^2)\ , \quad \mathcal{R}_6^{(10)} = 6081075\ , \quad \mathcal{S}_6^{(10)} = 0$$

$$\mathcal{P}_6^{(10),*}(x) = -(27\, x^9 + 3477\, x^7 + 135135\, x^5 + 2027025\, x^3 + 6081075\, x)\ ,$$

$$\mathcal{Q}_6^{(10),*}(x) = x^{10} + 387\, x^8 + 26259\, x^6 + 675675\, x^4 + 6081075\, x^2\ , \quad \mathcal{R}_6^{(10),*} = 6081075\ , \quad \mathcal{S}_6^{(10),*} = 0$$

$$\mathcal{P}_6^{(-3)}(x) = -\frac{14\,x^2 - 240}{x^6}\ , \quad \mathcal{Q}_6^{(-3)}(x) = -\frac{x^4 - 88\,x^2 + 480}{x^7}\ , \quad \mathcal{R}_6^{(-3)} = 0\ , \quad \mathcal{S}_6^{(-3)} = 0$$

$$\mathcal{P}_6^{(-3),*}(x) = -\frac{14\,x^2 + 240}{x^6}\ , \quad \mathcal{Q}_6^{(-3),*}(x) = \frac{x^4 + 88\,x^2 + 480}{x^7}\ , \quad \mathcal{R}_6^{(-3),*} = 0\ , \quad \mathcal{S}_6^{(-3),*} = 0$$

$$\mathcal{P}_6^{(-4)}(x) = \frac{x^6 - 3\,x^4 - 12240\,x^2 + 201600}{945\,x^7}\ , \quad \mathcal{Q}_6^{(-4)}(x) = -\frac{x^8 - x^6 + 954\,x^4 - 74880\,x^2 + 403200}{945\,x^8}\ ,$$

$$\mathcal{R}_6^{(-4)} = \frac{1}{945}\ , \quad \mathcal{S}_6^{(-4)} = 0$$

$$\mathcal{P}_6^{(-4),*}(x) = \frac{x^6 + 3\,x^4 - 12240\,x^2 - 201600}{945\,x^7}\ , \quad \mathcal{Q}_6^{(-4),*}(x) = \frac{x^8 + x^6 + 954\,x^4 + 74880\,x^2 + 403200}{945\,x^8}\ ,$$

$$\mathcal{R}_6^{(-4),*} = -\frac{1}{945}\ , \quad \mathcal{S}_6^{(-4),*} = 0$$

$$\mathcal{P}_6^{(-5)}(x) = -\frac{12\,x^2 - 192}{x^8}\ , \quad \mathcal{Q}_6^{(-5)}(x) = -\frac{x^4 - 72\,x^2 + 384}{x^9}\ , \quad \mathcal{R}_6^{(-5)} = 0\ , \quad \mathcal{S}_6^{(-5)} = 0$$

$$\mathcal{P}_6^{(-5),*}(x) = -\frac{12\,x^2 + 192}{x^8}\ , \quad \mathcal{Q}_6^{(-5),*}(x) = \frac{x^4 + 72\,x^2 + 384}{x^9}\ , \quad \mathcal{R}_6^{(-5),*} = 0\ , \quad \mathcal{S}_6^{(-5),*} = 0$$

$$\mathcal{P}_6^{(-6)}(x) = \frac{x^8 - 3\,x^6 + 45\,x^4 - 115920\,x^2 + 1814400}{10395\,x^9}\ ,$$

$$\mathcal{Q}_6^{(-6)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 10170\,x^4 - 685440\,x^2 + 3628800}{10395\,x^{10}}\ ,$$

$$\mathcal{R}_6^{(-6)} = \frac{1}{10395}\ , \quad \mathcal{S}_6^{(-6)} = 0$$

$$\mathcal{P}_6^{(-6),*}(x) = -\frac{x^8 + 3\,x^6 + 45\,x^4 + 115920\,x^2 + 1814400}{10395\,x^9}\ ,$$

$$\mathcal{Q}_6^{(-6),*}(x) = -\frac{x^{10} + x^8 + 9\,x^6 - 10170\,x^4 - 685440\,x^2 - 3628800}{10395\,x^{10}}\ ,$$

$$\mathcal{R}_6^{(-6),*} = \frac{1}{10395}\ , \quad \mathcal{S}_6^{(-6),*} = 0$$

$\mathbf{Z_7(x)}$:

$$\int J_7(x)\,dx = \frac{x^4 - 240\,x^2 + 3840}{x^4} J_0(x) - \frac{24\,x^4 - 1440\,x^2 + 7680}{x^5} J_1(x)$$

$$\int I_7(x)\,dx = \frac{x^4 + 240\,x^2 + 3840}{x^4} I_0(x) - \frac{24\,x^4 + 1440\,x^2 + 7680}{x^5} I_1(x)$$

$$\int x\,J_7(x)\,dx = \frac{x^4 - 256\,x^2 + 4608}{x^3} J_0(x) - \frac{32\,x^4 - 1664\,x^2 + 9216}{x^4} J_1(x) + 7\Lambda_0(x)$$

$$\int x\,I_7(x)\,dx = \frac{x^4 + 256\,x^2 + 4608}{x^3} I_0(x) - \frac{32\,x^4 + 1664\,x^2 + 9216}{x^4} I_1(x) + 7\Lambda_0^*(x)$$

$$\int x^2\,J_7(x)\,dx = \frac{x^4 - 288\,x^2 + 5760}{x^2} J_0(x) - \frac{26\,x^4 - 1920\,x^2 + 11520}{x^3} J_1(x)$$

$$\int x^2\,I_7(x)\,dx = \frac{x^4 + 288\,x^2 + 5760}{x^2} I_0(x) - \frac{26\,x^4 + 1920\,x^2 + 11520}{x^3} I_1(x)$$

$$\int x^3\,J_7(x)\,dx = \frac{x^4 - 315\,x^2 + 7680}{x} J_0(x) - \frac{27\,x^4 - 1920\,x^2 + 15360}{x^2} J_1(x) + 315\Lambda_0(x)$$

$$\int x^3\,I_7(x)\,dx = \frac{x^4 + 315\,x^2 + 7680}{x} I_0(x) - \frac{27\,x^4 + 1920\,x^2 + 15360}{x^2} I_1(x) - 315\Lambda_0^*(x)$$

$$\int x^4\,J_7(x)\,dx = (x^4 - 344\,x^2 + 9600) J_0(x) - \frac{28\,x^4 - 2608\,x^2 + 23040}{x} J_1(x)$$

$$\int x^4\, I_7(x)\, dx = (x^4+344\,x^2+9600) I_0(x) - \frac{28\,x^4+2608\,x^2+23040}{x} I_1(x)$$

$$\int \frac{J_7(x)\, dx}{x} = \frac{8\,x^4-1536\,x^2+23040}{7\,x^5} J_0(x) - \frac{x^6+160\,x^4-8832\,x^2+46080}{7\,x^6} J_1(x) + \frac{1}{7}\Lambda_0(x)$$

$$\int \frac{I_7(x)\, dx}{x} = \frac{8\,x^4+1536\,x^2+23040}{7\,x^5} I_0(x) + \frac{x^6-160\,x^4-8832\,x^2-46080}{7\,x^6} I_1(x) - \frac{1}{7}\Lambda_0^*(x)$$

$$\int \frac{J_7(x)\, dx}{x^2} = \frac{x^4-200\,x^2+2880}{x^6} J_0(x) - \frac{22\,x^4-1120\,x^2+5760}{x^7} J_1(x)$$

$$\int \frac{I_7(x)\, dx}{x^2} = \frac{x^4+200\,x^2+2880}{x^6} I_0(x) - \frac{22\,x^4+1120\,x^2+5760}{x^7} I_1(x)$$

$$\mathcal{P}_7^{(5)}(x) = x^5-375\,x^3+12645\,x\ , \quad \mathcal{Q}_7^{(5)}(x) = -(29\,x^4-3045\,x^2+46080)\ ,$$

$$\mathcal{R}_7^{(5)} = 10395\ , \quad \mathcal{S}_7^{(5)} = 0$$

$$\mathcal{P}_7^{(5),*}(x) = x^5+375\,x^3+12645\,x\ , \quad \mathcal{Q}_7^{(5),*}(x) = -(29\,x^4+3045\,x^2+46080)\ ,$$

$$\mathcal{R}_7^{(5),*} = 10395\ , \quad \mathcal{S}_7^{(5),*} = 0$$

$$\mathcal{P}_7^{(6)}(x) = x^6-408\,x^4+16704\,x^2-46080\ , \quad \mathcal{Q}_7^{(6)}(x) = -(30\,x^5-3552\,x^3+56448\,x)\ ,$$

$$\mathcal{R}_7^{(6)} = 0\ , \quad \mathcal{S}_7^{(6)} = 0$$

$$\mathcal{P}_7^{(6),*}(x) = x^6+408\,x^4+16704\,x^2+46080\ , \quad \mathcal{Q}_7^{(6),*}(x) = -(30\,x^5+3552\,x^3+56448\,x)\ ,$$

$$\mathcal{R}_7^{(6),*} = 0\ , \quad \mathcal{S}_7^{(6),*} = 0$$

$$\mathcal{P}_7^{(7)}(x) = x^7-443\,x^5+22005\,x^3-135135\,x\ , \quad \mathcal{Q}_7^{(7)}(x) = -(31\,x^6-4135\,x^4+89055\,x^2)\ ,$$

$$\mathcal{R}_7^{(7)} = 135135\ , \quad \mathcal{S}_7^{(7)} = 0$$

$$\mathcal{P}_7^{(7),*}(x) = x^7+443\,x^5+22005\,x^3+135135\,x\ , \quad \mathcal{Q}_7^{(7),*}(x) = -(31\,x^6+4135\,x^4+89055\,x^2)\ ,$$

$$\mathcal{R}_7^{(7),*} = -135135\ , \quad \mathcal{S}_7^{(7),*} = 0$$

$$\mathcal{P}_7^{(8)}(x) = x^8-480\,x^6+28800\,x^4-322560\,x^2\ , \quad \mathcal{Q}_7^{(8)}(x) = -(32\,x^7-4800\,x^5+138240\,x^3-645120\,x)\ ,$$

$$\mathcal{R}_7^{(8)} = 0\ , \quad \mathcal{S}_7^{(8)} = 0$$

$$\mathcal{P}_7^{(8),*}(x) = x^8+480\,x^6+28800\,x^4+322560\,x^2\ , \quad \mathcal{Q}_7^{(8),*}(x) = -(32\,x^7+4800\,x^5+138240\,x^3+645120\,x)\ ,$$

$$\mathcal{R}_7^{(8),*} = 0\ , \quad \mathcal{S}_7^{(8),*} = 0$$

$$\mathcal{P}_7^{(9)}(x) = x^9-519\,x^7+37365\,x^5-675675\,x^3+2027025\,x\ ,$$

$$\mathcal{Q}_7^{(9)}(x) = -(33\,x^8-5553\,x^6+209865\,x^4-2027025\,x^2)\ , \quad \mathcal{R}_7^{(9)} = -2027025\ , \quad \mathcal{S}_7^{(9)} = 0$$

$$\mathcal{P}_7^{(9),*}(x) = x^9+519\,x^7+37365\,x^5+675675\,x^3+2027025\,x\ ,$$

$$\mathcal{Q}_7^{(9),*}(x) = -(33\,x^8+5553\,x^6+209865\,x^4+2027025\,x^2)\ , \quad \mathcal{R}_7^{(9),*} = -2027025\ , \quad \mathcal{S}_7^{(9),*} = 0$$

$$\mathcal{P}_7^{(10)}(x) = x^{10}-560\,x^8+48000\,x^6-1290240\,x^4+10321920\,x^2\ ,$$

$$\mathcal{Q}_7^{(10)}(x) = -(34\,x^9-6400\,x^7+311040\,x^5-5160960\,x^3+20643840\,x)\ , \quad \mathcal{R}_7^{(10)} = 0\ , \quad \mathcal{S}_7^{(10)} = 0$$

$$\mathcal{P}_7^{(10),*}(x) = x^{10}+560\,x^8+48000\,x^6+1290240\,x^4+10321920\,x^2\ ,$$

$$\mathcal{Q}_7^{(10),*}(x) = -(34\,x^9+6400\,x^7+311040\,x^5+5160960\,x^3+20643840\,x)\ , \quad \mathcal{R}_7^{(10),*} = 0\ , \quad \mathcal{S}_7^{(10),*} = 0$$

$$\mathcal{P}_7^{(-3)}(x) = \frac{x^6+312\,x^4-57600\,x^2+806400}{315\,x^7}\ ,$$

$$\mathcal{Q}_7^{(-3)}(x) = -\frac{x^8-x^6+6624\,x^4-316800\,x^2+1612800}{315\,x^8}\ , \quad \mathcal{R}_7^{(-3)} = \frac{1}{315}\ , \quad \mathcal{S}_7^{(-3)} = 0$$

$$\mathcal{P}_7^{(-3),*}(x) = \frac{x^6 - 312\,x^4 - 57600\,x^2 - 806400}{315\,x^7}\ ,$$

$$\mathcal{Q}_7^{(-3),*}(x) = -\frac{x^8 + x^6 + 6624\,x^4 + 316800\,x^2 + 1612800}{315\,x^8}\ , \quad \mathcal{R}_7^{(-3),*} = \frac{1}{315}\ , \quad \mathcal{S}_7^{(-3),*} = 0$$

$$\mathcal{P}_7^{(-4)}(x) = \frac{x^4 - 168\,x^2 + 2304}{x^8}\ , \quad \mathcal{Q}_7^{(-4)}(x) = -\frac{20\,x^4 - 912\,x^2 + 4608}{x^9}\ , \quad \mathcal{R}_7^{(-4)} = 0\ , \quad \mathcal{S}_7^{(-4)} = 0$$

$$\mathcal{P}_7^{(-4),*}(x) = \frac{x^4 + 168\,x^2 + 2304}{x^8}\ , \quad \mathcal{Q}_7^{(-4),*}(x) = -\frac{20\,x^4 + 912\,x^2 + 4608}{x^9}\ , \quad \mathcal{R}_7^{(-4),*} = 0\ , \quad \mathcal{S}_7^{(-4),*} = 0$$

$$\mathcal{P}_7^{(-5)}(x) = \frac{x^8 - 3\,x^6 + 10440\,x^4 - 1612800\,x^2 + 21772800}{10395\,x^9}\ ,$$

$$\mathcal{Q}_7^{(-5)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 197280\,x^4 - 8668800\,x^2 + 43545600}{10395\,x^{10}}\ , \quad \mathcal{R}_7^{(-5)} = \frac{1}{10395}\ , \quad \mathcal{S}_7^{(-5)} = 0$$

$$\mathcal{P}_7^{(-5),*}(x) = \frac{x^8 + 3\,x^6 + 10440\,x^4 + 1612800\,x^2 + 21772800}{10395\,x^9}\ ,$$

$$\mathcal{Q}_7^{(-5),*}(x) = \frac{x^{10} + x^8 + 9\,x^6 - 197280\,x^4 - 8668800\,x^2 - 43545600}{10395\,x^{10}}\ , \quad \mathcal{R}_7^{(-5),*} = -\frac{1}{10395}\ , \quad \mathcal{S}_7^{(-5),*} = 0$$

$$\mathcal{P}_7^{(-6)}(x) = \frac{x^4 - 144\,x^2 + 1920}{x^{10}}\ , \quad \mathcal{Q}_7^{(-6)}(x) = -\frac{18\,x^4 - 768\,x^2 + 3840}{x^{11}}\ , \quad \mathcal{R}_7^{(-6)} = 0\ , \quad \mathcal{S}_7^{(-6)} = 0$$

$$\mathcal{P}_7^{(-6),*}(x) = \frac{x^4 + 144\,x^2 + 1920}{x^{10}}\ , \quad \mathcal{Q}_7^{(-6),*}(x) = -\frac{18\,x^4 + 768\,x^2 + 3840}{x^{11}}\ , \quad \mathcal{R}_7^{(-6),*} = 0\ , \quad \mathcal{S}_7^{(-6),*} = 0$$

$\mathbf{Z_8(x)}$:

$$\int J_8(x)\,dx = \frac{32\,x^4 - 3456\,x^2 + 46080}{x^5}J_0(x) - \frac{448\,x^4 - 18432\,x^2 + 92160}{x^6}J_1(x) + \Lambda_0(x)$$

$$\int I_8(x)\,dx = -\frac{32\,x^4 + 3456\,x^2 + 46080}{x^5}I_0(x) + \frac{448\,x^4 + 18432\,x^2 + 92160}{x^6}I_1(x) + \Lambda_0^*(x)$$

$$\int x\,J_8(x)\,dx = \frac{32\,x^4 - 3840\,x^2 + 53760}{x^4}J_0(x) + \frac{x^6 - 480\,x^4 + 21120\,x^2 - 107520}{x^5}J_1(x)$$

$$\int x\,I_8(x)\,dx = -\frac{32\,x^4 + 3840\,x^2 + 53760}{x^4}I_0(x) + \frac{x^6 + 480\,x^4 + 21120\,x^2 + 107520}{x^5}I_1(x)$$

$$\int x^2\,J_8(x)\,dx = \frac{33\,x^4 - 4224\,x^2 + 64512}{x^3}J_0(x) + \frac{x^6 - 576\,x^4 + 24576\,x^2 - 129024}{x^4}J_1(x) + 63\Lambda_0(x)$$

$$\int x^2\,I_8(x)\,dx = -\frac{33\,x^4 + 4224\,x^2 + 64512}{x^3}I_0(x) + \frac{x^6 + 576\,x^4 + 24576\,x^2 + 129024}{x^4}I_1(x) - 63\Lambda_0^*(x)$$

$$\int x^3\,J_8(x)\,dx = \frac{34\,x^4 - 4800\,x^2 + 80640}{x^2}J_0(x) + \frac{x^6 - 548\,x^4 + 28800\,x^2 - 161280}{x^3}J_1(x)$$

$$\int x^3\,I_8(x)\,dx = -\frac{34\,x^4 + 4800\,x^2 + 80640}{x^2}I_0(x) + \frac{x^6 + 548\,x^4 + 28800\,x^2 + 161280}{x^3}I_1(x)$$

$$\int x^4\,J_8(x)\,dx = \frac{35\,x^4 - 5385\,x^2 + 107520}{x}J_0(x) + \frac{x^6 - 585\,x^4 + 30720\,x^2 - 215040}{x^2}J_1(x) + 3465\Lambda_0(x)$$

$$\int x^4\,I_8(x)\,dx = -\frac{35\,x^4 + 5385\,x^2 + 107520}{x}I_0(x) + \frac{x^6 + 585\,x^4 + 30720\,x^2 + 215040}{x^2}I_1(x) + 3465\Lambda_0^*(x)$$

$$\int \frac{J_8(x)\,dx}{x} = \frac{30\,x^4 - 3120\,x^2 + 40320}{x^6}J_0(x) + \frac{x^6 - 420\,x^4 + 16320\,x^2 - 80640}{x^7}J_1(x)$$

$$\int \frac{I_8(x)\,dx}{x} = -\frac{30\,x^4 + 3120\,x^2 + 40320}{x^6}I_0(x) + \frac{x^6 + 420\,x^4 + 16320\,x^2 + 80640}{x^7}I_1(x)$$

$$\int \frac{J_8(x)\,dx}{x^2} = \frac{x^6 + 1824\,x^4 - 178560\,x^2 + 2257920}{63x^7}J_0(x) -$$

$$-\frac{x^8-64\,x^6+24768\,x^4-921600\,x^2+4515840}{63x^8}J_1(x)+\frac{1}{63}\Lambda_0(x)$$

$$\int\frac{I_8(x)\,dx}{x^2}=\frac{x^6-1824\,x^4-178560\,x^2-2257920}{63x^7}I_0(x)+$$

$$+\frac{x^8+64\,x^6+24768\,x^4+921600\,x^2+4515840}{63x^8}I_1(x)-\frac{1}{63}\Lambda_0^*(x)$$

$$\mathcal{P}_8^{(5)}(x)=36\,x^4-6048\,x^2+138240\;,\quad \mathcal{Q}_8^{(5)}(x)=\frac{x^6-624\,x^4+40896\,x^2-322560}{x}\;,$$

$$\mathcal{R}_8^{(5)}=0\;,\quad \mathcal{S}_8^{(5)}=0$$

$$\mathcal{P}_8^{(5),*}(x)=-(36\,x^4+6048\,x^2+138240)\;,\quad \mathcal{Q}_8^{(5),*}(x)=\frac{x^6+624\,x^4+40896\,x^2+322560}{x}\;,$$

$$\mathcal{R}_8^{(5),*}=0\;,\quad \mathcal{S}_8^{(5),*}=0$$

$$\mathcal{P}_8^{(6)}(x)=37\,x^5-6795\,x^3+187425\,x\;,\quad \mathcal{Q}_8^{(6)}(x)=x^6-665\,x^4+49185\,x^2-645120\;,$$

$$\mathcal{R}_8^{(6)}=135135\;,\quad \mathcal{S}_8^{(6)}=0$$

$$\mathcal{P}_8^{(6),*}(x)=-(37\,x^5+6795\,x^3+187425\,x)\;,\quad \mathcal{Q}_8^{(6),*}(x)=x^6+665\,x^4+49185\,x^2+645120\;,$$

$$\mathcal{R}_8^{(6),*}=-135135\;,\quad \mathcal{S}_8^{(6),*}=0$$

$$\mathcal{P}_8^{(7)}(x)=38\,x^6-7632\,x^4+256896\,x^2-645120\;,\quad \mathcal{Q}_8^{(7)}(x)=x^7-708\,x^5+59328\,x^3-836352\,x\;,$$

$$\mathcal{R}_8^{(7)}=0\;,\quad \mathcal{S}_8^{(7)}=0$$

$$\mathcal{P}_8^{(7),*}(x)=-(38\,x^6+7632\,x^4+256896\,x^2+645120)\;,\quad \mathcal{Q}_8^{(7),*}(x)=x^7+708\,x^5+59328\,x^3+836352\,x\;,$$

$$\mathcal{R}_8^{(7),*}=0\;,\quad \mathcal{S}_8^{(7),*}=0$$

$$\mathcal{P}_8^{(8)}(x)=39\,x^7-8565\,x^5+353115\,x^3-2027025\,x\;,\quad \mathcal{Q}_8^{(8)}(x)=x^8-753\,x^6+71625\,x^4-1381905\,x^2\;,$$

$$\mathcal{R}_8^{(8)}=2027025\;,\quad \mathcal{S}_8^{(8)}=0$$

$$\mathcal{P}_8^{(8),*}(x)=-(39\,x^7+8565\,x^5+353115\,x^3+2027025\,x)\;,\quad \mathcal{Q}_8^{(8),*}(x)=x^8+753\,x^6+71625\,x^4+1381905\,x^2\;,$$

$$\mathcal{R}_8^{(8),*}=2027025\;,\quad \mathcal{S}_8^{(8),*}=0$$

$$\mathcal{P}_8^{(9)}(x)=40\,x^8-9600\,x^6+483840\,x^4-5160960\,x^2\;,$$

$$\mathcal{Q}_8^{(9)}(x)=x^9-800\,x^7+86400\,x^5-2257920\,x^3+10321920\,x\;,\quad \mathcal{R}_8^{(9)}=0\;,\quad \mathcal{S}_8^{(9)}=0$$

$$\mathcal{P}_8^{(9),*}(x)=-(40\,x^8+9600\,x^6+483840\,x^4+5160960\,x^2)\;,$$

$$\mathcal{Q}_8^{(9),*}(x)=x^9+800\,x^7+86400\,x^5+2257920\,x^3+10321920\,x\;,\quad \mathcal{R}_8^{(9),*}=0\;,\quad \mathcal{S}_8^{(9),*}=0$$

$$\mathcal{P}_8^{(10)}(x)=41\,x^9-10743\,x^7+658245\,x^5-11486475\,x^3+34459425\,x\;,$$

$$\mathcal{Q}_8^{(10)}(x)=x^{10}-849\,x^8+104001\,x^6-3613785\,x^4+34459425\,x^2\;,\quad \mathcal{R}_8^{(10)}=-34459425\;,\quad \mathcal{S}_8^{(10)}=0$$

$$\mathcal{P}_8^{(10),*}(x)=-(41\,x^9+10743\,x^7+658245\,x^5+11486475\,x^3+34459425\,x)\;,$$

$$\mathcal{Q}_8^{(10),*}(x)=x^{10}+849\,x^8+104001\,x^6+3613785\,x^4+34459425\,x^2\;,\quad \mathcal{R}_8^{(10),*}=34459425\;,\quad \mathcal{S}_8^{(10),*}=0$$

$$\mathcal{P}_8^{(-3)}(x)=\frac{28\,x^4-2592\,x^2+32256}{x^8}\;,\quad \mathcal{Q}_8^{(-3)}(x)=\frac{x^6-368\,x^4+13248\,x^2-64512}{x^9}\;,$$

$$\mathcal{R}_8^{(-3)}=0\;,\quad \mathcal{S}_8^{(-3)}=0$$

$$\mathcal{P}_8^{(-3),*}(x)=-\frac{28\,x^4+2592\,x^2+32256}{x^8}\;,\quad \mathcal{Q}_8^{(-3),*}(x)=\frac{x^6+368\,x^4+13248\,x^2+64512}{x^9}\;,$$

$$\mathcal{R}_8^{(-3),*}=0\;,\quad \mathcal{S}_8^{(-3),*}=0$$

$$\mathcal{P}_8^{(-4)}(x)=\frac{x^8-3\,x^6+93600\,x^4-8265600\,x^2+101606400}{3465\,x^9}\;,$$

$$\mathcal{Q}_8^{(-4)}(x) = -\frac{x^{10} - x^8 - 3456\,x^6 + 1195200\,x^4 - 41932800\,x^2 + 203212800}{3465\,x^{10}}\ ,$$

$$\mathcal{R}_8^{(-4)} = \frac{1}{3465}\ , \quad \mathcal{S}_8^{(-4)} = 0$$

$$\mathcal{P}_8^{(-4),*}(x) = -\frac{x^8 + 3\,x^6 + 93600\,x^4 + 8265600\,x^2 + 101606400}{3465\,x^9}\ ,$$

$$\mathcal{Q}_8^{(-4),*}(x) = -\frac{x^{10} + x^8 - 3456\,x^6 - 1195200\,x^4 - 41932800\,x^2 - 203212800}{3465\,x^{10}}\ ,$$

$$\mathcal{R}_8^{(-4),*} = \frac{1}{3465}\ , \quad \mathcal{S}_8^{(-4),*} = 0$$

$$\mathcal{P}_8^{(-5)}(x) = \frac{26\,x^4 - 2208\,x^2 + 26880}{x^{10}}\ , \quad \mathcal{Q}_8^{(-5)}(x) = \frac{x^6 - 324\,x^4 + 11136\,x^2 - 53760}{x^{11}}\ ,$$

$$\mathcal{R}_8^{(-5)} = 0\ , \quad \mathcal{S}_8^{(-5)} = 0$$

$$\mathcal{P}_8^{(-5),*}(x) = -\frac{26\,x^4 + 2208\,x^2 + 26880}{x^{10}}\ , \quad \mathcal{Q}_8^{(-5),*}(x) = \frac{x^6 + 324\,x^4 + 11136\,x^2 + 53760}{x^{11}}\ ,$$

$$\mathcal{R}_8^{(-5),*} = 0\ , \quad \mathcal{S}_8^{(-5),*} = 0$$

$$\mathcal{P}_8^{(-6)}(x) = \frac{x^{10} - 3\,x^8 + 45\,x^6 + 3376800\,x^4 - 277603200\,x^2 + 3353011200}{135135\,x^{11}}\ ,$$

$$\mathcal{Q}_8^{(-6)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 135360\,x^6 + 41227200\,x^4 - 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\ ,$$

$$\mathcal{R}_8^{(-6)} = \frac{1}{135135}\ , \quad \mathcal{S}_8^{(-6)} = 0$$

$$\mathcal{P}_8^{(-6),*}(x) = \frac{x^{10} + 3\,x^8 + 45\,x^6 - 3376800\,x^4 - 277603200\,x^2 - 3353011200}{135135\,x^{11}}\ ,$$

$$\mathcal{Q}_8^{(-6),*}(x) = \frac{x^{12} + x^{10} + 9\,x^8 + 135360\,x^6 + 41227200\,x^4 + 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\ ,$$

$$\mathcal{R}_8^{(-6),*} = -\frac{1}{135135}\ , \quad \mathcal{S}_8^{(-6),*} = 0$$

$\underline{\mathbf{Z}_9(\mathbf{x})}$:

$$\int J_9(x)\,dx = -\frac{x^6 - 720\,x^4 + 53760\,x^2 - 645120}{x^6} J_0(x) + \frac{40\,x^6 - 8160\,x^4 + 268800\,x^2 - 1290240}{x^7} J_1(x)$$

$$\int I_9(x)\,dx = \frac{x^6 + 720\,x^4 + 53760\,x^2 + 645120}{x^6} I_0(x) - \frac{40\,x^6 + 8160\,x^4 + 268800\,x^2 + 1290240}{x^7} I_1(x)$$

$$\int x\,J_9(x)\,dx =$$

$$= -\frac{x^6 - 768\,x^4 + 59904\,x^2 - 737280}{x^5} J_0(x) + \frac{32\,x^6 - 8832\,x^4 + 304128\,x^2 - 1474560}{x^6} J_1(x) + 9\Lambda_0(x)$$

$$\int x\,I_9(x)\,dx =$$

$$= \frac{x^6 + 768\,x^4 + 59904\,x^2 + 737280}{x^5} I_0(x) - \frac{32\,x^6 + 8832\,x^4 + 304128\,x^2 + 1474560}{x^6} I_1(x) - 9\Lambda_0^*(x)$$

$$\int x^2\,J_9(x)\,dx = -\frac{x^6 - 800\,x^4 + 67200\,x^2 - 860160}{x^4} J_0(x) + \frac{42\,x^6 - 9600\,x^4 + 349440\,x^2 - 1720320}{x^5} J_1(x)$$

$$\int x^2\,I_9(x)\,dx = \frac{x^6 + 800\,x^4 + 67200\,x^2 + 860160}{x^4} I_0(x) - \frac{42\,x^6 + 9600\,x^4 + 349440\,x^2 + 1720320}{x^5} I_1(x)$$

$$\int x^3\,J_9(x)\,dx = -\frac{x^6 - 843\,x^4 + 75264\,x^2 - 1032192}{x^3} J_0(x) +$$

$$+\frac{43\,x^6-11136\,x^4+408576\,x^2-2064384}{x^4}J_1(x)+693\Lambda_0(x)$$

$$\int x^3\,I_9(x)\,dx=\frac{x^6+843\,x^4+75264\,x^2+1032192}{x^3}I_0(x)-$$

$$-\frac{43\,x^6+11136\,x^4+408576\,x^2+2064384}{x^4}I_1(x)+693\Lambda_0^*(x)$$

$$\int x^4\,J_9(x)\,dx=-\frac{x^6-888\,x^4+86400\,x^2-1290240}{x^2}J_0(x)+\frac{44\,x^6-11376\,x^4+483840\,x^2-2580480}{x^3}J_1(x)$$

$$\int x^4\,I_9(x)\,dx=\frac{x^6+888\,x^4+86400\,x^2+1290240}{x^2}I_0(x)-\frac{44\,x^6+11376\,x^4+483840\,x^2+2580480}{x^3}I_1(x)$$

$$\int\frac{J_9(x)\,dx}{x}=-\frac{8\,x^6-6144\,x^4+437760\,x^2-5160960}{9x^7}J_0(x)-$$

$$-\frac{x^8-352\,x^6+67968\,x^4-2165760\,x^2+10321920}{9x^8}J_1(x)+\frac{1}{9}\Lambda_0(x)$$

$$\int\frac{I_9(x)\,dx}{x}=\frac{8\,x^6+6144\,x^4+437760\,x^2+5160960}{9x^7}I_0(x)-$$

$$-\frac{x^8+352\,x^6+67968\,x^4+2165760\,x^2+10321920}{9x^8}I_1(x)+\frac{1}{9}\Lambda_0^*(x)$$

$$\int\frac{J_9(x)\,dx}{x^2}=-\frac{x^6-648\,x^4+44352\,x^2-516096}{x^8}J_0(x)+\frac{38\,x^6-7008\,x^4+217728\,x^2-1032192}{x^9}J_1(x)$$

$$\int\frac{I_9(x)\,dx}{x^2}=\frac{x^6+648\,x^4+44352\,x^2+516096}{x^8}I_0(x)-\frac{38\,x^6+7008\,x^4+217728\,x^2+1032192}{x^9}I_1(x)$$

$$\mathcal{P}_9^{(5)}(x)=-\frac{x^6-935\,x^4+98805\,x^2-1720320}{x}\,,\quad\mathcal{Q}_9^{(5)}(x)=\frac{45\,x^6-12405\,x^4+537600\,x^2-3440640}{x^2}\,,$$

$$\mathcal{R}_9^{(5)}=45045\,,\quad\mathcal{S}_9^{(5)}=0$$

$$\mathcal{P}_9^{(5),*}(x)=\frac{x^6+935\,x^4+98805\,x^2+1720320}{x}\,,\quad\mathcal{Q}_9^{(5),*}(x)=-\frac{45\,x^6+12405\,x^4+537600\,x^2+3440640}{x^2}\,,$$

$$\mathcal{R}_9^{(5),*}=-45045\,,\quad\mathcal{S}_9^{(5),*}=0$$

$$\mathcal{P}_9^{(6)}(x)=-(x^6-984\,x^4+113472\,x^2-2257920)\,,\quad\mathcal{Q}_9^{(6)}(x)=\frac{46\,x^6-13536\,x^4+710784\,x^2-5160960}{x}\,,$$

$$\mathcal{R}_9^{(6)}=0\,,\quad\mathcal{S}_9^{(6)}=0$$

$$\mathcal{P}_9^{(6),*}(x)=x^6+984\,x^4+113472\,x^2+2257920\,,\quad\mathcal{Q}_9^{(6),*}(x)=-\frac{46\,x^6+13536\,x^4+710784\,x^2+5160960}{x}\,,$$

$$\mathcal{R}_9^{(6),*}=0\,,\quad\mathcal{S}_9^{(6),*}=0$$

$$\mathcal{P}_9^{(7)}(x)=-(x^7-1035\,x^5+130725\,x^3-3133935\,x)\,,\quad\mathcal{Q}_9^{(7)}(x)=47\,x^6-14775\,x^4+876015\,x^2-10321920\,,$$

$$\mathcal{R}_9^{(7)}=2027025\,,\quad\mathcal{S}_9^{(7)}=0$$

$$\mathcal{P}_9^{(7),*}(x)=x^7+1035\,x^5+130725\,x^3+3133935\,x\,,\quad\mathcal{Q}_9^{(7),*}(x)=-(47\,x^6+14775\,x^4+876015\,x^2+10321920)\,,$$

$$\mathcal{R}_9^{(7),*}=2027025\,,\quad\mathcal{S}_9^{(7),*}=0$$

$$\mathcal{P}_9^{(8)}(x)=-(x^8-1088\,x^6+150912\,x^4-4432896\,x^2+10321920)\,,$$

$$\mathcal{Q}_9^{(8)}(x)=48\,x^7-16128\,x^5+1087488\,x^3-14026752\,x\,,\quad\mathcal{R}_9^{(8)}=0\,,\quad\mathcal{S}_9^{(8)}=0$$

$$\mathcal{P}_9^{(8),*}(x)=x^8+1088\,x^6+150912\,x^4+4432896\,x^2+10321920\,,$$

$$\mathcal{Q}_9^{(8),*}(x)=-(48\,x^7+16128\,x^5+1087488\,x^3+14026752\,x)\,,\quad\mathcal{R}_9^{(8),*}=0\,,\quad\mathcal{S}_9^{(8),*}=0$$

$$\mathcal{P}_9^{(9)}(x)=-(x^9-1143\,x^7+174405\,x^5-6325515\,x^3+34459425\,x)\,,$$

$$\mathcal{Q}_9^{(9)}(x) = 49\,x^8 - 17601\,x^6 + 1355865\,x^4 - 24137505\,x^2\ , \quad \mathcal{R}_9^{(9)} = 34459425\ , \quad \mathcal{S}_9^{(9)} = 0$$

$$\mathcal{P}_9^{(9),*}(x) = x^9 + 1143\,x^7 + 174405\,x^5 + 6325515\,x^3 + 34459425\,x\ ,$$

$$\mathcal{Q}_9^{(9),*}(x) = -(49\,x^8 + 17601\,x^6 + 1355865\,x^4 + 24137505\,x^2)\ ,$$

$$\mathcal{R}_9^{(9),*} = -34459425\ , \quad \mathcal{S}_9^{(9),*} = 0$$

$$\mathcal{P}_9^{(10)}(x) = -(x^{10} - 1200\,x^8 + 201600\,x^6 - 9031680\,x^4 + 92897280\,x^2)\ ,$$

$$\mathcal{Q}_9^{(10)}(x) = 50\,x^9 - 19200\,x^7 + 1693440\,x^5 - 41287680\,x^3 + 185794560\,x\ ,$$

$$\mathcal{R}_9^{(10)} = 0\ , \quad \mathcal{S}_9^{(10)} = 0$$

$$\mathcal{P}_9^{(10),*}(x) = x^{10} + 1200\,x^8 + 201600\,x^6 + 9031680\,x^4 + 92897280\,x^2\ ,$$

$$\mathcal{Q}_9^{(10),*}(x) = -(50\,x^9 + 19200\,x^7 + 1693440\,x^5 + 41287680\,x^3 + 185794560\,x)\ ,$$

$$\mathcal{R}_9^{(10),*} = 0\ , \quad \mathcal{S}_9^{(10),*} = 0$$

$$\mathcal{P}_9^{(-3)}(x) = \frac{x^8 - 696\,x^6 + 426240\,x^4 - 28224000\,x^2 + 325140480}{693\,x^9}\ ,$$

$$\mathcal{Q}_9^{(-3)}(x) = -\frac{x^{10} - x^8 - 25632\,x^6 + 4521600\,x^4 - 137733120\,x^2 + 650280960}{693\,x^{10}}\ ,$$

$$\mathcal{R}_9^{(-3)} = \frac{1}{693}\ , \quad \mathcal{S}_9^{(-3)} = 0$$

$$\mathcal{P}_9^{(-3),*}(x) = \frac{x^8 + 696\,x^6 + 426240\,x^4 + 28224000\,x^2 + 325140480}{693\,x^9}\ ,$$

$$\mathcal{Q}_9^{(-3),*}(x) = \frac{x^{10} + x^8 - 25632\,x^6 - 4521600\,x^4 - 137733120\,x^2 - 650280960}{693\,x^{10}}\ ,$$

$$\mathcal{R}_9^{(-3),*} = -\frac{1}{693}\ , \quad \mathcal{S}_9^{(-3),*} = 0$$

$$\mathcal{P}_9^{(-4)}(x) = -\frac{x^6 - 584\,x^4 + 37632\,x^2 - 430080}{x^{10}}\ , \quad \mathcal{Q}_9^{(-4)}(x) = \frac{36\,x^6 - 6096\,x^4 + 182784\,x^2 - 860160}{x^{11}}\ ,$$

$$\mathcal{R}_9^{(-4)} = 0\ , \quad \mathcal{S}_9^{(-4)} = 0$$

$$\mathcal{P}_9^{(-4),*}(x) = \frac{x^6 + 584\,x^4 + 37632\,x^2 + 430080}{x^{10}}\ , \quad \mathcal{Q}_9^{(-4),*}(x) = -\frac{36\,x^6 + 6096\,x^4 + 182784\,x^2 + 860160}{x^{11}}\ ,$$

$$\mathcal{R}_9^{(-4),*} = 0\ , \quad \mathcal{S}_9^{(-4),*} = 0$$

$$\mathcal{P}_9^{(-5)}(x) = \frac{x^{10} - 3\,x^8 - 45000\,x^6 + 24998400\,x^4 - 1574899200\,x^2 + 17882726400}{45045\,x^{11}}\ ,$$

$$\mathcal{Q}_9^{(-5)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 1576800\,x^6 + 257443200\,x^4 - 7620480000\,x^2 + 35765452800}{45045\,x^{12}}\ ,$$

$$\mathcal{R}_9^{(-5)} = \frac{1}{45045}\ , \quad \mathcal{S}_9^{(-5)} = 0$$

$$\mathcal{P}_9^{(-5),*}(x) = -\frac{x^{10} + 3\,x^8 - 45000\,x^6 - 24998400\,x^4 - 1574899200\,x^2 - 17882726400}{45045\,x^{11}}\ ,$$

$$\mathcal{Q}_9^{(-5),*}(x) = -\frac{x^{12} + x^{10} + 9\,x^8 + 1576800\,x^6 + 257443200\,x^4 + 7620480000\,x^2 + 35765452800}{45045\,x^{12}}\ ,$$

$$\mathcal{R}_9^{(-5),*} = \frac{1}{45045}\ , \quad \mathcal{S}_9^{(-5),*} = 0$$

$$\mathcal{P}_9^{(-6)}(x) = -\frac{x^6 - 528\,x^4 + 32640\,x^2 - 368640}{x^{12}}\ ,$$

$$\mathcal{Q}_9^{(-6)}(x) = \frac{34\,x^6 - 5376\,x^4 + 157440\,x^2 - 737280}{x^{13}}\ , \quad \mathcal{R}_9^{(-6)} = 0\ , \quad \mathcal{S}_9^{(-6)} = 0$$

$$\mathcal{P}_9^{(-6),*}(x) = \frac{x^6 + 528\,x^4 + 32640\,x^2 + 368640}{x^{12}}\ ,$$

$$\mathcal{Q}_9^{(-6),*}(x) = -\frac{34\,x^6 + 5376\,x^4 + 157440\,x^2 + 737280}{x^{13}}\ ,\quad \mathcal{R}_9^{(-6),*} = 0\ ,\quad \mathcal{S}_9^{(-6),*} = 0$$

$\mathbf{Z}_{10}(\mathbf{x})$:

$$\int J_{10}(x)\,dx = -\frac{48\,x^6 - 15744\,x^4 + 921600\,x^2 - 10321920}{x^7} J_0(x) - \\ -\frac{2\,x^8 - 1152\,x^6 + 154368\,x^4 - 4423680\,x^2 + 20643840}{x^8} J_1(x) + \Lambda_0(x)$$

$$\int I_{10}(x)\,dx = -\frac{48\,x^6 + 15744\,x^4 + 921600\,x^2 + 10321920}{x^7} I_0(x) + \\ +\frac{2\,x^8 + 1152\,x^6 + 154368\,x^4 + 4423680\,x^2 + 20643840}{x^8} I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_{10}(x)\,dx = -\frac{50\,x^6 - 16800\,x^4 + 1021440\,x^2 - 11612160}{x^6} J_0(x) - \\ -\frac{x^8 - 1200\,x^6 + 168000\,x^4 - 4945920\,x^2 + 23224320}{x^7} J_1(x)$$

$$\int x\,I_{10}(x)\,dx = -\frac{50\,x^6 + 16800\,x^4 + 1021440\,x^2 + 11612160}{x^6} I_0(x) + \\ +\frac{x^8 + 1200\,x^6 + 168000\,x^4 + 4945920\,x^2 + 23224320}{x^7} I_1(x)$$

$$\int x^2\,J_{10}(x)\,dx = -\frac{51\,x^6 - 18048\,x^4 + 1142784\,x^2 - 13271040}{x^5} J_0(x) \\ -\frac{x^8 - 1152\,x^6 + 183552\,x^4 - 5603328\,x^2 + 26542080}{x^6} J_1(x) + 99\Lambda_0(x)$$

$$\int x^2\,I_{10}(x)\,dx = -\frac{51\,x^6 + 18048\,x^4 + 1142784\,x^2 + 13271040}{x^5} I_0(x) + \\ +\frac{x^8 + 1152\,x^6 + 183552\,x^4 + 5603328\,x^2 + 26542080}{x^6} I_1(x) + 99\Lambda_0^*(x)$$

$$\int x^3\,J_{10}(x)\,dx = -\frac{52\,x^6 - 19200\,x^4 + 1290240\,x^2 - 15482880}{x^4} J_0(x) - \\ -\frac{x^8 - 1304\,x^6 + 201600\,x^4 - 6451200\,x^2 + 30965760}{x^5} J_1(x)$$

$$\int x^3\,I_{10}(x)\,dx = -\frac{52\,x^6 + 19200\,x^4 + 1290240\,x^2 + 15482880}{x^4} I_0(x) + \\ +\frac{x^8 + 1304\,x^6 + 201600\,x^4 + 6451200\,x^2 + 30965760}{x^5} I_1(x)$$

$$\int x^4\,J_{10}(x)\,dx = -\frac{53\,x^6 - 20559\,x^4 + 1462272\,x^2 - 18579456}{x^3} J_0(x) - \\ -\frac{x^8 - 1359\,x^6 + 231168\,x^4 - 7569408\,x^2 + 37158912}{x^4} J_1(x) + 9009\Lambda_0(x)$$

$$\int x^4\,I_{10}(x)\,dx = -\frac{53\,x^6 + 20559\,x^4 + 1462272\,x^2 + 18579456}{x^3} I_0(x) + \\ +\frac{x^8 + 1359\,x^6 + 231168\,x^4 + 7569408\,x^2 + 37158912}{x^4} I_1(x) - 9009\Lambda_0^*(x)$$

$$\int \frac{J_{10}(x)\,dx}{x} = -\frac{48\,x^6 - 14784\,x^4 + 838656\,x^2 - 9289728}{x^8} J_0(x) -$$

$$-\frac{x^8-1104\,x^6+142464\,x^4-3999744\,x^2+18579456}{x^9}J_1(x)$$

$$\int\frac{I_{10}(x)\,dx}{x}=-\frac{48\,x^6+14784\,x^4+838656\,x^2+9289728}{x^8}I_0(x)+$$

$$+\frac{x^8+1104\,x^6+142464\,x^4+3999744\,x^2+18579456}{x^9}I_1(x)$$

$$\int\frac{J_{10}(x)\,dx}{x^2}=\frac{x^8-4656\,x^6+1376640\,x^4-76124160\,x^2+836075520}{99x^9}J_0(x)-$$

$$-\frac{x^{10}+98\,x^8-104832\,x^6+13075200\,x^4-361267200\,x^2+1672151040}{99x^{10}}J_1(x)+\frac{1}{99}\Lambda_0(x)$$

$$\int\frac{I_{10}(x)\,dx}{x^2}=-\frac{x^8+4656\,x^6+1376640\,x^4+76124160\,x^2+836075520}{99x^9}I_0(x)-$$

$$-\frac{x^{10}-98\,x^8-104832\,x^6-13075200\,x^4-361267200\,x^2-1672151040}{99x^{10}}I_1(x)+\frac{1}{99}\Lambda_0^*(x)$$

$$\mathcal{P}_{10}^{(5)}(x)=-\frac{54\,x^6-22032\,x^4+1693440\,x^2-23224320}{x^2}\;,$$

$$\mathcal{Q}_{10}^{(5)}(x)=-\frac{x^8-1416\,x^6+245664\,x^4-9031680\,x^2+46448640}{x^3}\;,\quad \mathcal{R}_{10}^{(5)}=0\;,\quad \mathcal{S}_{10}^{(5)}=0$$

$$\mathcal{P}_{10}^{(5),*}(x)=-\frac{54\,x^6+22032\,x^4+1693440\,x^2+23224320}{x^2}\;,$$

$$\mathcal{Q}_{10}^{(5),*}(x)=\frac{x^8+1416\,x^6+245664\,x^4+9031680\,x^2+46448640}{x^3}\;,\quad \mathcal{R}_{10}^{(5),*}=0\;,\quad \mathcal{S}_{10}^{(5),*}=0$$

$$\mathcal{P}_{10}^{(6)}(x)=-\frac{55\,x^6-23625\,x^4+1965915\,x^2-30965760}{x}\;,$$

$$\mathcal{Q}_{10}^{(6)}(x)=-\frac{x^8-1475\,x^6+272475\,x^4-10321920\,x^2+61931520}{x^2}\;,$$

$$\mathcal{R}_{10}^{(6)}=675675\;,\quad \mathcal{S}_{10}^{(6)}=0$$

$$\mathcal{P}_{10}^{(6),*}(x)=-\frac{55\,x^6+23625\,x^4+1965915\,x^2+30965760}{x}\;,$$

$$\mathcal{Q}_{10}^{(6),*}(x)=\frac{x^8+1475\,x^6+272475\,x^4+10321920\,x^2+61931520}{x^2}\;,$$

$$\mathcal{R}_{10}^{(6),*}=675675\;,\quad \mathcal{S}_{10}^{(6),*}=0$$

$$\mathcal{P}_{10}^{(7)}(x)=-(56\,x^6-25344\,x^4+2299392\,x^2-41287680)\;,$$

$$\mathcal{Q}_{10}^{(7)}(x)=-\frac{x^8-1536\,x^6+302976\,x^4-13630464\,x^2+92897280}{x}\;,\quad \mathcal{R}_{10}^{(7)}=0\;,\quad \mathcal{S}_{10}^{(7)}=0$$

$$\mathcal{P}_{10}^{(7),*}(x)=-(56\,x^6+25344\,x^4+2299392\,x^2+41287680)\;,$$

$$\mathcal{Q}_{10}^{(7),*}(x)=\frac{x^8+1536\,x^6+302976\,x^4+13630464\,x^2+92897280}{x}\;,\quad \mathcal{R}_{10}^{(7),*}=0\;,\quad \mathcal{S}_{10}^{(7),*}=0$$

$$\mathcal{P}_{10}^{(8)}(x)=-(57\,x^7-27195\,x^5+2706165\,x^3-58437855\,x)\;,$$

$$\mathcal{Q}_{10}^{(8)}(x)=-(x^8-1599\,x^6+337575\,x^4-17150175\,x^2+185794560)\;,$$

$$\mathcal{R}_{10}^{(8)}=34459425\;,\quad \mathcal{S}_{10}^{(8)}=0$$

$$\mathcal{P}_{10}^{(8),*}(x)=-(57\,x^7+27195\,x^5+2706165\,x^3+58437855\,x)\;,$$

$$\mathcal{Q}_{10}^{(8),*}(x)=x^8+1599\,x^6+337575\,x^4+17150175\,x^2+185794560\;,$$

$$\mathcal{R}_{10}^{(8),*}=-34459425\;,\quad \mathcal{S}_{10}^{(8),*}=0$$

$$\mathcal{P}_{10}^{(9)}(x) = -(58\,x^8 - 29184\,x^6 + 3200256\,x^4 - 84953088\,x^2 + 185794560)\;,$$

$$\mathcal{Q}_{10}^{(9)}(x) = -(x^9 - 1664\,x^7 + 376704\,x^5 - 21832704\,x^3 + 262803456\,x)\;,\quad \mathcal{R}_{10}^{(9)} = 0\;,\quad \mathcal{S}_{10}^{(9)} = 0$$

$$\mathcal{P}_{10}^{(9),*}(x) = -(58\,x^8 + 29184\,x^6 + 3200256\,x^4 + 84953088\,x^2 + 185794560)\;,$$

$$\mathcal{Q}_{10}^{(9),*}(x) = x^9 + 1664\,x^7 + 376704\,x^5 + 21832704\,x^3 + 262803456\,x\;,\quad \mathcal{R}_{10}^{(9),*} = 0\;,\quad \mathcal{S}_{10}^{(9),*} = 0$$

$$\mathcal{P}_{10}^{(10)}(x) = -(59\,x^9 - 31317\,x^7 + 3797535\,x^5 - 125345745\,x^3 + 654729075\,x)\;,$$

$$\mathcal{Q}_{10}^{(10)}(x) = -(x^{10} - 1731\,x^8 + 420819\,x^6 - 28019355\,x^4 + 468934515\,x^2)\;,$$

$$\mathcal{R}_{10}^{(10)} = 654729075\;,\quad \mathcal{S}_{10}^{(10)} = 0$$

$$\mathcal{P}_{10}^{(10),*}(x) = -(59\,x^9 + 31317\,x^7 + 3797535\,x^5 + 125345745\,x^3 + 654729075\,x)\;,$$

$$\mathcal{Q}_{10}^{(10),*}(x) = x^{10} + 1731\,x^8 + 420819\,x^6 + 28019355\,x^4 + 468934515\,x^2\;,$$

$$\mathcal{R}_{10}^{(10),*} = 654729075\;,\quad \mathcal{S}_{10}^{(10),*} = 0$$

$$\mathcal{P}_{10}^{(-3)}(x) = -\frac{46\,x^6 - 13104\,x^4 + 709632\,x^2 - 7741440}{x^{10}}\;,$$

$$\mathcal{Q}_{10}^{(-3)}(x) = -\frac{x^8 - 1016\,x^6 + 122976\,x^4 - 3354624\,x^2 + 15482880}{x^{11}}\;,\quad \mathcal{R}_{10}^{(-3)} = 0\;,\quad \mathcal{S}_{10}^{(-3)} = 0$$

$$\mathcal{P}_{10}^{(-3),*}(x) = -\frac{46\,x^6 + 13104\,x^4 + 709632\,x^2 + 7741440}{x^{10}}\;,$$

$$\mathcal{Q}_{10}^{(-3),*}(x) = \frac{x^8 + 1016\,x^6 + 122976\,x^4 + 3354624\,x^2 + 15482880}{x^{11}}\;,\quad \mathcal{R}_{10}^{(-3),*} = 0\;,\quad \mathcal{S}_{10}^{(-3),*} = 0$$

$$\mathcal{P}_{10}^{(-4)}(x) = \frac{x^{10} - 3\,x^8 - 405360\,x^6 + 111484800\,x^4 - 5933813760\,x^2 + 64377815040}{9009\,x^{11}}\;,$$

$$\mathcal{Q}_{10}^{(-4)}(x) = -\frac{x^{12} - x^{10} + 9018\,x^8 - 8784000\,x^6 + 1035820800\,x^4 - 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\;,$$

$$\mathcal{R}_{10}^{(-4)} = \frac{1}{9009}\;,\quad \mathcal{S}_{10}^{(-4)} = 0$$

$$\mathcal{P}_{10}^{(-4),*}(x) = \frac{x^{10} + 3\,x^8 - 405360\,x^6 - 111484800\,x^4 - 5933813760\,x^2 - 64377815040}{9009\,x^{11}}\;,$$

$$\mathcal{Q}_{10}^{(-4),*}(x) = \frac{x^{12} + x^{10} + 9018\,x^8 + 8784000\,x^6 + 1035820800\,x^4 + 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\;,$$

$$\mathcal{R}_{10}^{(-4),*} = -\frac{1}{9009}\;,\quad \mathcal{S}_{10}^{(-4),*} = 0$$

$$\mathcal{P}_{10}^{(-5)}(x) = -\frac{44\,x^6 - 11712\,x^4 + 614400\,x^2 - 6635520}{x^{12}}\;,$$

$$\mathcal{Q}_{10}^{(-5)}(x) = -\frac{x^8 - 936\,x^6 + 107904\,x^4 - 2887680\,x^2 + 13271040}{x^{13}}\;,$$

$$\mathcal{R}_{10}^{(-5)} = 0\;,\quad \mathcal{S}_{10}^{(-5)} = 0$$

$$\mathcal{P}_{10}^{(-5),*}(x) = -\frac{44\,x^6 + 11712\,x^4 + 614400\,x^2 + 6635520}{x^{12}}\;,$$

$$\mathcal{Q}_{10}^{(-5),*}(x) = \frac{x^8 + 936\,x^6 + 107904\,x^4 + 2887680\,x^2 + 13271040}{x^{13}}\;,$$

$$\mathcal{R}_{10}^{(-5),*} = 0\;,\quad \mathcal{S}_{10}^{(-5),*} = 0$$

$$\mathcal{P}_{10}^{(-6)}(x) = \frac{x^{12} - 3\,x^{10} + 45\,x^8 - 29055600\,x^6 + 7506172800\,x^4 - 388949299200\,x^2 + 4184557977600}{675675\,x^{13}}\;,$$

$$\mathcal{Q}_{10}^{(-6)}(x) =$$

$$-\frac{x^{14}-x^{12}+9\,x^{10}+675450\,x^{8}-607420800\,x^{6}+68660524800\,x^{4}-1824038092800\,x^{2}+8369115955200}{675675\,x^{14}}\ ,$$

$$\mathcal{R}_{10}^{(-6)}=\frac{1}{675675}\ ,\quad \mathcal{S}_{10}^{(-6)}=0$$

$$\mathcal{P}_{10}^{(-6),*}(x)=-\frac{x^{12}+3\,x^{10}+45\,x^{8}+29055600\,x^{6}+7506172800\,x^{4}+388949299200\,x^{2}+4184557977600}{675675\,x^{13}}\ ,$$

$$\mathcal{Q}_{10}^{(-6),*}(x)=$$

$$-\frac{x^{14}+x^{12}+9\,x^{10}-675450\,x^{8}-607420800\,x^{6}-68660524800\,x^{4}-1824038092800\,x^{2}-8369115955200}{675675\,x^{14}}\ ,$$

$$\mathcal{R}_{10}^{(-6),*}=\frac{1}{675675}\ ,\quad \mathcal{S}_{10}^{(-6),*}=0$$

1.1.8. Higher antiderivatives:

$\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in I., page 7. See also [1], 11.2. .

$$\begin{aligned}
J_0(x) &= \frac{d^2}{dx^2}\left\{x^2 J_0(x) - xJ_1(x) + x\,\Phi(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}J_0(x) - \frac{x^2}{2}J_1(x) + \frac{x^2-1}{2}\,\Phi(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4-2x^2}{6}J_0(x) - \frac{x^3-4x}{6}J_1(x) + \frac{x^3-3x}{6}\,\Phi(x)\right\}\\
&= \frac{d^5}{dx^5}\left\{\frac{x^5-5x^3}{24}J_0(x) - \frac{x^4-7x^2}{24}J_1(x) + \frac{x^4-6x^2+9}{24}\,\Phi(x)\right\}\\
&= \frac{d^6}{dx^6}\left\{\frac{x^6-9x^4+32x^2}{120}J_0(x) - \frac{x^5-11x^3+64x}{120}J_1(x) + \frac{x^5-10x^3+45x}{120}\,\Phi(x)\right\}\\
&= \frac{d^7}{dx^7}\left\{\frac{x^7-14x^5+117x^3}{720}J_0(x) - \frac{x^6-16x^4+159x^2}{720}J_1(x) + \frac{x^6-15x^4+135x^2-225}{720}\,\Phi(x)\right\}\\
I_0(x) &= \frac{d^2}{dx^2}\left\{x^2 I_0(x) - xI_1(x) + x\,\Psi(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}I_0(x) - \frac{x^2}{2}I_1(x) + \frac{x^2+1}{2}\,\Psi(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6}I_0(x) - \frac{x^3+4x}{6}I_1(x) + \frac{x^3+3x}{6}\,\Psi(x)\right\}\\
&= \frac{d^5}{dx^5}\left\{\frac{x^5+5x^3}{24}I_0(x) - \frac{x^4+7x^2}{24}I_1(x) + \frac{x^4+6x^2+9}{24}\,\Psi(x)\right\}\\
&= \frac{d^6}{dx^6}\left\{\frac{x^6+9x^4+32x^2}{120}I_0(x) - \frac{x^5+11x^3+64x}{120}I_1(x) + \frac{x^5+10x^3+45x}{120}\,\Psi(x)\right\}\\
&= \frac{d^7}{dx^7}\left\{\frac{x^7+14x^5+117x^3}{720}I_0(x) - \frac{x^6+16x^4+159x^2}{720}I_1(x) + \frac{x^6+15x^4+135x^2+225}{720}\,\Psi(x)\right\}\\
K_0(x) &= \frac{d^2}{dx^2}\left\{x^2 K_0(x) - xK_1(x) + x\,\Psi_K(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}K_0(x) - \frac{x^2}{2}K_1(x) + \frac{x^2+1}{2}\,\Psi_K(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6}K_0(x) - \frac{x^3+4x}{6}K_1(x) + \frac{x^3+3x}{6}\,\Psi_K(x)\right\} \quad \ldots
\end{aligned}$$

The formulas for $K_0(x)$ are similar to such for $I_0(x)$.

Let

$$J_0(x) = \frac{d^n}{dx^n}\left\{\frac{A_n(x)\,J_0(x) - B_n(x)\,J_1(x) + C_n(x)\,\Phi(x)}{(n-1)!}\right\},$$

then holds

$$\begin{aligned}
A_8 &= x^8 - 20\,x^6 + 291\,x^4 - 1152\,x^2\\
B_8 &= x^7 - 22\,x^5 + 345\,x^3 - 2304\,x\\
C_8 &= x^7 - 21\,x^5 + 315\,x^3 - 1575\,x\\
\hline
A_9 &= x^9 - 27\,x^7 + 599\,x^5 - 5541\,x^3\\
B_9 &= x^8 - 29\,x^6 + 667\,x^4 - 7407\,x^2\\
C_9 &= x^8 - 28\,x^6 + 630\,x^4 - 6300\,x^2 + 11025\\
\hline
A_{10} &= x^{10} - 35\,x^8 + 1095\,x^6 - 17613\,x^4 + 73728\,x^2\\
B_{10} &= x^9 - 37\,x^7 + 1179\,x^5 - 20583\,x^3 + 147456\,x\\
C_{10} &= x^9 - 36\,x^7 + 1134\,x^5 - 18900\,x^3 + 99225\,x\\
\hline
A_{11} &= x^{11} - 44\,x^9 + 1842\,x^7 - 45180\,x^5 + 439605\,x^3\\
B_{11} &= x^{10} - 46\,x^8 + 1944\,x^6 - 49770\,x^4 + 581535\,x^2\\
C_{11} &= x^{10} - 45\,x^8 + 1890\,x^6 - 47250\,x^4 + 496125\,x^2 - 893025\\
\hline
A_{12} &= x^{12} - 54\,x^{10} + 2912\,x^8 - 100770\,x^6 + 1702215\,x^4 - 7372800\,x^2\\
B_{12} &= x^{11} - 56\,x^9 + 3034\,x^7 - 107640\,x^5 + 1973205\,x^3 - 14745600\,x\\
C_{12} &= x^{11} - 55\,x^9 + 2970\,x^7 - 103950\,x^5 + 1819125\,x^3 - 9823275\,x
\end{aligned}$$

$$A_{13} = x^{13} - 65\,x^{11} + 4386\,x^9 - 203202\,x^7 + 5231565\,x^5 - 52454925\,x^3$$
$$B_{13} = x^{12} - 67\,x^{10} + 4530\,x^8 - 213174\,x^6 + 5731245\,x^4 - 68891175\,x^2$$
$$C_{13} = x^{12} - 66\,x^{10} + 4455\,x^8 - 207900\,x^6 + 5457375\,x^4 - 58939650\,x^2 + 108056025$$
$$A_{14} = x^{14} - 77\,x^{12} + 6354\,x^{10} - 379386\,x^8 + 13778685\,x^6 - 239546025\,x^4 + 1061683200\,x^2$$
$$B_{14} = x^{13} - 79\,x^{11} + 6522\,x^9 - 393462\,x^7 + 14661765\,x^5 - 276270075\,x^3 + 2123366400\,x$$
$$C_{14} = x^{13} - 78\,x^{11} + 6435\,x^9 - 386100\,x^7 + 14189175\,x^5 - 255405150\,x^3 + 1404728325\,x$$
$$A_{15} = x^{15} - 90\,x^{13} + 8915\,x^{11} - 666348\,x^9 + 32399703\,x^7 - 858134970\,x^5 + 8776408725\,x^3$$
$$B_{15} = x^{14} - 92\,x^{12} + 9109\,x^{10} - 685728\,x^8 + 33896331\,x^6 - 937030500\,x^4 + 11465661375\,x^2$$
$$C_{15} = x^{14} - 91\,x^{12} + 9009\,x^{10} - 675675\,x^8 + 33108075\,x^6 - 893918025\,x^4 + 9833098275\,x^2 - \\ -18261468225$$
$$A_{16} = x^{16} - 104\,x^{14} + 12177\,x^{12} - 1113480\,x^{10} + 69776595\,x^8 - 2606612400\,x^6 + 46180633275\,x^4 - \\ -208089907200\,x^2$$
$$B_{16} = x^{15} - 106\,x^{13} + 12399\,x^{11} - 1139580\,x^9 + 72217215\,x^7 - 2767649850\,x^5 + 53076402225\,x^3 - \\ -416179814400\,x$$
$$C_{16} = x^{15} - 105\,x^{13} + 12285\,x^{11} - 1126125\,x^9 + 70945875\,x^7 - 2681754075\,x^5 + 49165491375\,x^3 - \\ -273922023375\,x$$
$$A_{17} = x^{17} - 119\,x^{15} + 16257\,x^{13} - 1785015\,x^{11} + 140033835\,x^9 - 7003021725\,x^7 + \\ +188956336275\,x^5 - 1959828398325\,x^3$$
$$B_{17} = x^{16} - 121\,x^{14} + 16509\,x^{12} - 1819485\,x^{10} + 143878995\,x^8 - 7315353675\,x^6 + \\ +205891548975\,x^4 - 2550046679775\,x^2$$
$$C_{17} = x^{16} - 120\,x^{14} + 16380\,x^{12} - 1801800\,x^{10} + 141891750\,x^8 - 7151344200\,x^6 + 196661965500\,x^4 - \\ -2191376187000\,x^2 + 4108830350625$$
$$A_{18} = x^{18} - 135\,x^{16} + 21281\,x^{14} - 2762727\,x^{12} + 265161195\,x^{10} - 17089901325\,x^8 + \\ +650296717875\,x^6 - 11675732422725\,x^4 + 53271016243200\,x^2$$
$$B_{18} = x^{17} - 137\,x^{15} + 21565\,x^{13} - 2807469\,x^{11} + 271036755\,x^9 - 17667841275\,x^7 + \\ +689522590575\,x^5 - 13385846919375\,x^3 + 106542032486400\,x$$
$$C_{18} = x^{17} - 136\,x^{15} + 21420\,x^{13} - 2784600\,x^{11} + 268017750\,x^9 - 17367550200\,x^7 + \\ +668650682700\,x^5 - 12417798393000\,x^3 + 69850115960625\,x$$
$$A_{19} = x^{19} - 152\,x^{17} + 27384\,x^{15} - 4148856\,x^{13} + 478163970\,x^{11} - 38580445800\,x^9 + \\ +1965171423600\,x^7 - 53722981355400\,x^5 + 563060968600725\,x^3$$
$$B_{19} = x^{18} - 154\,x^{16} + 27702\,x^{14} - 4206042\,x^{12} + 486902160\,x^{10} - 39605788350\,x^8 + \\ +2050859043450\,x^6 - 58451723167950\,x^4 + 730304613424575\,x^2$$
$$C_{19} = x^{18} - 153\,x^{16} + 27540\,x^{14} - 4176900\,x^{12} + 482431950\,x^{10} - 39076987950\,x^8 + \\ +2005952048100\,x^6 - 55880092768500\,x^4 + 628651043645625\,x^2 - 1187451971330625$$
$$A_{20} = x^{20} - 170\,x^{18} + 34710\,x^{16} - 6069258\,x^{14} + 827072928\,x^{12} - 81642319470\,x^{10} + \\ +5359374805050\,x^8 - 206598479046750\,x^6 + 3746290783676175\,x^4 - 17259809262796800\,x^2$$
$$B_{20} = x^{19} - 172\,x^{17} + 35064\,x^{15} - 6141348\,x^{13} + 839759706\,x^{11} - 83394724500\,x^9 + \\ +5536741392000\,x^7 - 218854228527900\,x^5 + 4287004731290925\,x^3 - 34519618525593600\,x$$
$$C_{20} = x^{19} - 171\,x^{17} + 34884\,x^{15} - 6104700\,x^{13} + 833291550\,x^{11} - 82495863450\,x^9 + \\ +5444726987700\,x^7 - 212344352520300\,x^5 + 3981456609755625\,x^3 - 22561587455281875\,x$$

To get the functions for $I_0(x)$ or $K_0(x)$ one has to change all '-' in the fractions to '+' .
The higher antiderivatives of $J_1(x)$, $I_1(x)$ and $K_1(x)$ follow from the previous tables and the formulas $J_1(x) = -J_0'(x)$ and $I_1(x) = I_0'(x),\ K_1(x) = -K_0'(x)$.

1.2.1. Integrals of the type $\int x^{n+1/2} \cdot Z_\nu(x)\,dx$

With the Lommel functions $s_{\mu,\nu}$ (see [7], 8.57, or [8], 10 -7) holds:

$$\int \sqrt{x}\, J_0(x)\,dx = \sqrt{x}\, J_1(x) - \frac{x}{4}\,\left[2\, s_{-1/2,1}(x)\, J_0(x) + s_{-3/2,0}(x)\, J_1(x)\right]\ ,$$

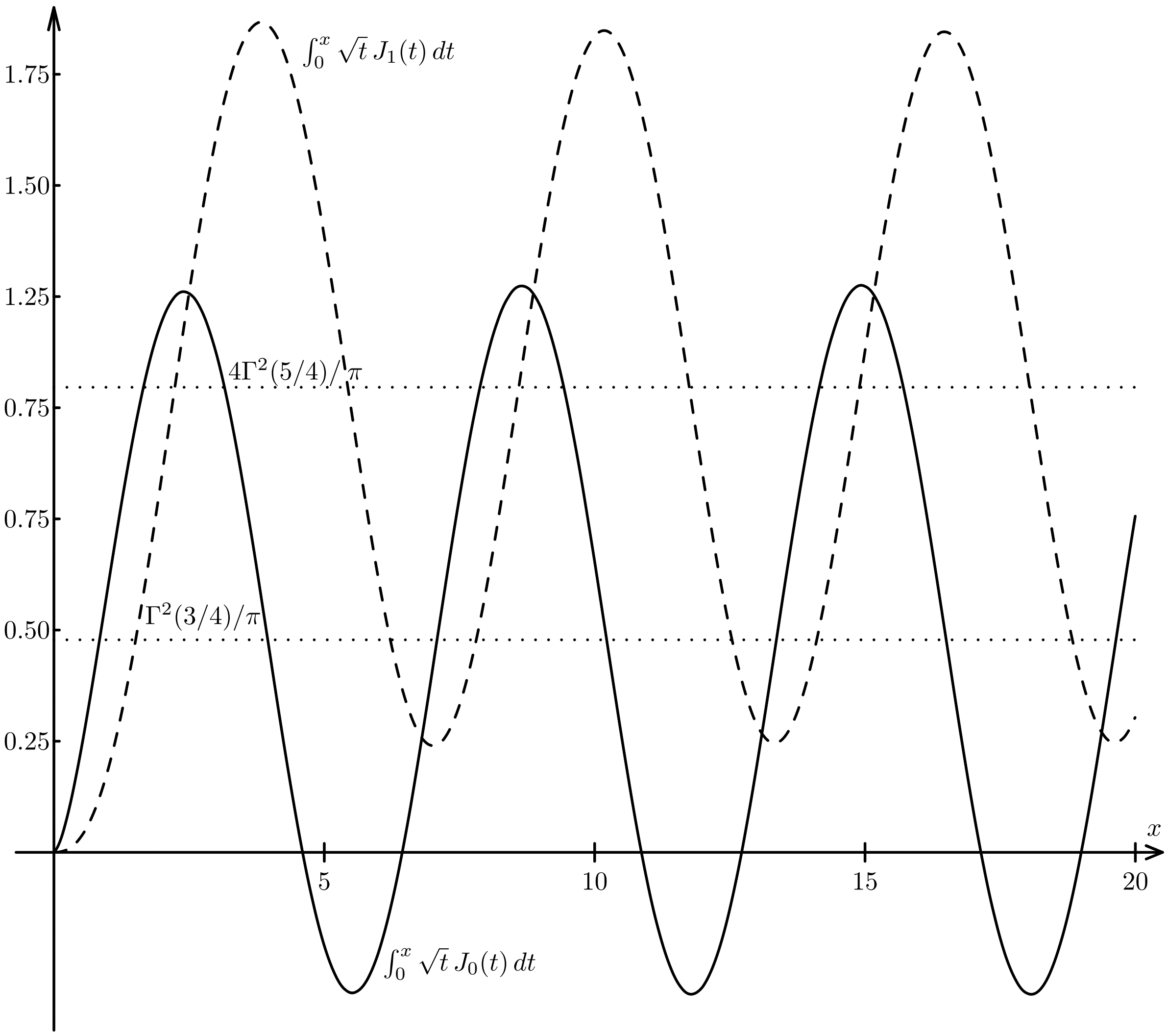
$$\int \sqrt{x}\, J_1(x)\,dx = \frac{x}{2}\,\left[s_{-1/2,0}(x)\, J_1(x) - 2\, s_{1/2,0}(x)\, J_0(x)\right].$$

$$x = t^2 \quad \Longrightarrow \quad \int x^{(2n-1)/2}\, J_\nu(x)\,dx = 2\int t^{2n}\, J_\nu(t^2)\,dt$$

Differential equations:

$$\int \sqrt{x}\, J_0(x)\,dx = y(x) \quad \Longrightarrow \quad x^2\, y''' + \left(x^2 + \frac{1}{4}\right) y' \;=\; 0$$

$$\int \sqrt{x}\, J_1(x)\,dx = z(x) \quad \Longrightarrow \quad x^2\, z''' + \left(x^2 - \frac{3}{4}\right) z' \;=\; 0$$



Function $J_0(x)$:

Approximation by Chebyshev polynomials, based on [2], 9.7.:

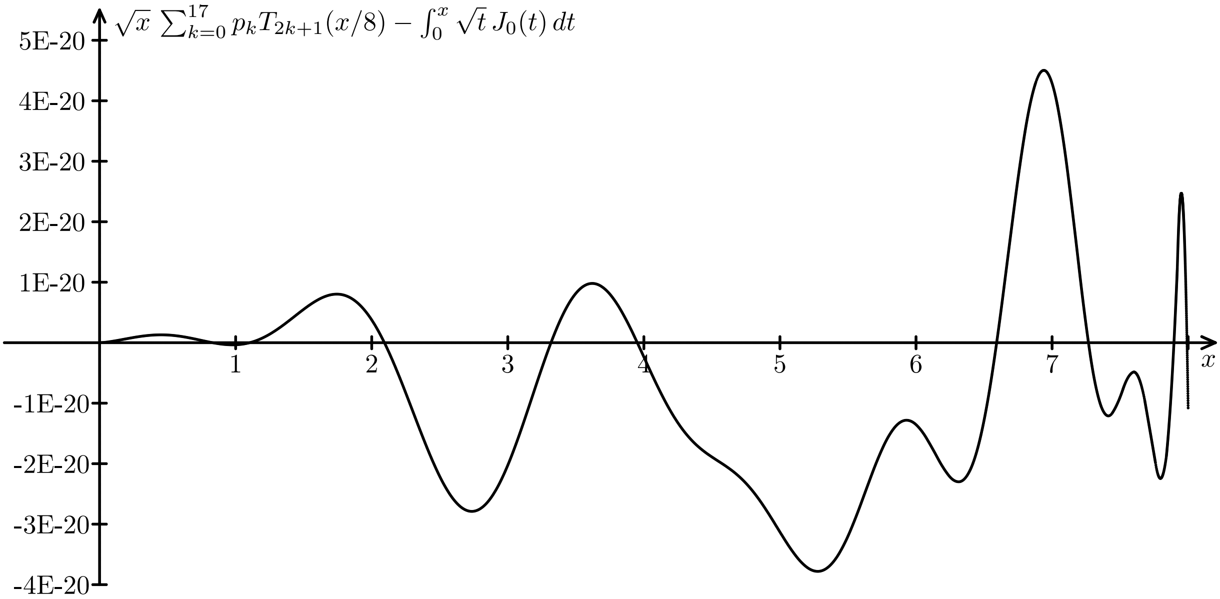
For $0 \le x \le 8$ holds

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} p_k\, T_{2k+1}\left(\frac{x}{8}\right)$$

with the following coefficients:

k	p_k	k	p_k
0	0.29396 17718 67412 06150	9	-0.00000 03762 56494 36038
1	-0.09593 33355 26137 75008	10	0.00000 00146 30781 61468
2	0.39583 39734 26816 07917	11	-0.00000 00004 68853 72762
3	-0.26902 16631 32696 96017	12	0.00000 00000 12605 05370
4	0.07963 03030 17678 07362	13	-0.00000 00000 00288 52942
5	-0.01366 63037 73087 91164	14	0.00000 00000 00005 69318
6	0.00155 43936 32776 04348	15	-0.00000 00000 00000 09787
7	-0.00012 67196 60682 08202	16	0.00000 00000 00000 00148
8	0.00000 77998 70507 77089	17	-0.00000 00000 00000 00002

This approximation differs from the true function as shown:



Asymptotic expansions for $x \to +\infty$:

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \;\sim\; \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{a_k}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right)$$

$$\frac{\Gamma^2(3/4)}{\pi} = 0.477\,988\,797\,486\,125$$

$$a_0 = 1\,,\quad a_1 = \frac{1}{8}\,,\quad a_2 = \frac{25}{128}\,,\quad a_3 = \frac{475}{1024}\,,\quad a_4 = \frac{49275}{32768}\,,\quad a_5 = \frac{1636335}{262144}\,,\quad a_6 = \frac{133308045}{4194304}\,,$$

$$a_7 = \frac{6456759075}{33554432}\,,\quad a_8 = \frac{2905671971475}{2147483648}\,,\quad a_9 = \frac{186381860485275}{17179869184}\,,\ \ldots$$

k	a_k	a_k/a_{k-1}	k	a_k	a_k/a_{k-1}
0	1.000 000 000	-	5	6.242 122 650	4.1510
1	0.125 000 000	0.1250	6	31.783 114 67	5.0917
2	0.195 312 500	1.5625	7	192.426 415 5	6.0544
3	0.463 867 188	2.3750	8	1 353.058 951	7.0316
4	1.503 753 662	3.2418	9	10 848.852 14	8.0180

Let

$$D_{0,n}(x) = \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{a_k}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt \,,$$

then its first maximum and minimum values of interest are $D_{0,n}(x_{i,n}^*)$.
In the case $x > x_{i,n}^*$ holds $|D_{0,n}(x)| < |D_{0,n}(x_{i,n}^*)|$.

$n=0,\ i=$	1	2	3	4	5	6	7	8	9	10
$x_{i,0}^*$	1.143	4.058	7.146	10.264	13.394	16.527	19.664	22.801	25.940	29.079
$10^3 D_{0,0}(x_i^*)$	50.87	-21.784	13.293	-9.4707	7.3310	-5.9717	5.0342	-4.3496	3.8281	-3.4179

$n=1,\ i=$	1	2	3	4	5	6	7	8	9	10
$x_{i,1}^*$	2.473	5.561	8.681	11.812	14.947	18.085	21.223	24.363	27.503	30.643
$10^4 D_{0,1}(x_i^*)$	158.930	-43.0361	19.1585	-10.6855	6.7779	-4.6705	3.4092	-2.5962	2.0420	-1.6478

$n=2,\ i=$	3	4	5	6	7	8	9	10	11	12
$x_{i,2}^*$	7.100	10.233	13.369	16.508	19.647	22.787	25.927	29.068	32.209	35.350
$10^5 D_{0,2}(x_i^*)$	-85.8809	31.2037	-14.5367	7.8808	-4.7310	3.0556	-2.0851	1.4850	-1.0945	0.8296

$n=3,\ i=$	4	5	6	7	8	9	10	11	12	13
$x_{i,3}^*$	11.793	14.932	18.072	21.213	24.353	27.494	30.635	33.777	36.918	40.059
$10^7 D_{0,3}(x_i^*)$	548.133	-222.407	106.176	-56.7759	33.0053	-20.4558	13.3362	-9.0584	6.3648	-4.6012

$n=4,\ i=$	5	6	7	8	9	10	11	12	13	14
$x_{i,4}^*$	16.499	19.649	22.780	25.922	29.063	32.204	35.345	38.487	41.628	44.770
$10^8 D_{0,4}(x_i^*)$	-369.198	158.653	-76.8752	40.7761	-23.2093	13.9787	-8.8181	5.7815	-3.9164	2.7284

For $8 \le x \le 30$ the special approximation holds:

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \;\approx\; 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right)$$

with

$$-5.0 \cdot 10^{-9} < 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt < 6 \cdot 10^{-9} \,.$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$
1	0.797 884 516 538	6	4.733 533 047 132
2	0.099 735 119 074	7	18.859 909 855 69
3	0.155 775 947 720	8	99.846 038 227 04
4	0.369 550 899 387	9	256.775 583 671 0
5	1.170 795 416 963	10	1 527.508 571 668

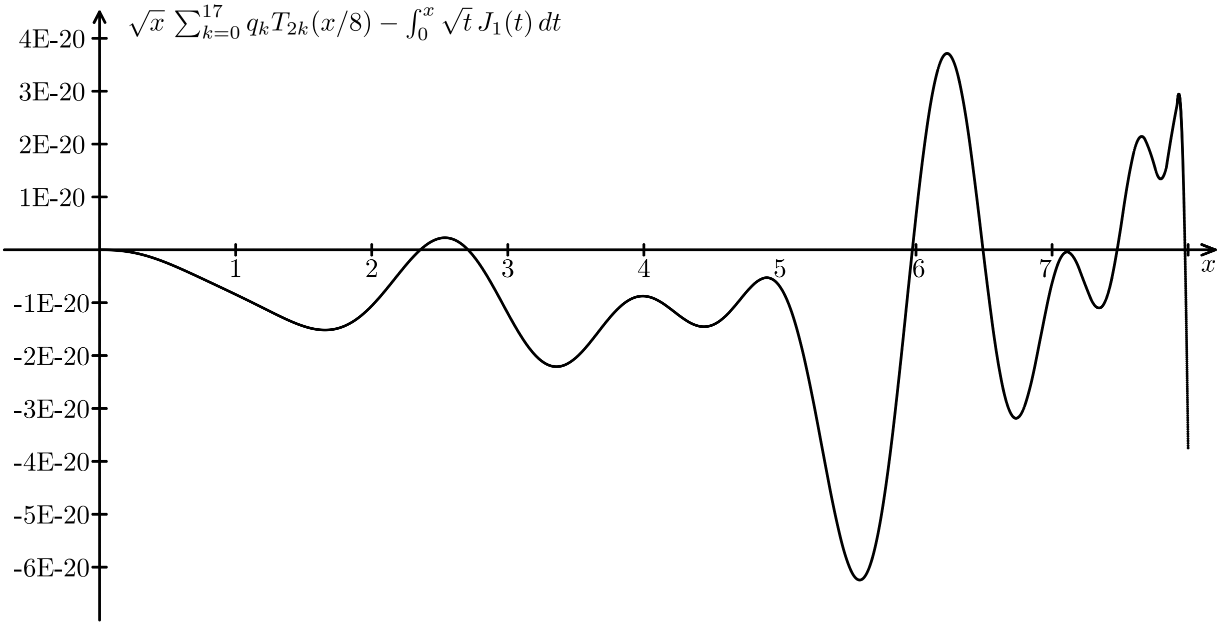
Function $J_1(x)$:
Approximation by Chebyshev polynomials, based on [2], 9.7.:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} q_k\, T_{2k}\left(\frac{x}{8}\right) , \qquad 0 \le x \le 8$$

with the coefficients:

k	q_k	k	q_k
0	0.37975 25427 04720 47384	9	0.00000 17101 75937 74175
1	-0.24153 71053 32677 35417	10	-0.00000 00740 94524 46089
2	-0.12554 99442 21699 83184	11	0.00000 00026 16214 44822
3	0.31360 55017 12763 75964	12	-0.00000 00000 76805 72461
4	-0.14432 03488 73845 84716	13	0.00000 00000 01905 60154
5	0.03274 98779 87894 78550	14	-0.00000 00000 00040 50286
6	-0.00458 83639 64558 05653	15	0.00000 00000 00000 74601
7	0.00044 24559 29648 31876	16	-0.00000 00000 00000 01203
8	-0.00003 13683 81557 99050	17	0.00000 00000 00000 00017

Difference between approximation and true function:



Asymptotic expansion for $x \to \infty$:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \sim \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{b_k}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right)$$

$$\frac{4\Gamma^2(5/4)}{\pi} = 1.046\,049\,620\,053\,102$$

$$b_0 = 1 , \quad b_1 = \frac{3}{8} , \quad b_2 = -\frac{63}{128} , \quad b_3 = \frac{1113}{1024} , \quad b_4 = -\frac{111573}{32768} , \quad b_5 = \frac{3643101}{262144} , \quad b_6 = -\frac{294285915}{4194304} ,$$
$$b_7 = \frac{14192615745}{33554432} , \quad b_8 = -\frac{6373074947085}{2147483648} , \quad b_9 = \frac{408344927902065}{17179869184} , \ \ldots$$

k	b_k	$\lvert b_k/b_{k-1} \rvert$	k	b_k	$\lvert b_k/b_{k-1} \rvert$
0	1.000 000 000	-	5	13.897 327 42	4.0815
1	0.375 000 000	0.3750	6	-70.163 229 70	5.0487
2	-0.492 187 500	1.3125	7	422.972 910 0	6.0284
3	1.086 914 063	2.2083	8	-2 967.694 284	7.0163
4	-3.404 937 744	3.1327	9	23 768.803 10	8.0092

Let

$$D_{1,n}(x) = \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{b_k}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_1(t)\, dt\,,$$

then its first maximum and minimum values of interest are $D_{1,n}(x^*_{i,n})$.
In the case $x > x^*_{i,n}$ holds $|D_{1,n}(x)| < |D_{1,n}(x^*_{i,n})|$.

$n=0,\ i=$	1	2	3	4	5	6	7	8	9	10
$x^*_{i,0}$	2.470	6.470	8.675	11.807	14.943	18.081	21.220	24.360	27.500	30.641
$10^3 D_{1,0}(x^*_i)$	-98.4511	50.5537	-33.4821	24.9168	-19.8073	16.4242	-14.0227	12.2312	-10.8443	9.7390

$n=1,\ i=$	2	3	4	5	6	7	8	9	10	11
$x^*_{i,1}$	3.974	7.097	10.230	13.367	15.506	16.645	22.786	25.926	29.067	32.208
$10^4 D_{1,1}(x^*_i)$	-194.456	70.4454	-35.5457	21.2582	-14.0945	10.0128	-7.4731	5.7878	-4.6133	3.7625

$n=2,\ i=$	3	4	5	6	7	8	9	10	11	12
$x^*_{i,2}$	8.654	12.654	14.931	18.071	21.212	24.353	27.494	30.635	33.776	36.917
$10^5 D_{1,2}(x^*_i)$	117.286	-48.9485	24.7672	-14.1826	8.8517	-5.8853	4.1071	-2.9778	2.2269	-1.7083

$n=3,\ i=$	4	5	6	7	8	9	10	11	12	13
$x^*_{i,3}$	13.358	16.498	19.639	22.780	29.921	29.063	32.204	35.345	38.487	41.628
$10^7 D_{0,3}(x^*_i)$	-773.343	342.802	-173.926	97.2278	-58.4641	37.2099	-24.7842	17.1337	-12.2177	8.9436

$n=4,\ i=$	4	5	6	7	8	9	10	11	12	13
$x^*_{i,4}$	11.785	14.926	18.067	21.208	24.349	27.491	30.632	33.774	36.915	40.057
$10^7 D_{1,4}(x^*_i)$	409.325	-133.167	53.0102	-24.2927	12.3500	-6.7990	3.9863	-2.4597	1.5831	-1.0557

For $8 \le x \le 30$ holds

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \;\approx\; 1.046\,049046\,618046\,299 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right)$$

with

$$-7.0 \cdot 10^{-9} < 1.046\,049\,046\,618 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt < 5 \cdot 10^{-9}\,.$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	-0.797 884 661 354	6	-10.914 764 692 58
2	-0.299 206 773 536	7	40.086 327 439 47
3	0.392 563 542 201	8	-264.350 611 778 9
4	-0.867 123 732 390	9	464.583 909 606 6
5	2.645 254 609 577	10	-5 043.243 969 567

Modified Bessel Functions:

$$\int_0^x \sqrt{t}\, I_0(t)\, dt \;=\; 2\sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+1}}{4^k \cdot (k!)^2 \cdot (4k+3)} \;=$$

$$= 2\sqrt{x}\left(\frac{x}{3} + \frac{x^3}{28} + \frac{x^5}{704} + \frac{x^7}{34\,560} + \frac{x^9}{2\,801\,664} + \frac{x^{11}}{339\,148\,800} + \frac{x^{13}}{57\,330\,892\,800} + \frac{x^{15}}{12\,901\,574\,246\,400} + \ldots\right)$$

$$\int_0^x \sqrt{t}\, I_1(t)\, dt \;=\; \sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+2}}{4^k \cdot (k!)^2 \cdot (4k+5) \cdot (k+1)} \;=$$

$$= \sqrt{x}\left(\frac{x^2}{5} + \frac{x^4}{72} + \frac{x^6}{2\,496} + \frac{x^8}{156\,672} + \frac{x^{10}}{15\,482\,880} + \frac{x^{12}}{2\,211\,840\,000} + \frac{x^{14}}{431\,043\,379\,200} + \ldots\right)$$

1.2.1. b) Integrals:

$$\int x^{3/2}\,J_0(x)\,dx = x^{3/2}J_1(x) - \frac{1}{2}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{3/2}\,J_1(x)\,dx = -x^{3/2}J_0(x) + \frac{3}{2}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{3/2}\,I_0(x)\,dx = x^{3/2}I_1(x) - \frac{1}{2}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{3/2}\,I_1(x)\,dx = x^{3/2}I_0(x) - \frac{3}{2}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{5/2}\,J_0(x)\,dx = \sqrt{x}\left[\frac{3x}{2}J_0(x) + x^2 J_1(x)\right] - \frac{9}{4}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{5/2}\,J_1(x)\,dx = \sqrt{x}\left[-x^2 J_0(x) + \frac{5}{2}xJ_1(x)\right] - \frac{5}{4}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{5/2}\,I_0(x)\,dx = \sqrt{x}\left[-\frac{3x}{2}I_0(x) + x^2 I_1(x)\right] + \frac{9}{4}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{5/2}\,I_1(x)\,dx = \sqrt{x}\left[x^2 J_0(x) - \frac{5}{2}xI_1(x)\right] + \frac{5}{4}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{7/2}\,J_0(x)\,dx = \sqrt{x}\left[\frac{5x^2}{2}J_0(x) + \left(x^3 - \frac{25}{4}\,x\right)J_1(x)\right] + \frac{25}{8}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{7/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^3 + \frac{21}{4}\,x\right)J_0(x) + \frac{7}{2}\,x^2 J_1(x)\right] - \frac{63}{8}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{7/2}\,I_0(x)\,dx = \sqrt{x}\left[-\frac{5x^2}{2}I_0(x) + \left(x^3 + \frac{25}{4}\,x\right)I_1(x)\right] - \frac{25}{8}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{7/2}\,I_1(x)\,dx = \sqrt{x}\left[\left(x^3 + \frac{21}{4}\,x\right)I_0(x) - \frac{7}{2}\,x^2 I_1(x)\right] - \frac{63}{8}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{9/2}\,J_0(x)\,dx = \sqrt{x}\left[\left(\frac{7}{2}\,x^3 - \frac{147}{8}\,x\right)J_0(x) + \left(x^4 - \frac{49}{4}\,x^2\right)J_1(x)\right] + \frac{441}{16}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{9/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^4 + \frac{45}{4}\,x^2\right)J_0(x) + \left(\frac{9}{2}\,x^3 - \frac{225}{8}\,x\right)J_1(x)\right] + \frac{225}{16}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{9/2}\,I_0(x)\,dx = \sqrt{x}\left[-\left(\frac{7}{2}\,x^3 + \frac{147}{8}\,x\right)I_0(x) + \left(x^4 + \frac{49}{4}\,x^2\right)I_1(x)\right] + \frac{441}{16}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{9/2}\,I_1(x)\,dx = \sqrt{x}\left[\left(x^4 + \frac{45}{4}\,x^2\right)I_0(x) - \left(\frac{9}{2}\,x^3 + \frac{225}{8}\,x\right)I_1(x)\right] + \frac{225}{16}\int \sqrt{x}\,I_1(x)\,dx$$

To find $\int x^{(2n+1)/2}\,I_\nu(x)\,dx$ with $n > 4$ use the recurrence formulas (see page 56).

$$\int x^{11/2}\,J_0(x)\,dx = \sqrt{x}\left[\left(\frac{9}{2}\,x^4 - \frac{405}{8}\,x^2\right)J_0(x) + \left(x^5 - \frac{81}{4}\,x^3 + \frac{2025}{16}\,x\right)J_1(x)\right] - \frac{2025}{32}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{11/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^5 + \frac{77}{4}\,x^3 - \frac{1617}{16}\,x\right)J_0(x) + \left(\frac{11}{2}\,x^4 - \frac{539}{8}\,x^2\right)J_1(x)\right] + \frac{4851}{32}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{13/2}\,J_0(x)\,dx = \sqrt{x}\left[\left(\frac{11}{2}\,x^5 - \frac{847}{8}\,x^3 + \frac{17787}{32}\,x\right)J_0(x) + \left(x^6 - \frac{121}{4}\,x^4 + \frac{5929}{16}\,x^2\right)J_1(x)\right] - \\ -\frac{53361}{64}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{13/2}\,J_1(x)\,dx=\sqrt{x}\left[\left(-x^6+\frac{117}{4}\,x^4-\frac{5265}{16}\,x^2\right)J_0(x)+\left(\frac{13}{2}\,x^5-\frac{1053}{8}\,x^3+\frac{26325}{32}\,x\right)J_1(x)\right]-$$
$$-\frac{26325}{64}\int\sqrt{x}\,J_1(x)\,dx$$

$$*E*\qquad\int x^{15/2}\,J_0(x)\,dx=\sqrt{x}\left[\left(\frac{13}{2}\,x^6-\frac{1521}{8}\,x^4+\frac{68445}{32}\,x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^7-\frac{169}{4}\,x^5+\frac{13689}{16}\,x^3-\frac{342225}{64}\,x\right)J_1(x)\right]+\frac{342225}{128}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int x^{15/2}\,J_1(x)\,dx=\sqrt{x}\left[\left(-x^7+\frac{165}{4}\,x^5-\frac{12705}{16}\,x^3+\frac{266805}{64}\,x\right)J_0(x)+\right.$$
$$\left.+\left(\frac{15}{2}\,x^6-\frac{1815}{8}\,x^4+\frac{88935}{32}\,x^2\right)J_1(x)\right]-\frac{800415}{128}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{17/2}\,J_0(x)\,dx=\sqrt{x}\left[\left(\frac{15}{2}\,x^7-\frac{2475}{8}\,x^5+\frac{190575}{32}\,x^3-\frac{4002075}{128}\,x\right)J_0(x)+\right.$$
$$\left.+\left(x^8-\frac{225}{4}\,x^6+\frac{27225}{16}\,x^4-\frac{1334025}{64}\,x^2\right)J_1(x)\right]+\frac{12006225}{256}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{17/2}\,J_1(x)\,dx=\sqrt{x}\left[\left(-x^8+\frac{221}{4}\,x^6-\frac{25857}{16}\,x^4+\frac{1163565}{64}\,x^2\right)J_0(x)+\right.$$
$$\left.+\left(\frac{17}{2}\,x^7-\frac{2873}{8}\,x^5+\frac{232713}{32}\,x^3-\frac{5817825}{128}\,x\right)J_1(x)\right]+\frac{5817825}{256}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int x^{19/2}\,J_0(x)\,dx=\sqrt{x}\left[\left(\frac{17}{2}\,x^8-\frac{3757}{8}\,x^6+\frac{439569}{32}\,x^4-\frac{19780605}{128}\,x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^9-\frac{289}{4}\,x^7+\frac{48841}{16}\,x^5-\frac{3956121}{64}\,x^3+\frac{98903025}{256}\,x\right)J_1(x)\right]-\frac{98903025}{512}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int x^{19/2}\,J_1(x)\,dx=\sqrt{x}\left[\left(-x^9+\frac{285}{4}\,x^7-\frac{47025}{16}\,x^5+\frac{3620925}{64}\,x^3-\frac{76039425}{256}\,x\right)J_0(x)+\right.$$
$$\left.+\left(\frac{19}{2}\,x^8-\frac{4275}{8}\,x^6+\frac{517275}{32}\,x^4-\frac{25346475}{128}\,x^2\right)J_1(x)\right]+\frac{228118275}{512}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{21/2}\,J_0(x)\,dx=$$
$$\sqrt{x}\left[\left(\frac{19}{2}\,x^9-\frac{5415}{8}\,x^7+\frac{893475}{32}\,x^5-\frac{68797575}{128}\,x^3+\frac{1444749075}{512}\,x\right)J_0(x)+\right.$$
$$\left.+\left(x^{10}-\frac{361}{4}\,x^8+\frac{81225}{16}\,x^6-\frac{9828225}{64}\,x^4+\frac{481583025}{256}\,x^2\right)J_1(x)\right]-$$
$$-\frac{4334247225}{1024}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{21/2}\,J_1(x)\,dx=\sqrt{x}\left[\left(-x^{10}+\frac{357}{4}\,x^8-\frac{78897}{16}\,x^6+\frac{9230949}{64}\,x^4-\frac{415392705}{256}\,x^2\right)J_0(x)+\right.$$
$$\left.+\left(\frac{21}{2}\,x^9-\frac{6069}{8}\,x^7+\frac{1025661}{32}\,x^5-\frac{83078541}{128}\,x^3+\frac{2076963525}{512}\,x\right)J_1(x)\right]-\frac{2076963525}{1024}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int x^{23/2}\,J_0(x)\,dx=\sqrt{x}\left[\left(\frac{21}{2}\,x^{10}-\frac{7497}{8}\,x^8+\frac{1656837}{32}\,x^6-\frac{193849929}{128}\,x^4+\frac{8723246805}{512}\,x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^{11}-\frac{441}{4}\,x^9+\frac{127449}{16}\,x^7-\frac{21538881}{64}\,x^5+\frac{1744649361}{256}\,x^3-\frac{43616234025}{1024}\,x\right)J_1(x)\right]+$$

$$+\frac{43616234025}{2048}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{23/2}\,J_1(x)\,dx =$$

$$= \sqrt{x}\left[\left(-x^{11}+\frac{437}{4}\,x^9-\frac{124545}{16}\,x^7+\frac{20549925}{64}\,x^5-\frac{1582344225}{256}\,x^3+\frac{33229228725}{1024}\,x\right)J_0(x)+\right.$$

$$\left.+\left(\frac{23}{2}\,x^{10}-\frac{8303}{8}\,x^8+\frac{1868175}{32}\,x^6-\frac{226049175}{128}\,x^4+\frac{11076409575}{512}\,x^2\right)J_1(x)\right]-$$

$$-\frac{99687686175}{2048}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{25/2}\,J_0(x)\,dx =$$

$$= \sqrt{x}\left[\left(\frac{23}{2}\,x^{11}-\frac{10051}{8}\,x^9+\frac{2864535}{32}\,x^7-\frac{472648275}{128}\,x^5+\frac{36393917175}{512}\,x^3-\frac{764272260675}{2048}\,x\right)J_0(x)+\right.$$

$$\left.+\left(x^{12}-\frac{529}{4}\,x^{10}+\frac{190969}{16}\,x^8-\frac{42968025}{64}\,x^6+\frac{5199131025}{256}\,x^4-\frac{254757420225}{1024}\,x^2\right)J_1(x)\right]+$$

$$+\frac{2292816782025}{4096}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{25/2}\,J_1(x)\,dx =$$

$$= \sqrt{x}\left[\left(-x^{12}+\frac{525}{4}\,x^{10}-\frac{187425}{16}\,x^8+\frac{41420925}{64}\,x^6-\frac{4846248225}{256}\,x^4+\frac{218081170125}{1024}\,x^2\right)J_0(x)+\right.$$

$$\left.+\left(\frac{25}{2}\,x^{11}-\frac{11025}{8}\,x^9+\frac{3186225}{32}\,x^7-\frac{538472025}{128}\,x^5+\frac{43616234025}{512}\,x^3-\frac{1090405850625}{2048}\,x\right)J_1(x)\right]+$$

$$+\frac{1090405850625}{4096}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{27/2}\,J_0(x)\,dx =$$

$$= \sqrt{x}\left[\left(\frac{25}{2}\,x^{12}-\frac{13125}{8}\,x^{10}+\frac{4685625}{32}\,x^8-\frac{1035523125}{128}\,x^6+\frac{121156205625}{512}\,x^4-\frac{5452029253125}{2048}\,x^2\right)J_0(x)+\right.$$

$$+\left(x^{13}-\frac{625}{4}\,x^{11}+\frac{275625}{16}\,x^9-\frac{79655625}{64}\,x^7+\frac{13461800625}{256}\,x^5-\frac{1090405850625}{1024}\,x^3+\right.$$

$$\left.\left.+\ \frac{27260146265625}{4096}\,x\right)J_1(x)\right]-\frac{27260146265625}{8192}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{27/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^{13}+\frac{621}{4}\,x^{11}-\frac{271377}{16}\,x^9+\frac{77342445}{64}\,x^7-\frac{12761503425}{256}\,x^5+\right.\right.$$

$$\left.\frac{982635763725}{1024}\,x^3-\frac{20635351038225}{4096}\,x\right)J_0(x)+\left(\frac{27}{2}\,x^{12}-\frac{14283}{8}\,x^{10}+\frac{5156163}{32}\,x^8-\frac{1160136675}{128}\,x^6+\right.$$

$$\left.\left.+\frac{140376537675}{512}\,x^4-\frac{6878450346075}{2048}\,x^2\right)J_1(x)\right]+\frac{61906053114675}{8192}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{29/2}\,J_0(x)\,dx = \sqrt{x}\left[\left(\frac{27}{2}\,x^{13}-\frac{16767}{8}\,x^{11}+\frac{7327179}{32}\,x^9-\frac{2088246015}{128}\,x^7+\frac{344560592475}{512}\,x^5-\right.\right.$$

$$-\frac{26531165620575}{2048}\,x^3+\left.\frac{557154478032075}{8192}\,x\right)J_0(x)+\left(x^{14}-\frac{729}{4}\,x^{12}+\frac{385641}{16}\,x^{10}-\frac{139216401}{64}\,x^8+\right.$$

$$+\frac{31323690225}{256}\,x^6-\frac{3790166517225}{1024}\,x^4+\frac{185718159344025}{4096}\,x^2\Big)\,J_1(x)\Big]-\frac{1671463434096225}{16384}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{29/2}\,J_1(x)\,dx=\sqrt{x}\left[\Big(-x^{14}+\frac{725}{4}\,x^{12}-\frac{380625}{16}\,x^{10}+\frac{135883125}{64}\,x^8-\frac{30030170625}{256}\,x^6+\right.$$

$$+\frac{3513529963125}{1024}\,x^4-\frac{158108848340625}{4096}\,x^2\Big)\,J_0(x)+\Big(\frac{29}{2}\,x^{13}-\frac{18125}{8}\,x^{11}+\frac{7993125}{32}\,x^9-\frac{2310013125}{128}\,x^7+$$

$$\left.+\frac{390392218125}{512}\,x^5-\frac{31621769668125}{2048}\,x^3+\frac{790544241703125}{8192}\,x\Big)\,J_1(x)\right]-\frac{790544241703125}{16384}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int\frac{J_0(x)}{\sqrt{x}}\,dx=2\sqrt{x}J_0(x)+2\int\sqrt{x}J_1(x)\,dx$$

$$\int\frac{J_1(x)}{\sqrt{x}}\,dx=-2\sqrt{x}J_1(x)+2\int\sqrt{x}J_0(x)\,dx$$

$$\int\frac{I_0(x)}{\sqrt{x}}\,dx=2\sqrt{x}I_0(x)-2\int\sqrt{x}J_1(x)\,dx$$

$$\int\frac{I_1(x)}{\sqrt{x}}\,dx=-2\sqrt{x}I_1(x)+2\int\sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{x}[-2J_0(x)+4xJ_1(x)]-4\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-3/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{3x}[4xJ_0(x)-2J_1(x)]+\frac{4}{3}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-3/2}\cdot I_0(x)\,dx=-\frac{\sqrt{x}}{x}[2I_0(x)+4xI_1(x)]+4\int\sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2}\cdot I_1(x)\,dx=\frac{\sqrt{x}}{3x}[4xI_0(x)-2I_1(x)]-\frac{4}{3}\int\sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{9x^2}\left[\left(-8x^2-6\right)J_0(x)+4xJ_1(x)\right]-\frac{8}{9}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-5/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{5x^2}[-4xJ_0(x)+(8x^2-2)J_1(x)]-\frac{8}{5}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-5/2}\cdot I_0(x)\,dx=\frac{\sqrt{x}}{9x^2}\left[\left(8x^2-6\right)I_0(x)-4xI_1(x)\right]-\frac{8}{9}\int\sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2}\cdot I_1(x)\,dx=\frac{\sqrt{x}}{5x^2}[-4xI_0(x)-(8x^2+2)I_1(x)]+\frac{8}{5}\int\sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{25\,x^3}[(8\,x^2-10)J_0(x)+(-16\,x^3+4\,x)J_1(x)]+\frac{16}{25}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-7/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{63\,x^3}[(-16\,x^3-12\,x)J_0(x)+(8\,x^2-18)J_1(x)]-\frac{16}{63}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-7/2}\cdot I_0(x)\,dx=-\frac{\sqrt{x}}{25\,x^3}[(8\,x^2+10)I_0(x)+(16\,x^3+4\,x)I_1(x)]+\frac{16}{25}\int\sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2}\cdot I_1(x)\,dx=\frac{\sqrt{x}}{63\,x^3}[(16\,x^3-12\,x)I_0(x)-(8\,x^2+18)I_1(x)]-\frac{16}{63}\int\sqrt{x}I_1(x)\,dx$$

To find $\int x^{-(2n+1)/2}\,I_\nu(x)\,dx$ with $n>4$ use the recurrence formulas (see page 56).

$$\int x^{-9/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{441\,x^4}\left[\left(32\,x^4+24\,x^2-126\right)J_0(x)+(-16\,x^3+36\,x)J_1(x)\right]+\frac{32}{441}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-9/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{225\,x^4}[(16\,x^3-20\,x)J_0(x)+(-32\,x^4+8\,x^2-50)J_1(x)]+\frac{32}{225}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-11/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{2025\,x^5}[(-32\,x^4+40\,x^2-450)J_0(x)+(64\,x^5-16\,x^3+100\,x)J_1(x)]-\frac{64}{2025}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-11/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{4851\,x^5}[(64\,x^5+48\,x^3-252\,x)J_0(x)+(-32\,x^4+72\,x^2-882)J_1(x)]+\frac{64}{4851}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-13/2}\cdot J_0(x)\,dx=$$

$$=\frac{\sqrt{x}}{53361\,x^6}\,[(-128\,x^6-96\,x^4+504\,x^2-9702)J_0(x)+(64\,x^5-144\,x^3+1764\,x)J_1(x)]-\frac{128}{53361}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-13/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{26325\,x^6}[(-64\,x^5+80\,x^3-900\,x)J_0(x)+$$

$$+(128\,x^6-32\,x^4+200\,x^2-4050)J_1(x)]-\frac{128}{26325}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-15/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{342225\,x^7}[(128\,x^6-160\,x^4+1800\,x^2-52650)J_0(x)+$$

$$+(-256\,x^7+64\,x^5-400\,x^3+8100\,x)J_1(x)]+\frac{256}{342225}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-15/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{800415\,x^7}[(-256\,x^7-192\,x^5+1008\,x^3-19404\,x)J_0(x)+$$

$$+(128\,x^6-288\,x^4+3528\,x^2-106722)J_1(x)]-\frac{256}{800415}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-17/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{12006225\,x^8}[(512\,x^8+384\,x^6-2016\,x^4+38808\,x^2-1600830)J_0(x)+$$

$$+(-256\,x^7+576\,x^5-7056\,x^3+213444\,x)J_1(x)]+\frac{512}{12006225}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-17/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{5817825\,x^8}[(256\,x^7-320\,x^5+3600\,x^3-105300\,x)J_0(x)+$$

$$+(-512\,x^8+128\,x^6-800\,x^4+16200\,x^2-684450)J_1(x)]+\frac{512}{5817825}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-19/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{98903025\,x^9}[(-512\,x^8+640\,x^6-7200\,x^4+210600\,x^2-11635650)J_0(x)+$$

$$+(1024\,x^9-256\,x^7+1600\,x^5-32400\,x^3+1368900\,x)J_1(x)]-\frac{1024}{98903025}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-19/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{228118275\,x^9}[(1024\,x^9+768\,x^7-4032\,x^5+77616\,x^3-3201660\,x)J_0(x)+$$

$$+(-512\,x^8+1152\,x^6-14112\,x^4+426888\,x^2-24012450)J_1(x)]+\frac{1024}{228118275}\int\sqrt{x}J_1(x)\,dx\qquad *E*$$

$$\int x^{-21/2}\cdot J_0(x)\,dx=$$

$$=\frac{\sqrt{x}}{4334247225\,x^{10}}[(-2048\,x^{10}-1536\,x^8+8064\,x^6-155232\,x^4+6403320\,x^2-456236550)J_0(x)+$$

$$+(1024\,x^9-2304\,x^7+28224\,x^5-853776\,x^3+48024900\,x)J_1(x)]-\frac{2048}{4334247225}+\frac{1024}{228118275}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-21/2}\cdot J_1(x)\,dx=\frac{\sqrt{x}}{2076963525\,x^{10}}[(-1024\,x^9+1280\,x^7-14400\,x^5+421200\,x^3-23271300\,x)J_0(x)+$$

$$+(2048\,x^{10}-512\,x^8+3200\,x^6-64800\,x^4+2737800\,x^2-197806050)J_1(x)]-\frac{2048}{2076963525}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-23/2}\cdot J_0(x)\,dx=$$

$$=\frac{\sqrt{x}}{43616234025\,x^{11}}[(2048\,x^{10}-2560\,x^8+28800\,x^6-842400\,x^4+46542600\,x^2-4153927050)J_0(x)+$$

$$+(-4096\,x^{11}+1024\,x^9-6400\,x^7+129600\,x^5-5475600\,x^3+395612100\,x)J_1(x)]+\frac{4096}{43616234025}\int\sqrt{x}J_0(x)\,dx$$

$$\int x^{-23/2}\cdot J_1(x)\,dx=$$

$$=\frac{\sqrt{x}}{99687686175\,x^{11}}[(-4096\,x^{11}-3072\,x^9+16128\,x^7-310464\,x^5+12806640\,x^3-912473100\,x)J_0(x)+$$

$$+(2048\,x^{10}-4608\,x^8+56448\,x^6-1707552\,x^4+96049800\,x^2-8668494450)J_1(x)]-\frac{4096}{99687686175}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-25/2}\cdot J_0(x)\,dx=\frac{\sqrt{x}}{2292816782025\,x^{12}}[(8192\,x^{12}+6144\,x^{10}-32256\,x^8+620928\,x^6-25613280\,x^4+$$

$$+1824946200\,x^2-199375372350)J_0(x)+(-4096\,x^{11}+9216\,x^9-112896\,x^7+3415104\,x^5-192099600\,x^3+$$

$$+17336988900\,x)J_1(x)]+\frac{8192}{2292816782025}\int\sqrt{x}J_1(x)\,dx$$

$$\int x^{-25/2}\cdot J_1(x)\,dx=$$

$$=\frac{\sqrt{x}}{1090405850625\,x^{12}}[(4096\,x^{11}-5120\,x^9+57600\,x^7-1684800\,x^5+93085200\,x^3-8307854100\,x)J_0(x)+$$

$$+(-8192\,x^{12}+2048\,x^{10}-12800\,x^8+259200\,x^6-10951200\,x^4+791224200\,x^2-87232468050)J_1(x)]+$$

$$+\frac{8192}{1090405850625}\int\sqrt{x}J_0(x)\,dx$$

1.2.1. c) Recurrence Formulas:

$$\int x^{n+5/2}\,J_0(x)\,dx=x^{n+3/2}\,\left[\left(n+\frac{3}{2}\right)\,J_0(x)+x\,J_1(x)\right]-\left(n+\frac{3}{2}\right)^2\int x^{n+1/2}\,J_0(x)\,dx$$

$$\int x^{n+5/2}\,J_1(x)\,dx=x^{n+3/2}\,\left[\left(n+\frac{5}{2}\right)\,J_1(x)-x\,J_0(x)\right]-\frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,J_1(x)\,dx$$

$$\int x^{n+5/2}\,I_0(x)\,dx=x^{n+3/2}\,\left[x\,I_1(x)-\left(n+\frac{3}{2}\right)\,I_0(x)\right]+\left(n+\frac{3}{2}\right)^2\int x^{n+1/2}\,I_0(x)\,dx$$

$$\int x^{n+5/2}\,I_1(x)\,dx=x^{n+3/2}\,\left[x\,I_0(x)-\left(n+\frac{5}{2}\right)\,I_1(x)\right]+\frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,I_1(x)\,dx$$

1.2.2. Integrals of the type $\int x^n\, e^{\pm x} \cdot \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu x \end{array} \right\} dx$

See also [1], 11.3.

a) Integrals with e^x :

$$\int e^x\, I_0(x)\, dx \;=\; xe^x\,[I_0(x) - I_1(x)] \;, \qquad \int \frac{e^x \cdot I_1(x)\, dx}{x} = e^x\,[I_0(x) - I_1(x)]$$

$$\int e^x\, K_0(x)\, dx = xe^x\,[K_0(x) + K_1(x)] \;, \qquad \int \frac{e^x\, K_1(x)\, dx}{x} = -e^x\,[K_0(x) + K_1(x)]$$

$$\int e^x\, I_1(x)\, dx \;=\; e^x\,[(1-x)I_0(x) + xI_1(x)]$$

$$\int e^x\, K_1(x)\, dx = -e^x\,[(1-x)K_0(x) - xK_1(x)]$$

$$\int xe^x\, I_0(x)\, dx \;=\; \frac{xe^x}{3}\,[xI_0(x) + (1-x)I_1(x)]$$

$$\int x\, e^x\, K_0(x)\, dx = \frac{xe^x}{3}\;[x\, K_0(x) + (x-1)\, K_1(x)]$$

$$\int x\, e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{3}\,[-xI_0(x) + (2+x)I_1(x)]$$

$$\int x\, e^x\, K_1(x)\, dx = \frac{xe^x}{3}\;[x\, K_0(x) + (x+2)\, K_1(x)]$$

$$\int x^2 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{15}\left[(2\,x + 3\,x^2)I_0(x) + (-4 + 4\,x - 3\,x^2)I_1(x)\right]$$

$$\int x^2\, e^x\, K_0(x)\, dx = \frac{xe^x}{15}\;\left[(3x^2 + 2x)\, K_0(x) + (3\,x^2 - 4\,x + 4)\, K_1(x)\right]$$

$$\int x^2 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{5}\left[(x - x^2)I_0(x) + (-2 + 2x + x^2)I_1(x)\right]$$

$$\int x^2\, e^x\, K_1(x)\, dx = \frac{xe^x}{5}\;\left[(x^2 - x)\, K_0(x) + (x^2 + 2x - 2)\, K_1(x)\right]$$

$$\int x^3 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{35}\left[(-6\,x + 6\,x^2 + 5\,x^3)I_0(x) + (12 - 12\,x + 9\,x^2 - 5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^x\, K_0(x)\, dx = \frac{xe^x}{35}\;\left[(5\,x^3 + 6\,x^2 - 6\,x)\, K_0(x) + (5\,x^3 - 9\,x^2 + 12\,x - 12)\, K_1(x)\right]$$

$$\int x^3 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{35}\left[(-8\,x + 8\,x^2 - 5\,x^3)I_0(x) + (16 - 16\,x + 12\,x^2 + 5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^x\, K_1(x)\, dx = \frac{xe^x}{35}\;\left[(5\,x^3 - 8\,x^2 + 8\,x)\, K_0(x) + (5\,x^3 + 12\,x^2 - 16\,x + 16)\, K_1(x)\right]$$

$$\int x^4 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{315}\left[(96\,x - 96\,x^2 + 60\,x^3 + 35\,x^4)I_0(x) + (-192 + 192\,x - 144\,x^2 + 80\,x^3 - 35\,x^4)I_1(x)\right]$$

$$\int x^4\, e^x\, K_0(x)\, dx = \frac{xe^x}{315}\;\left[(35\,x^4 + 60\,x^3 - 96\,x^2 + 96\,x)\, K_0(x) + (35\,x^4 - 80\,x^3 + 144\,x^2 - 192\,x + 192)\, K_1(x)\right]$$

$$\int x^4 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{63}\left[(24\,x - 24\,x^2 + 15\,x^3 - 7\,x^4)I_0(x) + (-48 + 48\,x - 36\,x^2 + 20\,x^3 + 7\,x^4)I_1(x)\right]$$

$$\int x^4\, e^x\, K_1(x)\, dx = \frac{xe^x}{63}\;\left[(7\,x^4 - 15\,x^3 + 24\,x^2 - 24\,x)\, K_0(x) + (7\,x^4 + 20\,x^3 - 36\,x^2 + 48\,x - 48)\, K_1(x)\right]$$

$$\int x^5 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{693}\left[(-480\,x + 480\,x^2 - 300\,x^3 + 140\,x^4 + 63\,x^5)I_0(x) + \right.$$
$$\left. + (960 - 960\,x + 720\,x^2 - 400\,x^3 + 175\,x^4 - 63\,x^5)I_1(x)\right]$$

$$\int x^5\, e^x\, K_0(x)\, dx = \frac{xe^x}{693}\, \big[(63\, x^5 + 140\, x^4 - 300\, x^3 + 480\, x^2 - 480\, x)\, K_0(x) +$$
$$+\, (63\, x^5 - 175\, x^4 + 400\, x^3 - 720\, x^2 + 960\, x - 960)\, K_1(x)\big]$$
$$\int x^5 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{231}\, \big[(-192\, x + 192\, x^2 - 120\, x^3 + 56\, x^4 - 21\, x^5) I_0(x) +$$
$$+ (384 - 384\, x + 288\, x^2 - 160\, x^3 + 70\, x^4 + 21\, x^5) I_1(x)\big]$$
$$\int x^5\, e^x\, K_1(x)\, dx = \frac{xe^x}{231}\, \big[(21\, x^5 - 56\, x^4 + 120\, x^3 - 192\, x^2 + 192\, x)\, K_0(x) +$$
$$+ (21\, x^5 + 70\, x^4 - 160\, x^3 + 288\, x^2 - 384\, x + 384)\, K_1(x)\big]$$
$$\int x^6 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{1001}\, \big[(1920\, x - 1920\, x^2 + 1200\, x^3 - 560\, x^4 + 210 x^5 + 77 x^6) I_0(x) +$$
$$+\, (-3840 + 3840\, x - 2880\, x^2 + 1600\, x^3 - 700\, x^4 + 252 x^5 - 77 x^6) I_1(x)\big]$$
$$\int x^6\, e^x\, K_0(x)\, dx = \frac{xe^x}{1001}\, \big[(77\, x^6 + 210\, x^5 - 560\, x^4 + 1200\, x^3 - 1920\, x^2 + 1920\, x)\, K_0(x) +$$
$$+\, (77\, x^6 - 252\, x^5 + 700\, x^4 - 1600\, x^3 + 2880\, x^2 - 3840\, x + 3840)\, K_1(x)\big]$$
$$\int x^6 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{429}\, \big[(960\, x - 960\, x^2 + 600\, x^3 - 280\, x^4 + 105\, x^5 - 33\, x^6) I_0(x) +$$
$$+ (-1920 + 1920\, x - 1440\, x^2 + 800\, x^3 - 350\, x^4 + 126\, x^5 + 33\, x^6) I_1(x)\big]$$
$$\int x^6\, e^x\, K_1(x)\, dx = \frac{xe^x}{429}\, \big[(33\, x^6 - 105\, x^5 + 280\, x^4 - 600\, x^3 + 960\, x^2 - 960\, x)\, K_0(x) +$$
$$+ (33\, x^6 + 126\, x^5 - 350\, x^4 + 800\, x^3 - 1440\, x^2 + 1920\, x - 1920)\, K_1(x)\big]$$
$$\int x^7 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{2145}\, \big[(-13440\, x + 13440\, x^2 - 8400\, x^3 + 3920\, x^4 - 1470\, x^5 + 462\, x^6 + 143\, x^7) I_0(x) +$$
$$+ (26880 - 26880\, x + 20160\, x^2 - 11200\, x^3 + 4900\, x^4 - 1764\, x^5 + 539\, x^6 - 143\, x^7) I_1(x)\big]$$
$$\int x^7\, e^x\, K_0(x)\, dx = \frac{xe^x}{2145}\, \big[(143\, x^7 + 462\, x^6 - 1470\, x^5 + 3920\, x^4 - 8400\, x^3 + 13440\, x^2 - 13440\, x)\, K_0(x) +$$
$$+ (143\, x^7 - 539\, x^6 + 1764\, x^5 - 4900\, x^4 + 11200\, x^3 - 20160\, x^2 + 26880\, x - 26880)\, K_1(x)\big]$$
$$\int x^7 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{2145}\, \big[(-15360\, x + 15360\, x^2 - 9600\, x^3 + 4480\, x^4 - 1680\, x^5 + 528\, x^6 - 143\, x^7) I_0(x) +$$
$$+ (30720 - 30720\, x + 23040\, x^2 - 12800\, x^3 + 5600\, x^4 - 2016\, x^5 + 616\, x^6 + 143\, x^7) I_1(x)\big]$$
$$\int x^7\, e^x\, K_1(x)\, dx = \frac{xe^x}{2145}\, \big[(143\, x^7 - 528\, x^6 + 1680\, x^5 - 4480\, x^4 + 9600\, x^3 - 15360\, x^2 + 15360\, x)\, K_0(x) +$$
$$+ (143\, x^7 + 616\, x^6 - 2016\, x^5 + 5600\, x^4 - 12800\, x^3 + 23040\, x^2 - 30720\, x + 30720)\, K_1(x)\big]$$

<u>Recurrence formulas:</u>

$$\int x^n e^x I_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x) I_0(x) - x I_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$
$$\int x^n e^x I_1(x)\, dx = \frac{x^n e^x}{2n+1}[(n+1-x) I_0(x) + x I_1(x)] - \frac{n(n+1)}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^x K_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x) K_0(x) + x K_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$
$$\int x^n e^x K_1(x)\, dx = \frac{x^n e^x}{2n+1}[(x-n-1) K_0(x) + x K_1(x)] + \frac{n(n+1)}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

b) Integrals with e^{-x} :

$$\int e^{-x}\, I_0(x)\, dx \;=\; xe^{-x}\,[I_0(x)+I_1(x)]\;, \qquad \int \frac{e^{-x}\cdot I_1(x)\,dx}{x} = e^{-x}\,[I_0(x)+I_1(x)]$$

$$\int e^{-x}\, K_0(x)\, dx = xe^{-x}\,[K_0(x)-K_1(x)]\;, \qquad \int \frac{e^{-x}\, K_1(x)\,dx}{x} = e^{-x}\,[K_0(x)-K_1(x)]$$

$$\int e^{-x}\, I_1(x)\, dx \;=\; e^{-x}\,[(1+x)I_0(x)+xI_1(x)]$$

$$\int e^{-x}\, K_1(x)\, dx = -e^{-x}\,[(1+x)\, K_0(x) - x\, K_1(x)]$$

$$\int xe^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{3}\,[xI_0(x)+(1+x)I_1(x)]$$

$$\int x\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{3}\;[x\, K_0(x)-(x+1)\, K_1(x)]$$

$$\int xe^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{3}\,[xI_0(x)+(-2+x)I_1(x)]$$

$$\int x\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{3}\;[-x\, K_0(x)+(x-2)\, K_1(x)]$$

$$\int x^2e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{15}\,\left[(-2\,x+3\,x^2)I_0(x)+(4+4\,x+3\,x^2)I_1(x)\right]$$

$$\int x^2\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{15}\;\left[(3\,x^2-2\,x)\, K_0(x)-(3\,x^2+4\,x+4)\, K_1(x)\right]$$

$$\int x^2e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{5}\,\left[(x+x^2)I_0(x)+(-2-2x+x^2)I_1(x)\right]$$

$$\int x^2\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{5}\;\left[-(x^2+x)\, K_0(x)+(x^2-2\,x-2)\, K_1(x)\right]$$

$$\int x^3e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{35}\,\left[(-6\,x-6\,x^2+5\,x^3)I_0(x)+(12+12\,x+9\,x^2+5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{35}\;\left[(5\,x^3-6\,x^2-6\,x)\, K_0(x)-(5\,x^3+9\,x^2+12\,x+12)\, K_1(x)\right]$$

$$\int x^3e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{35}\,\left[(8\,x+8\,x^2+5\,x^3)I_0(x)+(-16-16\,x-12\,x^2+5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{35}\;\left[-(5\,x^3+8\,x^2+8\,x)\, K_0(x)+(5\,x^3-12\,x^2-16\,x-16)\, K_1(x)\right]$$

$$\int x^4e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{315}\,\left[(-96\,x-96\,x^2-60\,x^3+35\,x^4)I_0(x)+(192+192\,x+144\,x^2+80\,x^3+35\,x^4)I_1(x)\right]$$

$$\int x^4\, e^{-x}\, K_0(x)\, dx =$$

$$= \frac{xe^{-x}}{315}\;\left[(35\,x^4-60\,x^3-96\,x^2-96\,x)\, K_0(x)-(35\,x^4+80\,x^3+144\,x^2+192\,x+192)\, K_1(x)\right]$$

$$\int x^4e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{63}\,\left[(24\,x+24\,x^2+15\,x^3+7\,x^4)I_0(x)+(-48-48\,x-36\,x^2-20\,x^3+7\,x^4)I_1(x)\right]$$

$$\int x^4\, e^{-x}\, K_1(x)\, dx =$$

$$= \frac{xe^{-x}}{63}\;\left[-(7\,x^4+15\,x^3+24\,x^2+24\,x)\, K_0(x)+(7\,x^4-20\,x^3-36\,x^2-48\,x-48)\, K_1(x)\right]$$

$$\int x^5 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{693}\Big[(-480\,x-480\,x^2-300\,x^3-140\,x^4+63\,x^5)I_0(x)+$$
$$+(960+960\,x+720\,x^2+400\,x^3+175\,x^4+63\,x^5)I_1(x)\Big]$$
$$\int x^5\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{693}\;\Big[(63\,x^5-140\,x^4-300\,x^3-480\,x^2-480\,x)\, K_0(x)-$$
$$-(63\,x^5+175\,x^4+400\,x^3+720\,x^2+960\,x+960)\, K_1(x)\Big]$$
$$\int x^5 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{231}\Big[(192\,x+192\,x^2+120\,x^3+56\,x^4+21\,x^5)I_0(x)+$$
$$+(-384-384\,x-288\,x^2-160\,x^3-70\,x^4+21\,x^5)I_1(x)\Big]$$
$$\int x^5\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{231}\;\Big[-(21\,x^5+56\,x^4+120\,x^3+192\,x^2+192\,x)\, K_0(x)+$$
$$+(21\,x^5-70\,x^4-160\,x^3-288\,x^2-384\,x-384)\, K_1(x)\Big]$$
$$\int x^6 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{1001}\Big[(-1920\,x-1920\,x^2-1200\,x^3-560\,x^4-210x^5+77x^6)I_0(x)+$$
$$+\;(3840+3840\,x+2880\,x^2+1600\,x^3+700\,x^4+252x^5+77x^6)I_1(x)\Big]$$
$$\int x^6\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{1001}\;\Big[(77\,x^6-210\,x^5-560\,x^4-1200\,x^3-1920\,x^2-1920\,x)\, K_0(x)-$$
$$-(77\,x^6+252\,x^5+700\,x^4+1600\,x^3+2880\,x^2+3840\,x+3840)\, K_1(x)\Big]$$
$$\int x^6 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{429}\Big[(960\,x+960\,x^2+600\,x^3+280\,x^4+105\,x^5+33\,x^6)I_0(x)+$$
$$+(-1920-1920\,x-1440\,x^2-800\,x^3-350\,x^4-126\,x^5+33\,x^6)I_1(x)\Big]$$
$$\int x^6\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{429}\;\Big[-(33\,x^6+105\,x^5+280\,x^4+600\,x^3+960\,x^2+960\,x)\, K_0(x)+$$
$$+(33\,x^6-126\,x^5-350\,x^4-800\,x^3-1440\,x^2-1920\,x-1920)\, K_1(x)\Big]$$
$$\int x^7 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{2145}\Big[(-13440\,x-13440\,x^2-8400\,x^3-3920\,x^4-1470\,x^5-462\,x^6+143\,x^7)I_0(x)+$$
$$+(26880+26880\,x+20160\,x^2+11200\,x^3+4900\,x^4+1764\,x^5+539\,x^6+143\,x^7)I_1(x)\Big]$$
$$\int x^7\, e^{-x}\, K_0(x)\, dx =$$
$$=\frac{xe^{-x}}{2145}\;\Big[(143\,x^7-462\,x^6-1470\,x^5-3920\,x^4-8400\,x^3-13440\,x^2-13440\,x)\, K_0(x)-$$
$$-(143\,x^7+539\,x^6+1764\,x^5+4900\,x^4+11200\,x^3+20160\,x^2+26880\,x+26880)\, K_1(x)\Big]$$
$$\int x^7 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{2145}\Big[(15360\,x+15360\,x^2+9600\,x^3+4480\,x^4+1680\,x^5+528\,x^6+143\,x^7)I_0(x)+$$
$$+(-30720-30720\,x-23040\,x^2-12800\,x^3-5600\,x^4-2016\,x^5-616\,x^6+143\,x^7)I_1(x)\Big]$$
$$\int x^7\, e^{-x}\, K_1(x)\, dx =$$
$$=\frac{xe^{-x}}{2145}\;\Big[-(143\,x^7+528\,x^6+1680\,x^5+4480\,x^4+9600\,x^3+15360\,x^2+15360\,x)\, K_0(x)+$$
$$+(143\,x^7-616\,x^6-2016\,x^5-5600\,x^4-12800\,x^3-23040\,x^2-30720\,x-30720)\, K_1(x)\Big]$$

Recurrence formulas:

$$\int x^n e^{-x} I_0(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(x-n)I_0(x) + xI_1(x)] + \frac{n^2}{2n+1}\int x^{n-1}e^{-x}I_0(x)\,dx \quad (*)$$

$$\int x^n e^{-x} I_1(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(n+1+x)I_0(x) + xI_1(x)] - \frac{n(n+1)}{2n+1}\int x^{n-1}e^{-x}I_0(x)\,dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^{-x} K_0(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(x-n)K_0(x) - xK_1(x)] + \frac{n^2}{2n+1}\int x^{n-1}e^{-x}K_0(x)\,dx$$

$$\int x^n e^{-x} K_1(x)\,dx = \frac{x^n e^{-x}}{2n+1}[-(x+n+1)K_0(x) + xK_1(x)] + \frac{n(n+1)}{2n+1}\int x^{n-1}e^{-x}K_0(x)\,dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

1.2.3. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sinh \\ \cosh \end{array} \right\} x \cdot I_\nu(x)\, dx$

$$\int \frac{\sinh x\, I_1(x)\, dx}{x} = x\, \cosh x\, I_0(x) - \sinh x\, I_1(x)$$

$$\int \frac{\cosh x\, I_1(x)\, dx}{x} = x\, \sinh x\, I_0(x) - \cosh x\, I_1(x)$$

$$\int \sinh x\, I_0(x)\, dx = x\, \sinh x\, I_0(x) - x\, \cosh x\, I_1(x)$$

$$\int \cosh x\, I_0(x)\, dx = x\, \cosh x\, I_0(x) - x\, \sinh x\, I_1(x)$$

$$\int \sinh x\, I_1(x)\, dx = \sinh x\, I_0(x) - x\, \cosh x\, I_0(x) + x\, \sinh x\, I_1(x)$$

$$\int \cosh x\, I_1(x)\, dx = -x\, \sinh x\, I_0(x) + \cosh x\, I_0(x) + x\, \cosh x\, I_1(x)$$

$$\int x\, \sinh x\, I_0(x)\, dx = \frac{x^2}{3}\, \sinh x\, I_0(x) + \frac{x}{3}\, \sinh x\, I_1(x) - \frac{x^2}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \cosh x\, I_0(x)\, dx = \frac{x^2}{3}\, \cosh x\, I_0(x) - \frac{x^2}{3}\, \sinh x\, I_1(x) + \frac{x}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \sinh x\, I_1(x)\, dx = -\frac{x^2}{3}\, \cosh x\, I_0(x) + \frac{x^2}{3}\, \sinh x\, I_1(x) + \frac{2x}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \cosh x\, I_1(x)\, dx = -\frac{x^2}{3}\, \sinh x\, I_0(x) + \frac{2x}{3}\, \sinh x\, I_1(x) + \frac{x^2}{3}\, \cosh x\, I_1(x)$$

$$\int x^2\, \sinh x\, I_0(x)\, dx = \frac{x^3}{5}\, \sinh x\, I_0(x) + \frac{2x^2}{15}\, \cosh x\, I_0(x) + \frac{4x^2}{15}\, \sinh x\, I_1(x) - \frac{3x^3 + 4x}{15}\, \cosh x\, I_1(x)$$

$$\int x^2\, \cosh x\, I_0(x)\, dx = \frac{2x^2}{15}\, \sinh x\, I_0(x) + \frac{x^3}{5}\, \cosh x\, I_0(x) - \frac{3\,x^3 + 4\,x}{15}\, \sinh x\, I_1(x) + \frac{4x^2}{15}\, \cosh x\, I_1(x)$$

$$\int x^2\, \sinh x\, I_1(x)\, dx = \frac{x^2}{5}\, \sinh x\, I_0(x) - \frac{x^3}{5}\, \cosh x\, I_0(x) + \frac{x^3 - 2\,x}{5}\, \sinh x\, I_1(x) + \frac{2x^2}{5}\, \cosh x\, I_1(x)$$

$$\int x^2\, \cosh x\, I_1(x)\, dx = -\frac{x^3}{5}\, \sinh x\, I_0(x) + \frac{x^2}{5}\, \cosh x\, I_0(x) + \frac{2x^2}{5}\, \sinh x\, I_1(x) + \frac{x^3 - 2\,x}{5}\, \cosh x\, I_1(x)$$

$$\int x^3\, \sinh x\, I_0(x)\, dx =$$
$$= \frac{5\,x^4 - 6\,x^2}{35}\, \sinh x\, I_0(x) + \frac{6x^3}{35}\, \cosh x\, I_0(x) + \frac{9\,x^3 + 12\,x}{35}\, \sinh x\, I_1(x) - \frac{5\,x^4 + 12\,x^2}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \cosh x\, I_0(x)\, dx =$$
$$= \frac{6x^3}{35}\, \sinh x\, I_0(x) + \frac{5\,x^4 - 6\,x^2}{35}\, \cosh x\, I_0(x) - \frac{5\,x^4 + 12\,x^2}{35}\, \sinh x\, I_1(x) + \frac{9\,x^3 + 12\,x}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \sinh x\, I_1(x)\, dx =$$
$$= \frac{8x^3}{35}\, \sinh x\, I_0(x) - \frac{5\,x^4 + 8\,x^2}{35}\, \cosh x\, I_0(x) + \frac{5\,x^4 - 16\,x^2}{35}\, \sinh x\, I_1(x) + \frac{12\,x^3 + 16\,x}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \cosh x\, I_1(x)\, dx =$$
$$= -\frac{5\,x^4 + 8\,x^2}{35}\, \sinh x\, I_0(x) + \frac{8x^3}{35}\, \cosh x\, I_0(x) + \frac{12\,x^3 + 16\,x}{35}\, \sinh x\, I_1(x) + \frac{5\,x^4 - 16\,x^2}{35}\, \cosh x\, I_1(x)$$

$$\int x^4 \sinh x\, I_0(x)\, dx = \frac{35\,x^5 - 96\,x^3}{315} \sinh x\, I_0(x) + \frac{20\,x^4 + 32\,x^2}{105} \cosh x\, I_0(x) +$$
$$+ \frac{80\,x^4 + 192\,x^2}{315} \sinh x\, I_1(x) - \frac{35x^5 + 144x^3 + 192x}{315} \cosh x\, I_1(x)$$
$$\int x^4 \cosh x\, I_0(x)\, dx = \frac{20\,x^4 + 32\,x^2}{105} \sinh x\, I_0(x) + \frac{35\,x^5 - 96\,x^3}{315} \cosh x\, I_0(x) -$$
$$- \frac{35\,x^5 + 144\,x^3 + 192\,x}{315} \sinh x\, I_1(x) + \frac{80\,x^4 + 192\,x^2}{315} \cosh x\, I_1(x)$$
$$\int x^4 \sinh x\, I_1(x)\, dx = \frac{5\,x^4 + 8\,x^2}{21} \sinh x\, I_0(x) - \frac{7\,x^5 + 24\,x^3}{63} \cosh x\, I_0(x) +$$
$$+ \frac{7\,x^5 - 36\,x^3 - 48\,x}{63} \sinh x\, I_1(x) + \frac{20\,x^4 + 48\,x^2}{63} \cosh x\, I_1(x)$$
$$\int x^4 \cosh x\, I_1(x)\, dx = -\frac{7\,x^5 + 24\,x^3}{63} \sinh x\, I_0(x) + \frac{5\,x^4 + 8\,x^2}{21} \cosh x\, I_0(x) +$$
$$+ \frac{20\,x^4 + 48\,x^2}{63} \sinh x\, I_1(x) + \frac{7\,x^5 - 36\,x^3 - 48\,x}{63} \cosh x\, I_1(x)$$

$$\int x^5 \sinh x\, I_0(x)\, dx = \frac{21\,x^6 - 100\,x^4 - 160\,x^2}{231} \sinh x\, I_0(x) + \frac{140\,x^5 + 480\,x^3}{693} \cosh x\, I_0(x) +$$
$$+ \frac{175\,x^5 + 720\,x^3 + 960\,x}{693} \sinh x\, I_1(x) - \frac{63\,x^6 + 400\,x^4 + 960\,x^2}{693} \cosh x\, I_1(x)$$
$$\int x^5 \cosh x\, I_0(x)\, dx = \frac{140\,x^5 + 480\,x^3}{693} \sinh x\, I_0(x) + \frac{21\,x^6 - 100\,x^4 - 160\,x^2}{231} \cosh x\, I_0(x) -$$
$$- \frac{63\,x^6 + 400\,x^4 + 960\,x^2}{693} \sinh x\, I_1(x) + \frac{175\,x^5 + 720\,x^3 + 960\,x}{693} \cosh x\, I_1(x)$$
$$\int x^5 \sinh x\, I_1(x)\, dx = \frac{56\,x^5 + 192\,x^3}{231} \sinh x\, I_0(x) - \frac{7\,x^6 + 40\,x^4 + 64\,x^2}{77} \cosh x\, I_0(x) +$$
$$+ \frac{21\,x^6 - 160\,x^4 - 384\,x^2}{231} \sinh x\, I_1(x) + \frac{70\,x^5 + 288\,x^3 + 384\,x}{231} \cosh x\, I_1(x)$$
$$\int x^5 \cosh x\, I_1(x)\, dx = -\frac{7\,x^6 + 40\,x^4 + 64\,x^2}{77} \sinh x\, I_0(x) + \frac{56\,x^5 + 192\,x^3}{231} \cosh x\, I_0(x) +$$
$$+ \frac{70\,x^5 + 288\,x^3 + 384\,x}{231} \sinh x\, I_1(x) + \frac{21\,x^6 - 160\,x^4 - 384\,x^2}{231} \cosh x\, I_1(x)$$

$$\int x^6 \sinh x\, I_0(x)\, dx = \frac{77\,x^7 - 560\,x^5 - 1920\,x^3}{1001} \sinh x\, I_0(x) + \frac{210\,x^6 + 1200\,x^4 + 1920\,x^2}{1001} \cosh x\, I_0(x) +$$
$$+ \frac{252\,x^6 + 1600\,x^4 + 3840\,x^2}{1001} \sinh x\, I_1(x) - \frac{77\,x^7 + 700\,x^5 + 2880\,x^3 + 3840\,x}{1001} \cosh x\, I_1(x)$$
$$\int x^6 \cosh x\, I_0(x)\, dx = \frac{210\,x^6 + 1200\,x^4 + 1920\,x^2}{1001} \sinh x\, I_0(x) + \frac{77\,x^7 - 560\,x^5 - 1920\,x^3}{1001} \cosh x\, I_0(x) -$$
$$- \frac{77\,x^7 + 700\,x^5 + 2880\,x^3 + 3840\,x}{1001} \sinh x\, I_1(x) + \frac{252\,x^6 + 1600\,x^4 + 3840\,x^2}{1001} \cosh x\, I_1(x)$$
$$\int x^6 \sinh x\, I_1(x)\, dx = \frac{35\,x^6 + 200\,x^4 + 320\,x^2}{143} \sinh x\, I_0(x) - \frac{33\,x^7 + 280\,x^5 + 960\,x^3}{429} \cosh x\, I_0(x) +$$
$$+ \frac{33\,x^7 - 350\,x^5 - 1440\,x^3 - 1920\,x}{429} \sinh x\, I_1(x) + \frac{126\,x^6 + 800\,x^4 + 1920\,x^2}{429} \cosh x\, I_1(x)$$
$$\int x^6 \cosh x\, I_1(x)\, dx = -\frac{33\,x^7 + 280\,x^5 + 960\,x^3}{429} \sinh x\, I_0(x) + \frac{35\,x^6 + 200\,x^4 + 320\,x^2}{143} \cosh x\, I_0(x) +$$
$$+ \frac{126\,x^6 + 800\,x^4 + 1920\,x^2}{429} \sinh x\, I_1(x) + \frac{33\,x^7 - 350\,x^5 - 1440\,x^3 - 1920\,x}{429} \cosh x\, I_1(x)$$

Recurrence formulas:

$$\int x^{n+1}\,\sinh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,[x\,\sinh x\cdot I_0(x)+(n+1)\,\cosh x\cdot I_0(x)-x\,\cosh x\cdot I_1(x)]-\frac{(n+1)^2}{2n+3}\int x^n\,\cosh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,[(n+1)\,\sinh x\cdot I_0(x)+x\,\cosh x\cdot I_0(x)-x\,\sinh x\cdot I_1(x)]-\frac{(n+1)^2}{2n+3}\int x^n\,\sinh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,[(n+2)\,\sinh x\cdot I_0(x)-x\,\cosh x\cdot I_0(x)+x\,\sinh x\cdot I_1(x)]-\frac{(n+1)(n+2)}{2n+3}\int x^n\,\sinh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,[-x\,\sinh x\cdot I_0(x)+(n+2)\,\cosh x\cdot I_0(x)+x\,\cosh x\cdot I_1(x)]-\frac{(n+1)}{2n+3}\int x^n\,\cosh x\cdot I_0(x)\,dx$$

1.2.4. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} x \cdot J_\nu(x)\,dx$

See also [1], 11.3.

$$\int \frac{\sin x \cdot J_0(x)\,dx}{x} = -\sin x\,J_1(x) - \cos x\,J_0(x)$$

$$\int \frac{\cos x \cdot J_0(x)\,dx}{x} = \sin x\,J_0(x) - \cos x\,J_1(x)$$

$$\int \sin x \cdot J_0(x)\,dx = x[\sin x \cdot J_0(x) - \cos x \cdot J_1(x)]$$

$$\int \cos x \cdot J_0(x)\,dx = x[\cos x \cdot J_0(x) + \sin x \cdot J_1(x)]$$

$$\int \sin x \cdot J_1(x)\,dx = (x\cos x - \sin x)J_0(x) + x\sin x \cdot J_1(x)$$

$$\int \cos x \cdot J_1(x)\,dx = -(x\sin x + \cos x)J_0(x) + x\cos x \cdot J_1(x)$$

$$\int x\sin x \cdot J_0(x)\,dx = \frac{x^2}{3}\sin x \cdot J_0(x) + \frac{x\sin x - x^2\cos x}{3} \cdot J_1(x)$$

$$\int x\cos x \cdot J_0(x)\,dx = \frac{x^2}{3}\cos x \cdot J_0(x) + \frac{x^2\sin x - x\cos x}{3} \cdot J_1(x)$$

$$\int x\sin x \cdot J_1(x)\,dx = \frac{x^2}{3} \cdot \cos x \cdot J_0(x) + \frac{x^2\sin x - 2x\cos x}{3} \cdot J_1(x)$$

$$\int x\cos x \cdot J_1(x)\,dx = -\frac{x^2}{3}\sin x \cdot J_0(x) + \frac{2x\sin x + x^2\cos x}{3} \cdot J_1(x)$$

$$\int x^2\sin x \cdot J_0(x)\,dx = \frac{1}{15}\left\{\left[3\,x^3\sin x - 2\,x^2\cos x\right] \cdot J_0(x) + \left[4\,x^2\sin x + (4\,x - 3\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2\cos x \cdot J_0(x)\,dx = \frac{1}{15}\left\{\left[2\,x^2\sin x + 3\,x^3\cos x\right] \cdot J_0(x) + \left[(-4\,x + 3\,x^3)\sin x + 4\,x^2\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2\sin x \cdot J_1(x)\,dx = \frac{1}{5}\left\{\left[-x^2\,\sin x + x^3\cos x\right] \cdot J_0(x) + \left[(2\,x + x^3)\sin x - 2\,x^2\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2\cos x \cdot J_1(x)\,dx = \frac{1}{5}\left\{\left[-x^3\sin x - x^2\cos x\right] \cdot J_0(x) + \left[2\,x^2\sin x + (2\,x + x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3\sin x \cdot J_0(x)\,dx = \frac{1}{35}\left\{\left[(6\,x^2 + 5\,x^4)\sin x - 6\,x^3\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-12\,x + 9\,x^3)\sin x + (12\,x^2 - 5\,x^4)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3\cos x \cdot J_0(x)\,dx = \frac{1}{35}\left\{\left[6\,x^3\sin x + (6\,x^2 + 5\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-12\,x^2 + 5\,x^4)\sin x + (-12\,x + 9\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3\sin x \cdot J_1(x)\,dx = \frac{1}{35}\left\{\left[-8x^3\,\sin x + (-8\,x^2 + 5\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(16\,x^2 + 5\,x^4)\sin x + (16\,x - 12\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3\cos x \cdot J_1(x)\,dx = \frac{1}{35}\left\{\left[(8\,x^2 - 5\,x^4)\sin x - 8\,x^3\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-16\,x + 12\,x^3)\sin x + (16\,x^2 + 5\,x^4)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^4\sin x \cdot J_0(x)\,dx = \frac{1}{315}\left\{\left[(96\,x^3 + 35\,x^5)\sin x + (96\,x^2 - 60\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-192\,x^2 + 80\,x^4)\sin x + (-192\,x + 144\,x^3 - 35\,x^5)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^4 \cos x \cdot J_0(x)\,dx \;=\; \frac{1}{315}\left\{\left[(-96\,x^2+60\,x^4)\sin x+(96\,x^3+35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(192\,x-144\,x^3+35\,x^5)\sin x+(-192\,x^2+80\,x^4)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^4 \sin x \cdot J_1(x)\,dx \;=\; \frac{1}{315}\left\{\left[(120x^2-75x^4)\sin x+(-120\,x^3+35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-240\,x+180\,x^3+35\,x^5)\sin x+(240\,x^2-100\,x^4)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^4 \cos x \cdot J_1(x)\,dx \;=\; \frac{1}{315}\left\{\left[(120\,x^3-35\,x^5)\sin x+(120\,x^2-75\,x^4)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-240\,x^2+100\,x^4)\sin x+(-240\,x+180\,x^3+35\,x^5)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^5 \sin x \cdot J_0(x)\,dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^2+300\,x^4+63\,x^6)\sin x+(480\,x^3-140\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(960\,x-720\,x^3+175\,x^5)\sin x+(-960\,x^2+400\,x^4-63\,x^6)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^5 \cos x \cdot J_0(x)\,dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^3+140\,x^5)\sin x+(-480\,x^2+300\,x^4+63\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(960\,x^2-400\,x^4+63\,x^6)\sin x+(960\,x-720\,x^3+175\,x^5)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^5 \sin x \cdot J_1(x)\,dx \;=\; \frac{1}{231}\left\{\left[(192x^3-56x^5)\sin x+(192\,x^2-120\,x^4+21\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-384\,x^2+160\,x^4+21\,x^6)\sin x+(-384\,x+288\,x^3-70\,x^5)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^5 \cos x \cdot J_1(x)\,dx \;=\; \frac{1}{231}\left\{\left[(-192\,x^2+120\,x^4-21\,x^6)\sin x+(192\,x^3-56\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(384\,x-288\,x^3+70\,x^5)\sin x+(-384\,x^2+160\,x^4+21\,x^6)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^6 \sin x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{1001}\left\{\left[(-1920\,x^3+560\,x^5+77\,x^7)\sin x+(-1920\,x^2+1200\,x^4-210\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(3840\,x^2-1600\,x^4+252\,x^6)\sin x+(3840\,x-2880\,x^3+700\,x^5-77\,x^7)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^6 \cos x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{1001}\left\{\left[(1920\,x^2-1200\,x^4+210\,x^6)\sin x+(-1920\,x^3+560\,x^5+77\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-3840\,x+2880\,x^3-700\,x^5+77\,x^7)\sin x+(3840\,x^2-1600\,x^4+252\,x^6)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^6 \sin x \cdot J_1(x)\,dx \;=$$
$$=\;\frac{1}{429}\left\{\left[(-960x^2+600x^4-105x^6)\sin x+(960\,x^3-280\,x^5+33\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(1920\,x-1440\,x^3+350\,x^5+33\,x^7)\sin x+(-1920\,x^2+800\,x^4-126\,x^6)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^6 \cos x \cdot J_1(x)\,dx \;=$$
$$=\;\frac{1}{429}\left\{\left[(-960\,x^3+280\,x^5-33\,x^7)\sin x+(-960\,x^2+600\,x^4-105\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(1920\,x^2-800\,x^4+126\,x^6)\sin x+(1920\,x-1440\,x^3+350\,x^5+33\,x^7)\cos x\right]\cdot J_1(x)\right\}$$
$$\int x^7 \sin x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{2145}\left\{\left[(13440\,x^2-8400\,x^4+1470\,x^6+143\,x^8)\sin x+(-13440\,x^3+3920\,x^5-462\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-26880\,x+20160\,x^3-4900\,x^5+539\,x^7)\sin x+(26880\,x^2-11200\,x^4+1764\,x^6-143\,x^8)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \cos x \cdot J_0(x)\, dx \;=$$

$$= \;\frac{1}{2145}\left\{\left[(13440\,x^3 - 3920\,x^5 + 462\,x^7)\sin x + (13440\,x^2 - 8400\,x^4 + 1470\,x^6 + 143\,x^8)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(-26880\,x^2 + 11200\,x^4 - 1764\,x^6 + 143\,x^8)\sin x + (-26880\,x + 20160\,x^3 - 4900\,x^5 + 539\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \sin x \cdot J_1(x)\, dx \;=$$

$$= \frac{1}{2145}\left\{\left[(-15360x^3 + 4480x^5 - 528x^7)\sin x + (-15360\,x^2 + 9600\,x^4 - 1680\,x^6 + 143\,x^8)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(30720\,x^2 - 12800\,x^4 + 2016\,x^6 + 143\,x^8)\sin x + (30720\,x - 23040\,x^3 + 5600\,x^5 - 616\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \cos x \cdot J_1(x)\, dx \;=$$

$$= \;\frac{1}{2145}\left\{\left[(15360\,x^2 - 9600\,x^4 + 1680\,x^6 - 143\,x^8)\sin x + (-15360\,x^3 + 4480\,x^5 - 528\,x^7)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(-30720\,x + 23040\,x^3 - 5600\,x^5 + 616\,x^7)\sin x + (30720\,x^2 - 12800\,x^4 + 2016\,x^6 + 143\,x^8)\cos x\right]\cdot J_1(x)\right\}$$

Recurrence formulas:

Let

$$S_n^{(\nu)} = \int x^n \,\sin x \cdot J_\nu(x)\, dx \quad , \qquad C_n^{(\nu)} = \int x^n \,\cos x \cdot J_\nu(x)\, dx$$

and

$$\sigma_n^{(\nu)} = x^n \,\sin x \cdot J_\nu(x) \quad , \qquad \gamma_n^{(\nu)} = x^n \,\cos x \cdot J_\nu(x)\;,$$

then holds

$$S_n^{(0)} \;=\; \frac{n^2 C_{n-1}^{(0)} - n\gamma_n^{(0)} + \sigma_{n+1}^{(0)} - \gamma_{n+1}^{(1)}}{2n+1}\;, \quad S_n^{(1)} \;=\; \frac{n(n+1)S_{n-1}^{(0)} - (n+1)\sigma_n^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1}\;,$$

$$C_n^{(0)} \;=\; \frac{n\sigma_n^{(0)} - n^2 S_{n-1}^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1}\;, \quad C_n^{(1)} \;=\; \frac{n(n+1)C_{n-1}^{(0)} - (n+1)\gamma_n^{(0)} - \sigma_{n+1}^{(0)} + \gamma_{n+1}^{(1)}}{2n+1}\;.$$

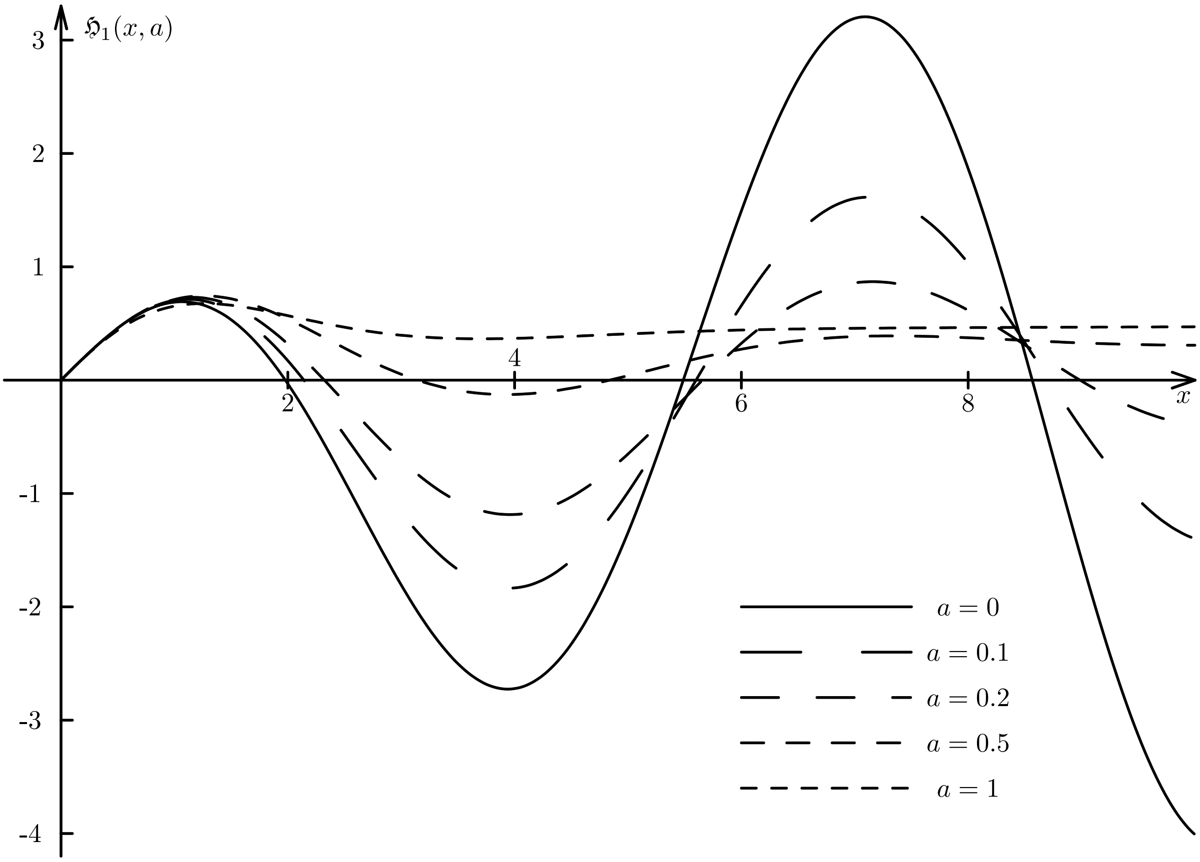
1.2.5. Integrals of the type $\int x^n \cdot e^{ax} \cdot Z_\nu(x)\,dx$

a) General facts:

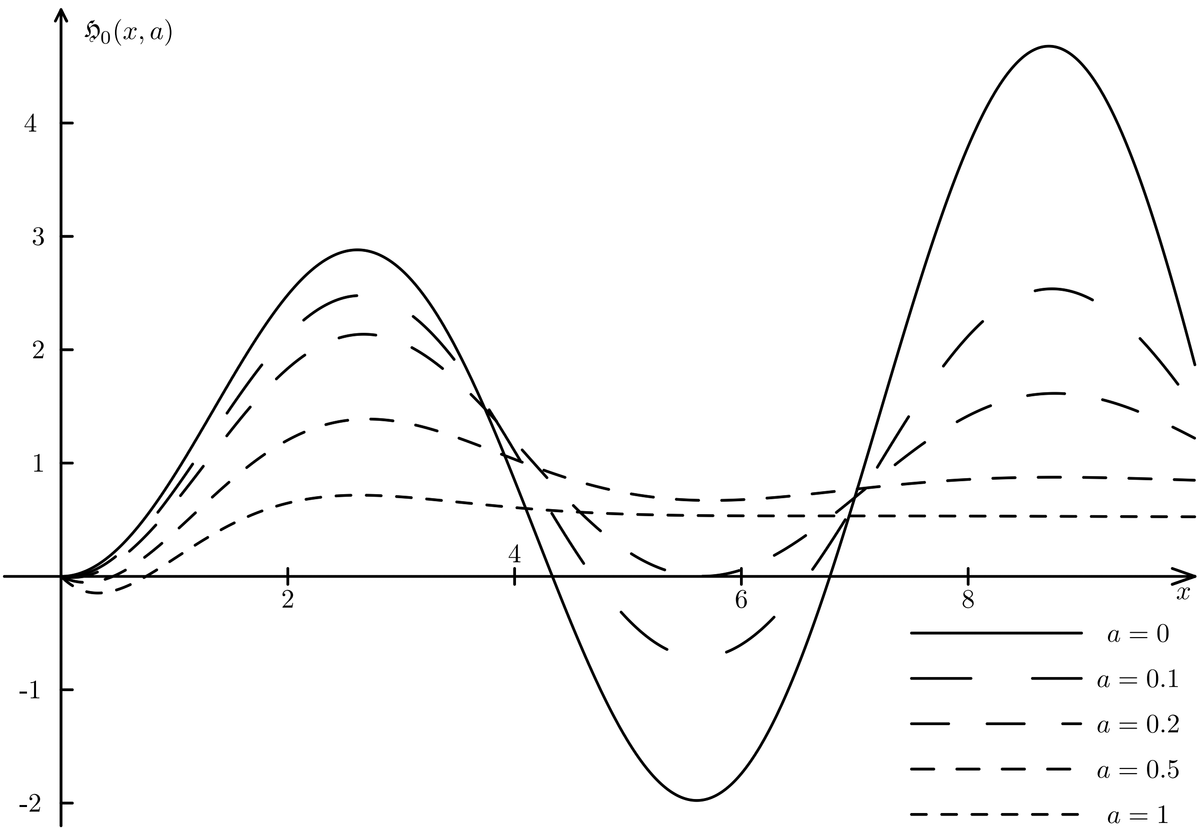
Holds
$$\int e^{ax}\,J_0(x)\,dx \;=\; \int e^{-a\cdot(-x)}\,J_0(-x)\,dx\;,$$
therefore the integral on the left hand side is discussed, assuming $x\ge 0$ and treating the cases $a>0$ and $a<0$ separately.

Let $\mathfrak{H}_\nu(x,a)$ denote the following functions:
$$\mathfrak{H}_1(x,a)\;=\;\sum_{k=1}^{\infty} b_k(a)\,x^k\quad,\qquad \mathfrak{H}_0(x,a)\;=\;\sum_{k=1}^{\infty} c_k(a)\,x^k$$
with
$$b_1(a)=1\;,\quad b_2(a)=0\;,\quad b_{k+2}(a)=-\frac{a(1+2k)b_{k+1}(a)+(1+a^2)b_k(a)}{k(k+2)}\;,\quad k\ge 1$$
and
$$c_k(a)=-(k+1)b_{k+1}(a)-a\,b_k(a)\;.$$
Then holds with $a\in\mathbb{R}$
$$\int_0^x e^{at}\,J_0(t)\,dt\;=\;e^{ax}\,\left[\,\mathfrak{H}_1(x,a)\,J_0(x)+\,\mathfrak{H}_0(x,a)\,J_1(x)\right]\;.$$
In the case $a=0$ one has with the Struve functions
$$\mathfrak{H}_1(x,0)=x-\frac{\pi x}{2}\,\mathbf{H}_1(x)\;,\qquad \mathfrak{H}_0(x,0)=\frac{\pi x}{2}\,\mathbf{H}_0(x)\;.$$
First terms of the power series:
$$\mathfrak{H}_1(x,a)=x-(a^2+1)\left[\frac{x^3}{3}-\frac{5a}{24}x^4+\frac{27\,a^2-8}{360}x^5-\frac{7a\,\left(8\,a^2-7\right)}{2\,880}x^6+\frac{400\,a^4-691\,a^2+64}{100\,800}x^7-\right.$$
$$-\frac{a\left(1\,080\,a^4-3076\,a^2+849\right)}{1\,612\,800}x^8+\frac{9\,800\,a^6-41\,484\,a^4+22\,767\,a^2-1024}{101\,606\,400}x^9-$$
$$-\frac{11a\,\left(1\,792\,a^6-10\,536\,a^4+9\,588\,a^2-1\,289\right)}{1\,625\,702\,400}x^{10}+$$
$$+\frac{217\,728\,a^8-1\,695\,080\,a^6+2\,303\,364\,a^4-617\,289\,a^2+16\,384}{160\,944\,537\,600}x^{11}-$$
$$\left.-\frac{13\,\left(67\,200\,a^8-668\,576\,a^6+1\,266\,744\,a^4-564\,120\,a^2+44\,815\right)a}{6\,437\,781\,504\,000}x^{12}+\ldots\right]$$
and
$$\mathfrak{H}_0(x,a)=-ax+(a^2+1)\left[x^2-\frac{a}{2}x^3+\frac{3\,a^2-2}{18}x^4-\frac{a\left(12\,a^2-23\right)}{288}x^5+\frac{60\,a^4-223\,a^2+32}{7\,200}x^6-\right.$$
$$-\frac{a\left(40\,a^4-242\,a^2+103\right)}{28\,800}x^7+\frac{280\,a^6-2\,494\,a^4+2\,103\,a^2-128}{1\,411\,200}x^8-$$
$$-\frac{a\left(2240\,a^6-27512\,a^4+38356\,a^2-6967\right)}{90\,316\,800}x^9+\frac{20160\,a^8-326008\,a^6+677076\,a^4-244839\,a^2+8192}{7\,315\,660\,800}x^{10}-$$
$$-\frac{\left(40\,320\,a^8-829\,424\,a^6+2\,397\,216\,a^4-1438\,890\,a^2+143\,995\right)}{146\,313\,216\,000}x^{11}+$$
$$\left.+\frac{443\,520\,a^{10}-11\,300\,944\,a^8+43\,320\,176\,a^6-38\,861\,430\,a^4+7\,756\,835\,a^2-163\,840}{17\,703\,899\,136\,000}x^{12}-\ldots\right]\;.$$

b) The case $a > 0$:



$\mathfrak{H}_1(x,a)$
3
2
1
-1
-2
-3
-4
2
4
6
8
x
$a = 0$
$a = 0.1$
$a = 0.2$
$a = 0.5$
$a = 1$



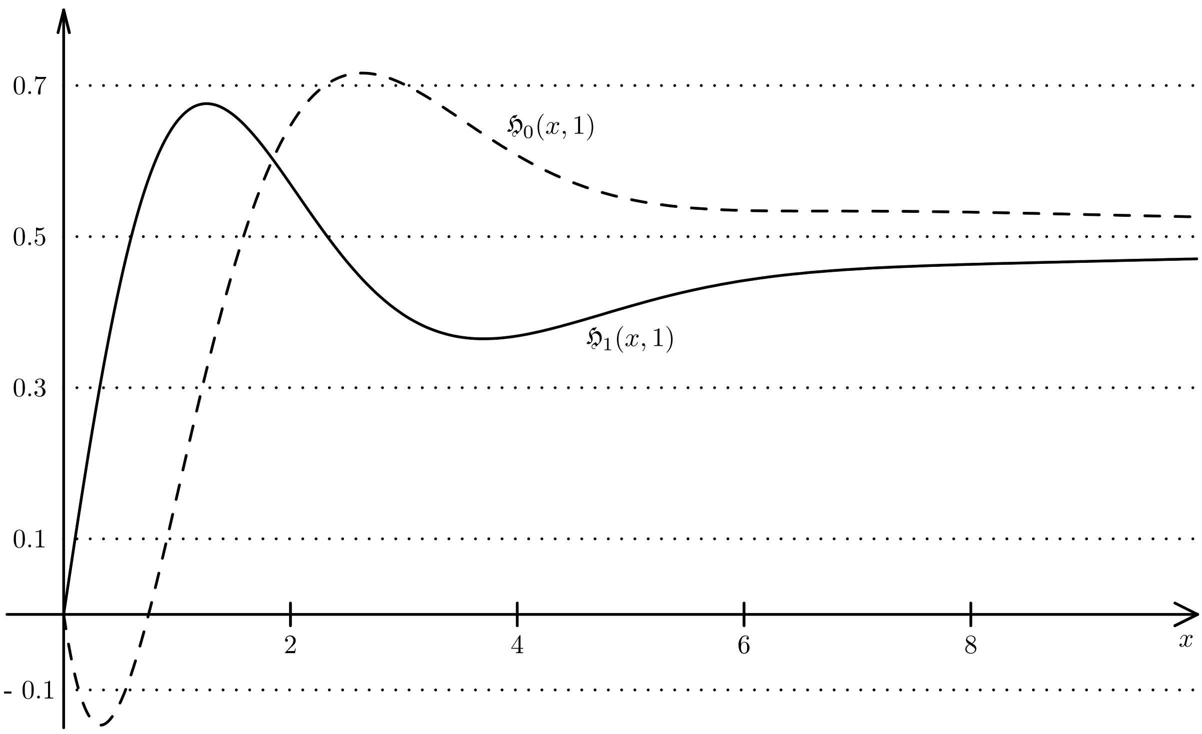
$\mathfrak{H}_0(x,a)$
4
3
2
1
-1
-2
2
4
6
8
x
$a = 0$
$a = 0.1$
$a = 0.2$
$a = 0.5$
$a = 1$

The special case $a = 1$:

$$\int_0^x e^t\, J_0(t)\, dt = e^x \left[\left(x - \frac{2x^3}{3} + \frac{5}{12}\, x^4 - \frac{19}{180}\, x^4 + \frac{7\, x^6}{1440}\, x^4 + \frac{227\, x^7}{50400} - \frac{1147\, x^8}{806400} + \frac{9941\, x^9}{50803200} + \ldots \right) J_0(x) + \right.$$

$$\left. + \left(-x + 2x^2 - x^3 + \frac{x^4}{9} + \frac{11}{144}\, x^5 - \frac{131\, x^6}{3600} + \frac{11\, x^7}{1600} - \frac{239\, x^8}{705600} - \frac{2039\, x^9}{15052800} + \ldots \right) J_1(x) \right]$$

k	$b_k(1)$	$b_k(1)$	$c_k(1)$	$c_k(1)$
1	1	1.000000000000000	-1	-1.000000000000000
2	0	0.000000000000000	2	2.000000000000000
3	$-2/3$	-0.666666666666667	-1	-1.000000000000000
4	$\frac{5}{12}$	0.416666666666667	$1/9$	0.111111111111111
5	$-\frac{19}{180}$	-0.105555555555556	$\frac{11}{144}$	0.076388888888889
6	$\frac{7}{1440}$	0.004861111111111	$-\frac{131}{3600}$	-0.036388888888889
7	$\frac{227}{50400}$	0.004503968253968	$\frac{11}{1600}$	0.006875000000000
8	$-\frac{1147}{806400}$	-0.001422371031746	$-\frac{239}{705600}$	-0.000338718820862
9	$\frac{9941}{50803200}$	0.000195676650290	$-\frac{2039}{15052800}$	-0.000135456526361
10	$-\frac{979}{162570240}$	-0.000006022012393	$\frac{134581}{3657830400}$	0.000036792575183
11	$-\frac{225107}{80472268800}$	-0.000002797323890	$-\frac{313217}{73156608000}$	-0.000004281458758
12	$\frac{1898819}{3218890752000}$	0.000000589898554	$\frac{1194317}{8851949568000}$	0.000000134921352
13	$-\frac{8554759}{153433792512000}$	-0.000000055755377	$\frac{16109741}{424893579264000}$	0.000000037914767
14	$\frac{14077813}{11047233060864000}$	0.000000001274329	$-\frac{517397957}{71807014895616000}$	-0.000000007205396
15	$\frac{18928541}{47871343263744000}$	0.000000000395404	$\frac{1217807483}{2010596417077248000}$	0.000000000605695
16	$-\frac{402561241}{6433908534647193600}$	-0.000000000062569	$-\frac{422808761}{30158946256158720000}$	-0.000000000014019
17	$\frac{36957033251}{8203233381675171840000}$	0.000000000004505	$-\frac{23427152899}{7720690241576632320000}$	-0.000000000003034
18	$-\frac{21450103637}{262503468213605498880000}$	-0.000000000000082	$\frac{76121087023}{171636883062742056960000}$	0.000000000000444
19	$-\frac{1614496500769}{847886202329945761382400000}$	-0.000000000000019	$-\frac{781266674809}{26775353757787760885760000}$	-0.000000000000029
20	$\frac{4906209165197}{203492688559186982731776000O0}$	0.000000000000002	$\frac{5157087816757}{966590270656138167975936000O}$	0.000000000000001



Asymptotic formulas for $x \to +\infty$ in the case $a > 0$:

$$\mathfrak{H}_1(x,a) \sim \frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\,x} - \frac{3a}{(1+a^2)^3\,x^2} - \frac{3\,(4\,a^2-1)}{(1+a^2)^4\,x^3} - \frac{15a\,\left(4\,a^2-3\right)}{(1+a^2)^5\,x^4} -$$

$$-\frac{360\,a^4-540\,a^2+45}{(1+a^2)^6\,x^5} - \frac{315a\,\left(8\,a^4-20\,a^2+5\right)}{(1+a^2)^7\,x^6} - \frac{20160\,a^6-75600\,a^4+37800\,a^2-1575}{(1+a^2)^8\,x^7} - \ldots$$

$$\mathfrak{H}_0(x,a) \sim \frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\,x} + \frac{2\,a^2-1}{(1+a^2)^3\,x^2} + \frac{3\,a\left(2\,a^2-3\right)}{(1+a^2)^4\,x^3} + \frac{24\,a^4-72\,a^2+9}{(1+a^2)^5\,x^4} +$$

$$+\frac{15\,a\left(8\,a^4-40\,a^2+15\right)}{(1+a^2)^6\,x^5} + \frac{720\,a^6-5400\,a^4+4050\,a^2-225}{(1+a^2)^7\,x^6} + \frac{315\,a\left(16\,a^6-168\,a^4+210\,a^2-35\right)}{(1+a^2)^8\,x^7} + \ldots$$

The greater a the better these formulas. They cannot be used with $a = 0$.
The following tables show some relative errors. x_k denotes consecutive maxima or minima of this difference.

$$D_0(x) = \frac{aJ_0(x)+J_1(x)}{1+a^2} - \int_0^x e^{a(t-x)}\,J_0(t)\,dt \quad :$$

$a = 0.1$		$a = 0.3$		$a = 1$		$a = 3$	
x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$
60.226	-4.072E-3	13.386	-3.089E-3	2.399	-1.085E-1	1.757	-1.101E-2
63.622	-1.710E-4	15.961	-1.866E-2	6.200	1.879E-2	5.495	2.073E-3
66.565	-2.720E-3	19.421	5.504E-3	9.308	-1.068E-2	8.759	-1.015E-3
69.863	4.271E-4	22.413	-7.812E-3	12.481	6.748E-3	11.956	6.306E-4
72.883	-1.942E-3	25.632	5.035E-3	15.639	-4.767E-3	15.129	-4.404E-4
76.120	6.932E-4	28.741	-4.770E-3	18.791	3.595E-3	18.290	3.299E-4
79.188	-1.478E-3	31.900	3.866E-3	21.941	-2.835E-3	21.446	-2.590E-4
82.387	7.898E-4	35.036	-3.439E-3	25.089	2.310E-3	24.598	2.104E-4
85.485	-1.188E-3	38.182	2.988E-3	28.235	-1.929E-3	27.748	-1.753E-4
88.661	8.025E-4	41.323	-2.666E-3	31.381	1.642E-3	30.896	1.490E-4
91.776	-9.978E-4	44.466	2.382E-3	34.525	-1.420E-3	34.043	-1.286E-4
91.776	-9.978E-4	47.608	-2.151E-3	37.670	1.244E-3	37.189	1.126E-4

$$D_1(x) = \left(\frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\,x}\right) J_0(x) + \left(\frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\,x}\right) J_1(x) - \int_0^x e^{a(t-x)}\,J_0(t)\,dt \quad :$$

$a = 0.1$		$a = 0.3$		$a = 1$		$a = 3$	
x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$
111.782	-1.972E-5	29.538	-2.864E-4	2.061	-5.622E-2	1.816	-3.195E-3
115.333	-4.255E-6	33.053	7.320E-5	6.836	2.161E-3	5.526	2.229E-4
118.158	-1.252E-5	36.018	-1.159E-4	9.729	-1.174E-3	8.831	-6.917E-5
121.548	-3.783E-7	39.248	7.100E-5	12.930	5.554E-4	12.050	3.163E-5
124.497	-8.467E-6	42.348	-6.756E-5	16.086	-3.201E-4	15.237	-1.750E-5
127.789	1.493E-6	45.509	5.293E-5	19.240	2.033E-4	18.408	1.087E-5
130.814	-6.122E-6	48.642	-4.618E-5	22.391	-1.384E-4	21.571	-7.286E-6
134.046	2.317E-6	51.788	3.894E-5	25.540	9.919E-5	24.728	5.164E-6
137.118	-4.708E-6	54.928	-3.381E-5	28.687	-7.393E-5	27.882	-3.817E-6
140.313	2.604E-6	58.070	2.933E-5	31.833	5.683E-5	31.033	2.915E-6

If $x \to +\infty$, then the follwing direct asymptotic formula holds in the case $a > 0$:

$$\int_0^x e^{at} \cdot J_0(t)\, dt \;\sim\; \frac{e^{ax}}{\sqrt{\pi x}} \left[\sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{\infty} \frac{\mu_k}{x^k} \cos x \right]$$

with

$$\lambda_0 = \frac{a+1}{a^2+1}\,, \quad \mu_0 = \frac{a-1}{a^2+1}$$

$$\lambda_1 = \frac{a^3+3\,a^2+9\,a-5}{8\,\left(a^2+1\right)^2}\,, \quad \mu_1 = \frac{-a^3+3\,a^2-9\,a-5}{8\,\left(a^2+1\right)^2}$$

$$\lambda_2 = \frac{-9\,a^5+15\,a^4+30\,a^3+270\,a^2-345\,a-129}{128\,\left(a^2+1\right)^3}$$

$$\mu_2 = \frac{-9\,a^5-15\,a^4+30\,a^3-270\,a^2-345\,a+129}{128\,\left(a^2+1\right)^3}$$

$$\lambda_3 = \frac{-75\,a^7-105\,a^6-105\,a^5+525\,a^4+5775\,a^3-12075\,a^2-9555\,a+2655}{1024\,\left(a^2+1\right)^4}$$

$$\mu_3 = \frac{75\,a^7-105\,a^6+105\,a^5+525\,a^4-5775\,a^3-12075\,a^2+9555\,a+2655}{1024\,\left(a^2+1\right)^4}$$

$$\lambda_4 = \left[32768\,\left(a^2+1\right)^5\right]^{-1} \cdot [3675\,a^9-4725\,a^8+11340\,a^7-8820\,a^6+92610\,a^5+727650\,a^4-$$
$$-1984500\,a^3-2407860\,a^2+1371195\,a+301035\,]$$

$$\mu_4 = \left[32768\,\left(a^2+1\right)^5\right]^{-1} \cdot [3675\,a^9+4725\,a^8+11340\,a^7+8820\,a^6+92610\,a^5-727650\,a^4-$$
$$-1984500\,a^3+2407860\,a^2+1371195\,a-301035\,]$$

$$\lambda_5 = \left[262144\,\left(a^2+1\right)^6\right]^{-1} \cdot [59535\,a^{11}+72765\,a^{10}+259875\,a^9+280665\,a^8+686070\,a^7+3056130\,a^6+$$
$$+30124710\,a^5-98232750\,a^4-157827285\,a^3+135748305\,a^2+60259815\,a-10896795\,]$$

$$\mu_5 = \left[262144\,\left(a^2+1\right)^6\right]^{-1} \cdot [-59535\,a^{11}+72765\,a^{10}-259875\,a^9+280665\,a^8-686070\,a^7+3056130\,a^6-$$
$$-30124710\,a^5-98232750\,a^4+157827285\,a^3+135748305\,a^2-60259815\,a-10896795\,]$$

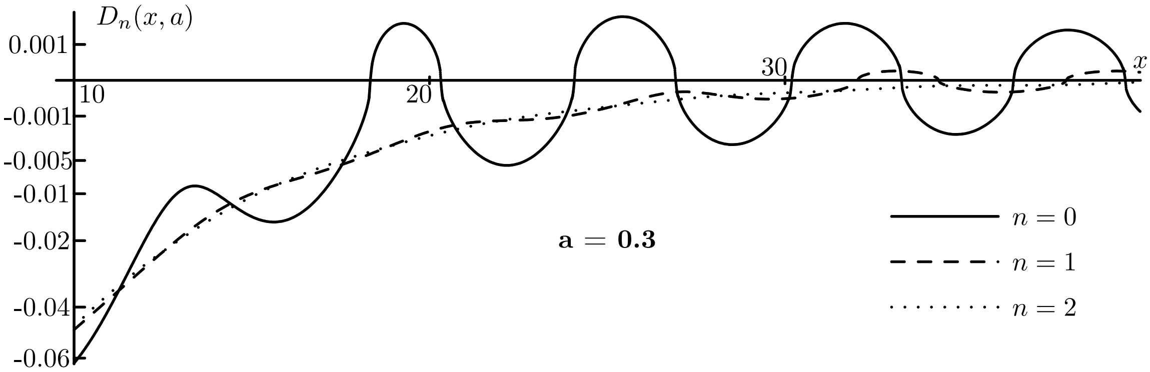
$$\lambda_6 = \left[4194304\,\left(a^2+1\right)^7\right]^{-1} \cdot [-2401245\,a^{13}+2837835\,a^{12}-13243230\,a^{11}+14864850\,a^{10}-34189155\,a^9+$$
$$+49054005\,a^8+160540380\,a^7+2871889020\,a^6-11331475155\,a^5-22569301755\,a^4+25820244450\,a^3+$$
$$-17234307090\,a^2-6264182925\,a-961319205\,]$$

$$\mu_6 = \left[4194304\,\left(a^2+1\right)^7\right]^{-1} \cdot [-2401245\,a^{13}-2837835\,a^{12}-13243230\,a^{11}-14864850\,a^{10}-34189155\,a^9-$$
$$-49054005\,a^8+160540380\,a^7-2871889020\,a^6-11331475155\,a^5+22569301755\,a^4+25820244450\,a^3-$$
$$-17234307090\,a^2-6264182925\,a+961319205\,]$$

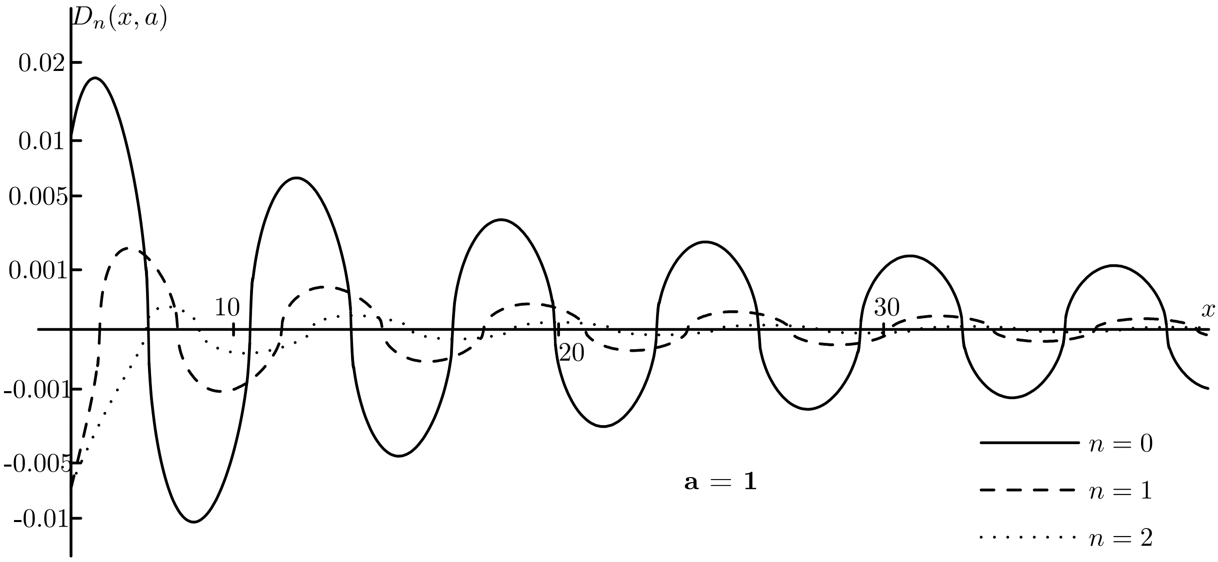
Let

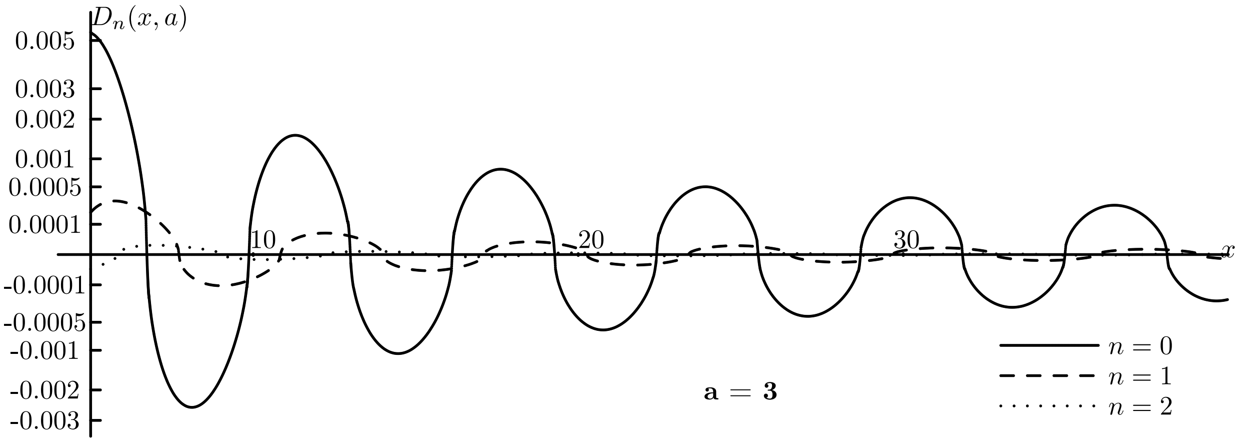
$$D_n(x,a) = \frac{1}{\sqrt{\pi x}} \left[\sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{n} \frac{\mu_k}{x^k} \cos x \right] - e^{-ax} \int_0^x e^{at} \cdot J_0(t)\, dt$$

describe the 'relative difference' between the asymptotic approximation and the true function. With $a = 0.3$, $a = 1$ and $a = 3$ one has the following behaviour at $10 \le x \le 40$:



Note that there is a quadratic scale on the D_n-axis.
The first zero of $D_2(x, 0.3)$ is near $x = 48$.





Furthermore, let $\mathfrak{H}_\nu^*(x,a)$ denote the following functions:

$$\mathfrak{H}_1^*(x,a) \;=\; \sum_{k=1}^{\infty} b_k^*(a)\, x^k \quad , \qquad \mathfrak{H}_0^*(x,a) \;=\; \sum_{k=1}^{\infty} c_k^*(a)\, x^k$$

with

$$b_1^*(a) = 1 \;, \quad b_2^*(a) = 0 \;, \quad b_{k+2}^*(a) = -\frac{a(1+2k)b_{k+1}^*(a) + (1-a^2)b_k^*(a)}{k(k+2)} \quad , \quad k \geq 1$$

and

$$c_k^*(a) = -(k+1)b_{k+1}^*(a) - a\, b_k^*(a) \;.$$

Then holds with $a \in \mathbb{R}$

$$\int_0^x e^{at}\, I_0(t)\, dt \;=\; e^{ax}\; [\; \mathfrak{H}_1^*(x,a)\, I_0(x) + \; \mathfrak{H}_0^*(x,a)\, I_1(x)] \;.$$

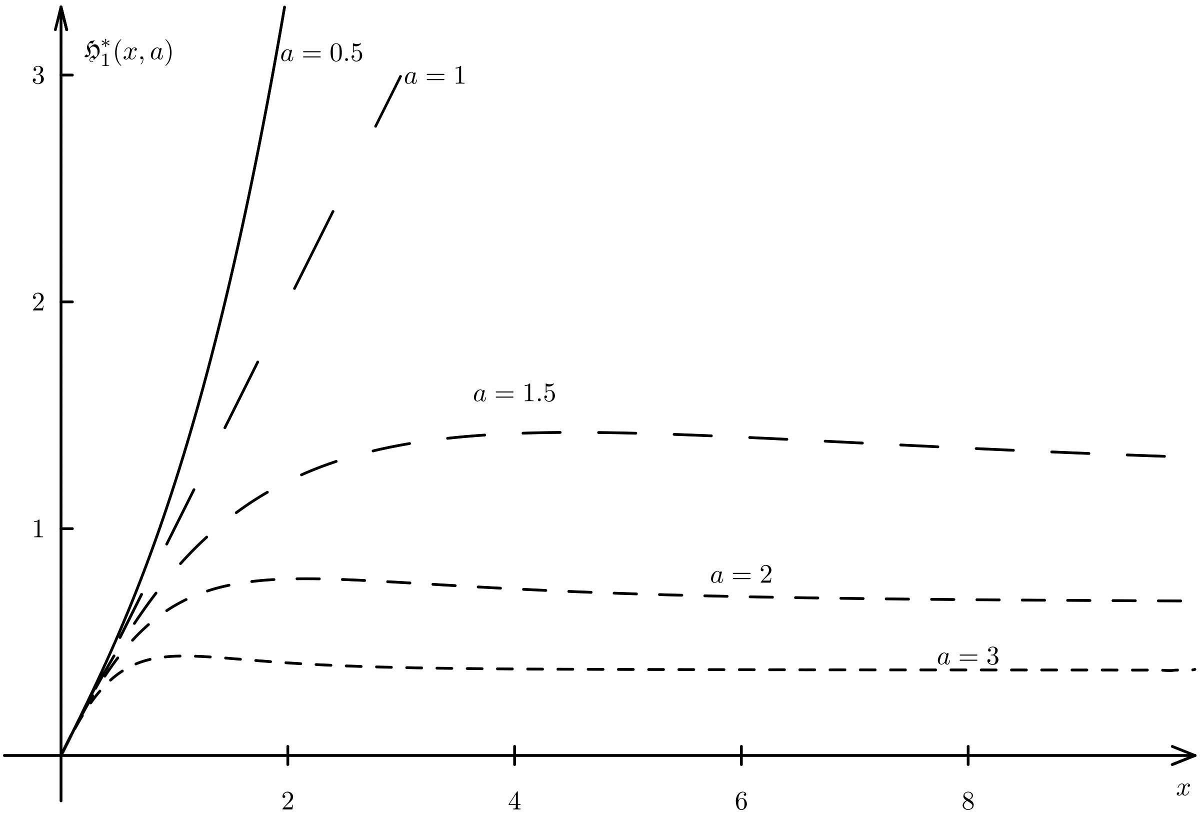
In the case $a=0$ one has with the Struve functions

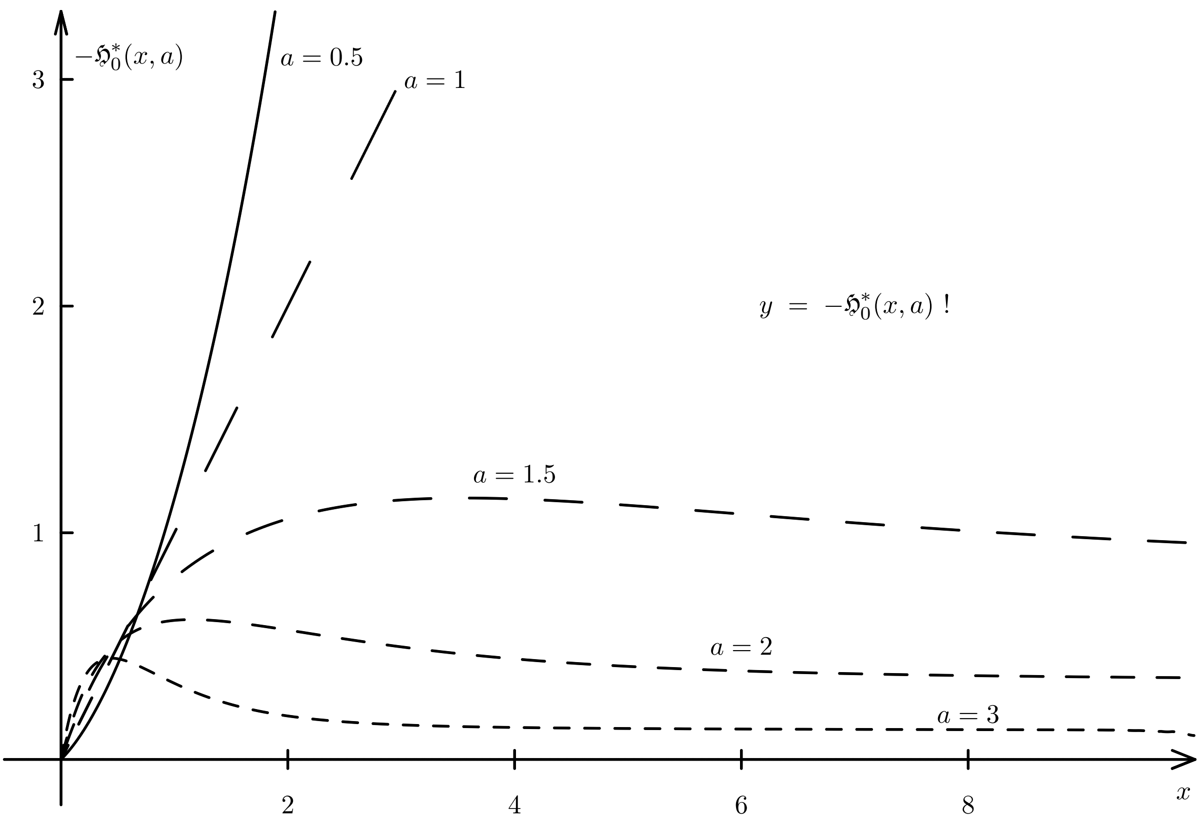
$$\mathfrak{H}_1^*(x,0) = x - \frac{\pi x}{2}\, \mathbf{L}_1(x) \;, \qquad \mathfrak{H}_0^*(x,0) = \frac{\pi x}{2}\, \mathbf{L}_0(x) \;.$$

With $a=1$ holds $\mathfrak{H}_1^*(x,1) = -\, \mathfrak{H}_0^*(x,1) = x$ (see page 57).
First terms of the power series:

$$\mathfrak{H}_1^*(x,a) = x - \frac{a^2-1}{3}\, x^3 + \frac{5\,a^3 - 5\,a}{24}\, x^4 - \frac{27\,a^4 - 19\,a^2 - 8}{360}\, x^5 + \frac{56\,a^5 - 7\,a^3 - 49\,a}{2880}\, x^6 -$$
$$- \frac{400\,a^6 + 291\,a^4 - 627\,a^2 - 64}{100800}\, x^7 + \frac{1080\,a^7 + 1996\,a^5 - 2227\,a^3 - 849\,a}{1612800}\, x^8 -$$
$$- \frac{9800\,a^8 + 31684\,a^6 - 18717\,a^4 - 21743\,a^2 - 1024}{101606400}\, x^9 +$$
$$+ \frac{19712\,a^9 + 96184\,a^7 - 10428\,a^5 - 91289\,a^3 - 14179\,a}{1625702400}\, x^{10} -$$
$$- \frac{217728\,a^{10} + 1477352\,a^8 + 608284\,a^6 - 1686075\,a^4 - 600905\,a^2 - 16384}{160944537600}\, x^{11} +$$
$$+ \frac{873600\,a^{11} + 7817888\,a^9 + 7776184\,a^7 - 9134112\,a^5 - 6750965\,a^3 - 582595\,a}{6437781504000}\, x^{12} - \ldots$$

$$\mathfrak{H}_0^*(x,a) = -ax + (a^2-1)x^2 - \frac{a^3 - a}{2}\, x^3 + \frac{3\,a^4 - a^2 - 2}{18}\, x^4 - \frac{12\,a^5 + 11\,a^3 - 23\,a}{288}\, x^5 +$$
$$+ \frac{60\,a^6 + 163\,a^4 - 191\,a^2 - 32}{7200}\, x^6 - \frac{40\,a^7 + 202\,a^5 - 139\,a^3 - 103\,a}{28800}\, x^7 +$$
$$+ \frac{280\,a^8 + 2214\,a^6 - 391\,a^4 - 1975\,a^2 - 128}{1411200}\, x^8 -$$
$$- \frac{2240\,a^9 + 25272\,a^7 + 10844\,a^5 - 31389\,a^3 - 6967\,a}{90316800}\, x^9 +$$
$$+ \frac{20160\,a^{10} + 305848\,a^8 + 351068\,a^6 - 432237\,a^4 - 236647\,a^2 - 8192}{7315660800}\, x^{10} -$$
$$- \frac{40320\,a^{11} + 789104\,a^9 + 1567792\,a^7 - 958326\,a^5 - 1294895\,a^3 - 143995\,a}{146313216000}\, x^{11} +$$
$$+ \frac{443520\,a^{12} + 10857424\,a^{10} + 32019232\,a^8 - 4458746\,a^6 - 31104595\,a^4 - 7592995\,a^2 - 163840}{17703899136000}\, x^{12} - \ldots$$





If $0 < a < 1$, then $\mathfrak{H}_1^*(x,a)$ and $-\mathfrak{H}_0^*(x,a)$ are growing rapidly with $x \to +\infty$.

Asymptotic behaviour for $x \to +\infty$ and $a > 1$:

$$\mathfrak{H}_1^*(x,a) \sim \frac{a}{a^2-1} + \frac{1}{(a^2-1)^2\,x} + \frac{3a}{(a^2-1)^3\,x^2} + \frac{12\,a^2+3}{(a^2-1)^4\,x^3} + \frac{15a\,\left(4\,a^2+3\right)}{(a^2-1)^5\,x^4} + \frac{360\,a^4+540\,a^2+45}{(a^2-1)^6\,x^5} +$$

$$+ \frac{315a\,\left(8\,a^4+20\,a^2+5\right)}{(a^2-1)^7\,x^6} + \frac{20160\,a^6+75600\,a^4+37800\,a^2+1575}{(a^2-1)^8\,x^7} + \ldots$$

$$\mathfrak{H}_0^*(x,a) \sim -\frac{1}{a^2-1} - \frac{a}{(a^2-1)^2\,x} - \frac{2\,a^2+1}{(a^2-1)^3\,x^2} - \frac{3\,a\left(2\,a^2+3\right)}{(a^2-1)^4\,x^3} - \frac{24\,a^4+72\,a^2+9}{(a^2-1)^5\,x^4} -$$

$$- \frac{15a\,\left(8\,a^4+40\,a^2+15\right)}{(a^2-1)^6\,x^5} - \frac{720\,a^6+5400\,a^4+4050\,a^2+225}{(a^2-1)^7\,x^6} -$$

$$- \frac{315\,\left(16\,a^6+168\,a^4+210\,a^2+35\right)a}{(a^2-1)^8\,x^7} - \ldots$$

Direct asymptotic formula for the case $x \to +\infty,\ a > 0$:

$$\int_0^x e^{at}\cdot I_0(t)\,dt \sim \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \sum_{k=0}^{\infty} \frac{\beta_k(a)}{x^k} = \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \left[1 + \frac{a+5}{8(1+a)\,x} + \right.$$

$$+ \frac{9\,a^2+42\,a+129}{128\,(1+a)^2\,x^2} + \frac{75\,a^3+405\,a^2+1065\,a+2655}{1024\,(1+a)^3\,x^3} +$$

$$+ \frac{3675\,a^4+23100\,a^3+67410\,a^2+133980\,a+301035}{32768\,(1+a)^4\,x^4} +$$

$$+ \frac{59535\,a^5+429975\,a^4+1426950\,a^3+3022110\,a^2+5120955\,a+10896795}{262144\,(1+a)^5\,x^5} +$$

$$+ \frac{2401245\,a^6+19646550\,a^5+73856475\,a^4+173596500\,a^3+301964355\,a^2+465051510\,a+961319205}{4194304\,(1+a)^6\,x^6} +$$

$$\left. + \frac{135135\,(370345+181955\,a+125205\,a^2+81815\,a^3+43435\,a^4+16569\,a^5+3927\,a^6+429\,a^7)}{33554432\,(1+a)^7\,x^7} + \ldots \right]$$

Some coefficients:

k	$\beta_k(0.1)$	$\beta_k(0.3)$	$\beta_k(1)$	$\beta_k(3)$	$\beta_k(10)$
1	0.579545	0.509615	0.375000	0.250000	0.170455
2	0.860602	0.658330	0.351563	0.164063	0.093556
3	2.029155	1.339262	0.512695	0.175781	0.094505
4	6.568555	3.717857	1.009369	0.265961	0.142222
5	27.098470	13.096614	2.498188	0.526314	0.285290
6	136.064853	55.981252	7.442518	1.296183	0.715146
7	805.747313	281.633987	25.915913	3.834025	2.150314

With fixed $x > 0$ and $a \to +\infty$ holds

$$\int_0^x e^{a\,t}\,J_0(t)\,dt \sim e^{ax}\left[\left(\frac{1}{a} - \frac{1}{a^3} - \frac{1}{a^4x} + \frac{x^2-3}{a^5x^2} + \frac{2\,x^2-12}{a^6x^3} - \frac{x^4-9\,x^2+60}{a^7x^4} - \frac{3\,x^4-51\,x^2+360}{a^8x^5} + \right.\right.$$

$$\left. \frac{x^6-18\,x^4+345\,x^2-2520}{a^9x^6} + \frac{4\,x^6-132\,x^4+2700\,x^2-20160}{a^{10}x^7} + \ldots\right) J_0(x) +$$

$$+ \left(\frac{1}{a^2} + \frac{1}{a^3x} - \frac{x^2-2}{a^4x^2} - \frac{2\,x^2-6}{a^5x^3} + \frac{x^4-7\,x^2+24}{a^6x^4} + \frac{3\,x^4-33\,x^2+120}{a^7x^5} - \frac{x^6-15\,x^4+192\,x^2-720}{a^8x^6} - \right.$$

$$\left.\left. - \frac{4\,x^6-96\,x^4+1320\,x^2-5040}{a^9x^7} + \frac{x^8-26\,x^6+729\,x^4-10440\,x^2+40320}{a^{10}x^8} + \ldots\right) J_1(x)\right]$$

and

$$\int_0^x e^{a\,t}\, I_0(t)\, dt \;\sim\; e^{ax}\,\left[\left(\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^4x}+\frac{x^2+3}{a^5x^2}+\frac{2\,x^2+12}{a^6x^3}+\frac{x^4+9\,x^2+60}{a^7x^4}+\frac{3\,x^4+51\,x^2+360}{a^8x^5}+\right.\right.$$

$$\left.\frac{x^6+18\,x^4+345\,x^2+2520}{a^9x^6}+\frac{4\,x^6+132\,x^4+2700\,x^2+20160}{a^{10}x^7}+\ldots\right)\, I_0(x)-$$

$$-\left(\frac{1}{a^2}+\frac{1}{a^3x}+\frac{x^2+2}{a^4x^2}+\frac{2\,x^2+6}{a^5x^3}+\frac{x^4+7\,x^2+24}{a^6x^4}+\frac{3\,x^4+33\,x^2+120}{a^7x^5}+\frac{x^6+15\,x^4+192\,x^2+720}{a^8x^6}+\right.$$

$$\left.\left.\frac{4\,x^6+96\,x^4+1320\,x^2+5040}{a^9x^7}+\frac{x^8+26\,x^6+729\,x^4+10440\,x^2+40320}{a^{10}x^8}\right)\, I_1(x)\right]\;.$$

c) The case $a<0$:

To express this fact clearly it is written $a=-\alpha$ with $\alpha>0$.
One has the Lipschitz integral (see [11], part I, §21, or the tables of Laplace transforms)

$$\int_0^\infty e^{a\,x}\, J_0(x)\, dx=\int_0^\infty e^{-\alpha\,x}\, J_0(x)\, dx=\frac{1}{\sqrt{1+\alpha^2}}=\frac{1}{\sqrt{1+a^2}}\;.$$

The representation

$$\int_0^x e^{a\,t}\, J_0(t)\, dt \;=\; e^{ax}\;[\;\mathfrak{H}_1(x,a)\, J_0(x)+\;\mathfrak{H}_0(x,a)\, J_1(x)]$$

keeps being true, but $\mathfrak{H}_0(x,a)$ and $\mathfrak{H}_1(x,a)$ are rapidly growing with x. For that reason other formulas are more applicable.

$$\int_0^x e^{-\alpha\,t}\, J_0(t)\, dt \;=\; \frac{1}{\sqrt{1+\alpha^2}}-e^{-\alpha\,x}\;\left[\frac{1}{\sqrt{1+\alpha^2}}-\sum_{k=1}^\infty \varphi_k(\alpha)\cdot x^{2k-1}\;\cdot\left(1+\frac{\alpha\,x}{2k}\right)\right]$$

with

$$\varphi_k(\alpha)=\frac{(-1)^{k-1}}{(2k-1)!\cdot a^2}\;\sum_{i=0}^\infty\frac{(-1)^i\cdot(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

Some first functions:

$$\varphi_1(\alpha)=\frac{\sqrt{1+\alpha^2}-\alpha}{\sqrt{1+\alpha^2}}$$

$$\varphi_2(\alpha)=\frac{1}{6}\cdot\frac{(2\,\alpha^2-1)\,\sqrt{1+\alpha^2}-2\,\alpha^3}{2\,\sqrt{1+\alpha^2}}$$

$$\varphi_3(\alpha)=\frac{1}{120}\cdot\frac{(8\,\alpha^4-4\,\alpha^2+3)\,\sqrt{1+\alpha^2}-8\,\alpha^5}{8\,\sqrt{1+\alpha^2}}$$

$$\varphi_4(\alpha)=\frac{1}{5040}\cdot\frac{(16\,\alpha^6-8\,\alpha^4+6\,\alpha^2-5)\,\sqrt{1+\alpha^2}-16\,\alpha^7}{16\,\sqrt{1+\alpha^2}}$$

$$\varphi_5(\alpha)=\frac{1}{362\,880}\cdot\frac{(128\,\alpha^8-64\,\alpha^6+48\,\alpha^4-40\,\alpha^2+35)\,\sqrt{1+\alpha^2}-128\,\alpha^9}{128\,\sqrt{1+\alpha^2}}$$

$$\varphi_6(\alpha)=\frac{1}{29\,916\,800}\cdot\frac{(256\,\alpha^{10}-128\,\alpha^8+96\,\alpha^6-80\,\alpha^4+70\,\alpha^2-63)\,\sqrt{1+\alpha^2}-256\,\alpha^{11}}{256\,\sqrt{1+\alpha^2}}$$

$$\varphi_7(\alpha)=\frac{1}{1\,307\,674\,368\,000}\cdot$$

$$\cdot\frac{+(1024\,\alpha^{12}-512\,\alpha^{10}+384\,\alpha^8-320\,\alpha^6+280\,\alpha^4-252\,\alpha^2+231)\,\sqrt{1+\alpha^2}-1024\,\alpha^{13}}{1024\,\sqrt{1+\alpha^2}}$$

Special values for the case $\alpha=1$:

$$\varphi_1(1)=\frac{\sqrt{2}}{2+2\sqrt{2}}=0.292893218813452\;,\quad \varphi_2(1)=\frac{1-\sqrt{2}}{12}=-0.034517796864425\;,$$

$$\varphi_3(1) = -\frac{1}{240}\sqrt{2} + \frac{7}{960} = 0.001399110156779\ , \quad \varphi_4(1) = -\frac{1}{10080}\sqrt{2} + \frac{1}{8960} = -0.000028691821664\ ,$$

$$\varphi_5(1) = -\frac{1}{725760}\sqrt{2} + \frac{107}{46448640} = 0.000000355022924\ ,$$

$$\varphi_6(1) = -\frac{1}{79833600}\sqrt{2} + \frac{151}{10218700800} = -0.000000002937686\ ,$$

$$\varphi_7(1) = -\frac{1}{12454041600}\sqrt{2} + \frac{167}{1275293859840} = 0.000000000017396\ ,$$

$$\varphi_8(1) = -\frac{1}{2615348736000}\sqrt{2} + \frac{1241}{2678117105664000} = -0.000000000000077\ ,$$

Asymptotic behaviour of $\varphi_k(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{1+\alpha^2}} \sim \alpha^{-1} - \frac{1}{2}\alpha^{-3} + \frac{3}{8}\alpha^{-5} - \frac{5}{16}\alpha^{-7} + \frac{35}{128}\alpha^{-9} + \ldots$$

$$\varphi_1(\alpha) \sim \frac{1}{2}\alpha^{-2} - \frac{3}{8}\alpha^{-4} + \frac{5}{16}\alpha^{-6} - \frac{35}{128}\alpha^{-8} + \ldots$$

$$3!\ \varphi_2(\alpha) \sim -\frac{3}{8}\alpha^{-2} + \frac{5}{16}\alpha^{-4} - \frac{35}{128}\alpha^{-6} + \frac{63}{256}\alpha^{-8} + \ldots$$

$$5!\ \varphi_3(\alpha) \sim \frac{5}{16}\alpha^{-2} - \frac{35}{128}\alpha^{-4} + \frac{63}{256}\alpha^{-6} - \frac{231}{1024}\alpha^{-8} \ldots$$

$$7!\ \varphi_4(\alpha) \sim -\frac{35}{128}\alpha^{-2} + \frac{63}{256}\alpha^{-4} - \frac{231}{1024}\alpha^{-6} + \frac{429}{2048}\alpha^{-8} + \ldots$$

$$9!\ \varphi_5(\alpha) \sim \frac{63}{256}\alpha^{-2} - \frac{231}{1024}\alpha^{-4} + \frac{429}{2048}\alpha^{-6} - \frac{6435}{32768}\alpha^{-8} + \ldots$$

$$11!\ \varphi_6(\alpha) \sim -\frac{231}{1024}\alpha^{-2} + \frac{429}{2048}\alpha^{-4} - \frac{6435}{32768}\alpha^{-6} + \frac{12155}{65536}\alpha^{-8} + \ldots$$

$$13!\ \varphi_7(\alpha) \sim \frac{429}{2048}\alpha^{-2} - \frac{6435}{32768}\alpha^{-4} + \frac{12155}{65536}\alpha^{-6} - \frac{46189}{262144}\alpha^{-8} + \ldots$$

Direct asymptotic formula for $x \to \infty$:

$$\int_0^x e^{-\alpha t} J_0(t)\, dt \sim \frac{1}{\sqrt{1+\alpha^2}} + \frac{e^{-\alpha x}}{\sqrt{\pi x}} \left\{ \left[-\frac{\alpha-1}{1+\alpha^2} - \frac{\alpha^3 - 3\alpha^2 + 9\alpha + 5}{8(1+\alpha^2)^2 x} + \right.\right.$$

$$+ \frac{3(3\alpha^5 + 5\alpha^4 - 10\alpha^3 + 90\alpha^2 + 115\alpha - 43)}{128(1+\alpha^2)^3 x^2} +$$

$$+ \frac{15(5\alpha^7 - 7\alpha^6 + 7\alpha^5 + 35\alpha^4 - 385\alpha^3 - 805\alpha^2 + 637\alpha + 177)}{1024(1+\alpha^2)^4 x^3} -$$

$$- \frac{105(35\alpha^9 + 45\alpha^8 + 108\alpha^7 + 84\alpha^6 + 882\alpha^5 - 6930\alpha^4 - 18900\alpha^3 + 22932\alpha^2 + 13059\alpha - 2867)}{32768(1+\alpha^2)^5 x^4} +$$

$$- \frac{945\, s_5}{262144(1+\alpha^2)^6 x^5} - \frac{34459425\, s_6}{17179869184(1+\alpha^2)^7 x^6} - \frac{135135\, s_7}{33554432(1+\alpha^2)^8 x^7} - \frac{2027025\, s_8}{2147483648(1+\alpha^2)^9 x^8} -$$

$$\left. - \frac{34459425\, s_9}{17179869184(1+\alpha^2)^{10} x^9} + \frac{654729075\, s_{10}}{274877906944(1+\alpha^2)^{11} x^{10}} + \ldots \right] \sin x +$$

$$+ \left[-\frac{1+\alpha}{1+\alpha^2} + \frac{\alpha^3 + 3\alpha^2 + 9\alpha - 5}{8(1+\alpha^2)^2 x} + \frac{3(3\alpha^5 - 5\alpha^4 - 10\alpha^3 - 90\alpha^2 + 115\alpha + 43)}{128(1+\alpha^2)^3 x^2} - \right.$$

$$- \frac{15(5\alpha^7 + 7\alpha^6 + 7\alpha^5 - 35\alpha^4 - 385\alpha^3 + 805\alpha^2 + 637\alpha - 177)}{(1+\alpha^2)^4 x^3} -$$

$$- \frac{105(35\alpha^9 - 45\alpha^8 + 108\alpha^7 - 84\alpha^6 + 882\alpha^5 + 6930\alpha^4 - 18900\alpha^3 - 22932\alpha^2 + 13059\alpha + 2867)}{32768(1+\alpha^2)^5 x^4} +$$

$$+\frac{945\,c_5}{262144\,(1+\alpha^2)^6\,x^5}+\frac{10395\,c_6}{4194304\,(1+\alpha^2)^7\,x^6}-\frac{135135\,c_7}{33554432\,(1+\alpha^2)^8\,x^7}-\frac{2027025\,c_8}{2147483648\,(1+\alpha^2)^9\,x^8}+$$

$$\left.+\frac{34459425\,c_9}{17179869184\,(1+\alpha^2)^{10}\,x^9}+\frac{654729075\,274877906944c_{10}}{274877906944\,(1+\alpha^2)^{11}\,x^{10}}+\ldots]\,\cos x\right\}$$

$$s_5=63\,\alpha^{11}-77\,\alpha^{10}+275\,\alpha^9-297\,\alpha^8+726\,\alpha^7-3234\,\alpha^6+31878\,\alpha^5+103950\,\alpha^4-167013\,\alpha^3-$$
$$-143649\,\alpha^2+63767\,\alpha+11531$$

$$c_5=63\,\alpha^{11}+77\,\alpha^{10}+275\,\alpha^9+297\,\alpha^8+726\,\alpha^7+3234\,\alpha^6+31878\,\alpha^5-103950\,\alpha^4-$$
$$-167013\,\alpha^3+143649\,\alpha^2+63767\,\alpha-11531$$

$$s_6=231\,\alpha^{13}+273\,\alpha^{12}+1274\,\alpha^{11}+1430\,\alpha^{10}+3289\,\alpha^9+4719\,\alpha^8-15444\,\alpha^7+276276\,\alpha^6+1090089\,\alpha^5-$$
$$-2171169\,\alpha^4-2483910\,\alpha^3+1657942\,\alpha^2+602615\,\alpha-92479$$

$$c_6=231\,\alpha^{13}-273\,\alpha^{12}+1274\,\alpha^{11}-1430\,\alpha^{10}+3289\,\alpha^9-4719\,\alpha^8-15444\,\alpha^7-276276\,\alpha^6+1090089\,\alpha^5+$$
$$+2171169\,\alpha^4-2483910\,\alpha^3-1657942\,\alpha^2+602615\,\alpha+92479$$

$$s_7=429\,\alpha^{15}-495\,\alpha^{14}+2835\,\alpha^{13}-3185\,\alpha^{12}+8385\,\alpha^{11}-9867\,\alpha^{10}+9295\,\alpha^9+57915\,\alpha^8-1151865\,\alpha^7-$$
$$-5450445\,\alpha^6+13054041\,\alpha^5+18629325\,\alpha^4-16564405\,\alpha^3-9039225\,\alpha^2+2780805\,\alpha+370345$$

$$c_7=429\,\alpha^{15}+495\,\alpha^{14}+2835\,\alpha^{13}+3185\,\alpha^{12}+8385\,\alpha^{11}+9867\,\alpha^{10}+9295\,\alpha^9-57915\,\alpha^8-1151865\,\alpha^7+$$
$$+5450445\,\alpha^6+13054041\,\alpha^5-18629325\,\alpha^4-16564405\,\alpha^3+9039225\,\alpha^2+2780805\,\alpha-370345$$

$$s_8=6435\,\alpha^{17}+7293\,\alpha^{16}+49368\,\alpha^{15}+55080\,\alpha^{14}+168980\,\alpha^{13}+190060\,\alpha^{12}+312936\,\alpha^{11}+252824\,\alpha^{10}+$$
$$+2601170\,\alpha^9-39163410\,\alpha^8-210913560\,\alpha^7+591783192\,\alpha^6+1014047892\,\alpha^5-1126379540\,\alpha^4-$$
$$-819264680\,\alpha^3+378189480\,\alpha^2+100843235\,\alpha-11857475$$

$$c_8=6435\,\alpha^{17}-7293\,\alpha^{16}+49368\,\alpha^{15}-55080\,\alpha^{14}+168980\,\alpha^{13}-190060\,\alpha^{12}+312936\,\alpha^{11}-252824\,\alpha^{10}+$$
$$+2601170\,\alpha^9+39163410\,\alpha^8-210913560\,\alpha^7-591783192\,\alpha^6+1014047892\,\alpha^5+1126379540\,\alpha^4-$$
$$-819264680\,\alpha^3-378189480\,\alpha^2+100843235\,\alpha+11857475$$

$$s_9=12155\,\alpha^{19}-13585\,\alpha^{18}+105963\,\alpha^{17}-117249\,\alpha^{16}+414732\,\alpha^{15}-458660\,\alpha^{14}+936700\,\alpha^{13}-990964\,\alpha^{12}+$$
$$+1771978\,\alpha^{11}-9884446\,\alpha^{10}+168589850\,\alpha^9+1001839410\,\alpha^8-3209765988\,\alpha^7-6422303316\,\alpha^6+$$
$$+8562147308\,\alpha^5+7783014460\,\alpha^4-4789707245\,\alpha^3-1916021465\,\alpha^2+450814995\,\alpha+47442055$$

$$c_9=12155\,\alpha^{19}+13585\,\alpha^{18}+105963\,\alpha^{17}+117249\,\alpha^{16}+414732\,\alpha^{15}+458660\,\alpha^{14}+936700\,\alpha^{13}+990964\,\alpha^{12}+$$
$$+1771978\,\alpha^{11}+9884446\,\alpha^{10}+168589850\,\alpha^9-1001839410\,\alpha^8-3209765988\,\alpha^7+6422303316\,\alpha^6+$$

$$+8562147308\,\alpha^5-7783014460\,\alpha^4-4789707245\,\alpha^3+1916021465\,\alpha^2+450814995\,\alpha-47442055$$

$$s_{10}=46189\,\alpha^{21}+51051\,\alpha^{20}+450450\,\alpha^{19}+494494\,\alpha^{18}+1988217\,\alpha^{17}+2177343\,\alpha^{16}+5191256\,\alpha^{15}+$$
$$+5620200\,\alpha^{14}+9265578\,\alpha^{13}+12403846\,\alpha^{12}-53260116\,\alpha^{11}+1416154740\,\alpha^{10}+9373133770\,\alpha^9-$$
$$-33702542874\,\alpha^8-77051011752\,\alpha^7+119870062312\,\alpha^6+130763372649\,\alpha^5-100583852145\,\alpha^4-$$
$$-53645367790\,\alpha^3+18934229790\,\alpha^2+3986102589\,\alpha-379582629$$

$$c_{10}=46189\,\alpha^{21}-51051\,\alpha^{20}+450450\,\alpha^{19}-494494\,\alpha^{18}+1988217\,\alpha^{17}-2177343\,\alpha^{16}+5191256\,\alpha^{15}-$$
$$-5620200\,\alpha^{14}+9265578\,\alpha^{13}-12403846\,\alpha^{12}-53260116\,\alpha^{11}-1416154740\,\alpha^{10}+9373133770\,\alpha^9+$$
$$+33702542874\,\alpha^8-77051011752\,\alpha^7-119870062312\,\alpha^6+130763372649\,\alpha^5+100583852145\,\alpha^4-$$
$$-53645367790\,\alpha^3-18934229790\,\alpha^2+3986102589\,\alpha+379582629$$

In the special case $\alpha=1$ holds

$$\int_0^x e^{-t}\,J_0(t)\,dt\;\sim\;\frac{1}{\sqrt{2}}+\frac{e^{-x}}{\sqrt{\pi x}}\;\left[\left(\sum_{k=0}^{\infty}\frac{s_k^*}{x^k}\right)\,\sin x+\left(\sum_{k=0}^{\infty}\frac{c_k^*}{x^k}\right)\,\cos x\right]$$

with

k	s_k^*	s_k^*	c_k^*	c_k^*
0	0	0	-1	-1
1	$-\frac{3}{8}$	-0.375000000000000	$\frac{1}{4}$	0.250000000000000
2	$\frac{15}{32}$	0.468750000000000	$\frac{21}{128}$	0.164062500000000
3	$-\frac{315}{1024}$	-0.307617187500000	$-\frac{405}{512}$	-0.791015625000000
4	$-\frac{3465}{4096}$	-0.845947265625000	$\frac{59325}{32768}$	1.810455322265625
5	$\frac{1507275}{262144}$	5.749797821044922	$-\frac{284445}{131072}$	-2.170143127441406
6	$-\frac{22837815}{1048576}$	-21.77983760833740	$-\frac{38887695}{4194304}$	-9.271548986434937
7	$\frac{1422025605}{33554432}$	42.37966552376747	$\frac{1693106415}{16777216}$	100.9170064330101
8	$\frac{29462808375}{134217728}$	219.5150284096599	$-\frac{1167021130275}{2147483648}$	-543.4365618391894
9	$-\frac{56125340496225}{17179869184}$	-3266.924788257165	$\frac{11825475336675}{8589934592}$	1376.666517075500
10	$\frac{1515749532221925}{68719476736}$	22057.05870033016	$\frac{2498294907783675}{274877906944}$	9088.743928382155

The Laplace transform of $I_0(x)$ exists only in the case $\alpha>1$:

$$*E*\qquad\int_0^{\infty}e^{a\,x}\,I_0(x)\,dx=\int_0^{\infty}e^{-\alpha\,x}\,I_0(x)\,dx=\frac{1}{\sqrt{\alpha^2-1}}=\frac{1}{\sqrt{a^2-1}}\,.$$

If $\alpha>1$ one has

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt\;=\;\frac{1}{\sqrt{\alpha^2-1}}-e^{-\alpha\,x}\;\left[\frac{1}{\sqrt{\alpha^2-1}}+\sum_{k=1}^{\infty}\varphi_k^*(\alpha)\cdot x^{2k-1}\cdot\left(1+\frac{\alpha\,x}{2k}\right)\right]$$

with

$$\varphi_k^*(\alpha)=\frac{1}{(2k-1)!\cdot a^2}\sum_{i=0}^{\infty}\frac{(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

This series fails to converge in the case $0<\alpha\le 1$.
If $\alpha=1$, then see page 57 for solutions with elementary functions.

Some first functions $\varphi_k^*(\alpha)$:

$$\varphi_1^*(\alpha) = \frac{\alpha - \sqrt{\alpha^2-1}}{\sqrt{\alpha}^2-1}$$

$$\varphi_2^*(\alpha) = \frac{1}{6}\cdot\frac{2\alpha^3 \ - (2\,\alpha^2+1)\,\sqrt{\alpha^2-1}}{2\,\sqrt{\alpha^2-1}}$$

$$\varphi_3^*(\alpha) = \frac{1}{120}\cdot\frac{8\alpha^5 - \ (8\,\alpha^4+4\,\alpha^2+3)\,\sqrt{\alpha^2-1}}{8\,\sqrt{\alpha^2-1}}$$

$$\varphi_4^*(\alpha) = \frac{1}{5\,040}\cdot\frac{16\alpha^7 - \ (16\,\alpha^6+8\,\alpha^4+6\,\alpha^2+5)\,\sqrt{\alpha^2-1}}{16\,\sqrt{\alpha^2-1}}$$

$$\varphi_5^*(\alpha) = \frac{1}{362\,880}\cdot\frac{128\alpha^9 - \ (128\,\alpha^8+64\,\alpha^6+48\,\alpha^4+40\,\alpha^2+35)\,\sqrt{\alpha^2-1}}{128\,\sqrt{\alpha^2-1}}$$

$$\varphi_6^*(\alpha) = \frac{1}{39\,916\,800}\cdot\frac{(256\,\alpha^{10}+128\,\alpha^8+96\,\alpha^6+80\,\alpha^4+70\,\alpha^2+63)\,\sqrt{\alpha^2-1}}{256\,\sqrt{\alpha^2-1}}$$

Asymptotic behaviour of $\varphi_k^*(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{\alpha^2-1}} \ \sim \ \alpha^{-1} + \frac{1}{2}\,\alpha^{-3} + \frac{3}{8}\,\alpha^{-5} + \frac{5}{16}\,\alpha^{-7} + \frac{35}{128}\,\alpha^{-9} + \ldots$$

$$\varphi_1^*(\alpha) \ \sim \ \frac{1}{2}\,\alpha^{-2} + \frac{3}{8}\,\alpha^{-4} + \frac{5}{16}\,\alpha^{-6} + \frac{35}{128}\,\alpha^{-8} + \ldots$$

$$6\,\varphi_2^*(\alpha) \ \sim \ \frac{3}{8}\,\alpha^{-2} + \frac{5}{16}\,\alpha^{-4} + \frac{35}{128}\,\alpha^{-6} + \frac{63}{256}\,\alpha^{-8} + \ldots$$

$$120\,\varphi_3^*(\alpha) \ \sim \ \frac{5}{16}\,\alpha^{-2} + \frac{35}{128}\,\alpha^{-4} + \frac{63}{256}\,\alpha^{-6} + \frac{231}{1024}\,\alpha^{-8} + \ldots$$

$$5\,040\,\varphi_4^*(\alpha) \ \sim \ \frac{35}{128}\,\alpha^{-2} + \frac{63}{256}\,\alpha^{-4} + \frac{231}{1024}\,\alpha^{-6} + \frac{429}{2048}\,\alpha^{-8} + \ldots$$

$$362\,880\,\varphi_5^*(\alpha) \ \sim \ \frac{63}{256}\,\alpha^{-2} + \frac{231}{1024}\,\alpha^{-4} + \frac{429}{2048}\,\alpha^{-6} + \frac{6435}{32768}\,\alpha^{-8} + \ldots$$

$$39\,916\,800\,\varphi_6^*(\alpha) \ \sim \ \frac{231}{1024}\,\alpha^{-2} + \frac{429}{2048}\,\alpha^{-4} + \frac{6435}{32768}\,\alpha^{-6} + \frac{12155}{65536}\,\alpha^{-8} + \ldots$$

$$6\,227\,020\,800\,\varphi_7^*(\alpha) \ \sim \ \frac{429}{2048}\,\alpha^{-2} + \frac{6435}{32768}\,\alpha^{-4} + \frac{12155}{65536}\,\alpha^{-6} + \frac{46189}{262144}\,\alpha^{-8} + \ldots$$

If $0 < \alpha \le 1 \Longleftrightarrow -1 \le a < 0$ and x is not very large, then $\mathfrak{H}_\nu^*(x,a)$ may be used.
With $0 < \alpha < 1$, $x >> 1$ one has the direct asymptotic formula

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt \sim \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}}\sum_{k=0}^{\infty}\frac{w_k(\alpha)}{x^k} = \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}}\left[\frac{1}{1-\alpha} - \frac{\alpha-5}{8\,(1-\alpha)^2\,x} - \frac{9\,\alpha^2-42\,\alpha+129}{128\,(1-\alpha)^3\,x^2} - \right.$$

$$-\frac{75\,\alpha^3-405\,\alpha^2+1065\,\alpha-2655}{1024\,(1-\alpha)^4\,x^3} - \frac{3675\,\alpha^4-23100\,\alpha^3+67410\,\alpha^2-133980\,\alpha+301035}{32768\,(1-\alpha)^5\,x^4} -$$

$$-\frac{59535\,\alpha^5-429975\,\alpha^4+1426950\,\alpha^3-3022110\,\alpha^2+5120955\,\alpha-10896795}{262144\,(1-\alpha)^6\,x^5} -$$

$$-\frac{10395}{4194304}\cdot\frac{z_6}{(1-\alpha)^7\,x^6} - \frac{135135}{33554432}\cdot\frac{z_7}{(1-\alpha)^8\,x^7} - \frac{2027025}{2147483648}\cdot\frac{z_8}{(1-\alpha)^9\,x^8} -$$

$$\left. -\frac{34459425}{17179869184}\cdot\frac{z_9}{(1-\alpha)^{10}\,x^9} - \frac{654729075}{274877906944}\cdot\frac{z_{10}}{(1-\alpha)^{11}x^{10}} + \ldots\right]$$

$$z_6 = 231\,\alpha^6 - 1890\,\alpha^5 + 7105\,\alpha^4 - 16700\,\alpha^3 + 29049\,\alpha^2 - 44738\,\alpha + 92479$$

$$z_7 = 429\,\alpha^7 - 3927\,\alpha^6 + 16569\,\alpha^5 - 43435\,\alpha^4 + 81815\,\alpha^3 - 125205\,\alpha^2 + 181955\,\alpha - 370345$$

$$z_8 = 6435\,\alpha^8 - 65208\,\alpha^7 + 305844\,\alpha^6 - 890568\,\alpha^5 + 1840370\,\alpha^4 - 2978440\,\alpha^3 + 4186740\,\alpha^2 - 5874040\,\alpha + 11857475$$

$$z_9 = 12155\,\alpha^9 - 135135\,\alpha^8 + 698412\,\alpha^7 - 2244396\,\alpha^6 + 5093802\,\alpha^5 - 8893010\,\alpha^4 + 12934780\,\alpha^3 - 17184540\,\alpha^2 + \\ + 23605555\,\alpha - 47442055$$

$$z_{10} = 46189\,\alpha^{10} - 559130\,\alpha^9 + 3159585\,\alpha^8 - 11129976\,\alpha^7 + 27654858\,\alpha^6 - 52390044\,\alpha^5 + 80843770\,\alpha^4 - \\ - 109020920\,\alpha^3 + 139554825\,\alpha^2 - 189306330\,\alpha + 379582629$$

The first functions $w_k(\alpha)$:

x	$w_0(\alpha)$	$w_1(\alpha)$	$w_2(\alpha)$	$w_3(\alpha)$	$w_4(\alpha)$	$w_5(\alpha)$	$w_6(\alpha)$	$w_7(\alpha)$
0.1	1.1111E+00	7.5617E-01	1.3384E+00	3.7992E+00	1.4899E+01	7.4749E+01	4.5743E+02	3.3056E+03
0.2	1.2500E+00	9.3750E-01	1.8457E+00	5.8594E+00	2.5775E+01	1.4527E+02	9.9943E+02	8.1226E+03
0.3	1.4286E+00	1.1990E+00	2.6697E+00	9.6392E+00	4.8356E+01	3.1119E+02	2.4459E+03	2.2714E+04
0.4	1.6667E+00	1.5972E+00	4.1102E+00	1.7248E+01	1.0080E+02	7.5638E+02	6.9345E+03	7.5126E+04
0.5	2.0000E+00	2.2500E+00	6.8906E+00	3.4600E+01	2.4242E+02	2.1822E+03	2.4006E+04	3.1208E+05
0.6	2.5000E+00	3.4375E+00	1.3066E+01	8.1848E+01	7.1645E+02	8.0606E+03	1.1084E+05	1.8011E+06
0.7	3.3333E+00	5.9722E+00	3.0095E+01	2.5104E+02	2.9292E+03	4.3938E+04	8.0554E+05	1.7453E+07
0.8	5.0000E+00	1.3125E+01	9.8789E+01	1.2352E+03	2.1617E+04	4.8639E+05	1.3376E+07	4.3471E+08
0.9	1.0000E+01	5.1250E+01	7.6945E+02	1.9237E+04	6.7330E+05	3.0298E+07	1.6664E+09	1.0832E+11

With fixed $x >$ and $a = -\alpha \to -\infty$ holds

$$\int_0^x e^{-\alpha\,t}\,J_0(t)\,dt \;\sim\; \frac{1}{a} - \frac{1}{2a^3} + \frac{3}{8a^5} - \frac{5}{16a^7} + \frac{35}{128a^9} - \frac{63}{256a^{11}} + \ldots =$$

$$= \frac{1}{a} - \frac{0.5}{a^3} + \frac{0.375}{a^5} - \frac{0.3125}{a^7} + \frac{0.2734375}{a^9} - \frac{0.24609375}{a^{11}} + \ldots$$

and

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt \;\sim\; \frac{1}{a} + \frac{1}{2a^3} + \frac{3}{8a^5} + \frac{5}{16a^7} + \frac{35}{128a^9} + \frac{63}{256a^{11}} + \ldots\,.$$

d) Integrals:

Concerning the case '$a = \pm 1$ *and modified Bessel function*' see page 57.

$$\int e^{ax}\,J_1(x)\,dx = -e^{ax}\,J_0(x) + a\int e^{ax}\,J_0(x)\,dx$$

$$\int e^{ax}\,I_1(x)\,dx = e^{ax}\,I_0(x) - a\int e^{ax}\,I_0(x)\,dx$$

$$\int xe^{ax}J_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2+1}\,J_0(x) + \frac{x}{a^2+1}\,J_1(x)\right] - \frac{a}{a^2+1}\int e^{ax}\,J_0(x)\,dx$$

$$\int xe^{ax}I_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2-1}\,I_0(x) - \frac{x}{a^2-1}\,I_1(x)\right] - \frac{a}{a^2-1}\int e^{ax}I_0(x)\,dx$$

$$\int xe^{ax}J_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2+1}\,J_0(x) + \frac{ax}{1+a^2}\,J_1(x)\right] + \frac{1}{a^2+1}\int e^{ax}J_0(x)\,dx$$

$$\int xe^{ax}I_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2-1}\,I_0(x) + \frac{ax}{a^2-1}\,I_1(x)\right] + \frac{1}{a^2-1}\int e^{ax}I_0(x)\,dx$$

$$\int x^2e^{ax}J_0(x)\,dx = e^{ax}\left[\frac{a(a^2+1)x^2+(-2a^2+1)x}{(a^2+1)^2}J_0(x) + \frac{(a^2+1)x^2-3ax}{(a^2+1)^2}J_1(x)\right] + \frac{2a^2-1}{(a^2+1)^2}\int e^{ax}J_0(x)\,dx$$

$$\int x^2e^{ax}I_0(x)\,dx = e^{ax}\left[\frac{a(a^2-1)x^2-(2a^2+1)x}{(a^2-1)^2}\,I_0(x) + \frac{-(a^2-1)x^2+3ax}{(a^2-1)^2}\,I_1(x)\right] + \frac{2\,a^2+1}{(a^2-1)^2}\int e^{ax}I_0(x)\,dx$$

$$\int x^2e^{ax}J_1(x)\,dx = e^{ax}\left[\frac{-(a^2+1)x^2+3\,ax}{(a^2+1)^2}J_0(x) + \frac{a(a^2+1)x^2+(2-a^2)x}{(a^2+1)^2}J_1(x)\right] - \frac{3a}{(a^2+1)^2}\int e^{ax}J_0(x)\,dx$$

$$\int x^2e^{ax}I_1(x)\,dx = e^{ax}\left[\frac{-(a^2-1)x^2+3\,ax}{(a^2-1)^2}\,I_0(x) + \frac{a(a^2-1)x^2-(a^2+2)x}{(a^2-1)^2}I_1(x)\right] - \frac{3a}{(a^2-1)^2}\int e^{ax}I_0(x)\,dx$$

$$\int x^3 e^{ax} J_0(x)\,dx = e^{ax}\left[\frac{a\left(1+a^2\right)^2 x^3-\left(3\,a^2-2\right)\left(1+a^2\right)x^2+3\,a\left(2\,a^2-3\right)x}{(a^2+1)^3}J_0(x)+\right.$$

$$\left.+\frac{\left(1+a^2\right)^2 x^3-5\,a\left(1+a^2\right)x^2+(11\,a^2-4)x}{(a^2+1)^3}J_1(x)\right]-\frac{3a(2a^2-3)}{(a^2+1)^3}\int e^{ax}J_0(x)\,dx$$

$$\int x^3 e^{ax} I_0(x)\,dx = e^{ax}\left[\frac{a\left(a^2-1\right)^2 x^3+-\left(3\,a^2+2\right)\left(a^2-1\right)x^2+3\,a\left(2\,a^2+3\right)x}{(a^2-1)^3}I_0(x)+\right.$$

$$\left.+\frac{-\left(a^2-1\right)^2 x^3+5\,a\left(a^2-1\right)x^2-(11\,a^2+4)x}{(a^2-1)^3}I_1(x)\right]-\frac{3a\,(2a^2+3)}{(a^2-1)^3}\int e^{ax}I_0(x)\,dx$$

$$\int x^3 e^{ax} J_1(x)\,dx = e^{ax}\left[\frac{-\left(1+a^2\right)^2 x^3+5\,a\left(1+a^2\right)x^2-3\,\left(4\,a^2-1\right)x}{(a^2+1)^3}J_0(x)+\right.$$

$$\left.+\frac{a\left(1+a^2\right)^2 x^3-\left(2\,a^2-3\right)\left(1+a^2\right)x^2+a\left(2\,a^2-13\right)x}{(a^2+1)^3}J_1(x)\right]+\frac{12a^2-3}{(a^2+1)^3}\int e^{ax}J_0(x)\,dx$$

$$\int x^3 e^{ax} I_1(x)\,dx = e^{ax}\left[\frac{-\left(a^2-1\right)^2 x^3+5\,a\left(a^2-1\right)x^2-3(4a^2+1)x}{(a^2-1)^3}I_0(x)+\right.$$

$$\left.+\frac{a\left(a^2-1\right)^2 x^3-(2a^2+3)(a^2-1)x^2+a(2a^2+13)x}{(a^2-1)^3}I_1(x)\right]-\frac{3(4a^2+1)}{(a^2-1)^3}\int e^{ax}I_0(x)\,dx$$

Let

$$\int x^n e^{ax} J_\nu(x)\,dx = e^{ax}\left[\frac{P_n^{(\nu)}}{(a^2+1)^n}J_0(x)+\frac{Q_n^{(\nu)}}{(a^2+1)^n}J_1(x)\right]+\frac{R_n^{(\nu)}}{(a^2+1)^n}\int e^{ax}J_0(x)\,dx$$

and

$$\int x^n e^{ax} I_\nu(x)\,dx = e^{ax}\left[\frac{\mathfrak{P}_n^{(\nu)}}{(a^2-1)^n}I_0(x)+\frac{\mathfrak{Q}_n^{(\nu)}}{(a^2-1)^n}I_1(x)\right]+\frac{\mathfrak{R}_n^{(\nu)}}{(a^2-1)^n}\int e^{ax}I_0(x)\,dx\;,$$

then holds

$$P_4^{(0)} = a\left(1+a^2\right)^3 x^4-\left(4\,a^2-3\right)\left(1+a^2\right)^2 x^3+a\left(12\,a^2-23\right)\left(1+a^2\right)x^2+(-24\,a^4+72\,a^2-9)x$$

$$Q_4^{(0)} = \left(1+a^2\right)^3 x^4-7\,a\left(1+a^2\right)^2 x^3+\left(26\,a^2-9\right)\left(1+a^2\right)x^2-5\,a\left(10\,a^2-11\right)x$$

$$R_4^{(0)} = 24\,a^4-72\,a^2+9$$

$$\mathfrak{P}_4^{(0)} = a\left(a^2-1\right)^3 x^4-\left(4\,a^2+3\right)\left(a^2-1\right)^2 x^3+a\left(12\,a^2+23\right)\left(a^2-1\right)x^2-(24\,a^4+72\,a^2+9)x$$

$$\mathfrak{Q}_4^{(0)} = -\left(a^2-1\right)^3 x^4+7a\,\left(a^2-1\right)^2 x^3-\left(26\,a^2+9\right)\left(a^2-1\right)x^2+5\,a\left(10\,a^2+11\right)x$$

$$\mathfrak{R}_4^{(0)} = 24\,a^4+72\,a^2+9$$

$$P_4^{(1)} = -\left(1+a^2\right)^3 x^4+7\,a\left(1+a^2\right)^2 x^3-\left(27\,a^2-8\right)\left(1+a^2\right)x^2+15\,a\left(4\,a^2-3\right)x$$

$$Q_4^{(1)} = a\left(1+a^2\right)^3 x^4-\left(3\,a^2-4\right)\left(1+a^2\right)^2 x^3+a\left(6\,a^2-29\right)\left(1+a^2\right)x^2+(-6\,a^4+83\,a^2-16)x$$

$$R_4^{(1)} = -15a\,\left(4\,a^2-3\right)$$

$$\mathfrak{P}_4^{(1)} = -\left(a^2-1\right)^3 x^4+7\,a\left(a^2-1\right)^2 x^3-\left(27\,a^2+8\right)\left(a^2-1\right)x^2+15\,a\left(4\,a^2+3\right)x$$

$$\mathfrak{Q}_4^{(1)} = a\left(a^2-1\right)^3 x^4-\left(4+3\,a^2\right)\left(a^2-1\right)^2 x^3+a\left(6\,a^2+29\right)\left(a^2-1\right)x^2-(6\,a^4+83\,a^2+16)x$$

$$\mathfrak{R}_4^{(1)} = -15a\ \left(4\,a^2+3\right)$$

$$P_5^{(0)} = a\left(1+a^2\right)^4 x^5 - \left(5\,a^2-4\right)\left(a^2+1\right)^3 x^4 + a\left(20\,a^2-43\right)\left(a^2+1\right)^2 x^3 -$$
$$-\left(a^2+1\right)\left(60\,a^4-223\,a^2+32\right)x^2 + 15\ \left(8\,a^4-40\,a^2+15\right)ax$$
$$Q_5^{(0)} = \left(a^2+1\right)^4 x^5 - 9\,a\left(a^2+1\right)^3 x^4 + \left(47\,a^2-16\right)\left(a^2+1\right)^2 x^3 -$$
$$-7\,a\left(22\,a^2-23\right)\left(a^2+1\right)x^2 + (274\,a^4-607\,a^2+64)x$$
$$R_5^{(0)} = -15a\ \left(8\,a^4-40\,a^2+15\right)$$

$$\mathfrak{P}_5^{(0)} = a\left(a^2-1\right)^4 x^5 - \left(4+5\,a^2\right)\left(a^2-1\right)^3 x^4 + a\left(20\,a^2+43\right)\left(a^2-1\right)^2 x^3 -$$
$$-\left(60\,a^4+223\,a^2+32\right)\left(a^2-1\right)x^2 + 15\,a\left(8\,a^4+40\,a^2+15\right)x$$
$$\mathfrak{Q}_5^{(0)} = -\left(a^2-1\right)^4 x^5 + 9\,a\left(a^2-1\right)^3 x^4 - \left(16+47\,a^2\right)\left(a^2-1\right)^2 x^3 + 7\,a\left(22\,a^2+23\right)\left(a^2-1\right)x^2 -$$
$$-(274\,a^4+607\,a^2+64)x$$
$$\mathfrak{R}_5^{(0)} = -15a\ \left(8\,a^4+40\,a^2+15\right)$$

$$P_5^{(1)} = -\left(a^2+1\right)^4 x^5 + 9\ \left(a^2+1\right)^3 ax^4 - 3\ \left(16\,a^2-5\right)\left(a^2+1\right)^2 x^3 + 21\,a\left(8\,a^2-7\right)\left(a^2+1\right)x^2 +$$
$$+(-360\,a^4+540\,a^2-45)x$$
$$Q_5^{(1)} = a\left(a^2+1\right)^4 x^5 - \left(-5+4\,a^2\right)\left(a^2+1\right)^3 x^4 + 3\,a\left(4\,a^2-17\right)\left(a^2+1\right)^2 x^3 -$$
$$-3\ \left(a^2+1\right)\left(8\,a^4-82\,a^2+15\right)x^2 + 3\,a\left(8\,a^4-194\,a^2+113\right)x$$
$$R_5^{(1)} = 360\,a^4-540\,a^2+45$$

$$\mathfrak{P}_5^{(1)} = -\left(a^2-1\right)^4 x^5 + 9\,a\left(a^2-1\right)^3 x^4 - 3\ \left(16\,a^2+5\right)\left(a^2-1\right)^2 x^3 + 21\,a\left(8\,a^2+7\right)\left(a^2-1\right)x^2 -$$
$$-(360\,a^4+540\,a^2+45)x$$
$$\mathfrak{Q}_5^{(1)} = a\left(a^2-1\right)^4 x^5 - \left(4\,a^2+5\right)\left(a^2-1\right)^3 x^4 + 3a\ \left(4\,a^2+17\right)\left(a^2-1\right)^2 x^3 -$$
$$-3\ \left(a^2-1\right)\left(8\,a^4+82\,a^2+15\right)x^2 + 3\,a\left(8\,a^4+194\,a^2+113\right)x$$
$$\mathfrak{R}_5^{(1)} = 360\,a^4+540\,a^2+45$$

$$P_6^{(0)} = a\left(a^2+1\right)^5 x^6 - \left(6\,a^2-5\right)\left(a^2+1\right)^4 x^5 + 3\,a\left(10\,a^2-23\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\ \left(40\,a^4-166\,a^2+25\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(40\,a^4-242\,a^2+103\right)x^2 +$$
$$+(-720\,a^6+5400\,a^4-4050\,a^2+225)x$$
$$Q_6^{(0)} = \left(a^2+1\right)^5 x^6 - 11\,a\left(a^2+1\right)^4 x^5 + \left(74\,a^2-25\right)\left(a^2+1\right)^3 x^4 - 9\,a\left(38\,a^2-39\right)\left(a^2+1\right)^2 x^3 +$$
$$+9\ \left(a^2+1\right)\left(116\,a^4-244\,a^2+25\right)x^2 - 63\ \left(28\,a^4-104\,a^2+33\right)ax$$
$$R_6^{(0)} = 720\,a^6-5400\,a^4+4050\,a^2-225$$

$$\mathfrak{P}_6^{(0)} = a\left(a^2-1\right)^5 x^6 - \left(6\,a^2+5\right)\left(a^2-1\right)^4 x^5 + 3\,a\left(10\,a^2+23\right)\left(a^2-1\right)^3 x^4 -$$
$$-3\ \left(40\,a^4+166\,a^2+25\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(40\,a^4+242\,a^2+103\right)\left(a^2-1\right)x^2 - (720\,a^6+5400\,a^4+4050\,a^2+225)x$$
$$\mathfrak{Q}_6^{(0)} = -\left(a^2-1\right)^5 x^6 + 11a\ \left(a^2-1\right)^4 x^5 - \left(74\,a^2+25\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(38\,a^2+39\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\ \left(116\,a^4+244\,a^2+25\right)\left(a^2-1\right)x^2 + 63\,a\left(28\,a^4+104\,a^2+33\right)x$$

$$\mathfrak{R}_6^{(0)} = 720\,a^6 + 5400\,a^4 + 4050\,a^2 + 225$$

$$P_6^{(1)} = -\left(a^2+1\right)^5 x^6 + 11\,a\left(a^2+1\right)^4 x^5 - 3\,\left(25\,a^2-8\right)\left(a^2+1\right)^3 x^4 + 9\,a\left(40\,a^2-37\right)\left(a^2+1\right)^2 x^3 -$$

$$-3\,\left(a^2+1\right)\left(400\,a^4-691\,a^2+64\right)x^2 + 315\,\left(8\,a^4-20\,a^2+5\right)ax$$

$$Q_6^{(1)} = a\left(a^2+1\right)^5 x^6 - \left(5\,a^2-6\right)\left(a^2+1\right)^4 x^5 + a\left(20\,a^2-79\right)\left(a^2+1\right)^3 x^4 -$$

$$-3\,\left(20\,a^4-179\,a^2+32\right)\left(a^2+1\right)^2 x^3 + 3\,a\left(a^2+1\right)\left(40\,a^4-718\,a^2+397\right)x^2 +$$

$$+(-120\,a^6+4554\,a^4-5337\,a^2+384)x$$

$$R_6^{(1)} = -315\,\left(8\,a^4-20\,a^2+5\right)a$$

$$\mathfrak{P}_6^{(1)} = -\left(a^2-1\right)^5 x^6 + 11\,a\left(a^2-1\right)^4 x^5 - 3\,\left(25\,a^2+8\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(40\,a^2+37\right)\left(a^2-1\right)^2 x^3 -$$

$$-3\,\left(400\,a^4+691\,a^2+64\right)\left(a^2-1\right)x^2 + 315\,a\left(8\,a^4+20\,a^2+5\right)x$$

$$\mathfrak{Q}_6^{(1)} = a\left(a^2-1\right)^5 x^6 - \left(5\,a^2+6\right)\left(a^2-1\right)^4 x^5 + a\left(20\,a^2+79\right)\left(a^2-1\right)^3 x^4 -$$

$$-3\,\left(20\,a^4+179\,a^2+32\right)\left(a^2-1\right)^2 x^3 + 3\,a\left(a^2-1\right)\left(40\,a^4+718\,a^2+397\right)x^2 -$$

$$-(120\,a^6+4554\,a^4+5337\,a^2+384)x$$

$$\mathfrak{R}_6^{(1)} = -315\,\left(8\,a^4+20\,a^2+5\right)a$$

$$P_7^{(0)} = a\left(1+a^2\right)^6 x^7 - \left(7\,a^2-6\right)\left(a^2+1\right)^5 x^6 + a\left(42\,a^2-101\right)\left(a^2+1\right)^4 x^5 -$$

$$-3\,\left(70\,a^4-311\,a^2+48\right)\left(a^2+1\right)^3 x^4 + 3\,a\left(280\,a^4-1882\,a^2+841\right)\left(a^2+1\right)^2 x^3 +$$

$$-9\,\left(a^2+1\right)\left(280\,a^6-2494\,a^4+2103\,a^2-128\right)x^2 + 315\,\left(16\,a^6-168\,a^4+210\,a^2-35\right)ax$$

$$Q_7^{(0)} = \left(a^2+1\right)^6 x^7 - 13\,a\left(a^2+1\right)^5 x^6 + \left(107\,a^2-36\right)\left(a^2+1\right)^4 x^5 - 11\,\left(58\,a^2-59\right)\left(a^2+1\right)^3 ax^4 +$$

$$+9\,\left(306\,a^4-631\,a^2+64\right)\left(a^2+1\right)^2 x^3 - 9\,\left(a^2+1\right)\left(892\,a^4-3144\,a^2+969\right)ax^2 +$$

$$+(13068\,a^6-73188\,a^4+46575\,a^2-2304)x$$

$$R_7^{(0)} = -315\,\left(16\,a^6-168\,a^4+210\,a^2-35\right)a$$

$$\mathfrak{P}_7^{(0)} = a\left(a^2-1\right)^6 x^7 - \left(7\,a^2+6\right)\left(a^2-1\right)^5 x^6 + a\left(101+42\,a^2\right)\left(a^2-1\right)^4 x^5 -$$

$$-3\,\left(70\,a^4+311\,a^2+48\right)\left(a^2-1\right)^3 x^4 + 3\,a\left(280\,a^4+1882\,a^2+841\right)\left(a^2-1\right)^2 x^3 -$$

$$-9\,\left(280\,a^6+2494\,a^4+2103\,a^2+128\right)\left(a^2-1\right)x^2 + 315\,a\left(16\,a^6+168\,a^4+210\,a^2+35\right)x$$

$$\mathfrak{Q}_7^{(0)} = -\left(a^2-1\right)^6 x^7 + 13\,a\left(a^2-1\right)^5 x^6 - \left(107\,a^2+36\right)\left(a^2-1\right)^4 x^5 + 11\,a\left(59+58\,a^2\right)\left(a^2-1\right)^3 x^4 -$$

$$-9\,\left(306\,a^4+631\,a^2+64\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(892\,a^4+3144\,a^2+969\right)\left(a^2-1\right)x^2 -$$

$$-(13068\,a^6+73188\,a^4+46575\,a^2+2304)x$$

$$\mathfrak{R}_7^{(0)} = -315a\,\left(16\,a^6+168\,a^4+210\,a^2+35\right)$$

$$P_7^{(1)} = -\left(1+a^2\right)^6 x^7 + 13\,a\left(a^2+1\right)^5 x^6 - \left(108\,a^2-35\right)\left(a^2+1\right)^4 x^5 + 33\,a\left(20\,a^2-19\right)\left(a^2+1\right)^3 x^4 -$$

$$-3\,\left(1000\,a^4-1828\,a^2+175\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(1080\,a^4-3076\,a^2+849\right)x^2 +$$

$$+(-20160\,a^6+75600\,a^4-37800\,a^2+1575)x$$

$$Q_7^{(1)} = a\left(a^2+1\right)^6 x^7 - \left(6\,a^2-7\right)\left(a^2+1\right)^5 x^6 + a\left(30\,a^2-113\right)\left(a^2+1\right)^4 x^5 -$$

$$-\left(120\,a^4-992\,a^2+175\right)\left(a^2+1\right)^3x^4+9\,a\left(40\,a^4-624\,a^2+337\right)\left(a^2+1\right)^2x^3-$$
$$-9\,\left(a^2+1\right)\left(80\,a^6-2248\,a^4+2502\,a^2-175\right)x^2+9\,a\left(80\,a^6-4408\,a^4+8654\,a^2-1873\right)x$$
$$R_7^{(1)}=20160\,a^6-75600\,a^4+37800\,a^2-1575$$

$$\mathfrak{P}_7^{(1)}=-\left(a^2-1\right)^6x^7+13\,a\left(a^2-1\right)^5x^6-\left(108\,a^2+35\right)\left(a^2-1\right)^4x^5+$$
$$+33\,a\left(20\,a^2+19\right)\left(a^2-1\right)^3x^4-3\,\left(1000\,a^4+1828\,a^2+175\right)\left(a^2-1\right)^2x^3+$$
$$+9\,a\left(1080\,a^4+3076\,a^2+849\right)\left(a^2-1\right)-(20160\,a^6+75600\,a^4+37800\,a^2+1575)x$$
$$\mathfrak{Q}_7^{(1)}=a\left(a^2-1\right)^6x^7-\left(6\,a^2+7\right)\left(a^2-1\right)^5x^6+a\left(30\,a^2+113\right)\left(a^2-1\right)^4x^5-$$
$$-\left(120\,a^4+992\,a^2+175\right)\left(a^2-1\right)^3x^4+9\,a\left(40\,a^4+624\,a^2+337\right)\left(a^2-1\right)^2x^3-$$
$$-9\,\left(80\,a^6+2248\,a^4+2502\,a^2+175\right)\left(a^2-1\right)x^2+9\,a\left(80\,a^6+4408\,a^4+8654\,a^2+1873\right)x$$
$$\mathfrak{R}_7^{(1)}=20160\,a^6+75600\,a^4+37800\,a^2+1575$$

$$P_8^{(0)}=a\left(a^2+1\right)^7x^8-\left(8\,a^2-7\right)\left(a^2+1\right)^6x^7+a\left(56\,a^2-139\right)\left(a^2+1\right)^5x^6-$$
$$-\left(336\,a^4-1564\,a^2+245\right)\left(a^2+1\right)^4x^5+3\,a\left(560\,a^4-4028\,a^2+1847\right)\left(a^2+1\right)^3x^4-$$
$$-3\,\left(2240\,a^6-22056\,a^4+19524\,a^2-1225\right)\left(a^2+1\right)^2x^3+$$
$$+9\,a\left(a^2+1\right)\left(2240\,a^6-27512\,a^4+38356\,a^2-6967\right)x^2+$$
$$+(-40320\,a^8+564480\,a^6-1058400\,a^4+352800\,a^2-11025)x$$
$$Q_8^{(0)}=\left(a^2+1\right)^7x^8-15\,a\left(a^2+1\right)^6x^7+\left(146\,a^2-49\right)\left(a^2+1\right)^5x^6-13\,a\left(82\,a^2-83\right)\left(a^2+1\right)^4x^5+$$
$$+\left(5944\,a^4-12136\,a^2+1225\right)\left(a^2+1\right)^3x^4-99\,a\left(248\,a^4-856\,a^2+261\right)\left(a^2+1\right)^2x^3+$$
$$+9\,\left(a^2+1\right)\left(7696\,a^6-40888\,a^4+25266\,a^2-1225\right)x^2-9\,a\left(12176\,a^6-95912\,a^4+101978\,a^2-15159\right)x$$
$$R_8^{(0)}=40320\,a^8-564480\,a^6+1058400\,a^4-352800\,a^2+11025$$

$$\mathfrak{P}_8^{(0)}=a\left(a^2-1\right)^7x^8-\left(8\,a^2+7\right)\left(a^2-1\right)^6x^7+a\left(56\,a^2+139\right)\left(a^2-1\right)^5x^6-$$
$$-\left(336\,a^4+1564\,a^2+245\right)\left(a^2-1\right)^4x^5+3\,a\left(560\,a^4+4028\,a^2+1847\right)\left(a^2-1\right)^3x^4-$$
$$-3\,\left(2240\,a^6+22056\,a^4+19524\,a^2+1225\right)\left(a^2-1\right)^2x^3+$$
$$+9\,a\left(2240\,a^6+27512\,a^4+38356\,a^2+6967\right)\left(a^2-1\right)x^2-$$
$$-(40320\,a^8+564480\,a^6+1058400\,a^4+352800\,a^2+11025)x$$
$$\mathfrak{Q}_8^{(0)}=-\left(a^2-1\right)^7x^8+15\,a\left(a^2-1\right)^6x^7-\left(146\,a^2+49\right)\left(a^2-1\right)^5x^6+13\,a\left(82\,a^2+83\right)\left(a^2-1\right)^4x^5-$$
$$-\left(5944\,a^4+12136\,a^2+1225\right)\left(a^2-1\right)^3x^4+99\,a\left(248\,a^4+856\,a^2+261\right)\left(a^2-1\right)^2x^3-$$
$$-9\,\left(7696\,a^6+40888\,a^4+25266\,a^2+1225\right)\left(a^2-1\right)x^2+$$
$$+9a\,\left(12176\,a^6+95912\,a^4+101978\,a^2+15159\right)x$$
$$\mathfrak{R}_8^{(0)}=40320\,a^8+564480\,a^6+1058400\,a^4+352800\,a^2+11025$$

$$P_8^{(1)}=-\left(a^2+1\right)^7x^8+15\,a\left(a^2+1\right)^6x^7-3\,\left(49\,a^2-16\right)\left(a^2+1\right)^5x^6+39\,a\left(28\,a^2-27\right)\left(a^2+1\right)^4x^5-$$
$$-9\,\left(700\,a^4-1317\,a^2+128\right)\left(a^2+1\right)^3x^4+99\,a\left(280\,a^4-844\,a^2+241\right)\left(a^2+1\right)^2x^3-$$
$$-9\,\left(a^2+1\right)\left(9800\,a^6-41484\,a^4+22767\,a^2-1024\right)x^2+2835\,a\left(64\,a^6-336\,a^4+280\,a^2-35\right)x$$

$$Q_8^{(1)} = a\left(a^2+1\right)^7 x^8 - \left(7\,a^2-8\right)\left(a^2+1\right)^6 x^7 + 3\,a\left(14\,a^2-51\right)\left(a^2+1\right)^5 x^6 -$$
$$-3\,\left(70\,a^4-549\,a^2+96\right)\left(a^2+1\right)^4 x^5 + 3\,a\left(280\,a^4-4016\,a^2+2139\right)\left(a^2+1\right)^3 x^4 -$$
$$-9\,\left(280\,a^6-6816\,a^4+7407\,a^2-512\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(560\,a^6-22872\,a^4+42666\,a^2-8977\right)x^2 +$$
$$+(-5040\,a^8+382248\,a^6-1130706\,a^4+490599\,a^2-18432)x$$
$$R_8^{(1)} = -2835a\,\left(64\,a^6-336\,a^4+280\,a^2-35\right)$$

$$\mathfrak{P}_8^{(1)} = -\left(a^2-1\right)^7 x^8 + 15\,a\left(a^2-1\right)^6 x^7 - 3\,\left(49\,a^2+16\right)\left(a^2-1\right)^5 x^6 + 39\,a\left(28\,a^2+27\right)\left(a^2-1\right)^4 x^5 -$$
$$-9\,\left(700\,a^4+1317\,a^2+128\right)\left(a^2-1\right)^3 x^4 + 99\,a\left(280\,a^4+844\,a^2+241\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\,\left(9800\,a^6+41484\,a^4+22767\,a^2+1024\right)\left(a^2-1\right)x^2 + 2835\,a\left(64\,a^6+336\,a^4+280\,a^2+35\right)x$$
$$\mathfrak{Q}_8^{(1)} = a\left(a^2-1\right)^7 x^8 - \left(7\,a^2+8\right)\left(a^2-1\right)^6 x^7 + 3\,a\left(14\,a^2+51\right)\left(a^2-1\right)^5 x^6 -$$
$$-3\,\left(70\,a^4+549\,a^2+96\right)\left(a^2-1\right)^4 x^5 + 3\,a\left(280\,a^4+4016\,a^2+2139\right)\left(a^2-1\right)^3 x^4 -$$
$$-9\,\left(280\,a^6+6816\,a^4+7407\,a^2+512\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(560\,a^6+22872\,a^4+42666\,a^2+8977\right)\left(a^2-1\right)x^2 -$$
$$-(5040\,a^8+382248\,a^6+1130706\,a^4+490599\,a^2+18432)x$$
$$\mathfrak{R}_8^{(1)} = -2835a\,\left(64\,a^6+336\,a^4+280\,a^2+35\right)$$

Recurrence formulas: Let

$$\mathbf{J}_n^{(\nu)} = \int x^n e^{ax} J_\nu(x)\,dx\,, \qquad \mathbf{I}_n^{(\nu)} = \int x^n e^{ax} I_\nu(x)\,dx\,,$$

then holds

$$\mathbf{J}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aJ_0(x)+J_1(x)] - (n+1)\,a\,\mathbf{J}_n^{(0)} - n\mathbf{J}_n^{(1)}}{a^2+1}\,,$$
$$\mathbf{J}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aJ_1(x)-J_0(x)] + (n+1)\,\mathbf{J}_n^{(0)} - a\,n\,\mathbf{J}_n^{(1)}}{a^2+1}$$

and

$$\mathbf{I}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aI_0(x)-I_1(x)] - (n+1)\,a\,\mathbf{I}_n^{(0)} + n\mathbf{I}_n^{(1)}}{a^2-1}\,,$$
$$\mathbf{I}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aI_1(x)-I_0(x)] + (n+1)\,\mathbf{I}_n^{(0)} - a\,n\,\mathbf{I}_n^{(1)}}{a^2-1}\,.$$

e) Special Cases:

$$\int x^2 \exp\left(\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[\sqrt{2}x\,J_0(x) + 2(x-\sqrt{2})\,J_1(x)\right]\exp\left(\frac{x}{\sqrt{2}}\right)$$
$$\int x^2 \exp\left(-\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[-\sqrt{2}x\,J_0(x) + 2(x+\sqrt{2})\,J_1(x)\right]\exp\left(-\frac{x}{\sqrt{2}}\right)$$
$$\int x^3 \exp\left(\sqrt{\frac{3}{2}}\,x\right) J_0(x)\,dx = \frac{x}{5}\left[(\sqrt{6}x^2-2\,x)\,J_0(x) + (x^2-\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}\,x\right)$$
$$\int x^3 \exp\left(-\sqrt{\frac{3}{2}}\,x\right) J_0(x)\,dx = \frac{x}{5}\left[-(\sqrt{6}x^2+2\,x)\,J_0(x) + (x^2+\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}\,x\right)$$
$$\int x^3 \exp\left(\frac{x}{2}\right) J_0(x)\,dx = \frac{2x}{5}\left[-(2x^2-4x)\,J_0(x) + (x^2+4x-8)\,J_1(x)\right]\exp\left(\frac{x}{2}\right)$$

$$\int x^3 \exp\left(-\frac{x}{2}\right) J_0(x)\,dx = -\frac{2x}{5}\left[(2x^2+4x)\,J_0(x) + (x^2-4x-8)\,J_1(x)\right]\,\exp\left(-\frac{x}{2}\right)$$

$$w \quad = \quad \pm\,\frac{\sqrt{6\pm\sqrt{30}}}{2} \quad = \quad \pm\,\{0.361\,516;\ 1.693\,903\}$$

$$\int x^4\,e^{wx}\,J_0(x)\,dx = \frac{x}{5w(4w^2-3)}\left\{\left[(12\,w^2-3)x^3 - 24wx^2 + 24x\right]\,J_0(x) + \right.$$
$$\left. + \left[(8\,w^3-12\,w)x^3 + (-24\,w^2+12)x^2 + 48wx - 48\right]\,J_1(x)\right\}\,e^{wx}$$

$$w \quad = \quad \pm\,\frac{\sqrt{3}}{2} \quad = \quad \pm\,0.866\,025$$

$$\int x^4\,e^{wx}\,J_1(x)\,dx = \frac{x}{8\,w^4-24\,w^2+3}\left\{\left[(12\,w^2-3)x^3 - 24wx^2 + 24x\right]\,J_0(x) + \right.$$
$$\left. + \left[(8\,w^3-12\,w)x^3 + (-24\,w^2+12)x^2 + 48wx - 48\right]\,J_1(x)\right\}\,e^{wx}$$

$$w \quad = \quad \pm\,\frac{\sqrt{10\pm\sqrt{70}}}{2} \quad = \quad \pm\,\{0.639\,023;\ 2.142\,814\}$$

$$\int x^5\,e^{wx}\,J_0(x)\,dx = \frac{x}{3(8\,w^4-12\,w^2+1)}\left\{\left[(16\,w^3-12\,w)x^4 + (-48\,w^2+12)x^3 + 96wx - 96x\right]\,J_0(x) + \right.$$
$$\left. + \left[(8\,w^4-24\,w^2+3)x^4 + (-32\,w^3+48\,w)x^3 + (96\,w^2-48)x^2 - 192wx + 192\right]\,J_1(x)\right\}\,e^{wx}$$

$$w \quad = \quad \pm\,\frac{\sqrt{3\pm\sqrt{7}}}{2} \quad = \quad \pm\,\{0.297\,594;\ 1.188\,039\}$$

$$\int x^5\,e^{wx}\,J_1(x)\,dx = \frac{x}{w(8\,w^4-40\,w^2+15)}\left\{\left[(16\,w^3-12\,w)x^4 + (-48\,w^2+12)x^3 + 96wx^2 - 96x\right]\,J_0(x) + \right.$$
$$\left. + \left[(8\,w^4-24\,w^2+3)x^4 + (-32\,w^3+48\,w)x^3 + (96\,w^2-48)x^2 - 192wx + 192\right]\,J_1(x)\right\}\,e^{wx}$$

$$16w^6 - 120w^4 + 90w^2 - 5 = 0 \quad\Longrightarrow\quad w \in \{\pm\,0.245\,717\,164,\ \pm\,0.881\,375\,831,\ \pm\,2.581\,239\,958\}$$

$$\int x^6\,e^{wx}\,J_0(x)\,dx = \frac{x}{7w(8w^4-20w^2+5)}\cdot$$
$$\cdot\left\{\left[(40\,w^4-60\,w^2+5)x^5 + (-160\,w^3+120\,w)x^4 + (480\,w^2-120)x^3 - 960wx^2 + 960x\right]\,J_0(x) + \right.$$
$$+ \left[(16\,w^5-80\,w^3+30\,w)x^5 + (-80\,w^4+240\,w^2-30)x^4 + (320\,w^3-480\,w)x^3 + \right.$$
$$\left.\left. + (-960\,w^2+480)x^2 + 1920wx - 1920\right]\,J_1(x)\right\}\,e^{wx}$$

$$w \quad = \quad \pm\,\frac{\sqrt{5\pm\sqrt{15}}}{2} \quad = \quad \pm\,\{0.530\,805,\ 1.489\,378\}$$

$$\int x^6\,e^{wx}\,J_1(x)\,dx = \frac{x}{16w^6-120w^4+90w^2-5}\left\{\left[(40\,w^4-60\,w^2+5)x^5 + (-160\,w^3+120\,w)x^4 + \right.\right.$$
$$\left. +\ (480\,w^2-120)x^3 - 960wx^2 + 960w\right]\,J_0(x) +$$
$$\left[(16\,w^5-80\,w^3+30\,w)x^5 + (-80\,w^4+240\,w^2-30)x^4 + (320\,w^3-480\,w)x^3\right.$$
$$\left.\left. +\ (-960\,w^2+480)x^2 + 1920x - 1920\right]\,J_1(x)\right\}\,e^{wx}$$

$$16w^6 - 168w^4 + 210w^2 - 35 = 0 \quad \Rightarrow \quad w \in \{\pm 0.444\,060\,144,\ \pm 1.105\,247\,947,\ \pm 3.013\,509\,178\}$$

$$\int x^7\, e^{wx}\, J_0(x)\, dx = \frac{x}{64w^6 - 240w^4 + 120w^2 - 5} \Big\{ \big[(48\, w^5 - 120\, w^3 + 30\, w) x^6 + (-240\, w^4 + 360\, w^2 - 30) x^5 +$$

$$+ (960\, w^3 - 720\, w) x^4 + (-2880\, w^2 + 720) x^3 + 5760 w x^2 - 5760 x \big]\ J_0(x) +$$

$$+ \big[(16\, w^6 - 120\, w^4 + 90\, w^2 - 5) x^6 + (-96\, w^5 + 480\, w^3 - 180\, w) x^5 + (480\, w^4 - 1440\, w^2 + 180) x^4 +$$

$$+\ (-1920\, w^3 + 2880\, w) x^3 + (5760\, w^2 - 2880) x^2 - 11520 w x + 11520 \big]\ J_1(x) \Big\}\ e^{wx}$$

$$64w^6 - 240w^4 + 120w^2 - 5 = 0 \quad \Rightarrow \quad w \in \{\pm 0.214\,039\,849,\ \pm 0.733\,975\,352,\ \pm 1.779\,175\,968\}$$

$$\int x^7\, e^{wx}\, J_1(x)\, dx = \frac{x}{w(16w^6 - 168w^4 + 210w^2 - 35)} \cdot$$

$$\cdot \Big\{ \big[(48\, w^5 - 120\, w^3 + 30\, w) x^6 + (-240\, w^4 + 360\, w^2 - 30) x^5 + (960\, w^3 - 720\, w) x^4$$

$$+ (-2880\, w^2 + 720) x^3 + 5760 w x^2 - 5760 \big]\ J_0(x) +$$

$$+ \big[(16\, w^6 - 120\, w^4 + 90\, w^2 - 5) x^6 + (-96\, w^5 + 480\, w^3 - 180\, w) x^5 + (480\, w^4 - 1440\, w^2 + 180) x^4 +$$

$$+\ (-1920\, w^3 + 2880\, w) x^3 + (5760\, w^2 - 2880) x^2 - 11520 w x + 11520 \big]\ J_1(x) \Big\}\ e^{wx}$$

1.2.6. Integrals of the type $\int x^{-n-1/2} \sin x \, J_\nu(x)\, dx$ and $\int x^{-n-1/2} \cos x \, J_\nu(x)\, dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p = 1, 2$.

The next four integrals are special cases of the general integral 1.8.2.7 from [4].

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{3/2}} = \frac{(4x\cos x - 2\sin x)J_0(x) + 4x\sin x\, J_1(x)}{\sqrt{x}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{3/2}} = \frac{(-4x\sin x - 2\cos x)J_0(x) + 4x\cos x\, J_1(x)}{\sqrt{x}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{3/2}} = \frac{4x\sin x\, J_0(x) - (4x\cos x + 2\sin x)\, J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{3/2}} = \frac{4x\cos\, xJ_0(x) + (4x\sin x - 2\cos x)J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{5/2}} = \frac{[(-32\,x^2 - 6)\sin x - 12\,x\cos x]\, J_0(x) + [4\,x\sin x + 32\,x^2\cos x]\, J_1(x)}{9\,x^{3/2}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{5/2}} = \frac{[12\,x\sin x - (32\,x^2 + 6)\cos x]\, J_0(x) + [-32\,x^2\sin x + 4\,x\cos x]\, J_1(x)}{9\,x^{3/2}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{5/2}} = \frac{[-12\,x\sin x + 32\,x^2\cos x]\, J_0(x) + [(32\,x^2 - 6)\sin x - 4\,x\cos x]\, J_1(x)}{15\,x^{3/2}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{5/2}} = \frac{[-32\,x^2\sin x - 12\,x\cos x]\, J_0(x) + [4\,x\sin x + (32\,x^2 - 6)\cos x]\, J_1(x)}{15\,x^{3/2}}$$

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(192\,x^2 - 90)\sin x + (-512\,x^3 - 60\,x)\cos x]\, J_0(x) + [(-512\,x^3 + 36\,x)\sin x + 64\,x^2\cos x]\, J_1(x)}{225\,x^{5/2}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(512\,x^3 + 60\,x)\sin x + (192\,x^2 - 90)\cos x]\, J_0(x) + [-64\,x^2\sin x + (-512\,x^3 + 36\,x)\cos x]\, J_1(x)}{225\,x^{5/2}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(-512\,x^3 - 60\,x)\sin x - 192\,x^2\cos x]\, J_0(x) + [(64\,x^2 - 90)\sin x + (512\,x^3 - 36\,x)\cos x]\, J_1(x)}{315\,x^{5/2}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{7/2}} =$$

$$= \frac{[192\,x^2\sin x + (-512\,x^3 - 60\,x)\cos x]\, J_0(x) + [(-512\,x^3 + 36\,x)\sin x + (64\,x^2 - 90)\cos x]\, J_1(x)}{315\,x^{5/2}}$$

Let

$$\int \frac{\sin x\, J_\nu(x)\, dx}{x^{n+1/2}} = \frac{[P_n^{(s,\nu)}(x)\, \sin x + Q_n^{(s,\nu)}(x)\cos x]\, J_0(x) + [R_n^{(s,\nu)}(x)\, \sin x + S_n^{(s,\nu)}(x)\, \cos x]\, J_1(x)}{N_n^{(s,\nu)}\, x^{n-1/2}}$$

and

$$\int \frac{\cos x\, J_\nu(x)\, dx}{x^{n+1/2}} = \frac{[P_n^{(c,\nu)}(x)\, \sin x + Q_n^{(c,\nu)}(x)\cos x]\, J_0(x) + [R_n^{(c,\nu)}(x)\, \sin x + S_n^{(c,\nu)}(x)\, \cos x]\, J_1(x)}{N_n^{(c,\nu)}\, x^{n-1/2}}\ ,$$

then holds

$$P_4^{(s,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\;,\quad Q_4^{(s,0)}(x) = 1536\,x^3 - 420\,x\;,\quad R_4^{(s,0)}(x) = -512\,x^3 + 300\,x\;,$$

$$S_4^{(s,0)}(x) = -4096\,x^4 + 288\,x^2\;,\quad N_4^{(s,0)} = 3675$$

$$P_4^{(c,0)}(x) = -1536\,x^3 + 420\,x\;,\quad Q_4^{(c,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\;,\quad R_4^{(c,0)}(x) = 4096\,x^4 - 288\,x^2\;,$$

$$S_4^{(c,0)}(x) = -512\,x^3 + 300\,x\;,\quad N_4^{(c,0)} = 3675$$

$$P_4^{(s,1)}(x) = 1536\,x^3 - 420\,x\;,\quad Q_4^{(s,1)}(x) = -4096\,x^4 - 480\,x^2\;,\quad R_4^{(s,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\;,$$

$$S_4^{(s,1)}(x) = 512\,x^3 - 300\,x\;,\quad N_4^{(s,1)} = 4725$$

$$P_4^{(c,1)}(x) = 4096\,x^4 + 480\,x^2\;,\quad Q_4^{(c,1)}(x) = 1536\,x^3 - 420\,x\;,\quad R_4^{(c,1)}(x) = -512\,x^3 + 300\,x\;,$$

$$S_4^{(c,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\;,\quad N_4^{(c,1)} = 4725$$

$$P_5^{(s,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\;,\quad Q_5^{(s,0)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;,$$

$$R_5^{(s,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;,\quad S_5^{(s,0)}(x) = -16384\,x^4 + 9600\,x^2\;,\quad N_5^{(s,0)} = 297675$$

$$P_5^{(c,0)}(x) = -131072\,x^5 - 15360\,x^3 + 18900\,x\;,\quad Q_5^{(c,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\;,$$

$$R_5^{(c,0)}(x) = 16384\,x^4 - 9600\,x^2\;,\quad S_5^{(c,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;,\quad N_5^{(c,0)} = 297675$$

$$P_5^{(s,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;,\quad Q_5^{(s,1)}(x) = 49152\,x^4 - 13440\,x^2\;,$$

$$R_5^{(s,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\;,\quad S_5^{(s,1)}(x) = -131072\,x^5 + 9216\,x^3 - 14700\,x\;,\quad N_5^{(s,1)} = 363825$$

$$P_5^{(c,1)}(x) = -49152\,x^4 + 13440\,x^2\;,\quad Q_5^{(c,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;,$$

$$R_5^{(c,1)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;,\quad S_5^{(c,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\;,\quad N_5^{(c,1)} = 363825$$

$$P_6^{(s,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770\;,\quad Q_6^{(s,0)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;,$$

$$R_6^{(s,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;,\quad S_6^{(s,0)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2\;,\quad N_6^{(s,0)} = 7203735$$

$$P_6^{(c,0)}(x) = 393216\,x^5 - 107520\,x^3 + 291060\,x\;,\quad Q_6^{(c,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770\;,$$

$$R_6^{(c,0)}(x) = -1048576\,x^6 + 73728\,x^4 - 117600\,x^2\;,\quad S_6^{(c,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;,$$

$$N_6^{(c,0)} = 7203735$$

$$P_6^{(s,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;,\quad Q_6^{(s,1)}(x) = 1048576\,x^6 + 122880\,x^4 - 151200\,x^2\;,$$

$$R_6^{(s,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\;,\quad S_6^{(s,1)}(x) = -131072\,x^5 + 76800\,x^3 - 238140\,x\;,$$

$$N_6^{(s,1)} = 8513505$$

$$P_6^{(c,1)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2\;,\quad Q_6^{(c,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;,$$

$$R_6^{(c,1)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;,\quad S_6^{(c,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\;,$$

$$N_6^{(c,1)} = 8513505$$

$$P_7^{(s,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\;,$$

$$Q_7^{(s,0)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\;,$$

$$R_7^{(s,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\;,$$

$$S_7^{(s,0)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2\;,\quad N_7^{(s,0)} = 405810405$$

$$P_7^{(c,0)}(x) = 16777216\,x^7 + 1966080\,x^5 - 2419200\,x^3 + 11351340\,x\ ,$$
$$Q_7^{(c,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\ ,$$
$$R_7^{(c,0)}(x) = -2097152\,x^6 + 1228800\,x^4 - 3810240\,x^2\ ,$$
$$S_7^{(c,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\ , \quad N_7^{(c,0)} = 405810405$$
$$P_7^{(s,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\ ,$$
$$Q_7^{(s,1)}(x) = -6291456\,x^6 + 1720320\,x^4 - 4656960\,x^2\ ,$$
$$R_7^{(s,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\ ,$$
$$S_7^{(s,1)}(x) = 16777216\,x^7 - 1179648\,x^5 + 1881600\,x^3 - 9604980\,x\ , \quad N_7^{(s,1)} = 468242775$$
$$P_7^{(c,1)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2\ ,$$
$$Q_7^{(c,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\ ,$$
$$R_7^{(c,1)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\ ,$$
$$S_7^{(c,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\ , \quad N_7^{(c,1)} = 468242775$$

$$P_8^{(s,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\ ,$$
$$Q_8^{(s,0)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$R_8^{(s,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ ,$$
$$S_8^{(s,0)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2\ , \quad N_8^{(s,0)} = 13043905875$$
$$P_8^{(c,0)}(x) = -50331648\,x^7 + 13762560\,x^5 - 37255680\,x^3 + 267567300\,x\ ,$$
$$Q_8^{(c,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\ ,$$
$$R_8^{(c,0)}(x) = 134217728\,x^8 - 9437184\,x^6 + 15052800\,x^4 - 76839840\,x^2\ ,$$
$$S_8^{(c,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ , \quad N_8^{(c,0)} = 13043905875$$
$$P_8^{(s,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$Q_8^{(s,1)}(x) = -134217728\,x^8 - 15728640\,x^6 + 19353600\,x^4 - 90810720\,x^2\ ,$$
$$R_8^{(s,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\ ,$$
$$S_8^{(s,1)}(x) = 16777216\,x^7 - 9830400\,x^5 + 30481920\,x^3 - 231891660\,x\ , \quad N_8^{(s,1)} = 14783093325$$
$$P_8^{(c,1)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2\ ,$$
$$Q_8^{(c,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$R_8^{(c,1)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ ,$$
$$S_8^{(c,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\ ,$$
$$N_8^{(c,1)} = 14783093325$$

Recurrence relations:

$$\text{Let} \qquad I_n^{(s,\nu)} = \int \frac{\sin x J_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{and} \qquad I_n^{(c,\nu)} = \int \frac{\cos x J_\nu(x)\,dx}{x^{n+1/2}}\ , \qquad \text{then holds}$$

$$I_{n+1}^{(s,0)} = \frac{2}{2n+1}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\sin x \cdot J_0(x)}{x^{n+1/2}}\right]\ , \qquad I_{n+1}^{(s,1)} = \frac{2}{2n+3}\left[I_n^{(s,0)} + I_n^{(c,1)} - \frac{\sin x \cdot J_1(x)}{x^{n+1/2}}\right]$$
$$I_{n+1}^{(c,0)} = \frac{2}{2n+1}\left[-I_n^{(s,0)} - I_n^{(c,1)} - \frac{\cos x \cdot J_0(x)}{x^{n+1/2}}\right]\ , \qquad I_{n+1}^{(c,1)} = \frac{2}{2n+3}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\cos x \cdot J_1(x)}{x^{n+1/2}}\right]$$

1.2.7. Integrals of the Type $\int x^{-n-1/2}\, e^{\pm x} \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

a) Integrals $\int x^{-n-1/2}\, e^{x}\, Z_\nu(x)\, dx$:

See also [4], 1.11.2. and 1.12.2. .

$$\int \frac{e^x\, I_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, I_0(x) - 4x\, I_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, K_0(x) + 4x\, K_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{3/2}} = \frac{4x\, I_0(x) - (2+4x)\, I_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{3/2}} = \frac{-4x\, I_0(x) - (2+4x)\, K_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, I_0(x) - (4x+32x^2)\, I_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, K_0(x) + (4x+32x^2)\, K_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{5/2}} = \frac{(-12x+32x^2)\, I_0(x) - (6+4x+32x^2)\, I_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{5/2}} = \frac{(12x-32x^2)\, K_0(x) - (6+4x+32x^2)\, K_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, I_0(x) + (512\, x^3 + 64\, x^2 + 36\, x)\, I_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x)\, I_0(x) - (512\, x^3 + 64\, x^2 + 36\, x + 90)\, I_1(x)}{315\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, K_0(x) + (512\, x^3 + 64\, x^2 + 36\, x)\, K_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{7/2}} = \frac{-(512\, x^3 - 192\, x^2 - 60\, x)\, K_0(x) + (512\, x^3 + 64\, x^2 + 36\, x + 90)\, K_1(x)}{315\, x^{5/2}}\, e^x$$

Let[1]

$$\int \frac{e^x\, I_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, I_0(x) - Q_n^{(0,+)}(x)\, I_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x \,,$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(1,+)}(x)\, I_0(x) - Q_n^{(1,+)}(x)\, I_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x \,,$$

then holds

$$\int \frac{e^x\, K_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, K_0(x) + Q_n^{(0,+)}(x)\, K_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x \,,$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{n+1/2}} = \frac{-P_n^{(1,+)}(x)\, K_0(x) - Q_n^{(1,+)}(x)\, K_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x \,.$$

$$P_4^{(0,+)}(x) = 4096\, x^4 - 1536\, x^3 - 480\, x^2 - 420\, x - 1050 \,,$$

$$Q_4^{(0,+)}(x) = 4096\, x^4 + 512\, x^3 + 288\, x^2 + 300\, x \,, \quad N_4^{(0,+)} = 3675$$

$$P_4^{(1,+)}(x) = 4096\, x^4 - 1536\, x^3 - 480\, x^2 - 420\, x \,,$$

[1] Note that there are signs '-' in the numerators !

$$Q_4^{(1,+)}(x) = 4096\,x^4 + 512\,x^3 + 288\,x^2 + 300\,x + 1050\;, \quad N_4^{(1,+)} = 4725$$

$$P_5^{(0,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x - 66150\;,$$
$$Q_5^{(0,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x\;, \quad N_5^{(0,+)} = 297675$$
$$P_5^{(1,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x\;,$$
$$Q_5^{(1,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x + 66150\;, \quad N_5^{(1,+)} = 363825$$

$$P_6^{(0,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x - 1309770$$
$$Q_6^{(0,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x\;,$$
$$N_6^{(0,+)} = 7203735$$
$$P_6^{(1,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x\;,$$
$$Q_6^{(1,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x + 1309770\;,$$
$$N_6^{(1,+)} = 8513505$$

$$P_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x - 62432370\;,$$
$$Q_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x\;,$$
$$N_7^{(0,+)} = 405810405$$
$$P_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x\;,$$
$$Q_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x + 62432370\;,$$
$$N_7^{(1,+)} = 468242775$$

<u>Recurrence relations:</u>

$$\text{With} \qquad \mathbf{I}_n^{(\nu,+)} = \int \frac{e^x\,I_\nu(x)\,dx}{x^{n+1/2}} \quad \text{and} \quad \mathbf{K}_n^{(\nu,+)} = \int \frac{e^x\,K_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{holds}$$

$$\mathbf{I}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\,I_0(x)}{x^{n-1/2}}\right]\;, \qquad \mathbf{I}_n^{(1,+)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\,I_1(x)}{x^{n-1/2}}\right]$$
$$\mathbf{K}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,+)} - \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\,K_0(x)}{x^{n-1/2}}\right]\;, \qquad \mathbf{K}_n^{(1,+)} = \frac{2}{2n+1}\left[-\mathbf{K}_{n-1}^{(0,+)} + \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\,K_1(x)}{x^{n-1/2}}\right]$$

<u>b) Integrals $\int x^{-n-1/2}\,e^{-x}\,Z_\nu(x)\,dx$:</u>

The next two integrals are special cases of the formula 1.11.2.1 from [4].

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{3/2}} = -\,\frac{(4x+2)\,I_0(x) + 4x\,I_1(x)}{\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{3/2}} = \frac{4x\,K_0(x) + (4x-2)\,K_1(x)}{3\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{3/2}} = \frac{-(4x+2)\,K_0(x) + 4x\,K_1(x)}{\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{3/2}} = \frac{-4x\,K_0(x) + (4x-2)\,K_1(x)}{3\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x-6)\,I_0(x) + (32\,x^2-4\,x)\,I_1(x)}{9\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{5/2}} = \frac{(-32\,x^2-12\,x)\,I_0(x) + (-32\,x^2+4\,x-6)\,I_1(x)}{15\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x-6)\,K_0(x) - (32\,x^2-4\,x)\,K_1(x)}{9\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x)\,K_0(x) + (-32\,x^2+4\,x-6)\,K_1(x)}{15\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3-192\,x^2+60\,x-90)\,I_0(x) + (-512\,x^3+64\,x^2-36\,x)\,I_1(x)}{225\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{7/2}} = \frac{(512\,x^3+192\,x^2-60\,x)\,I_0(x) + (512\,x^3-64\,x^2+36\,x-90)\,I_1(x)}{315\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3-192\,x^2+60\,x-90)\,K_0(x) - (-512\,x^3+64\,x^2-36\,x)\,K_1(x)}{225\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{7/2}} = \frac{-(512\,x^3+192\,x^2-60\,x)\,K_0(x) + (512\,x^3-64\,x^2+36\,x-90)\,K_1(x)}{315\,x^{5/2}}\,e^{-x}$$

Let

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,I_0(x) + Q_n^{(0,-)}(x)\,I_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^{x}\ ,$$

$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(1,-)}(x)\,I_0(x) + Q_n^{(1,-)}(x)\,I_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^{x}\ ,$$

then holds

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,K_0(x) - Q_n^{(0,-)}(x)\,K_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^{x}\ ,$$

$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{n+1/2}} = \frac{-\,P_n^{(1,-)}(x)\,K_0(x) + Q_n^{(1,-)}(x)\,K_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^{x}\ .$$

$$P_4^{(0,-)}(x) = 4096\,x^4 + 1536\,x^3 - 480\,x^2 + 420\,x - 1050\ ,$$

$$Q_4^{(0,-)}(x) = 4096\,x^4 - 512\,x^3 + 288\,x^2 - 300\,x\ , \quad N_4^{(0,-)} = 3675$$

$$P_4^{(1,-)}(x) = -4096\,x^4 - 1536\,x^3 + 480\,x^2 - 420\,x\ ,$$

$$Q_4^{(1,-)}(x) = -4096\,x^4 + 512\,x^3 - 288\,x^2 + 300\,x - 1050\ , \quad N_4^{(1,-)} = 4725$$

$$P_5^{(0,-)}(x) = -131072\,x^5 - 49152\,x^4 + 15360\,x^3 - 13440\,x^2 + 18900\,x - 66150\ ,$$

$$Q_5^{(0,-)}(x) = -131072\,x^5 + 16384\,x^4 - 9216\,x^3 + 9600\,x^2 - 14700\,x\ , \quad N_5^{(0,-)} = 297675$$

$$P_5^{(1,-)}(x) = 131072\,x^5 + 49152\,x^4 - 15360\,x^3 + 13440\,x^2 - 18900\,x\ ,$$

$$Q_5^{(1,-)}(x) = 131072\,x^5 - 16384\,x^4 + 9216\,x^3 - 9600\,x^2 + 14700\,x - 66150\ , \quad N_5^{(1,-)} = 363825$$

$$P_6^{(0,-)}(x) = 1048576\,x^6 + 393216\,x^5 - 122880\,x^4 + 107520\,x^3 - 151200\,x^2 + 291060\,x - 1309770\ ,$$
$$Q_6^{(0,-)}(x) = 1048576\,x^6 - 131072\,x^5 + 73728\,x^4 - 76800\,x^3 + 117600\,x^2 - 238140\,x\ ,$$
$$N_6^{(0,-)} = 7203735$$
$$P_6^{(1,-)}(x) = -1048576\,x^6 - 393216\,x^5 + 122880\,x^4 - 107520\,x^3 + 151200\,x^2 - 291060\,x\ ,$$
$$Q_6^{(1,-)}(x) = -1048576\,x^6 + 131072\,x^5 - 73728\,x^4 + 76800\,x^3 - 117600\,x^2 + 238140\,x - 1309770\ ,$$
$$N_6^{(1,-)} = 8513505$$

$$P_7^{(0,-)}(x) =$$
$$= -16777216\,x^7 - 6291456\,x^6 + 1966080\,x^5 - 1720320\,x^4 + 2419200\,x^3 - 4656960\,x^2 + 11351340\,x - 62432370\ ,$$
$$Q_7^{(0,-)}(x) = -16777216\,x^7 + 2097152\,x^6 - 1179648\,x^5 + 1228800\,x^4 - 1881600\,x^3 + 3810240\,x^2 - 9604980\,x\ ,$$
$$N_7^{(0,-)} = 405810405$$
$$P_7^{(1,-)}(x) = 16777216\,x^7 + 6291456\,x^6 - 1966080\,x^5 + 1720320\,x^4 - 2419200\,x^3 + 4656960\,x^2 - 11351340\,x\ ,$$
$$Q_7^{(1,-)}(x) =$$
$$= 16777216\,x^7 - 2097152\,x^6 + 1179648\,x^5 - 1228800\,x^4 + 1881600\,x^3 - 3810240\,x^2 + 9604980\,x - 62432370\ ,$$
$$N_7^{(1,-)} = 468242775$$

Recurrence relations:

$$\text{With} \qquad \mathbf{I}_n^{(\nu,-)} = \int \frac{e^{-x}\,I_\nu(x)\,dx}{x^{n+1/2}} \quad \text{and} \quad \mathbf{K}_n^{(\nu,-)} = \int \frac{e^{-x}\,K_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{holds}$$
$$\mathbf{I}_n^{(0,-)} = \frac{2}{2n-1}\left[-\mathbf{I}_{n-1}^{(0,-)} + \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{I}_n^{(1,-)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,-)} - \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_1(x)}{x^{n-1/2}}\right]$$
$$\mathbf{K}_n^{(0,-)} = -\frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{K}_n^{(1,-)} = -\frac{2}{2n+1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_1(x)}{x^{n-1/2}}\right]$$

c) General formulas

Let
$$P_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \vartheta_k^{(n,\nu,\pm)}\,x^k \qquad \text{and} \quad Q_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \eta_k^{(n,\nu,\pm)}\,x^k\ ,$$
then holds
$$\int \frac{e^{\pm x}\,I_\nu(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(\nu,\pm)}(x)\,I_0(x)\ +\ Q_n^{(\nu,\pm)}(x)\,I_1(x)}{x^{n-1/2}}\ .$$
I. $\nu = 0,\ e^x$:
$$\vartheta_0^{(n,0,+)} = -\frac{2}{2n-1}\ , \quad \vartheta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)(2n-3)}\ ,$$
$$\vartheta_2^{(n,0,+)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,+)}\ ,$$
$$\vartheta_3^{(n,0,+)} = -\frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,+)}\ ,$$
$$\vartheta_4^{(n,0,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,+)}\ , \quad \dots\ ,$$

which gives $\vartheta_0^{(n,0,+)} = -\,2/(2n-1)$ and

$$\vartheta_k^{(n,0,+)} = -\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)^2}\ ,\quad \eta_2^{(n,0,+)} = -\frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,+)}\ ,$$

$$\eta_3^{(n,0,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,+)}\ ,\ \dots\ ,$$

which gives $\eta_0^{(n,0,+)} = 0$ and

$$\eta_k^{(n,0,+)} = -\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\quad k>0\ .$$

II. $\nu=1,\ e^x$:

$$\vartheta_0^{(n,1,+)} = 0\ ,\quad \vartheta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,1,+)} = -\frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,+)}\ ,$$

$$\vartheta_3^{(n,1,+)} = -\frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,+)}\ ,$$

$$\vartheta_4^{(n,1,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,+)}\ ,\quad \dots\ ,$$

which gives $\vartheta_0^{(n,1,+)} = 0$ and

$$\vartheta_k^{(n,1,+)} = -\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_0^{(n,1,+)} = -\frac{2}{2n+1}\ ,\quad \eta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-1)}\ ,$$

$$\eta_2^{(n,1,+)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,+)}\ ,$$

$$\eta_3^{(n,1,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_1^{(n,2,+)}\ ,\ \dots\ ,$$

which gives $\eta_0^{(n,1,+)} = -2/(2n+1)$ and

$$\eta_k^{(n,0,+)} = -\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\quad k>1\ .$$

III. $\nu=0,\ e^{-x}$:

$$\vartheta_0^{(n,0,-)} = -2/(2n-1)\ ,\quad \vartheta_1^{(n,0,-)} = \frac{2^2}{(2n-1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,0,-)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = -\,\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,-)}\ ,$$

$$\vartheta_3^{(n,0,-)} = \frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,-)}\ ,$$

$$\vartheta_4^{(n,0,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,-)}\ ,\quad \dots\ ,$$

which gives $\vartheta_0^{(n,0,-)} = -2/(2n-1)$ and

$$\vartheta_k^{(n,0,-)} = \frac{(-2)^{k-1} \cdot \Gamma(n-k-\frac{1}{2}) \cdot \Gamma(n-k+\frac{3}{2}) \cdot (n-1)!}{\Gamma^2(n+\frac{1}{2}) \cdot (n-k)!} \ , \ k > 0 \ .$$

Furthermore

$$\eta_1^{(n,0,-)} = -\frac{2^2}{(2n-1)^2} \ , \quad \eta_2^{(n,0,-)} = \frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = -\frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,-)} \ ,$$

$$\eta_3^{(n,0,-)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = -\frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,-)} \ , \ \ldots \ ,$$

which gives $\eta_0^{(n,0,-)} = 0$ and

$$\eta_k^{(n,0,-)} = -\frac{(-2)^{k-1} \cdot \Gamma^2(n-k+\frac{1}{2}) \cdot (n-1)!}{\Gamma^2(n+\frac{1}{2}) \cdot (n-k)!} \ , \quad k > 0 \ .$$

IV. $\nu = 1, \ e^{-x}$:

$$\vartheta_0^{(n,1,-)} = 0 \ , \quad \vartheta_1^{(n,1,-)} = -\frac{2^2}{(2n+1)(2n-3)} \ ,$$

$$\vartheta_2^{(n,1,-)} = \frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = -\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,-)} \ ,$$

$$\vartheta_3^{(n,1,-)} = \frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = -\frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,-)} \ ,$$

$$\vartheta_4^{(n,1,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = -\frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,-)} \ , \quad \ldots \ ,$$

which gives $\vartheta_0^{(n,1,-)} = 0$ and

$$\vartheta_k^{(n,1,-)} = -\frac{(-2)^{k-1} \cdot \Gamma(n-k-\frac{1}{2}) \cdot \Gamma(n-k+\frac{3}{2}) \cdot (n-1)!}{\Gamma(n+\frac{3}{2}) \cdot \Gamma(n-\frac{1}{2}) \cdot (n-k)!} \ , \ k > 0 \ .$$

Furthermore

$$\eta_0^{(n,1,-)} = -\frac{2}{2n+1} \ , \quad \eta_1^{(n,1,-)} = \frac{2^2}{(2n+1)(2n-1)} \ ,$$

$$q\eta_2^{(n,1,-)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,-)} \ ,$$

$$\eta_3^{(n,1,-)} = \frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,1,-)} \ , \ \ldots \ ,$$

which gives $\eta_0^{(n,1,-)} = -2/(2n+1)$ and

$$\eta_k^{(n,1,-)} = \frac{(-2)^{k-1} \cdot \Gamma^2(n-k+\frac{1}{2}) \cdot (n-1)!}{\Gamma(n+\frac{3}{2}) \cdot \Gamma(n-\frac{1}{2}) \cdot (n-k)!} \ .$$

When $n > 0$, then the functions $P_n^{(\nu,\pm)}(x)$ and $Q_n^{(\nu,\pm)}(x)$ are polynomials. In the case $n \le 0$ they are power series with the radius of convergence $R = +\infty$. Their coefficients may be found by the given recurrence relations.

$$P_0^{(0,+)} = 2-\frac{4}{3}\,x+\frac{32}{15}\,x^2-\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4-\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6-\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8- -\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(0,+)} = -4\,x+\frac{32}{9}\,x^2-\frac{512}{225}\,x^3+\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5+\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7+\frac{134217728}{13043905875}\,x^8- -\frac{8589934592}{3769688797875}\,x^9+\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(0,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{64}{315}\,x^2-\frac{512}{4725}\,x^3+\frac{16384}{363825}\,x^4-\frac{131072}{8513505}\,x^5+\frac{2097152}{468242775}\,x^6-\frac{16777216}{14783093325}\,x^7+ +\frac{1073741824}{4213181597625}\,x^8-\frac{8589934592}{167122870039125}\,x^9+\frac{137438953472}{14606538841419525}\,x^{10}+\dots$$

$$Q_{-1}^{(0,+)} = -\frac{4}{9}\,x+\frac{64}{225}\,x^2-\frac{512}{3675}\,x^3+\frac{16384}{297675}\,x^4-\frac{131072}{7203735}\,x^5+\frac{2097152}{405810405}\,x^6-\frac{16777216}{13043905875}\,x^7+ +\frac{1073741824}{3769688797875}\,x^8-\frac{8589934592}{151206406225875}\,x^9+\frac{137438953472}{13336405029122175}\,x^{10}+\dots$$

$$P_{-2}^{(0,+)} = \frac{2}{5}-\frac{4}{35}\,x+\frac{32}{525}\,x^2-\frac{1024}{40425}\,x^3+\frac{8192}{945945}\,x^4-\frac{131072}{52026975}\,x^5+\frac{1048576}{1642565925}\,x^6-\frac{67108864}{468131288625}\,x^7+ +\frac{536870912}{18569207782125}\,x^8-\frac{8589934592}{1622948760157725}\,x^9+\frac{68719476736}{77458918098436875}\,x^{10}+\dots$$

$$Q_{-2}^{(0,+)} = -\frac{4}{25}\,x+\frac{96}{1225}\,x^2-\frac{1024}{33075}\,x^3+\frac{8192}{800415}\,x^4-\frac{131072}{45090045}\,x^5+\frac{1048576}{1449322875}\,x^6-\frac{67108864}{418854310875}\,x^7+ +\frac{536870912}{16800711802875}\,x^8-\frac{8589934592}{1481822781013575}\,x^9+\frac{68719476736}{71262204650561925}\,x^{10}+\dots$$

$$P_0^{(1,+)} = \frac{4}{3}\,x-\frac{32}{15}\,x^2+\frac{512}{315}\,x^3-\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5-\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7-\frac{134217728}{14783093325}\,x^8+ +\frac{8589934592}{4213181597625}\,x^9-\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(1,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{32}{245}\,x^2-\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4-\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6-\frac{67108864}{251312586525}\,x^7+ +\frac{536870912}{10080427081725}\,x^8--\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

$$P_{-1}^{(1,+)} = -\frac{4}{5}\,x+\frac{64}{105}\,x^2-\frac{512}{1575}\,x^3+\frac{16384}{121275}\,x^4-\frac{131072}{2837835}\,x^5+\frac{2097152}{156080925}\,x^6-\frac{16777216}{4927697775}\,x^7+ +\frac{1073741824}{1404393865875}\,x^8-\frac{8589934592}{55707623346375}\,x^9+\frac{137438953472}{4868846280473175}\,x^{10}+\dots$$

$$Q_{-1}^{(1,+)} = 2-\frac{4}{3}\,x+\frac{64}{75}\,x^2-\frac{512}{1225}\,x^3+\frac{16384}{99225}\,x^4-\frac{131072}{2401245}\,x^5+\frac{2097152}{135270135}\,x^6-\frac{16777216}{4347968625}\,x^7+ +\frac{1073741824}{1256562932625}\,x^8-\frac{8589934592}{50402135408625}\,x^9+\frac{137438953472}{4445468343040725}\,x^{10}+\dots$$

$$P_{-2}^{(1,+)} = -\frac{4}{21}\,x+\frac{32}{315}\,x^2-\frac{1024}{24255}\,x^3+\frac{8192}{567567}\,x^4-\frac{131072}{31216185}\,x^5+\frac{1048576}{985539555}\,x^6-\frac{67108864}{280878773175}\,x^7+ +\frac{536870912}{11141524669275}\,x^8-\frac{8589934592}{973769256094635}\,x^9+\frac{68719476736}{46475350859062125}\,x^{10}+\dots$$

$$Q_{-2}^{(1,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{32}{245}\,x^2-\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4-\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6-\frac{67108864}{251312586525}\,x^7+$$

$$+\frac{536870912}{10080427081725}\,x^8-\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

$$P_0^{(0,-)}=2+\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$
$$Q_0^{(0,-)}=-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$
$$P_{-1}^{(0,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{64}{315}\,x^2+\frac{512}{4725}\,x^3+\frac{16384}{363825}\,x^4+\frac{131072}{8513505}\,x^5+\frac{2097152}{468242775}\,x^6+\frac{16777216}{14783093325}\,x^7+$$
$$+\frac{1073741824}{4213181597625}\,x^8+\frac{8589934592}{167122870039125}\,x^9+\frac{137438953472}{14606538841419525}\,x^{10}+\dots$$
$$Q_{-1}^{(0,-)}=-\frac{4}{9}\,x-\frac{64}{225}\,x^2-\frac{512}{3675}\,x^3-\frac{16384}{297675}\,x^4-\frac{131072}{7203735}\,x^5-\frac{2097152}{405810405}\,x^6-\frac{16777216}{13043905875}\,x^7-$$
$$-\frac{1073741824}{3769688797875}\,x^8-\frac{8589934592}{151206406225875}\,x^9-\frac{137438953472}{13336405029122175}\,x^{10}+\dots$$
$$P_{-2}^{(0,-)}=\frac{2}{5}+\frac{4}{35}\,x+\frac{32}{525}\,x^2+\frac{1024}{40425}\,x^3+\frac{8192}{945945}\,x^4+\frac{131072}{52026975}\,x^5+\frac{1048576}{1642565925}\,x^6+\frac{67108864}{468131288625}\,x^7+$$
$$+\frac{536870912}{18569207782125}\,x^8+\frac{8589934592}{1622948760157725}\,x^9+\frac{68719476736}{77458918098436875}\,x^{10}+\dots$$
$$Q_{-2}^{(0,-)}=-\frac{4}{25}\,x-\frac{96}{1225}\,x^2-\frac{1024}{33075}\,x^3-\frac{8192}{800415}\,x^4-\frac{131072}{45090045}\,x^5-\frac{1048576}{1449322875}\,x^6-\frac{67108864}{418854310875}\,x^7-$$
$$-\frac{536870912}{16800711802875}\,x^8-\frac{8589934592}{1481822781013575}\,x^9-\frac{68719476736}{71262204650561925}\,x^{10}+\dots$$

$$P_0^{(1,-)}=\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$
$$Q_0^{(1,-)}=-2-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$
$$P_{-1}^{(1,-)}=-\frac{4}{5}\,x-\frac{64}{105}\,x^2-\frac{512}{1575}\,x^3-\frac{16384}{121275}\,x^4-\frac{131072}{2837835}\,x^5-\frac{2097152}{156080925}\,x^6-\frac{16777216}{4927697775}\,x^7-$$
$$-\frac{1073741824}{1404393865875}\,x^8-\frac{8589934592}{55707623346375}\,x^9-\frac{137438953472}{4868846280473175}\,x^{10}+\dots$$
$$Q_{-1}^{(1,-)}=2+\frac{4}{3}\,x+\frac{64}{75}\,x^2+\frac{512}{1225}\,x^3+\frac{16384}{99225}\,x^4+\frac{131072}{2401245}\,x^5+\frac{2097152}{135270135}\,x^6+\frac{16777216}{4347968625}\,x^7+$$
$$-\frac{1073741824}{1256562932625}\,x^8+\frac{8589934592}{50402135408625}\,x^9+\frac{137438953472}{4445468343040725}\,x^{10}+\dots$$
$$P_{-2}^{(1,-)}=-\frac{4}{21}\,x-\frac{32}{315}\,x^2-\frac{1024}{24255}\,x^3-\frac{8192}{567567}\,x^4-\frac{131072}{31216185}\,x^5-\frac{1048576}{985539555}\,x^6-\frac{67108864}{280878773175}\,x^7-$$
$$-\frac{536870912}{11141524669275}\,x^8-\frac{8589934592}{973769256094635}\,x^9-\frac{68719476736}{46475350859062125}\,x^{10}+\dots$$
$$Q_{-2}^{(1,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{32}{245}\,x^2+\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4+\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6+\frac{67108864}{251312586525}\,x^7+$$
$$+\frac{536870912}{10080427081725}\,x^8+\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

1.2.8. Integrals of the Type $\int x^{-n-1/2} \left\{ \begin{array}{c} \sinh x \\ \cosh x \end{array} \right\} \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

$$\int x^{-3/2} \sinh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, I_0(x) + 2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \sinh x\, I_0(x) - \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, K_0(x) + 2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \sinh x\, K_0(x) + \sinh x\, K_1(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, I_0(x) - \cosh x\, I_0(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, K_0(x) - \cosh x\, K_0(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x\, I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - 16x^2 \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{/2}} \left[-6x \sinh x\, I_0(x) + 16x^2 \cosh x\, I_0(x) - (16x^2 + 3) \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, K_0(x) - 6x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + 16x^2 \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[6x \sinh x\, K_0(x) - 16x^2 \cosh x\, K_0(x) - (16x^2 + 3) \sinh x\, K_1(x) - 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[-6x \sinh x\, I_0(x) + (16x^2 - 3) \cosh x\, I_0(x) - 16x^2 \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[16x^2 \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - (16x^2 + 3) \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2}\cosh x\cdot K_0(x)\,dx =$$

$$= \frac{2}{9x^{3/2}}\,\left[-6x\sinh x\,K_0(x)+(16x^2-3)\cosh x\,K_0(x)+16x^2\sinh x\,K_1(x)+2x\cosh x\,K_1(x)\right]$$

$$\int x^{-5/2}\cosh x\cdot K_1(x)\,dx =$$

$$= \frac{2}{15x^{3/2}}\,\left[-16\,x^2\sinh x\,K_0(x)+6\,x\cosh x\,K_0(x)-2\,x\sinh x\,K_1(x)-\left(3+16\,x^2\right)\,\cosh x\,K_1(x)\right]$$

Let

$$\int x^{-(2n+1)/2}\sinh x\cdot I_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(s0)}(x)\sinh x\,I_0(x)+Q_n^{(s0)}(x)\cosh x\,I_0(x)+R_n^{(s0)}(x)\sinh x\,I_1(x)+S_n^{(s0)}(x)\cosh x\,I_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\sinh x\cdot I_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[P_n^{(s1)}(x)\sinh x\,I_0(x)+Q_n^{(s1)}(x)\cosh x\,I_0(x)+R_n^{(s1)}(x)\sinh x\,I_1(x)+S_n^{(s1)}(x)\cosh x\,I_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(c0)}(x)\sinh x\,I_0(x)+Q_n^{(c0)}(x)\cosh x\,I_0(x)+R_n^{(c0)}(x)\sinh x\,I_1(x)+S_n^{(c0)}(x)\cosh x\,I_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot I_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[P_n^{(c1)}(x)\sinh x\,I_0(x)+Q_n^{(c1)}(x)\cosh x\,I_0(x)+R_n^{(c1)}(x)\sinh x\,I_1(x)+S_n^{(c1)}(x)\cosh x\,I_1(x)\right]\ ,$$

then holds

$$\int x^{-(2n+1)/2}\sinh x\cdot K_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(s0)}(x)\sinh x\,K_0(x)+Q_n^{(s0)}(x)\cosh x\,K_0(x)-R_n^{(s0)}(x)\sinh x\,K_1(x)-S_n^{(s0)}(x)\cosh x\,K_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\sinh x\cdot K_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[-\,P_n^{(s1)}(x)\sinh x\,K_0(x)-Q_n^{(s1)}(x)\cosh x\,K_0(x)+R_n^{(s1)}(x)\sinh x\,K_1(x)+S_n^{(s1)}(x)\cosh x\,K_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot K_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(c0)}(x)\sinh x\,K_0(x)+Q_n^{(c0)}(x)\cosh x\,K_0(x)-R_n^{(c0)}(x)\sinh x\,K_1(x)-S_n^{(c0)}(x)\cosh x\,K_1(x)\right]\ ,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot K_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[-\,P_n^{(c1)}(x)\sinh x\,K_0(x)-Q_n^{(c1)}(x)\cosh x\,K_0(x)+R_n^{(c1)}(x)\sinh x\,K_1(x)+S_n^{(c1)}(x)\cosh x\,K_1(x)\right]\ .$$

$M_3=225\,,\ P_3^{(s0)}(x)=-192\,x^2-90\,,\ Q_3^{(s0)}(x)=512\,x^3-60\,x\,,\ R_3^{(s0)}(x)=-512\,x^3-36\,x\ \ S_3^{(s0)}(x)=-64\,x^2$
$P_3^{(c0)}(x)=512\,x^3-60\,x\,,\ Q_3^{(c0)}(x)=-192\,x^2-90\,,\ R_3^{(c0)}(x)=-64\,x^2\,,\ S_3^{(c0)}(x)=-512\,x^3-36\,x$

$N_3=315\,,\ P_3^{(s1)}(x)=512\,x^3-60\,x\,,\ Q_3^{(s1)}(x)=-192\,x^2\,,\ R_3^{(s1)}(x)=-64\,x^2-90\,,\ S_3^{(s1)}(x)=-512\,x^3-36\,x$

$$P_3^{(c1)}(x) = -192\,x^2\,,\; Q_3^{(c1)}(x) = 512\,x^3 - 60\,x\,,\; R_3^{(c1)}(x) = -512\,x^3 - 36\,x\,,\; S_3^{(c1)}(x) = -64\,x^2 - 90$$

$$M_4 = 3675\,,\; P_4^{(s0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,\; Q_4^{(s0)}(x) = -1536\,x^3 - 420\,x$$
$$R_4^{(s0)}(x) = -512\,x^3 - 300\,x\,,\; S_4^{(s0)}(x) = -4096\,x^4 - 288\,x^2$$
$$P_4^{(c0)}(x) = -1536\,x^3 - 420\,x\,,\; Q_4^{(c0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,$$
$$R_4^{(c0)}(x) = -4096\,x^4 - 288\,x^2\,,\; S_4^{(c0)}(x) = -512\,x^3 - 300\,x$$

$$N_4 = 4725\,,\; P_4^{(s1)}(x) = -1536\,x^3 - 420\,x\,,\; Q_4^{(s1)}(x) = 4096\,x^4 - 480\,x^2\,,$$
$$R_4^{(s1)}(x) = -4096\,x^4 - 288\,x^2 - 1050\,,\; S_4^{(s1)}(x) = -512\,x^3 - 300\,x$$
$$P_4^{(c1)}(x) = 4096\,x^4 - 480\,x^2\,,\; Q_4^{(c1)}(x) = -1536\,x^3 - 420\,x\,,$$
$$R_4^{(c1)}(x) = -512\,x^3 - 300\,x\,,\; S_4^{(c1)}(x) = -4096\,x^4 - 288\,x^2 - 1050$$

$$M_5 = 297675\,,\; P_5^{(s0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,\; Q_5^{(s0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$
$$R_5^{(s0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\; S_5^{(s0)}(x) = -16384\,x^4 - 9600\,x^2$$
$$P_5^{(c0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\; Q_5^{(c0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,$$
$$R_5^{(c0)}(x) = -16384\,x^4 - 9600\,x^2\,,\; S_5^{(c0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$

$$N_5 = 363825\,,\; P_5^{(s1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\; Q_5^{(s1)}(x) = -49152\,x^4 - 13440\,x^2\,,$$
$$R_5^{(s1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150\,,\; S_5^{(s1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$
$$P_5^{(c1)}(x) = -49152\,x^4 - 13440\,x^2\,,\; Q_5^{(c1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$
$$R_5^{(c1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\; S_5^{(c1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150$$

$$M_6 = 7203735\,,$$
$$P_6^{(s0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,\; Q_6^{(s0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$
$$R_6^{(s0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\; S_6^{(s0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2$$
$$P_6^{(c0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\; Q_6^{(c0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,$$
$$R_6^{(c0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2\,,\; S_6^{(c0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$

$$N_6 = 8513505\,,\; P_6^{(s1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\; Q_6^{(s1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,$$
$$R_6^{(s1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770\,,\; S_6^{(s1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$
$$P_6^{(c1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,\; Q_6^{(c1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$
$$R_6^{(c1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\; S_6^{(c1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770$$

$$M_7 = 405810405\,,\; P_7^{(s0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$
$$Q_7^{(s0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$R_7^{(s0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$
$$S_7^{(s0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2$$

$$P_7^{(c0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$Q_7^{(c0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$
$$R_7^{(c0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2\,,$$
$$S_7^{(c0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$

$$N_7 = 468242775\,,\ P_7^{(s1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$Q_7^{(s1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$
$$R_7^{(s1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370\,,$$
$$S_7^{(s1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$
$$P_7^{(c1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$
$$Q_7^{(c1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$R_7^{(c1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$
$$S_7^{(c1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370$$

$$M_8 = 13043905875\,,\ P_8^{(s0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$
$$Q_8^{(s0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$R_8^{(s0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$
$$S_8^{(s0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2$$
$$P_8^{(c0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$Q_8^{(c0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$
$$R_8^{(c0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2\,,$$
$$S_8^{(c0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$

$$N_8 = 14783093325\,,\ P_8^{(s1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$Q_8^{(s1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$
$$R_8^{(s1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450\,,$$
$$S_8^{(s1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$
$$P_8^{(c1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$
$$Q_8^{(c1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$R_8^{(c1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$
$$S_8^{(c1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450$$

Recurrence Relations:

$$\int x^{-(n+1/2)} \sinh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx -$$
$$- \frac{2x^{-n+1/2}}{(2n-1)^2} \left[(2n-1) \sinh x \cdot I_0(x) - 2x \cosh x \cdot I_0(x) + 2x \sinh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx -$$
$$- \frac{2x^{-n+1/2}}{(2n-1)^2} \left[(2n-1) \cosh x \cdot I_0(x) + 2x \cosh x \cdot I_1(x) - 2x \sinh x \cdot I_0(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx +$$
$$+ \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \sinh x \cdot I_0(x) - 2x \cosh x \cdot I_1(x) - (2n-1) \sinh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx +$$
$$+ \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \cosh x \cdot I_0(x) - 2x \sinh x \cdot I_1(x) - (2n-1) \cosh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx +$$
$$+ \frac{2x^{-n+1/2}}{(2n-1)^2} \left[-(2n-1) \sinh x \cdot K_0(x) + 2x \cosh x \cdot K_0(x) + 2x \sinh x \cdot K_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx +$$
$$+ \frac{2x^{-n+1/2}}{(2n-1)^2} \left[-(2n-1) \cosh x \cdot K_0(x) + 2x \cosh x \cdot K_1(x) + 2x \sinh x \cdot K_0(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx -$$
$$- \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \sinh x \cdot K_0(x) + 2x \cosh x \cdot K_1(x) + (2n-1) \sinh x \cdot K_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx -$$
$$- \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \cosh x \cdot K_0(x) + 2x \sinh x \cdot K_1(x) + (2n-1) \cosh x \cdot K_1(x) \right]$$

1.2.9. Integrals of the type $\int x^{2n+1} \ln x \cdot Z_0(x)\,dx$

$$\int x \ln x \cdot J_0(x)\,dx = J_0(x) + x \ln x \cdot J_1(x)$$

$$\int x \ln x \cdot I_0(x)\,dx = -I_0(x) + x \ln x \cdot I_1(x)$$

$$\int x^3 \ln x \cdot J_0(x)\,dx = \left(x^2 - 4 + 2x^2 \ln x\right) J_0(x) + \left[-4x + \left(x^3 - 4x\right)\ln x\right] J_1(x)$$

$$\int x^3 \ln x \cdot I_0(x)\,dx = \left(-x^2 - 4 - 2x^2 \ln x\right) I_0(x) + \left[4x + \left(x^3 + 4x\right)\ln x\right] I_1(x)$$

$$\int x^5 \ln x \cdot J_0(x)\,dx =$$

$$= \left[x^4 - 32\,x^2 + 64 + \left(4\,x^4 - 32\,x^2\right)\ln x\right] J_0(x) + \left[-8\,x^3 + 96\,x + \left(x^5 - 16\,x^3 + 64\,x\right)\ln x\right] J_1(x)$$

$$\int x^5 \ln x \cdot I_0(x)\,dx =$$

$$= \left[-x^4 - 32\,x^2 - 64 + \left(-4\,x^4 - 32\,x^2\right)\ln x\right] I_0(x) + \left[8\,x^3 + 96\,x + \left(x^5 + 16\,x^3 + 64\,x\right)\ln x\right] I_1(x)$$

$$\int x^7 \ln x \cdot J_0(x)\,dx = \left[x^6 - 84\,x^4 + 1536\,x^2 - 2304 + \left(6\,x^6 - 144\,x^4 + 1152\,x^2\right)\ln x\right] J_0(x) +$$

$$+ \left[-12\,x^5 + 480\,x^3 - 4224\,x + \left(x^7 - 36\,x^5 + 576\,x^3 - 2304\,x\right)\ln x\right] J_1(x)$$

$$\int x^7 \ln x \cdot I_0(x)\,dx = \left[-x^6 - 84\,x^4 - 1536\,x^2 - 2304 + \left(-6\,x^6 - 144\,x^4 - 1152\,x^2\right)\ln x\right] I_0(x) +$$

$$+ \left[12\,x^5 + 480\,x^3 + 4224\,x + \left(x^7 + 36\,x^5 + 576\,x^3 + 2304\,x\right)\ln x\right] I_1(x)$$

$$\int x^9 \ln x \cdot J_0(x)\,dx =$$

$$= \left[x^8 - 160\,x^6 + 7680\,x^4 - 116736\,x^2 + 147456 + \left(8\,x^8 - 384\,x^6 + 9216\,x^4 - 73728\,x^2\right)\ln x\right] J_0(x) +$$

$$+ \left[-16\,x^7 + 1344\,x^5 - 39936\,x^3 + 307200\,x + \left(x^9 - 64\,x^7 + 2304\,x^5 - 36864\,x^3 + 147456\,x\right)\ln x\right] J_1(x)$$

$$\int x^9 \ln x \cdot I_0(x)\,dx =$$

$$= \left[-x^8 - 160\,x^6 - 7680\,x^4 - 116736\,x^2 - 147456 + \left(-8\,x^8 - 384\,x^6 - 9216\,x^4 - 73728\,x^2\right)\ln x\right] I_0(x) +$$

$$+ \left[16\,x^7 + 1344\,x^5 + 39936\,x^3 + 307200\,x + \left(x^9 + 64\,x^7 + 2304\,x^5 + 36864\,x^3 + 147456\,x\right)\ln x\right] I_1(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\,dx = (P_n(x) + Q_n(x)\ln x) J_0(x) + (R_n(x) + S_n(x)\ln x) J_1(x)\ ,$$

$$\int x^n \ln x \cdot I_0(x)\,dx = (P_n^*(x) + Q_n^*(x)\ln x) I_0(x) + (R_n^*(x) + S_n^*(x)\ln x) I_1(x)\ ,$$

then holds:

$$P_{11} = x^{10} - 260\,x^8 + 23680\,x^6 - 952320\,x^4 + 13148160\,x^2 - 14745600$$

$$Q_{11} = 10\,x^{10} - 800\,x^8 + 38400\,x^6 - 921600\,x^4 + 7372800\,x^2$$

$$R_{11} = -20\,x^9 + 2880\,x^7 - 180480\,x^5 + 4730880\,x^3 - 33669120\,x$$

$$S_{11} = x^{11} - 100\,x^9 + 6400\,x^7 - 230400\,x^5 + 3686400\,x^3 - 14745600\,x$$

$$P_{11}^* = -x^{10} - 260\,x^8 - 23680\,x^6 - 952320\,x^4 - 13148160\,x^2 - 14745600$$

$$Q_{11}^* = -10\,x^{10} - 800\,x^8 - 38400\,x^6 - 921600\,x^4 - 7372800\,x^2$$

$$R_{11}^* = 20\,x^9 + 2880\,x^7 + 180480\,x^5 + 4730880\,x^3 + 33669120\,x$$

$$S_{11}^* = x^{11} + 100\,x^9 + 6400\,x^7 + 230400\,x^5 + 3686400\,x^3 + 14745600\,x$$

$$P_{13} = x^{12} - 384\,x^{10} + 56640\,x^8 - 4331520\,x^6 + 159252480\,x^4 - 2070282240\,x^2 + 2123366400$$
$$Q_{13} = 12\,x^{12} - 1440\,x^{10} + 115200\,x^8 - 5529600\,x^6 + 132710400\,x^4 - 1061683200\,x^2$$
$$R_{13} = -24\,x^{11} + 5280\,x^9 - 568320\,x^7 + 31518720\,x^5 - 769720320\,x^3 + 5202247680\,x$$
$$S_{13} = x^{13} - 144\,x^{11} + 14400\,x^9 - 921600\,x^7 + 33177600\,x^5 - 530841600\,x^3 + 2123366400\,x$$
$$P^*_{13} = -x^{12} - 384\,x^{10} - 56640\,x^8 - 4331520\,x^6 - 159252480\,x^4 - 2070282240\,x^2 - 2123366400$$
$$Q^*_{13} = -12\,x^{12} - 1440\,x^{10} - 115200\,x^8 - 5529600\,x^6 - 132710400\,x^4 - 1061683200\,x^2$$
$$R^*_{13} = 24\,x^{11} + 5280\,x^9 + 568320\,x^7 + 31518720\,x^5 + 769720320\,x^3 + 5202247680\,x$$
$$S^*_{13} = x^{13} + 144\,x^{11} + 14400\,x^9 + 921600\,x^7 + 33177600\,x^5 + 530841600\,x^3 + 2123366400\,x$$

$$P_{15} = x^{14} - 532\,x^{12} + 115584\,x^{10} - 14327040\,x^8 + 1003806720\,x^6 -$$
$$-34929377280\,x^4 + 435502448640\,x^2 - 416179814400$$
$$Q_{15} = 14\,x^{14} - 2352\,x^{12} + 282240\,x^{10} - 22579200\,x^8 +$$
$$+1083801600\,x^6 - 26011238400\,x^4 + 208089907200\,x^2$$
$$R_{15} = -28\,x^{13} + 8736\,x^{11} - 1438080\,x^9 + 137195520\,x^7 -$$
$$-7106641920\,x^5 + 165728747520\,x^3 - 1079094804480\,x$$
$$S_{15} = x^{15} - 196\,x^{13} + 28224\,x^{11} - 2822400\,x^9 + 180633600\,x^7 -$$
$$-6502809600\,x^5 + 104044953600\,x^3 - 416179814400\,x$$
$$P^*_{15} = -x^{14} - 532\,x^{12} - 115584\,x^{10} - 14327040\,x^8 - 1003806720\,x^6 -$$
$$-34929377280\,x^4 - 435502448640\,x^2 - 416179814400$$
$$Q^*_{15} = -14\,x^{14} - 2352\,x^{12} - 282240\,x^{10} - 22579200\,x^8 -$$
$$-1083801600\,x^6 - 26011238400\,x^4 - 208089907200\,x^2$$
$$R^*_{15} = 28\,x^{13} + 8736\,x^{11} + 1438080\,x^9 + 137195520\,x^7 +$$
$$+7106641920\,x^5 + 165728747520\,x^3 + 1079094804480\,x$$
$$S^*_{15} = x^{15} + 196\,x^{13} + 28224\,x^{11} + 2822400\,x^9 + 180633600\,x^7 +$$
$$+6502809600\,x^5 + 104044953600\,x^3 + 416179814400\,x$$

Recurrence formulas:

$$\int x^{2n+1} \cdot \ln x \cdot J_0(x)\,dx =$$
$$= x^{2n} \ln x [2nJ_0(x) + xJ_1(x)] - 2n \int x^{2n-1} J_0(x)\,dx - \int x^{2n} J_1(x)\,dx - 4n^2 \int x^{2n-1} \cdot \ln x \cdot J_0(x)\,dx$$
$$\int x^{2n+1} \cdot \ln x \cdot I_0(x)\,dx =$$
$$= x^{2n} \ln x [xI_1(x) - 2nI_0(x)] + 2n \int x^{2n-1} I_0(x)\,dx - \int x^{2n} I_1(x)\,dx + 4n^2 \int x^{2n-1} \cdot \ln x \cdot I_0(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

1.2.10. Integrals of the type $\int x^{2n} \ln x \cdot Z_1(x)\,dx$

$$\int \ln x \cdot J_1(x)\,dx = -\ln x \cdot J_0(x) + \int \frac{J_0(x)}{x}\,dx$$
$$\int \ln x \cdot I_1(x)\,dx = \ln x \cdot I_0(x) - \int \frac{I_0(x)}{x}\,dx$$

Concerning the integrals on the right hand side see 1.1.3, page 13.

$$\int x^2 \ln x \cdot J_1(x)\,dx = \left(2 - x^2 \ln x\right) J_0(x) + x\,(1 + 2\ln x)\, J_1(x)$$

$$\int x^2 \ln x \cdot I_1(x)\,dx = \left(2 + x^2 \ln x\right) I_0(x) - x\,(1 + 2\ln x)\, I_1(x)$$

$$\int x^4 \ln x \cdot J_1(x)\,dx = \left[6\,x^2 - 16 + \left(-x^4 + 8\,x^2\right)\ln x\right] J_0(x) + \left[x^3 - 20\,x + \left(4\,x^3 - 16\,x\right)\ln x\right] J_1(x)$$

$$\int x^4 \ln x \cdot I_1(x)\,dx = \left[6\,x^2 + 16 + \left(x^4 + 8\,x^2\right)\ln x\right] I_0(x) + \left[-x^3 - 20\,x + \left(-4\,x^3 - 16\,x\right)\ln x\right] I_1(x)$$

$$\int x^6 \ln x \cdot J_1(x)\,dx = \left[10\,x^4 - 224\,x^2 + 384 + \left(-x^6 + 24\,x^4 - 192\,x^2\right)\ln x\right] J_0(x) +$$
$$+ \left[x^5 - 64\,x^3 + 640\,x + \left(6\,x^5 - 96\,x^3 + 384\,x\right)\ln x\right] J_1(x)$$

$$\int x^6 \ln x \cdot I_1(x)\,dx = \left[10\,x^4 + 224\,x^2 + 384 + \left(x^6 + 24\,x^4 + 192\,x^2\right)\ln x\right] I_0(x) +$$
$$+ \left[-x^5 - 64\,x^3 - 640\,x + \left(-6\,x^5 - 96\,x^3 - 384\,x\right)\ln x\right] I_1(x)$$

$$\int x^8 \ln x \cdot J_1(x)\,dx = \left[14\,x^6 - 816\,x^4 + 13440\,x^2 - 18432 + \left(-x^8 + 48\,x^6 - 1152\,x^4 + 9216\,x^2\right)\ln x\right] J_0(x) +$$
$$+ \left[x^7 - 132\,x^5 + 4416\,x^3 - 36096\,x + \left(8\,x^7 - 288\,x^5 + 4608\,x^3 - 18432\,x\right)\ln x\right] J_1(x)$$

$$\int x^8 \ln x \cdot I_1(x)\,dx = \left[14\,x^6 + 816\,x^4 + 13440\,x^2 + 18432 + \left(x^8 + 48\,x^6 + 1152\,x^4 + 9216\,x^2\right)\ln x\right] I_0(x) +$$
$$+ \left[-x^7 - 132\,x^5 - 4416\,x^3 - 36096\,x + \left(-8\,x^7 - 288\,x^5 - 4608\,x^3 - 18432\,x\right)\ln x\right] I_1(x)$$

Let

$$\int x^n \ln x \cdot J_1(x)\,dx = [P_n(x) + Q_n(x)\ln x] J_0(x) + [R_n(x) + S_n(x)\ln x] J_1(x)\;,$$
$$\int x^n \ln x \cdot I_1(x)\,dx = [P_n^*(x) + Q_n^*(x)\ln x] I_0(x) + [R_n^*(x) + S_n^*(x)\ln x] I_1(x)\;,$$

then holds:

$$P_{10}(x) = 18\,x^8 - 1984\,x^6 + 86016\,x^4 - 1241088\,x^2 + 1474560$$
$$Q_{10}(x) = -x^{10} + 80\,x^8 - 3840\,x^6 + 92160\,x^4 - 737280\,x^2$$
$$R_{10}(x) = x^9 - 224\,x^7 + 15744\,x^5 - 436224\,x^3 + 3219456\,x$$
$$S_{10}(x) = 10\,x^9 - 640\,x^7 + 23040\,x^5 - 368640\,x^3 + 1474560\,x$$
$$P_{10}^*(x) = 18\,x^8 + 1984\,x^6 + 86016\,x^4 + 1241088\,x^2 + 1474560$$
$$Q_{10}^*(x) = x^{10} + 80\,x^8 + 3840\,x^6 + 92160\,x^4 + 737280\,x^2$$
$$R_{10}^*(x) = -x^9 - 224\,x^7 - 15744\,x^5 - 436224\,x^3 - 3219456\,x$$
$$S_{10}^*(x) = -10\,x^9 - 640\,x^7 - 23040\,x^5 - 368640\,x^3 - 1474560\,x$$

$$P_{12}(x) = 22\,x^{10} - 3920\,x^8 + 322560\,x^6 - 12349440\,x^4 + 165150720\,x^2 - 176947200$$
$$Q_{12}(x) = -x^{12} + 120\,x^{10} - 9600\,x^8 + 460800\,x^6 - 11059200\,x^4 + 88473600\,x^2$$

$$R_{12}(x) = x^{11} - 340\,x^9 + 40960\,x^7 - 2396160\,x^5 + 60456960\,x^3 - 418775040\,x$$
$$S_{12}(x) = 12\,x^{11} - 1200\,x^9 + 76800\,x^7 - 2764800\,x^5 + 44236800\,x^3 - 176947200\,x$$
$$P^*_{12}(x) = 22\,x^{10} + 3920\,x^8 + 322560\,x^6 + 12349440\,x^4 + 165150720\,x^2 + 176947200$$
$$Q^*_{12}(x) = x^{12} + 120\,x^{10} + 9600\,x^8 + 460800\,x^6 + 11059200\,x^4 + 88473600\,x^2$$
$$R^*_{12}(x) = -x^{11} - 340\,x^9 - 40960\,x^7 - 2396160\,x^5 - 60456960\,x^3 - 418775040\,x$$
$$S^*_{12}(x) = -12\,x^{11} - 1200\,x^9 - 76800\,x^7 - 2764800\,x^5 - 44236800\,x^3 - 176947200\,x$$

$$P_{14}(x) = 26\,x^{12} - 6816\,x^{10} + 908160\,x^8 - 66170880\,x^6 +$$
$$+2362245120\,x^4 - 30045634560\,x^2 + 29727129600$$
$$Q_{14}(x) = -x^{14} + 168\,x^{12} - 20160\,x^{10} + 1612800\,x^8 -$$
$$-77414400\,x^6 + 1857945600\,x^4 - 14863564800\,x^2$$
$$R_{14}(x) = x^{13} - 480\,x^{11} + 88320\,x^9 - 8878080\,x^7 +$$
$$+474439680\,x^5 - 11306926080\,x^3 + 74954833920\,x$$
$$S_{14}(x) = 14\,x^{13} - 2016\,x^{11} + 201600\,x^9 - 12902400\,x^7 +$$
$$+464486400\,x^5 - 7431782400\,x^3 + 29727129600\,x$$
$$P^*_{14}(x) = 26\,x^{12} + 6816\,x^{10} + 908160\,x^8 + 66170880\,x^6 +$$
$$+2362245120\,x^4 + 30045634560\,x^2 + 29727129600$$
$$Q^*_{14}(x) = x^{14} + 168\,x^{12} + 20160\,x^{10} + 1612800\,x^8 +$$
$$+77414400\,x^6 + 1857945600\,x^4 + 14863564800\,x^2$$
$$R^*_{14}(x) = -x^{13} - 480\,x^{11} - 88320\,x^9 - 8878080\,x^7 -$$
$$-474439680\,x^5 - 11306926080\,x^3 - 74954833920\,x$$
$$S^*_{14}(x) = -14\,x^{13} - 2016\,x^{11} - 201600\,x^9 - 12902400\,x^7 -$$
$$-464486400\,x^5 - 7431782400\,x^3 - 29727129600\,x$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_1(x)\,dx = x^{2n+1} \ln x[(2n+2)J_1(x) - xJ_0(x)] +$$
$$+\int x^{2n+1} J_0(x)\,dx - (2n+2)\int x^{2n} J_1(x)\,dx - 4n(n+1)\int x^{2n} \cdot \ln x \cdot J_1(x)\,dx$$
$$\int x^{2n+2} \cdot \ln x \cdot I_1(x)\,dx = x^{2n+1} \ln x[xI_0(x) - (2n+2)I_1(x)] -$$
$$-\int x^{2n+1} I_0(x)\,dx + (2n+2)\int x^{2n} I_1(x)\,dx + 4n(n+1)\int x^{2n} \cdot \ln x \cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

1.2.11. Integrals of the type $\int x^{2n+\nu} \ln x \cdot Z_\nu(x)\, dx$

a) The Functions Λ_k and Λ_k^*, $k = 0, 1$:

Let

$$\Lambda_0(x) \;=\; \sum_{k=0}^{\infty} \alpha_k\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} \cdot (k!)^2 \cdot (2k+1)}\; x^{2k+1} = \int_0^x J_0(t)\, dt = x\, J_0(x) + \Phi(x)\;.$$

($\Phi(x)$ and further on $\Psi(x)$ defined as on page 7)

$$\Lambda_0(x) = x - \frac{x^3}{12} + \frac{x^5}{320} - \frac{x^7}{16\,128} + \frac{x^9}{1\,327\,104} - \frac{x^{11}}{162\,201\,600} + \frac{x^{13}}{27\,603\,763\,200} - \frac{x^{15}}{6\,242\,697\,216\,000} + \dots$$

k	α_k	$1/\alpha_k$
0	1.0000000000000000E+00	1
1	-8.3333333333333329E-02	-12
2	3.1250000000000002E-03	320
3	-6.2003968253968251E-05	-16128
4	7.5352044753086416E-07	1327104
5	-6.1651672979797980E-09	-162201600
6	3.6226944592830012E-11	27603763200
7	-1.6018716996829596E-13	-6242697216000
8	5.5211570531352012E-16	1811214552268800
9	-1.5246859958300586E-18	-655872751986278400
10	3.4486945143775135E-21	289964795614986240000
11	-6.5058017249306315E-24	-153708957370763182080000
12	1.0391211088430869E-26	96235173310390861824000000
13	-1.4232975959389203E-29	-70259375330450160400465920000
14	1.6902284962328838E-32	59163598426411660994999746560000
15	-1.7568683294176930E-35	-56919461934375356612430790656000000
16	1.6117104111016952E-38	-56919461934375356612430790656000000
17	-1.3145438350557573E-41	-76072016263922024994318857577431040000000
18	9.5948102742224525E-45	104223009253931112643645081672942092288000000
19	-6.3038564554696844E-48	-158633053760659041611878822148470455926784000000
20	3.7477195390749646E-51	266828931453826490506134634177940048943513600000000

Values of this function may be found in [1], Table 11.1 .

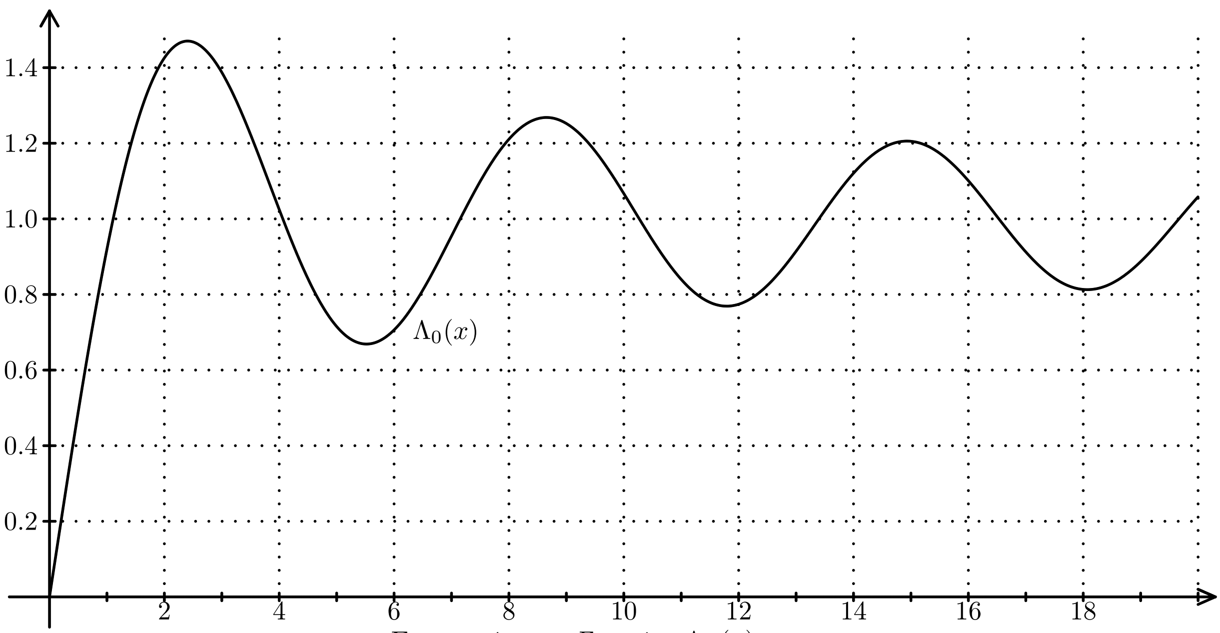


FIGURE 1 : *Function* $\Lambda_0\,(x)$

Approximations with Chebyshev polynomials are given in [1], table 9.3 .

The maxima and minima of $\Lambda_0(x)$ are situated in the zeros of $J_0(x)$:

k	1	2	3	4	5	6	7	8
max	1.470300	1.268168	1.205654	1.172888	1.151982	1.137178	1.125991	1.117157
min	0.668846	0.769119	0.812831	0.838567	0.855986	0.868771	0.878666	0.886617

Asymptotic expansion:

$$\Lambda_0(x) \sim 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

Recurrence relation:

$$\lambda_{k+1} = -\frac{2k+1}{16(k+1)} \left[(12k+10)\lambda_k + (4k^2-1)\lambda_{k-1}\right]$$

If $k > 1$, then up to $k \approx 30$ holds

$$\lambda_k \approx (-1)^k \, \Gamma(s_k) \quad \text{with} \quad s_k = k + \frac{1}{2} - \frac{1}{3\sqrt{k}} \; .$$

Coefficients of the asymptotic formula:

k	λ_k	λ_k	$q_k = \lvert\lambda_k/\lambda_{k-1}\rvert$	$\lvert\lambda_k\rvert/\Gamma(s_k)$
0	1	1	-	-
1	$-\frac{5}{8}$	-0.625	0.625	-
2	$\frac{129}{128}$	1.007812500	1.612500000	0.882203509
3	$-\frac{2655}{1024}$	-2.592773438	2.572674419	0.958684418
4	$\frac{301035}{32768}$	9.186859131	3.543255650	0.992044824
5	$-\frac{10896795}{262144}$	-41.56797409	4.524720963	1.007474317
6	$\frac{961319205}{4194304}$	229.1963589	5.513772656	1.014554883
7	$-\frac{50046571575}{33554432}$	-1491.504060	6.507538198	1.017456390
8	$\frac{24035398261875}{2147483648}$	11192.35450	7.504072427	1.018151550
9	$-\frac{1634825936118375}{17179869184}$	-95159.39374	8.502178320	1.017633944
10	$\frac{248523783571238175}{274877906944}$	904124.2577	9.501156135	1.016430975
11	$-\frac{20877210220441199625}{2199023255552}$	-9493856.042	10.50060980	1.014835786
12	$\frac{7683027147736313147775}{70368744177664}$	109182382.6	11.50032001	1.014835786

Roughly spoken, the item

$$\frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

in the asymptotic series should not be used if $|x| < q_k$.

Let

$$d_n(x) = 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right) - \Lambda_0(x) \; .$$

The following table gives some consecutive maxima and minima of interest of this functions:

$n=0$		$n=1$		$n=2$		$n=3$	
x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$
3.953	-5.390E-2	5.510	-9.219-3	3.936	1.020E-2	5.501	2.128E-3
7.084	2.479E-2	8.647	3.315E-3	7.074	-1.751-3	8.642	-3.501E-4
10.221	-1.475E-2	11.787	-1.592E-3	10.214	5.360E-4	11.783	9.561E-5
13.360	1.000E-2	14.927	9.002E-4	13.355	-2.197E-4	14.924	-3.469E-5
16.500	-7.337E-3	18.068	-5.647E-4	16.496	1.076E-4	18.065	1.511E-5

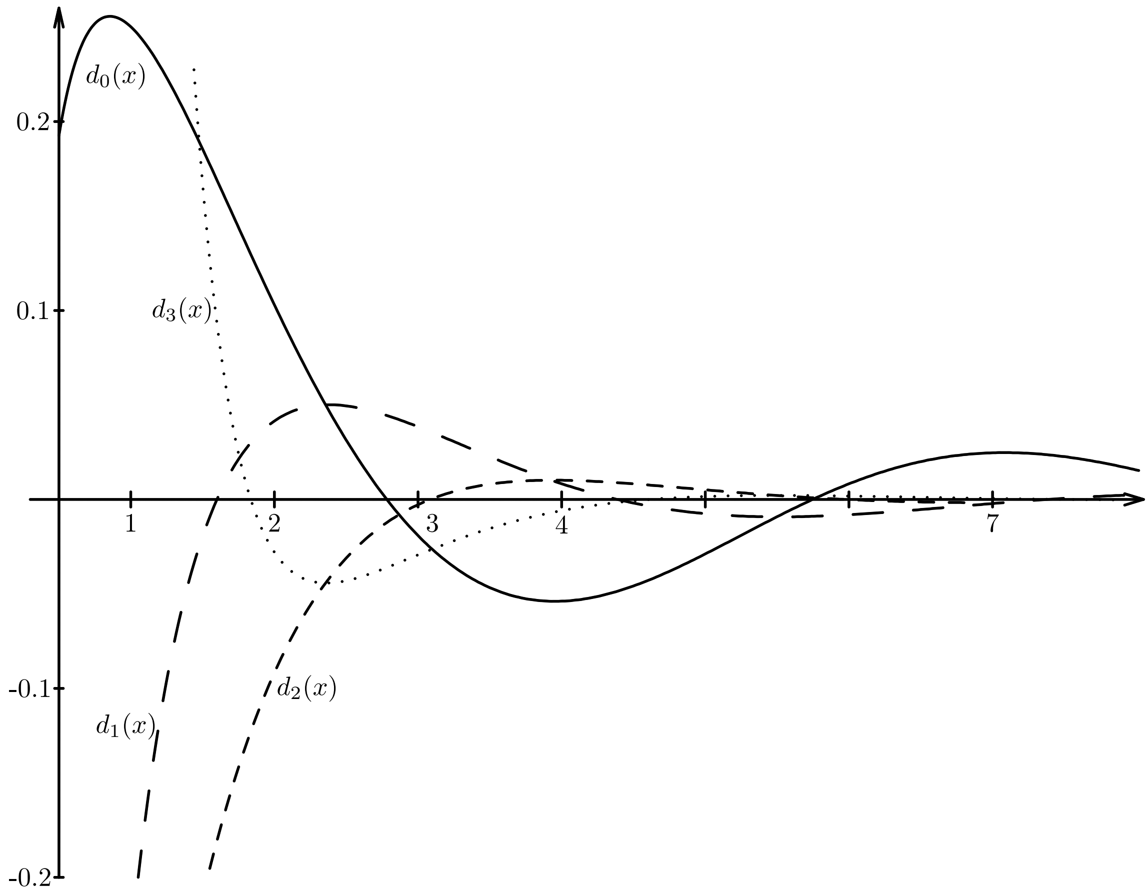


FIGURE 1 : *Differences* $d_0(x)\ \dots\ d_3(x)$

Let

$$\Lambda_0^*(x) = \sum_{k=0}^{\infty} |\alpha_k|\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}\cdot (k!)^2 \cdot (2k+1)}\, x^{2k+1} = \int_0^x I_0(t)\, dt = x\, I_0(x) + \Psi(x)\ .$$

Asymptotic expansion (see $\Lambda_0(x)$):

$$\Lambda_0^*(x) = \frac{e^x}{\sqrt{2\pi x}} \left[1 + \frac{5}{8x} + \frac{129}{128x^2} + \frac{2655}{1024x^3} + \dots\right]$$

Furthermore, let

$$\Lambda_1(x) = \sum_{k=0}^{\infty} \beta_k\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\cdot (k!)^2 \cdot (2k+1)^2}\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{\alpha_k}{2k+1}\, x^{2k+1}\ .$$

$\Lambda_1(x)$ can be written as a hypergeometric function.
One has

$$\Lambda_0(x) = x\,\Lambda_1'(x)\,,\ \Lambda_1(0) = 0 \quad \Longleftrightarrow \quad \Lambda_1(x) = \int_0^x \frac{\Lambda_0(t)\, dt}{t}\ .$$

$$\Lambda_1(x) = x - \frac{x^3}{36} + \frac{x^5}{1\,600} - \frac{x^7}{112\,896} + \frac{x^9}{11\,943\,936} - \frac{x^{11}}{1\,784\,217\,600} + \frac{x^{13}}{358\,848\,921\,600} - \frac{x^{15}}{93\,640\,458\,240\,000} + \dots$$

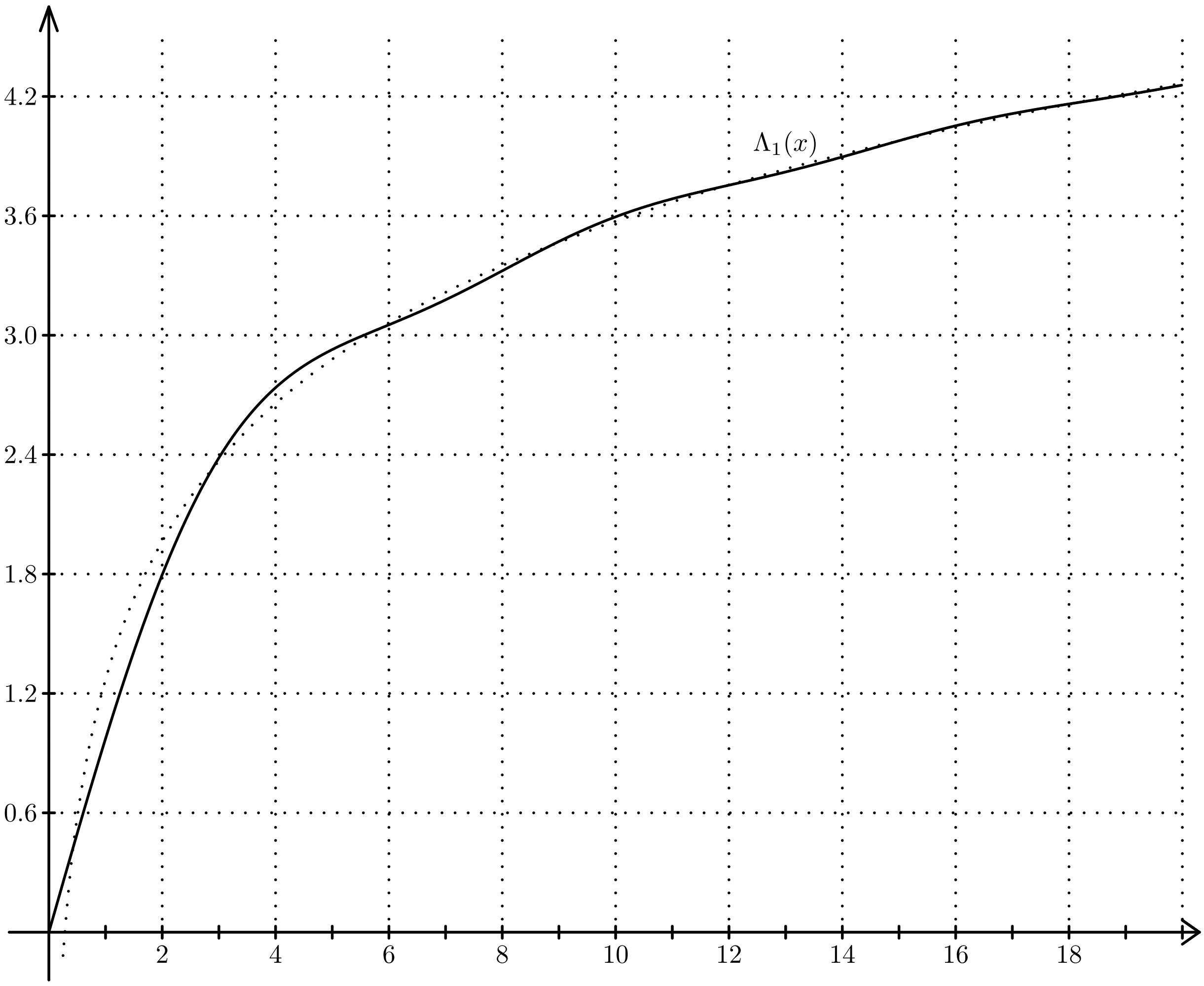


FIGURE 2 : *Function* $\Lambda_1(x)$, *dots:* $C + \ln 2x$

Asymptotic series with Euler's constant $\mathbf{C} = 0.577\ 215\ 664\ 901\ 533$:

$$\Lambda_1(x) \sim C + \ln 2x - \sqrt{\frac{2}{\pi x}} \sum_{k=1}^{\infty} \frac{\mu_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

k	μ_k	μ_k	$\lvert\mu_k/\mu_{k-1}\rvert$
1	1	1	1
2	$-\frac{17}{8}$	-2.125	2.125
3	$\frac{809}{128}$	6.320 312 500	2.974
4	$-\frac{25307}{1024}$	-24.713 867 187 500	3.910
5	$\frac{3945243}{32768}$	120.399 261 474 609	4.871
6	$-\frac{184487487}{262144}$	-703.763 912 200 928	5.845
7	$\frac{20148017853}{4194304}$	4 803.661 788 225 174	6.826
8	$-\frac{1258927642755}{33554432}$	-37 518.967 472 166	7.810
9	$\frac{708892035920595}{2147483648}$	330 103.578 008 988	8.798
10	$-\frac{55510620666083595}{17179869184}$	-3 231 143.384 827 510	9.788
11	$\frac{9574308055473282135}{274877906944}$	34 831 129.798 379	10.780
12	$-\frac{901713551323983156045}{2199023255552}$	-410 051 848.723 007	11.773

Some consecutive maxima and minima of the differences

$$\delta_n(x) \;=\; C + \ln 2x - \sqrt{\frac{2}{\pi x}}\sum_{k=1}^{n} \frac{\mu_k}{x^k}\,\sin\left(x + \frac{3-2k}{4}\,\pi\right) - \Lambda_1\,(x)$$

$n=1$	x	2.4704	5.569	8.689	11.819	14.953	18.090
	$\delta_n(x)$	9.374E-2	-1.855E-2	6.824E-3	-3.309E-3	1.880E-3	-1.182E-3
$n=2$	x	3.991	7.115	10.246	13.380	16.517	19.655
	$\delta_n(x)$	2.312E-2	-4.117E-3	1.279E-3	-5.284E-4	2.598E-4	-1.436E-4
$n=3$	x	5.541	8.673	11.808	14.945	18.083	21.222
	$\delta_n(x)$	5.439E-3	-9.148E-4	2.525E-4	-9.214E-5	4.028E-5	-1.997E-5
$n=4$	x	10.237	13.374	16.512	19.651	22.791	25.931
	$\delta_n(x)$	-2.043E-4	5.163E-5	-1.707E-5	6.771E-6	-3.061E-6	1.527E-6

Let

$$\Lambda_1^*(x) \;=\; \sum_{k=0}^{\infty} |\beta_k|\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}\cdot(k!)^2\cdot(2k+1)^2}\;.$$

b) Basic Integrals:

$$\int \ln x \cdot J_0(x)\,dx \;=\; \Lambda_0\,(x)\cdot \ln x - \Lambda_1\,(x) + c$$

$$\int \ln x \cdot I_0(x)\,dx \;=\; \Lambda_0^*\,(x)\cdot \ln x - \Lambda_1^*\,(x) + c$$

In particular, let

$$\int_0^x \ln t \cdot J_0(t)\,dt \;=\; F\,(x) \quad \text{and} \quad \int_0^x \ln t \cdot I_0(t)\,dt \;=\; F^*\,(x)\;.$$

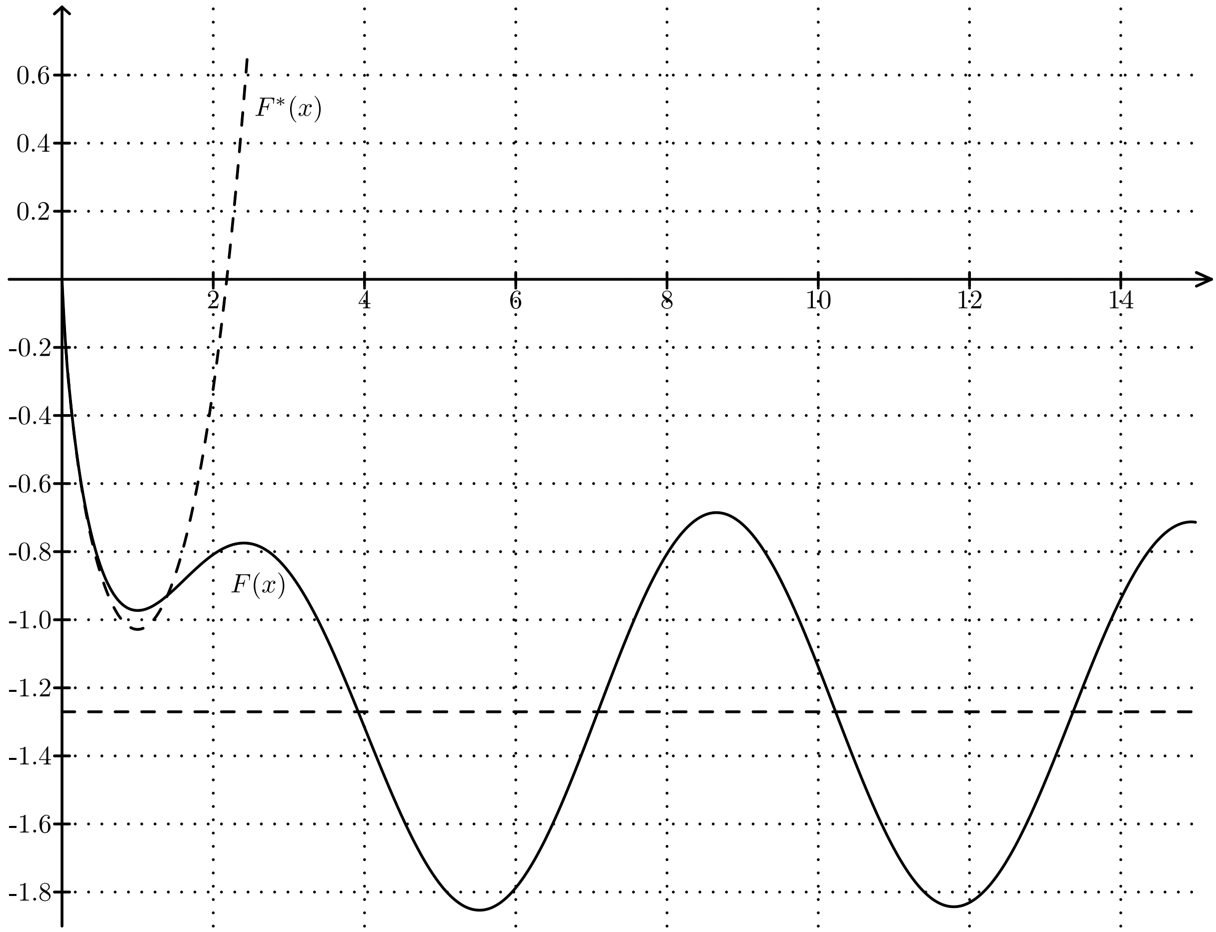


FIGURE 3 : *Functions* $F(x)$ and $F^*(x)$

Holds ([7], 6.772) with Euler's constant $\mathbf{C} = 0.577...$

$$\lim_{x\to\infty} F(x) \;=\; -\ln 2 - \mathbf{C} \;=\; -1.270\,362\,845\,461\,478\,170\;.$$

Asymptotic expansion:

$$F(x) \;\sim\; -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{\infty} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right)\right] =$$

$$= -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) - \frac{5\ln x + 8}{8x}\,\sin\left(x + \frac{\pi}{4}\right) + \frac{129\ln x + 272}{128x^2}\,\sin\left(x + \frac{3\pi}{4}\right) - \right.$$

$$\left. - \frac{2\,655\ln x + 6\,472}{1\,024x^3}\,\sin\left(x + \frac{5\pi}{4}\right) + \frac{301\,035\ln x + 809\,824}{32\,768x^4}\,\sin\left(x + \frac{7\pi}{4}\right) + \ldots\right]$$

Let

$$\Delta_n(x) = -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{n} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right)\right] - F(x)$$

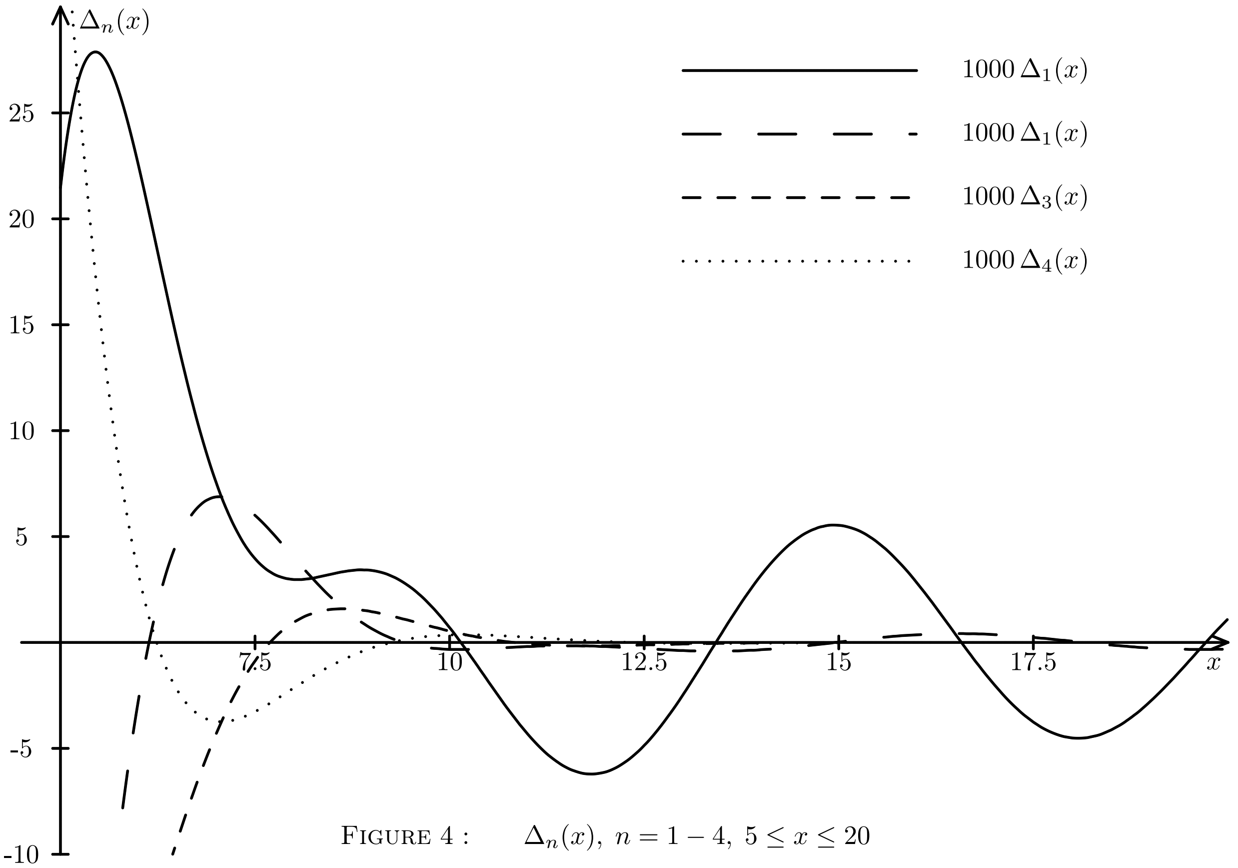


FIGURE 4 : $\Delta_n(x),\ n = 1-4,\ 5 \le x \le 20$

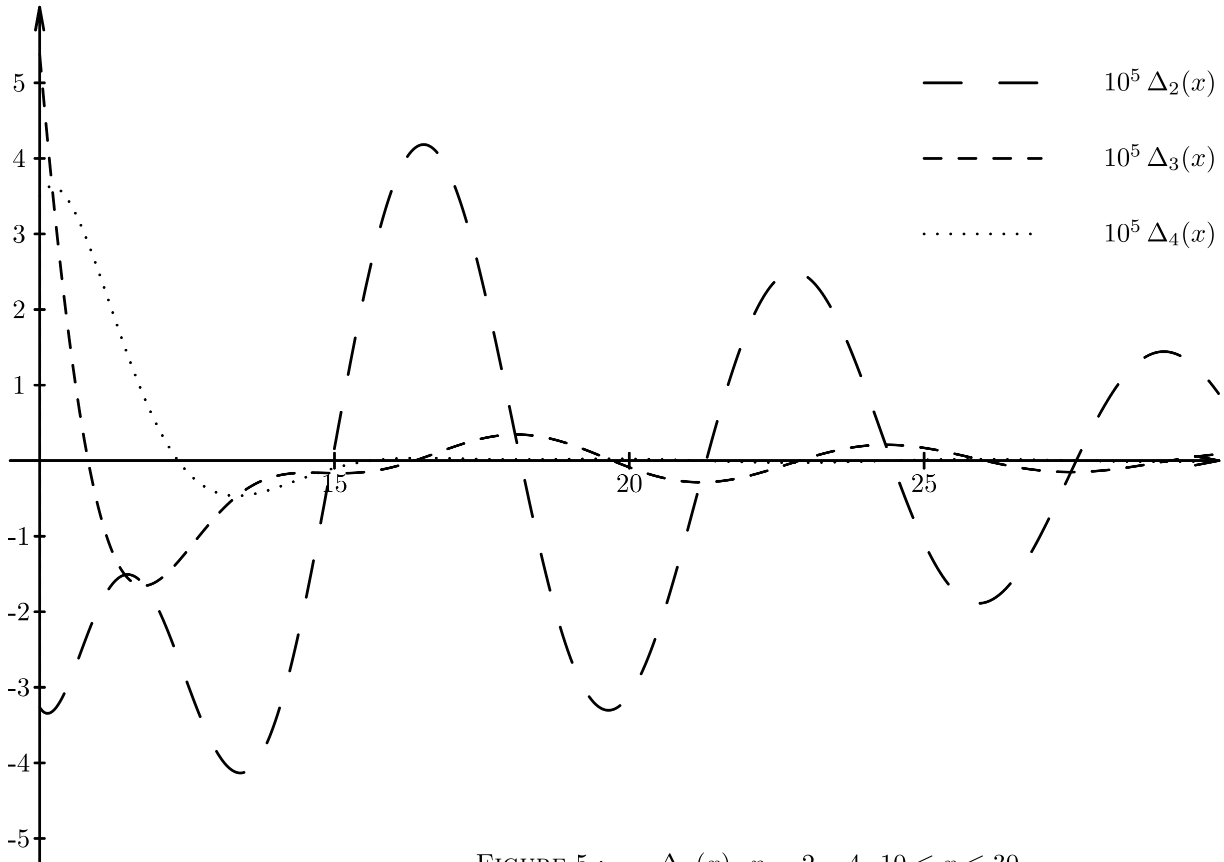


FIGURE 5 : $\Delta_n(x),\ n = 2-4,\ 10 \le x \le 30$

Some consecutive maxima and minima of the differences $\Delta_n(x)$:

$n=1$	x	11.822	14.944	18.079	21.217	24.356	27.496
	$\Delta_n(x)$	-6.213E-4	5.541E-4	-4.520E-4	3.644E-4	-2.959E-4	2.431E-4
$n=2$	x	7.046	10.134	13.404	16.513	19.647	22.785
	$\Delta_n(x)$	6.878E-4	-3.344E-5	-4.133E-5	4.183E-5	-3.303E-5	2.498E-5
$n=3$	x	18.083	21.215	24.354	27.494	30.634	33.775
	$\Delta_n(x)$	3.458-6	-2.863E-6	2.100E-6	-1.507E-6	1.087E-6	-7.949E-7
$n=4$	x	22.785	25.923	29.063	32.204	35.345	38.486
	$\Delta_n(x)$	-2.648E-7	1.982E-7	-1.385E-7	9.570E-8	-6.666E-8	4.709E-8

c) Integrals of $x^{2n} \ln x \cdot Z_0(x)$:

$$\int \ln x \cdot J_0(x)\,dx = [x\,J_0(x) + \Phi(x)] \cdot \ln x - \Lambda_1(x)$$

$$\int \ln x \cdot I_0(x)\,dx = [x\,I_0(x) + \Psi(x)] \cdot \ln x - \Lambda_1^*(x)$$

$$\int x^2 \ln x \cdot J_0(x)\,dx = [x^2\,J_1(x) - \Phi(x)] \cdot \ln x - xJ_0(x) + \Lambda_1(x) - 2\Phi(x)$$

$$\int x^2 \ln x \cdot I_0(x)\,dx = [x^2\,I_1(x) + \Psi(x)] \cdot \ln x + xI_0(x) - \Lambda_1^*(x) + 2\Psi(x)$$

$$\int x^4 \ln x \cdot J_0(x)\,dx =$$

$$= [3\,x^3\,J_0(x) + (x^4 - 9\,x^2)\,J_1(x) + 9\Phi(x)]\,\cdot \ln x + (x^3 + 9\,x)\,J_0(x) - 6\,x^2\,J_1(x) - 9\Lambda_1(x) + 24\Phi(x)$$

$$\int x^4 \ln x \cdot I_0(x)\,dx =$$

$$= [-3\,x^3\,I_0(x) + (x^4 + 9\,x^2)\,I_1(x) + 9\Psi(x)]\,\cdot \ln x + (-x^3 + 9\,x)\,I_0(x) + 6\,x^2\,I_1(x) - 9\Lambda_1^*(x) + 24\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\,dx =$$

$$= [P_n(x)\,J_0(x) + Q_n(x)\,J_1(x) + p_n\,\Phi(x)] \cdot \ln x + R_n(x)\,J_0(x) + S_n(x)\,J_1(x) - p_n\,\Lambda_1(x) + q_n\,\Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\,dx =$$

$$= [P_n^*(x)\,I_0(x) + Q_n^*(x)\,I_1(x) + p_n^*\,\Psi(x)] \cdot \ln x + R_n^*(x)\,I_0(x) + S_n^*(x)\,I_1(x) - p_n^*\,\Lambda_1^*(x) + q_n^*\,\Psi(x)\ ,$$

then holds

$$P_6(x) = 5\,x^5 - 75\,x^3\ , \quad Q_6(x) = x^6 - 25\,x^4 + 225\,x^2\ , \quad R_6(x) = x^5 - 55\,x^3 - 225\,x$$

$$S_6(x) = -10\,x^4 + 240\,x^2\ , \quad p_6 = -225\ , \quad q_6 = -690$$

$$P_6^*(x) = -5\,x^5 - 75\,x^3\ , \quad Q_6^*(x) = x^6 + 25\,x^4 + 225\,x^2\ , \quad R_6^*(x) = -x^5 - 55\,x^3 + 225\,x$$

$$S_6^*(x) = 10\,x^4 + 240\,x^2\ , \quad p_6^* = 225\ , \quad q_6^* = 690$$

$$P_8(x) = 7\,x^7 - 245\,x^5 + 3675\,x^3\ , \quad Q_8(x) = x^8 - 49\,x^6 + 1225\,x^4 - 11025\,x^2$$

$$R_8(x) = x^7 - 119\,x^5 + 3745\,x^3 + 11025\,x\ , \quad S_8(x) = -14\,x^6 + 840\,x^4 - 14910\,x^2\ , \quad p_8 = 11025\ , \quad q_8 = 36960$$

$$P_8^*(x) = -7\,x^7 - 245\,x^5 - 3675\,x^3\ , \quad Q_8^*(x) = x^8 + 49\,x^6 + 1225\,x^4 + 11025\,x^2$$

$$R_8^*(x) = -x^7 - 119\,x^5 - 3745\,x^3 + 11025\,x\ , \quad S_8^*(x) = 14\,x^6 + 840\,x^4 + 14910\,x^2\ , \quad p_8^* = 11025\ , \quad q_8^* = 36960$$

$$P_{10}(x) = 9\,x^9 - 567\,x^7 + 19845\,x^5 - 297675\,x^3 \ , \quad Q_{10}(x) = x^{10} - 81\,x^8 + 3969\,x^6 - 99225\,x^4 + 893025\,x^2$$
$$R_{10}(x) = x^9 - 207\,x^7 + 14049\,x^5 - 369495\,x^3 - 893025\,x \ , \quad S_{10}(x) = -18\,x^8 + 2016\,x^6 - 90090\,x^4 + 1406160\,x^2$$
$$p_{10} = -893025 \ , \quad q_{10} = -3192210$$
$$P^*_{10}(x) = -9\,x^9 - 567\,x^7 - 19845\,x^5 - 297675\,x^3 \ , \quad Q^*_{10}(x) = x^{10} + 81\,x^8 + 3969\,x^6 + 99225\,x^4 + 893025\,x^2$$
$$R^*_{10}(x) = -x^9 - 207\,x^7 - 14049\,x^5 - 369495\,x^3 + 893025\,x \ , \quad S^*_{10}(x) = 18\,x^8 + 2016\,x^6 + 90090\,x^4 + 1406160\,x^2$$
$$p^*_{10} = 893025 \ , \quad q^*_{10} = 3192210$$

$$P_{12}(x) = 11\,x^{11} - 1089\,x^9 + 68607\,x^7 - 2401245\,x^5 + 36018675\,x^3$$
$$Q_{12}(x) = x^{12} - 121\,x^{10} + 9801\,x^8 - 480249\,x^6 + 12006225\,x^4 - 108056025\,x^2$$
$$R_{12}(x) = x^{11} - 319\,x^9 + 37521\,x^7 - 2136519\,x^5 + 51257745\,x^3 + 108056025\,x$$
$$S_{12}(x) = -22\,x^{10} + 3960\,x^8 - 331254\,x^6 + 13083840\,x^4 - 189791910\,x^2$$
$$p_{12} = 108056025 \ , \quad q_{12} = 405903960$$
$$P^*_{12}(x) = -11\,x^{11} - 1089\,x^9 - 68607\,x^7 - 2401245\,x^5 - 36018675\,x^3$$
$$Q^*_{12}(x) = x^{12} + 121\,x^{10} + 9801\,x^8 + 480249\,x^6 + 12006225\,x^4 + 108056025\,x^2$$
$$R^*_{12}(x) = -x^{11} - 319\,x^9 - 37521\,x^7 - 2136519\,x^5 - 51257745\,x^3 + 108056025\,x$$
$$Q^*_{12}(x) = 22\,x^{10} + 3960\,x^8 + 331254\,x^6 + 13083840\,x^4 + 189791910\,x^2$$
$$p^*_{12} = 108056025 \ , \quad q^*_{12} = 405903960$$

$$P_{14}(x) = 13\,x^{13} - 1859\,x^{11} + 184041\,x^9 - 11594583\,x^7 + 405810405\,x^5 - 6087156075\,x^3$$
$$Q_{14}(x) = x^{14} - 169\,x^{12} + 20449\,x^{10} - 1656369\,x^8 + 81162081\,x^6 - 2029052025\,x^4 + 18261468225\,x^2$$
$$R_{14}(x) = x^{13} - 455\,x^{11} + 82225\,x^9 - 8124831\,x^7 + 423504081\,x^5 - 9599044455\,x^3 - 18261468225\,x$$
$$S_{14}(x) = -26\,x^{12} + 6864\,x^{10} - 924066\,x^8 + 68468400\,x^6 - 2523330810\,x^4 + 34884289440\,x^2$$
$$p_{14} = -18261468225 \ , \quad q_{14} = -71407225890$$
$$P^*_{14}(x) = -13\,x^{13} - 1859\,x^{11} - 184041\,x^9 - 11594583\,x^7 - 405810405\,x^5 - 6087156075\,x^3$$
$$Q^*_{14}(x) = x^{14} + 169\,x^{12} + 20449\,x^{10} + 1656369\,x^8 + 81162081\,x^6 + 2029052025\,x^4 + 18261468225\,x^2$$
$$R^*_{14}(x) = -x^{13} - 455\,x^{11} - 82225\,x^9 - 8124831\,x^7 - 423504081\,x^5 - 9599044455\,x^3 + 18261468225\,x$$
$$S^*_{14}(x) = 26\,x^{12} + 6864\,x^{10} + 924066\,x^8 + 68468400\,x^6 + 2523330810\,x^4 + 34884289440\,x^2$$
$$p^*_{14} = 18261468225 \ , \quad q^*_{14} = 71407225890$$

$$P_{16}(x) = 15\,x^{15} - 2925\,x^{13} + 418275\,x^{11} - 41409225\,x^9 + 2608781175\,x^7 - 91307341125\,x^5 + 1369610116875\,x^3$$
$$Q_{16}(x) = x^{16} - 225\,x^{14} + 38025\,x^{12} - 4601025\,x^{10} + 372683025\,x^8 - 18261468225\,x^6 +$$
$$+ 456536705625\,x^4 - 4108830350625\,x^2$$
$$R_{16}(x) = x^{15} - 615\,x^{13} + 158145\,x^{11} - 24021855\,x^9 + 2175924465\,x^7 - 107462730375\,x^5 +$$
$$+ 2342399684625\,x^3 + 4108830350625\,x$$
$$S_{16}(x) = -30\,x^{14} + 10920\,x^{12} - 2157870\,x^{10} + 257605920\,x^8 - 17840252430\,x^6 +$$
$$+ 628620993000\,x^4 - 8396809170750\,x^2$$
$$p_{16} = 4108830350625 \ , \quad q_{16} = 16614469872000$$

$$P_{16}^*(x) = -15\,x^{15} - 2925\,x^{13} - 418275\,x^{11} - 41409225\,x^9 - 2608781175\,x^7 - 91307341125\,x^5 - 1369610116875\,x^3$$

$$Q_{16}^*(x) = x^{16} + 225\,x^{14} + 38025\,x^{12} + 4601025\,x^{10} + 372683025\,x^8 + 18261468225\,x^6 +$$
$$+456536705625\,x^4 + 4108830350625\,x^2$$

$$R_{16}^*(x) = -x^{15} - 615\,x^{13} - 158145\,x^{11} - 24021855\,x^9 - 2175924465\,x^7 - 107462730375\,x^5 -$$
$$-2342399684625\,x^3 + 4108830350625\,x$$

$$S_{16}^*(x) = 30\,x^{14} + 10920\,x^{12} + 2157870\,x^{10} + 257605920\,x^8 + 17840252430\,x^6 +$$
$$+628620993000\,x^4 + 8396809170750\,x^2$$

$$p_{16}^* = 4108830350625\,, \quad q_{16}^* = 16614469872000$$

$$P_{18}(x) = 17\,x^{17} - 4335\,x^{15} + 845325\,x^{13} - 120881475\,x^{11} + 11967266025\,x^9 - 753937759575\,x^7 +$$
$$+26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}(x) = x^{18} - 289\,x^{16} + 65025\,x^{14} - 10989225\,x^{12} + 1329696225\,x^{10} - 107705394225\,x^8 +$$
$$+5277564317025\,x^6 - 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}(x) = x^{17} - 799\,x^{15} + 277185\,x^{13} - 59925255\,x^{11} + 8350229745\,x^9 - 717540730335\,x^7 +$$
$$+34161178676625\,x^5 - 723520252830375\,x^3 - 1187451971330625\,x$$

$$S_{18}(x) = -34\,x^{16} + 16320\,x^{14} - 4448730\,x^{12} + 780059280\,x^{10} - 87119333730\,x^8 + 5776722871920\,x^6 -$$
$$-197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18} = -1187451971330625\,, \quad q_{18} = -4941282024929250$$

$$P_{18}^*(x) = -17\,x^{17} - 4335\,x^{15} - 845325\,x^{13} - 120881475\,x^{11} - 11967266025\,x^9 - 753937759575\,x^7 -$$
$$-26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}^*(x) = x^{18} + 289\,x^{16} + 65025\,x^{14} + 10989225\,x^{12} + 1329696225\,x^{10} + 107705394225\,x^8 +$$
$$+5277564317025\,x^6 + 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}^*(x) = -x^{17} - 799\,x^{15} - 277185\,x^{13} - 59925255\,x^{11} - 8350229745\,x^9 - 717540730335\,x^7 -$$
$$-34161178676625\,x^5 - 723520252830375\,x^3 + 1187451971330625\,x$$

$$S_{18}^*(x) = 34\,x^{16} + 16320\,x^{14} + 4448730\,x^{12} + 780059280\,x^{10} + 87119333730\,x^8 + 5776722871920\,x^6 +$$
$$+197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18}^* = 1187451971330625\,, \quad q_{18}^* = 4941282024929250$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_0(x)\,dx = x^{2n+1} \ln x[(2n+1)J_0(x) + xJ_1(x)] -$$
$$-(2n+1)\int x^{2n} J_0(x)\,dx - \int x^{2n+1} J_1(x)\,dx - (2n+1)^2 \int x^{2n} \cdot \ln x \cdot J_0(x)\,dx$$

$$\int x^{2n+2} \cdot \ln x \cdot I_0(x)\,dx = -x^{2n+1} \ln x[(2n+1)I_0(x) - xI_1(x)] +$$
$$+(2n+1)\int x^{2n} I_0(x)\,dx - \int x^{2n+1} I_1(x)\,dx + (2n+1)^2 \int x^{2n} \cdot \ln x \cdot I_0(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

d) Integrals of $x^{2n+1} \ln x \cdot Z_1(x)$:

$$\int x \ln x \cdot J_1(x)\,dx = \Phi(x)\cdot \ln x + x\,J_0(x) - \Lambda_1(x) + \Phi(x)$$

$$\int x \ln x \cdot I_1(x)\,dx = -\Psi(x)\cdot \ln x - x\,I_0(x) + \Lambda_1^*(x) - \Psi(x)$$

$$\int x^3 \ln x \cdot J_1(x)\,dx = [-x^3 J_0(x) + 3x^2 J_1(x) - 3\,\Phi(x)] \ln x - 3x\,J_0(x) + x^2 J_1(x) + 3\,\Lambda_1(x) - 7\,\Phi(x)$$

$$\int x^3 \ln x \cdot I_1(x)\,dx = [x^3 I_0(x) - 3x^2 I_1(x) - 3\,\Psi(x)] \ln x - 3x\,I_0(x) - x^2 I_1(x) + 3\,\Lambda_1^*(x) - 7\,\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\,dx =$$

$$= [P_n(x)\,J_0(x) + Q_n(x)\,J_1(x) + p_n\,\Phi(x)]\cdot \ln x + R_n(x)\,J_0(x) + S_n(x)\,J_1(x) - p_n\,\Lambda_1(x) + q_n\,\Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\,dx =$$

$$= [P_n^*(x)\,I_0(x) + Q_n^*(x)\,I_1(x) + p_n^*\,\Psi(x)]\cdot \ln x + R_n^*(x)\,I_0(x) + S_n^*(x)\,I_1(x) - p_n^*\,\Lambda_1^*(x) + q_n^*\,\Psi(x)\;,$$

then holds

$$P_5(x) = -x^5 + 15\,x^3\;,\quad Q_5(x) = 5\,x^4 - 45\,x^2\;,\quad R_5(x) = 8\,x^3 + 45\,x\;,\quad S_5(x) = x^4 - 39\,x^2\;,$$
$$p_5 = 45\;,\quad q_5 = 129$$
$$P_5^*(x) = x^5 + 15\,x^3\;,\quad Q_5^*(x) = -5\,x^4 - 45\,x^2\;,\quad R_5^*(x) = 8\,x^3 - 45\,x\;,\quad S_5^*(x) = -x^4 - 39\,x^2$$
$$p_5^* = -45\;,\quad q_5^* = -129$$

$$P_7(x) = -x^7 + 35\,x^5 - 525\,x^3\;,\quad Q_7(x) = 7\,x^6 - 175\,x^4 + 1575\,x^2$$
$$R_7(x) = 12\,x^5 - 460\,x^3 - 1575\,x\;,\quad S_7(x) = x^6 - 95\,x^4 + 1905\,x^2\;,\quad p_7 = -1575\;,\quad q_7 = -5055$$
$$P_7^*(x) = x^7 + 35\,x^5 + 525\,x^3\;,\quad Q_7^*(x) = -7\,x^6 - 175\,x^4 - 1575\,x^2$$
$$R_7^*(x) = 12\,x^5 + 460\,x^3 - 1575\,x\;,\quad S_7^*(x) = -x^6 - 95\,x^4 - 1905\,x^2\;,\quad p_7^* = -1575\;,\quad q_7^* - 5055$$

$$P_9(x) = -x^9 + 63\,x^7 - 2205\,x^5 + 33075\,x^3\;,\quad Q_9(x) = 9\,x^8 - 441\,x^6 + 11025\,x^4 - 99225\,x^2$$
$$R_9(x) = 16\,x^7 - 1316\,x^5 + 37380\,x^3 + 99225\,x\;,\quad S_9(x) = x^8 - 175\,x^6 + 8785\,x^4 - 145215\,x^2$$
$$p_9 = 99225\;,\quad q_9 = 343665$$
$$P_9^*(x) = x^9 + 63\,x^7 + 2205\,x^5 + 33075\,x^3\;,\quad Q_9^*(x) = -9\,x^8 - 441\,x^6 - 11025\,x^4 - 99225\,x^2$$
$$R_9^*(x) = 16\,x^7 + 1316\,x^5 + 37380\,x^3 - 99225\,x\;,\quad S_9^*(x) = -x^8 - 175\,x^6 - 8785\,x^4 - 145215\,x^2$$
$$p_9^* = -99225\;,\quad q_9^* = -343665$$

$$P_{11}(x) = -x^{11} + 99\,x^9 - 6237\,x^7 + 218295\,x^5 - 3274425\,x^3$$
$$Q_{11}(x) = 11\,x^{10} - 891\,x^8 + 43659\,x^6 - 1091475\,x^4 + 9823275\,x^2$$
$$R_{11}(x) = 20\,x^9 - 2844\,x^7 + 174384\,x^5 - 4362120\,x^3 - 9823275\,x$$
$$S_{11}(x) = x^{10} - 279\,x^8 + 26145\,x^6 - 1090215\,x^4 + 16360785\,x^2$$
$$p_{11} = -9823275\;,\quad q_{11} = -36007335$$

$$P^*_{11}(x) = x^{11} + 99\,x^9 + 6237\,x^7 + 218295\,x^5 + 3274425\,x^3$$
$$Q^*_{11}(x) = -11\,x^{10} - 891\,x^8 - 43659\,x^6 - 1091475\,x^4 - 9823275\,x^2$$
$$R^*_{11}(x) = 20\,x^9 + 2844\,x^7 + 174384\,x^5 + 4362120\,x^3 - 9823275\,x$$
$$R^*_{11}(x) = -x^{10} - 279\,x^8 - 26145\,x^6 - 1090215\,x^4 - 16360785\,x^2$$
$$p^*_{11} = -9823275\ , \quad q^*_{11} = -36007335$$

$$P_{13}(x) = -x^{13} + 143\,x^{11} - 14157\,x^9 + 891891\,x^7 - 31216185\,x^5 + 468242775\,x^3$$
$$Q_{13}(x) = 13\,x^{12} - 1573\,x^{10} + 127413\,x^8 - 6243237\,x^6 + 156080925\,x^4 - 1404728325\,x^2$$
$$R_{13}(x) = 24\,x^{11} - 5236\,x^9 + 556380\,x^7 - 30175992\,x^5 + 702369360\,x^3 + 1404728325\,x$$
$$S_{13}(x) = x^{12} - 407\,x^{10} + 61281\,x^8 - 4786551\,x^6 + 182096145\,x^4 - 2575350855\,x^2$$
$$p_{13} = 1404728325\ , \quad q_{13} = 5384807505$$
$$P^*_{13}(x) = x^{13} + 143\,x^{11} + 14157\,x^9 + 891891\,x^7 + 31216185\,x^5 + 468242775\,x^3$$
$$R^*_{13}(x) = -13\,x^{12} - 1573\,x^{10} - 127413\,x^8 - 6243237\,x^6 - 156080925\,x^4 - 1404728325\,x^2$$
$$R^*_{13}(x) = 24\,x^{11} + 5236\,x^9 + 556380\,x^7 + 30175992\,x^5 + 702369360\,x^3 - 1404728325\,x$$
$$S^*_{13}(x) = -x^{12} - 407\,x^{10} - 61281\,x^8 - 4786551\,x^6 - 182096145\,x^4 - 2575350855\,x^2$$
$$p^*_{13} = -1404728325\ , \quad q^*_{13} = -5384807505$$

$$P_{15}(x) = -x^{15} + 195\,x^{13} - 27885\,x^{11} + 2760615\,x^9 - 173918745\,x^7 +$$
$$+ 6087156075\,x^5 - 91307341125\,x^3$$
$$Q_{15}(x) = 15\,x^{14} - 2535\,x^{12} + 306735\,x^{10} - 24845535\,x^8 + 1217431215\,x^6 -$$
$$- 30435780375\,x^4 + 273922023375\,x^2$$
$$R_{15}(x) = 28\,x^{13} - 8684\,x^{11} + 1417416\,x^9 - 133467048\,x^7 + 6758371620\,x^5 -$$
$$- 150072822900\,x^3 - 273922023375\,x$$
$$S_{15}(x) = x^{14} - 559\,x^{12} + 123409\,x^{10} - 15517359\,x^8 + 1108188081\,x^6 -$$
$$- 39879014175\,x^4 + 541525809825\,x^2$$
$$p_{15} = -273922023375\ , \quad q_{15} = -1089369856575$$
$$P^*_{15}(x) = x^{15} + 195\,x^{13} + 27885\,x^{11} + 2760615\,x^9 + 173918745\,x^7 +$$
$$+ 6087156075\,x^5 + 91307341125\,x^3$$
$$Q^*_{15}(x) = -15\,x^{14} - 2535\,x^{12} - 306735\,x^{10} - 24845535\,x^8 - 1217431215\,x^6 -$$
$$- 30435780375\,x^4 - 273922023375\,x^2$$
$$R^*_{15}(x) = 28\,x^{13} + 8684\,x^{11} + 1417416\,x^9 + 133467048\,x^7 + 6758371620\,x^5 +$$
$$+ 150072822900\,x^3 - 273922023375\,x$$
$$S^*_{15}(x) = -x^{14} - 559\,x^{12} - 123409\,x^{10} - 15517359\,x^8 - 1108188081\,x^6 -$$
$$- 39879014175\,x^4 - 541525809825\,x^2$$
$$p^*_{15} = -273922023375\ , \quad q^*_{15} = -1089369856575$$

$$P_{17}(x) = -x^{17} + 255\,x^{15} - 49725\,x^{13} + 7110675\,x^{11} - 703956825\,x^9 + 44349279975\,x^7 -$$
$$- 1552224799125\,x^5 + 23283371986875\,x^3$$

$$Q_{17}(x) = 17\,x^{16} - 3825\,x^{14} + 646425\,x^{12} - 78217425\,x^{10} + 6335611425\,x^8 - 310444959825\,x^6 +$$
$$+7761123995625\,x^4 - 69850115960625\,x^2$$
$$R_{17}(x) = 32\,x^{15} - 13380\,x^{13} + 3106740\,x^{11} - 449780760\,x^9 + 39599497080\,x^7 - 1918173757500\,x^5 +$$
$$+41190404755500\,x^3 + 69850115960625\,x$$
$$S_{17}(x) = x^{16} - 735\,x^{14} + 223665\,x^{12} - 41284815\,x^{10} + 4751983665\,x^8 - 321545759535\,x^6 +$$
$$+11143093586625\,x^4 - 146854586253375\,x^2$$
$$p_{17} = 69850115960625\ , \quad q_{17} = 286554818174625$$
$$P^*_{17}(x) = x^{17} + 255\,x^{15} + 49725\,x^{13} + 7110675\,x^{11} + 703956825\,x^9 + 44349279975\,x^7 + 1552224799125\,x^5 +$$
$$+23283371986875\,x^3$$
$$Q^*_{17}(x) = -17\,x^{16} - 3825\,x^{14} - 646425\,x^{12} - 78217425\,x^{10} - 6335611425\,x^8 - 310444959825\,x^6 -$$
$$-7761123995625\,x^4 - 69850115960625\,x^2$$
$$R^*_{17}(x) = 32\,x^{15} + 13380\,x^{13} + 3106740\,x^{11} + 449780760\,x^9 + 39599497080\,x^7 + 1918173757500\,x^5 +$$
$$+41190404755500\,x^3 - 69850115960625\,x$$
$$S^*_{17}(x) = -x^{16} - 735\,x^{14} - 223665\,x^{12} - 41284815\,x^{10} - 4751983665\,x^8 - 321545759535\,x^6 -$$
$$-11143093586625\,x^4 - 146854586253375\,x^2$$
$$p^*_{17} = -69850115960625\ , \quad q^*_{17} = -286554818174625$$

$$P_{19}(x) = -x^{19} + 323\,x^{17} - 82365\,x^{15} + 16061175\,x^{13} - 2296748025\,x^{11} + 227378054475\,x^9 -$$
$$-14324817431925\,x^7 + 501368610117375\,x^5 - 7520529151760625\,x^3$$
$$Q_{19}(x) = 19\,x^{18} - 5491\,x^{16} + 1235475\,x^{14} - 208795275\,x^{12} + 25264228275\,x^{10} - 2046402490275\,x^8 +$$
$$+100273722023475\,x^6 - 2506843050586875\,x^4 + 22561587455281875\,x^2$$
$$R_{19}(x) = 36\,x^{17} - 19516\,x^{15} + 6111840\,x^{13} - 1259461320\,x^{11} + 170621631180\,x^9 - 14387211635940\,x^7 +$$
$$+675450216441000\,x^5 - 14142702127554000\,x^3 - 22561587455281875\,x$$
$$S_{19}(x) = x^{18} - 935\,x^{16} + 375105\,x^{14} - 95515095\,x^{12} + 16150822545\,x^{10} - 1762972735095\,x^8 +$$
$$+115035298883505\,x^6 - 3878619692322375\,x^4 + 49948635534422625\,x^2$$
$$p_{19} = -22561587455281875\ , \quad q_{19} = -95071810444986375$$
$$P^*_{19}(x) = x^{19} + 323\,x^{17} + 82365\,x^{15} + 16061175\,x^{13} + 2296748025\,x^{11} + 227378054475\,x^9 +$$
$$+14324817431925\,x^7 + 501368610117375\,x^5 + 7520529151760625\,x^3$$
$$Q^*_{19}(x) = -19\,x^{18} - 5491\,x^{16} - 1235475\,x^{14} - 208795275\,x^{12} - 25264228275\,x^{10} - 2046402490275\,x^8 -$$
$$-100273722023475\,x^6 - 2506843050586875\,x^4 - 22561587455281875\,x^2$$
$$R^*_{19}(x) = 36\,x^{17} + 19516\,x^{15} + 6111840\,x^{13} + 1259461320\,x^{11} + 170621631180\,x^9 + 14387211635940\,x^7 +$$
$$+675450216441000\,x^5 + 14142702127554000\,x^3 - 22561587455281875\,x$$
$$S^*_{19}(x) = -x^{18} - 935\,x^{16} - 375105\,x^{14} - 95515095\,x^{12} - 16150822545\,x^{10} - 1762972735095\,x^8 -$$
$$-115035298883505\,x^6 - 3878619692322375\,x^4 - 49948635534422625\,x^2$$
$$p^*_{19} = -22561587455281875\ , \quad q^*_{19} = -95071810444986375$$

Recurrence formulas:

$$\int x^{2n+1}\cdot\ln x\cdot J_1(x)\,dx = x^{2n}\ln x\,[(2n+1)J_1(x) - xJ_0(x)] +$$
$$+\int x^{2n}J_0(x)\,dx - (2n+1)\int x^{2n-1}J_1(x)\,dx - (4n^2-1)\int x^{2n-1}\cdot\ln x\cdot J_1(x)\,dx$$
$$\int x^{2n+1}\cdot\ln x\cdot I_1(x)\,dx = -x^{2n}\ln x\,[xI_0(x) - (2n+1)I_1(x)] -$$
$$-\int x^{2n}I_0(x)\,dx + (2n+1)\int x^{2n-1}I_1(x)\,dx + (4n^2-1)\int x^{2n-1}\cdot\ln x\cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu x(x)\,dx$ are described before.

1.2.12. Integrals of the type $\int x^n e^{\pm x} \ln x \cdot Z_\nu(x)\,dx$

n = 0:

$$\int e^x \ln x\, I_0(x)\,dx = e^x \{(1-2x)\, I_0(x) + 2x\, I_1(x) + \ln x\, [x\, I_0(x) - x\, I_1(x)]\}$$

$$\int e^{-x} \ln x\, I_0(x)\,dx = e^{-x} \{-(1+2x)\, I_0(x) - 2x\, I_1(x) + \ln x\, [x\, I_0(x) + x\, I_1(x)]\}$$

$$\int e^x \ln x\, K_0(x)\,dx = e^x \{(1-2x)\, K_0(x) - 2x\, K_1(x) + \ln x\, [x\, K_0(x) + x\, K_1(x)]\}$$

$$\int e^{-x} \ln x\, K_0(x)\,dx = e^{-x} \{-(1+2x)\, K_0(x) + 2x\, K_1(x) + \ln x\, [x\, K_0(x) - x\, K_1(x)]\}$$

n = 1:

$$\int x\, e^x \ln x\, I_0(x)\,dx =$$

$$= \frac{e^x}{9} \{(-2\,x^2 + 3\,x - 3)\, I_0(x) + (2\,x^2 - 2\,x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) - (3\,x^2 - 3\,x)\, I_1(x)]\}$$

$$\int x\, e^{-x} \ln x\, I_0(x)\,dx =$$

$$= \frac{e^{-x}}{9} \{-(2\,x^2 + 3\,x + 3)\, I_0(x) - (2\,x^2 + 2\,x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3\,x^2 + 3\,x)\, I_1(x)]\}$$

$$\int x\, e^x \ln x\, K_0(x)\,dx =$$

$$= \frac{e^x}{9} \{(-2\,x^2 + 3\,x - 3)\, K_0(x) + (-2\,x^2 + 2\,x)\, K_1(x) + \ln x\, [3\,x^2\, K_0(x) + (3\,x^2 - 3\,x)\, K_1(x)]\}$$

$$\int x\, e^{-x} \ln x\, K_0(x)\,dx =$$

$$= \frac{e^{-x}}{9} \{-(2\,x^2 + 3\,x + 3)\, K_0(x) + (2\,x^2 + 2\,x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) - (3\,x^2 + 3\,x)\, K_1(x)]\}$$

$$\int x\, e^x \ln x\, I_1(x)\,dx =$$

$$= \frac{e^x}{9} \{(2\,x^2 + 6\,x - 6)\, I_0(x) - (2\,x^2 + 7\,x)\, I_1(x) + \ln x\, [-3x^2\, I_0(x) + (3\,x^2 + 6\,x)\, I_1(x)]\}$$

$$\int x\, e^{-x} \ln x\, I_1(x)\,dx =$$

$$= \frac{e^{-x}}{9} \{(-2x^2 + 6x + 6)\, I_0(x) - (2x^2 - 7x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3x^2 - 6x)\, I_1(x)]\}$$

$$\int x\, e^x \ln x\, K_1(x)\,dx =$$

$$= \frac{e^x}{9} \{-(2x^2 + 6x - 6)\, K_0(x) - (2x^2 + 7x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) + (3x^2 + 6x)\, K_1(x)]\}$$

$$\int x\, e^{-x} \ln x\, K_1(x)\,dx =$$

$$= \frac{e^{-x}}{9} \{(2x^2 - 6x - 6)\, K_0(x) - (2x^2 - 7x)\, K_1(x) - \ln x\, [3x^2\, K_0(x) - (3x^2 - 3x)\, K_1(x)]\}$$

n = 2:

$$\int x^2\, e^x \ln x\, I_0(x)\,dx = \frac{e^x}{225} \{(-18\,x^3 + 13\,x^2 - 60\,x + 60)\, I_0(x) + (18\,x^3 - 4\,x^2 + 4\,x)\, I_1(x) +$$

$$+ \ln x\, [(45\,x^3 + 30\,x^2)\, I_0(x) + (-45\,x^3 + 60\,x^2 - 60\,x)\, I_1(x)]\}$$

$$\int x^2\, e^{-x}\, \ln x\, I_0(x)\, dx = \frac{e^{-x}}{225}\, \{-(18\, x^3 + 13\, x^2 + 60\, x + 60)\, I_0(x) - (18\, x^3 + 4\, x^2 + 4\, x)\, I_1(x) +$$
$$+ \ln x\, [(45\, x^3 - 30\, x^2)\, I_0(x) + (45\, x^3 + 60\, x^2 + 60\, x)\, I_1(x)]\, \}$$
$$\int x^2\, e^{x}\, \ln x\, K_0(x)\, dx = \frac{e^{x}}{225}\, \{(-18\, x^3 + 13\, x^2 - 60\, x + 60)\, K_0(x) + (-18\, x^3 + 4\, x^2 - 4\, x)\, K_1(x) +$$
$$+ \ln x\, [(45\, x^3 + 30\, x^2)\, K_0(x) + (45\, x^3 - 60\, x^2 + 60\, x)\, K_1(x)]\, \}$$
$$\int x^2\, e^{-x}\, \ln x\, K_0(x)\, dx = \frac{e^{-x}}{225}\, \{-(18\, x^3 + 13\, x^2 + 60\, x + 60)\, K_0(x) + (18\, x^3 + 4\, x^2 + 4\, x)\, K_1(x) +$$
$$+ \ln x\, [(45\, x^3 - 30\, x^2)\, K_0(x) - (45\, x^3 + 60\, x^2 + 60\, x)\, K_1(x)]\, \}$$
$$\int x^2\, e^{x}\, \ln x\, I_1(x)\, dx = \frac{e^{x}}{75}\, \{(6\, x^3 + 4\, x^2 - 30\, x + 30)\, I_0(x) + (-6\, x^3 - 7\, x^2 + 7\, x)\, I_1(x) +$$
$$+ \ln x\, [(-15\, x^3 + 15\, x^2)\, I_0(x) + (15\, x^3 + 30\, x^2 - 30\, x)\, I_1(x)]\, \}$$
$$\int x^2\, e^{-x}\, \ln x\, I_1(x)\, dx = \frac{e^{-x}}{75}\, \{(-6\, x^3 + 4\, x^2 + 30\, x + 30)\, I_0(x) + (-6\, x^3 + 7\, x^2 + 7\, x)\, I_1(x) +$$
$$+ \ln x\, [(15\, x^3 + 15\, x^2)\, I_0(x) + (15\, x^3 - 30\, x^2 - 30\, x)\, I_1(x)]\, \}$$
$$\int x^2\, e^{x}\, \ln x\, K_1(x)\, dx = \frac{e^{x}}{75}\, \{(-6\, x^3 - 4\, x^2 + 30\, x - 30)\, K_0(x) + (-6\, x^3 - 7\, x^2 + 7\, x)\, K_1(x) +$$
$$+ \ln x\, [(15\, x^3 - 15\, x^2)\, K_0(x) + (15\, x^3 + 30\, x^2 - 30\, x)\, K_1(x)]\, \}$$
$$\int x^2\, e^{-x}\, \ln x\, K_1(x)\, dx = \frac{e^{-x}}{75}\, \{(6\, x^3 - 4\, x^2 - 30\, x - 30)\, K_0(x) + (-6\, x^3 + 7\, x^2 + 7\, x)\, K_1(x) +$$
$$+ \ln x\, [(-15\, x^3 - 15\, x^2)\, K_0(x) + (15\, x^3 - 30\, x^2 - 30\, x)\, K_1(x)]\, \}$$

n = 3:

$$\int x^3\, e^{x}\, \ln x\, I_0(x)\, dx =$$
$$= \frac{e^{x}}{1225}\, \{(-50\, x^4 + 31\, x^3 - 171\, x^2 + 420\, x - 420)\, I_0(x) + (50\, x^4 - 6\, x^3 - 132\, x^2 + 132\, x)\, I_1(x) +$$
$$+ \ln x\, [(175\, x^4 + 210\, x^3 - 210\, x^2)\, I_0(x) + (-175\, x^4 + 315\, x^3 - 420\, x^2 + 420\, x)\, I_1(x)]\, \}$$
$$\int x^3\, e^{-x}\, \ln x\, I_0(x)\, dx =$$
$$= \frac{e^{-x}}{1225}\, \{-(50\, x^4 + 31\, x^3 + 171\, x^2 + 420\, x + 420)\, I_0(x) - (50\, x^4 + 6\, x^3 - 132\, x^2 - 132\, x)\, I_1(x) +$$
$$+ \ln x\, [(175\, x^4 - 210\, x^3 - 210\, x^2)\, I_0(x) + (175\, x^4 + 315\, x^3 + 420\, x^2 + 420\, x)\, I_1(x)]\, \}$$
$$\int x^3\, e^{x}\, \ln x\, K_0(x)\, dx =$$
$$= \frac{e^{x}}{1225}\, \{(-50\, x^4 + 31\, x^3 - 171\, x^2 + 420\, x - 420)\, K_0(x) + (-50\, x^4 + 6\, x^3 + 132\, x^2 - 132\, x)\, K_1(x) +$$
$$+ \ln x\, [(175\, x^4 + 210\, x^3 - 210\, x^2)\, K_0(x) + (175\, x^4 - 315\, x^3 + 420\, x^2 - 420\, x)\, K_1(x)]\, \}$$
$$\int x^3\, e^{-x}\, \ln x\, K_0(x)\, dx =$$
$$= \frac{e^{-x}}{1225}\, \{-(50\, x^4 + 31\, x^3 + 171\, x^2 + 420\, x + 420)\, K_0(x) + (50\, x^4 + 6\, x^3 - 132\, x^2 - 132\, x)\, K_1(x) +$$
$$+ \ln x\, [(175\, x^4 - 210\, x^3 - 210\, x^2)\, K_0(x) - (175\, x^4 + 315\, x^3 + 420\, x^2 + 420\, x)\, K_1(x)]\, \}$$
$$\int x^3\, e^{x}\, \ln x\, I_1(x)\, dx =$$
$$= \frac{e^{x}}{3675}\, \{(150\, x^4 + 54\, x^3 - 614\, x^2 + 1680\, x - 1680)\, I_0(x) + (-150\, x^4 - 129\, x^3 - 388\, x^2 + 388\, x)\, I_1(x) +$$

$$+\ln x\,[(-525\,x^4+840\,x^3-840\,x^2)\,I_0(x)+(525\,x^4+1260\,x^3-1680\,x^2+1680\,x)\,I_1(x)]\,\}$$

$$\int x^3\,e^{-x}\,\ln x\,I_1(x)\,dx=$$

$$=\frac{e^{-x}}{3675}\,\{(-150\,x^4+54\,x^3+614\,x^2+1680\,x+1680)\,I_0(x)+(-150\,x^4+129\,x^3-388\,x^2-388\,x)\,I_1(x)+$$

$$+\ln x\,[(525\,x^4+840\,x^3+840\,x^2)\,I_0(x)+(525\,x^4-1260\,x^3-1680\,x^2-1680\,x)\,I_1(x)]\,\}$$

$$\int x^3\,e^{x}\,\ln x\,K_1(x)\,dx=$$

$$=\frac{e^{x}}{3675}\,\{(-150\,x^4-54\,x^3+614\,x^2-1680\,x+1680)\,K_0(x)+(-150\,x^4-129\,x^3-388\,x^2+388\,x)\,K_1(x)+$$

$$+\ln x\,[(525\,x^4-840\,x^3+840\,x^2)\,K_0(x)+(525\,x^4+1260\,x^3-1680\,x^2+1680\,x)\,K_1(x)]\,\}$$

$$\int x^3\,e^{-x}\,\ln x\,K_1(x)\,dx=$$

$$=\frac{e^{-x}}{3675}\,\{(150\,x^4-54\,x^3-614\,x^2-1680\,x-1680)\,K_0(x)+(-150\,x^4+129\,x^3-388\,x^2-388\,x)\,K_1(x)+$$

$$+\ln x\,[(-525\,x^4-840\,x^3-840\,x^2)\,K_0(x)+(525\,x^4-1260\,x^3-1680\,x^2-1680\,x)\,K_1(x)]\,\}$$

n = 4:

$$\int x^4\,e^{x}\,\ln x\,I_0(x)\,dx=\frac{e^{x}}{99225}\,\{(-2450\,x^5+1425\,x^4-12864\,x^3+33024\,x^2-60480\,x+60480)\,I_0(x)+$$

$$+(2450\,x^5-200\,x^4-11736\,x^3+35808\,x^2-35808\,x)\,I_1(x)+\ln x\,[(11025\,x^5+18900\,x^4-30240\,x^3+30240\,x^2)\,I_0(x)+$$

$$+(-11025\,x^5+25200\,x^4-45360\,x^3+60480\,x^2-60480\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{99225}\,\{-(2450\,x^5+1425\,x^4+12864\,x^3+33024\,x^2+60480\,x+60480)\,I_0(x)-$$

$$-(2450\,x^5+200\,x^4-11736\,x^3-35808\,x^2-35808\,x)\,I_1(x)+\ln x\,[(11025\,x^5-18900\,x^4-30240\,x^3-30240\,x^2)\,I_0(x)+$$

$$+(11025\,x^5+25200\,x^4+45360\,x^3+60480\,x^2+60480\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,K_0(x)\,dx=\frac{e^{x}}{99225}\,\{(-2450\,x^5+1425\,x^4-12864\,x^3+33024\,x^2-60480\,x+60480)\,K_0(x)+$$

$$+(-2450\,x^5+200\,x^4+11736\,x^3-35808\,x^2+35808\,x)\,K_1(x)+\ln x\,[(11025\,x^5+18900\,x^4-30240\,x^3+$$

$$+30240\,x^2)\,K_0(x)+(11025\,x^5-25200\,x^4+45360\,x^3-60480\,x^2+60480\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,K_0(x)\,dx=\frac{e^{-x}}{99225}\,\{-(2450\,x^5+1425\,x^4+12864\,x^3+33024\,x^2+60480\,x+60480)\,K_0(x)+$$

$$+(2450\,x^5+200\,x^4-11736\,x^3-35808\,x^2-35808\,x)\,K_1(x)+\ln x\,[(11025\,x^5-18900\,x^4-30240\,x^3-$$

$$-30240\,x^2)\,K_0(x)-(11025\,x^5+25200\,x^4+45360\,x^3+60480\,x^2+60480\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,I_1(x)\,dx=\frac{e^{x}}{19845}\,\{(490\,x^5+120\,x^4-2838\,x^3+7878\,x^2-15120\,x+15120)\,I_0(x)+$$

$$+(-490\,x^5-365\,x^4-2367\,x^3+8196\,x^2-8196\,x)\,I_1(x)+\ln x\,[(-2205\,x^5+4725\,x^4-7560\,x^3+7560\,x^2)\,I_0(x)+$$

$$+(2205\,x^5+6300\,x^4-11340\,x^3+15120\,x^2-15120\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,I_1(x)\,dx=\frac{e^{-x}}{19845}\,\{(-490\,x^5+120\,x^4+2838\,x^3+7878\,x^2+15120\,x+15120)\,I_0(x)+$$

$$+(-490\,x^5+365\,x^4-2367\,x^3-8196\,x^2-8196\,x)\,I_1(x)+\ln x\,[(2205\,x^5+4725\,x^4+7560\,x^3+7560\,x^2)\,I_0(x)+$$

$$+(2205\,x^5-6300\,x^4-11340\,x^3-15120\,x^2-15120\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,K_1(x)\,dx=\frac{e^{x}}{19845}\,\{(-490\,x^5-120\,x^4+2838\,x^3-7878\,x^2+15120\,x-15120)\,K_0(x)+$$

$$+(-490\,x^5-365\,x^4-2367\,x^3+8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(2205\,x^5-4725\,x^4+7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5+6300\,x^4-11340\,x^3+15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{19845}\,\{(490\,x^5-120\,x^4-2838\,x^3-7878\,x^2-15120\,x-15120)\,K_0(x)+$$
$$+(-490\,x^5+365\,x^4-2367\,x^3-8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(-2205\,x^5-4725\,x^4-7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5-6300\,x^4-11340\,x^3-15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

n = 5:

$$\int x^5\,e^{x}\,\ln x\,I_0(x)\,dx=$$
$$=\frac{e^{x}}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,I_0(x)+$$
$$+(7938\,x^6-490\,x^5-58280\,x^4+237960\,x^3-539040\,x^2+539040\,x)\,I_1(x)+$$
$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,I_0(x)+$$
$$+(-43659\,x^6+121275\,x^5-277200\,x^4+498960\,x^3-665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_0(x)\,dx=$$
$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,I_0(x)-$$
$$-(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,I_1(x)+$$
$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,I_0(x)+$$
$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{x}\,\ln x\,K_0(x)\,dx=$$
$$=\frac{e^{x}}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,K_0(x)+$$
$$+(-7938\,x^6+490\,x^5+58280\,x^4-237960\,x^3+539040\,x^2-539040\,x)\,K_1(x)+$$
$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,K_0(x)+$$
$$+(43659\,x^6-121275\,x^5+277200\,x^4-498960\,x^3+665280\,x^2-665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,K_0(x)\,dx=$$
$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,K_0(x)+$$
$$+(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,K_1(x)+$$
$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,K_0(x)-$$
$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{x}\,\ln x\,I_1(x)\,dx=$$
$$=\frac{e^{x}}{800415}\,\{(13230\,x^6+2450\,x^5-108210\,x^4+405984\,x^3-849504\,x^2+1330560\,x-1330560)\,I_0(x)+$$
$$+(-13230\,x^6-9065\,x^5-98080\,x^4+442656\,x^3-1033728\,x^2+1033728\,x)\,I_1(x)+$$
$$+\ln x\,[(-72765\,x^6+194040\,x^5-415800\,x^4+665280\,x^3-665280\,x^2)\,I_0(x)+$$
$$+(72765\,x^6+242550\,x^5-554400\,x^4+997920\,x^3-1330560\,x^2+1330560\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_1(x)\,dx=$$

$$= \frac{e^{-x}}{800415}\{(-13230\,x^6+2450\,x^5+108210\,x^4+405984\,x^3+849504\,x^2+1330560\,x+1330560)\,I_0(x)+$$
$$+(-13230\,x^6+9065\,x^5-98080\,x^4-442656\,x^3-1033728\,x^2-1033728\,x)\,I_1(x)+$$
$$+\ln x\,[(72765\,x^6+194040\,x^5+415800\,x^4+665280\,x^3+665280\,x^2)\,I_0(x)+$$
$$+(72765\,x^6-242550\,x^5-554400\,x^4-997920\,x^3-1330560\,x^2-1330560\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^x\,\ln x\,K_1(x)\,dx =$$
$$= \frac{e^{x}}{800415}\{(-13230\,x^6-2450\,x^5+108210\,x^4-405984\,x^3+849504\,x^2-1330560\,x+1330560)\,K_0(x)+$$
$$+(-13230\,x^6-9065\,x^5-98080\,x^4+442656\,x^3-1033728\,x^2+1033728\,x)\,K_1(x)+$$
$$+\ln x\,[(72765\,x^6-194040\,x^5+415800\,x^4-665280\,x^3+665280\,x^2)\,K_0(x)+$$
$$+(72765\,x^6+242550\,x^5-554400\,x^4+997920\,x^3-1330560\,x^2+1330560\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,K_1(x)\,dx =$$
$$= \frac{e^{-x}}{800415}\{(13230\,x^6-2450\,x^5-108210\,x^4-405984\,x^3-849504\,x^2-1330560\,x-1330560)\,K_0(x)+$$
$$+(-13230\,x^6+9065\,x^5-98080\,x^4-442656\,x^3-1033728\,x^2-1033728\,x)\,K_1(x)+$$
$$+\ln x\,[(-72765\,x^6-194040\,x^5-415800\,x^4-665280\,x^3-665280\,x^2)\,K_0(x)+$$
$$+(72765\,x^6-242550\,x^5-554400\,x^4-997920\,x^3-1330560\,x^2-1330560\,x)\,K_1(x)]\,\}$$

n = 6:

$$\int x^6\,e^x\,\ln x\,I_0(x)\,dx =$$
$$= \frac{e^{x}}{9018009}\{(-106722\,x^7+58653\,x^6-1137388\,x^5+5114220\,x^4-14236800\,x^3+25768320\,x^2-34594560\,x+$$
$$+34594560)\,I_0(x)+(106722\,x^7-5292\,x^6-1106420\,x^5+5617760\,x^4-17030880\,x^3+34239360\,x^2-$$
$$-34239360\,x)\,I_1(x)+\ln x\,[(693693\,x^7+1891890\,x^6-5045040\,x^5+10810800\,x^4-17297280\,x^3+17297280\,x^2)\,I_0(x)+$$
$$+(-693693\,x^7+2270268\,x^6-6306300\,x^5+14414400\,x^4-25945920\,x^3+34594560\,x^2-34594560\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,I_0(x)\,dx =$$
$$= \frac{e^{-x}}{9018009}\{-(106722\,x^7+58653\,x^6+1137388\,x^5+5114220\,x^4+14236800\,x^3+25768320\,x^2+34594560\,x+$$
$$+34594560)\,I_0(x)-(106722\,x^7+5292\,x^6-1106420\,x^5-5617760\,x^4-17030880\,x^3-34239360\,x^2-$$
$$-34239360\,x)\,I_1(x)+\ln x\,[(693693\,x^7-1891890\,x^6-5045040\,x^5-10810800\,x^4-17297280\,x^3-17297280\,x^2)\,I_0(x)+$$
$$+(693693\,x^7+2270268\,x^6+6306300\,x^5+14414400\,x^4+25945920\,x^3+34594560\,x^2+34594560\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,K_0(x)\,dx =$$
$$= \frac{e^{x}}{9018009}\{(-106722\,x^7+58653\,x^6-1137388\,x^5+5114220\,x^4-14236800\,x^3+25768320\,x^2-34594560\,x+$$
$$+34594560)\,K_0(x)+(-106722\,x^7+5292\,x^6+1106420\,x^5-5617760\,x^4+17030880\,x^3-34239360\,x^2+$$
$$+34239360\,x)\,K_1(x)+\ln x\,[(693693\,x^7+1891890\,x^6-5045040\,x^5+10810800\,x^4-17297280\,x^3+$$
$$+17297280\,x^2)\,K_0(x)+(693693\,x^7-2270268\,x^6+6306300\,x^5-14414400\,x^4+25945920\,x^3-34594560\,x^2+$$
$$+34594560\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,K_0(x)\,dx =$$

$$= \frac{e^{-x}}{9018009} \{-(106722\,x^7 + 58653\,x^6 + 1137388\,x^5 + 5114220\,x^4 + 14236800\,x^3 + 25768320\,x^2 + 34594560\,x + $$
$$+ 34594560)\,K_0(x) + (106722\,x^7 + 5292\,x^6 - 1106420\,x^5 - 5617760\,x^4 - 17030880\,x^3 - 34239360\,x^2 - $$
$$- 34239360\,x)\,K_1(x) + \ln x\,[(693693\,x^7 - 1891890\,x^6 - 5045040\,x^5 - 10810800\,x^4 - 17297280\,x^3 - $$
$$- 17297280\,x^2)\,K_0(x) - (693693\,x^7 + 2270268\,x^6 + 6306300\,x^5 + 14414400\,x^4 + 25945920\,x^3 + 34594560\,x^2 + $$
$$+ 34594560\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{3864861} \{(45738\,x^7 + 6804\,x^6 - 508634\,x^5 + 2428410\,x^4 - 6912480\,x^3 + 12678240\,x^2 - $$
$$- 17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 - $$
$$- 16707840\,x)\,I_1(x) + \ln x\,[(-297297\,x^7 + 945945\,x^6 - 2522520\,x^5 + 5405400\,x^4 - 8648640\,x^3 + 8648640\,x^2)\,I_0(x) + $$
$$+ (297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{3864861} \{(-45738\,x^7 + 6804\,x^6 + 508634\,x^5 + 2428410\,x^4 + 6912480\,x^3 + 12678240\,x^2 + $$
$$+ 17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 - $$
$$- 16707840\,x)\,I_1(x) + \ln x\,[(297297\,x^7 + 945945\,x^6 + 2522520\,x^5 + 5405400\,x^4 + 8648640\,x^3 + 8648640\,x^2)\,I_0(x) + $$
$$+ (297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,K_1(x)\,dx = \frac{e^x}{3864861} \{(-45738\,x^7 - 6804\,x^6 + 508634\,x^5 - 2428410\,x^4 + 6912480\,x^3 - 12678240\,x^2 + $$
$$+ 17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 - $$
$$- 16707840\,x)\,K_1(x) + \ln x\,[(297297\,x^7 - 945945\,x^6 + 2522520\,x^5 - 5405400\,x^4 + 8648640\,x^3 - 8648640\,x^2)\,K_0(x) + $$
$$+ (297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,K_1(x)\,dx = \frac{e^{-x}}{3864861} \{(45738\,x^7 - 6804\,x^6 - 508634\,x^5 - 2428410\,x^4 - 6912480\,x^3 - 12678240\,x^2 - $$
$$- 17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 - $$
$$- 16707840\,x)\,K_1(x) + \ln x\,[(-297297\,x^7 - 945945\,x^6 - 2522520\,x^5 - 5405400\,x^4 - 8648640\,x^3 - 8648640\,x^2)\,K_0(x) + $$
$$+ (297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,K_1(x)]\,\}$$

n = 7:

$$\int x^7\,e^x\,\ln x\,I_0(x)\,dx = \frac{e^x}{13803075} \{(-122694\,x^8 + 66429\,x^7 - 1734705\,x^6 + 9530780\,x^5 - 33807900\,x^4 + $$
$$+ 84362880\,x^3 - 142020480\,x^2 + 172972800\,x - 172972800)\,I_0(x) + (122694\,x^8 - 5082\,x^7 - 1703268\,x^6 + $$
$$+ 10336900\,x^5 - 39071200\,x^4 + 104922720\,x^3 - 197554560\,x^2 + 197554560\,x)\,I_1(x) + $$
$$+ \ln x\,[(920205\,x^8 + 2972970\,x^7 - 9459450\,x^6 + 25225200\,x^5 - 54054000\,x^4 + 86486400\,x^3 - 86486400\,x^2)\,I_0(x) + $$
$$+ (-920205\,x^8 + 3468465\,x^7 - 11351340\,x^6 + 31531500\,x^5 - 72072000\,x^4 + 129729600\,x^3 - 172972800\,x^2 + $$
$$+ 172972800\,x)\,I_1(x)]\,\}$$

$$\int x^7\,e^{-x}\,\ln x\,I_0(x)\,dx = \frac{e^{-x}}{13803075} \{(-122694\,x^8 - 66429\,x^7 - 1734705\,x^6 - 9530780\,x^5 - 33807900\,x^4 - $$
$$- 84362880\,x^3 - 142020480\,x^2 - 172972800\,x - 172972800)\,I_0(x) + (-122694\,x^8 - 5082\,x^7 + 1703268\,x^6 + $$
$$+ 10336900\,x^5 + 39071200\,x^4 + 104922720\,x^3 + 197554560\,x^2 + 197554560\,x)\,I_1(x) + $$
$$+ \ln x\,[(920205\,x^8 - 2972970\,x^7 - 9459450\,x^6 - 25225200\,x^5 - 54054000\,x^4 - 86486400\,x^3 - 86486400\,x^2)\,I_0(x) + $$
$$+ (920205\,x^8 + 3468465\,x^7 + 11351340\,x^6 + 31531500\,x^5 + 72072000\,x^4 + 129729600\,x^3 + 172972800\,x^2 + $$
$$+ 172972800\,x)\,I_1(x)]\,\}$$

$$\int x^7\, e^x\, \ln x\, K_0(x)\, dx = \frac{e^x}{13803075}\, \{(-122694\, x^8 + 66429\, x^7 - 1734705\, x^6 + 9530780\, x^5 - 33807900\, x^4 +$$
$$+84362880\, x^3 - 142020480\, x^2 + 172972800\, x - 172972800)\, K_0(x) + (-122694\, x^8 + 5082\, x^7 + 1703268\, x^6 -$$
$$-10336900\, x^5 + 39071200\, x^4 - 104922720\, x^3 + 197554560\, x^2 - 197554560\, x)\, K_1(x) +$$
$$+\ln x\, [(920205\, x^8 + 2972970\, x^7 - 9459450\, x^6 + 25225200\, x^5 - 54054000\, x^4 + 86486400\, x^3 - 86486400\, x^2)\, K_0(x) +$$
$$+(920205\, x^8 - 3468465\, x^7 + 11351340\, x^6 - 31531500\, x^5 + 72072000\, x^4 - 129729600\, x^3 + 172972800\, x^2 -$$
$$-172972800\, x)\, K_1(x)]\, \}$$

$$\int x^7\, e^{-x}\, \ln x\, K_0(x)\, dx = \frac{e^{-x}}{13803075}\, \{-(122694\, x^8 + 66429\, x^7 + 1734705\, x^6 + 9530780\, x^5 + 33807900\, x^4 +$$
$$+84362880\, x^3 + 142020480\, x^2 + 172972800\, x + 172972800)\, K_0(x) + (122694\, x^8 + 5082\, x^7 - 1703268\, x^6 -$$
$$-10336900\, x^5 - 39071200\, x^4 - 104922720\, x^3 - 197554560\, x^2 - 197554560\, x)\, K_1(x) +$$
$$+\ln x\, [(920205\, x^8 - 2972970\, x^7 - 9459450\, x^6 - 25225200\, x^5 - 54054000\, x^4 - 86486400\, x^3 - 86486400\, x^2)\, K_0(x) -$$
$$-(920205\, x^8 + 3468465\, x^7 + 11351340\, x^6 + 31531500\, x^5 + 72072000\, x^4 + 129729600\, x^3 + 172972800\, x^2 +$$
$$+172972800\, x)\, K_1(x)]\, \}$$

$$\int x^7\, e^x\, \ln x\, I_1(x)\, dx = \frac{e^x}{96621525}\, \{(858858\, x^8 + 106722\, x^7 - 12526290\, x^6 + 72642640\, x^5 - 262741200\, x^4 +$$
$$+662547840\, x^3 - 1123808640\, x^2 + 1383782400\, x - 1383782400)\, I_0(x) + (-858858\, x^8 - 536151\, x^7 - 12004524\, x^6 +$$
$$+78190700\, x^5 - 302273600\, x^4 + 820848960\, x^3 - 1555726080\, x^2 + 1555726080\, x)\, I_1(x) +$$
$$+\ln x\, [(-6441435\, x^8 + 23783760\, x^7 - 75675600\, x^6 + 201801600\, x^5 - 432432000\, x^4 + 691891200\, x^3 -$$
$$-691891200\, x^2)\, I_0(x) + (6441435\, x^8 + 27747720\, x^7 - 90810720\, x^6 + 252252000\, x^5 - 576576000\, x^4 +$$
$$-1037836800\, x^3 - 1383782400\, x^2 + 1383782400\, x)\, I_1(x)]\, \}$$

$$\int x^7\, e^{-x}\, \ln x\, I_1(x)\, dx = \frac{e^{-x}}{96621525}\, \{(-858858\, x^8 + 106722\, x^7 + 12526290\, x^6 + 72642640\, x^5 + 262741200\, x^4 +$$
$$+662547840\, x^3 + 1123808640\, x^2 + 1383782400\, x + 1383782400)\, I_0(x) + (-858858\, x^8 + 536151\, x^7 - 12004524\, x^6 -$$
$$-78190700\, x^5 - 302273600\, x^4 - 820848960\, x^3 - 1555726080\, x^2 - 1555726080\, x)\, I_1(x) +$$
$$+\ln x\, [(6441435\, x^8 + 23783760\, x^7 + 75675600\, x^6 + 201801600\, x^5 + 432432000\, x^4 + 691891200\, x^3 +$$
$$+691891200\, x^2)\, I_0(x) + (6441435\, x^8 - 27747720\, x^7 - 90810720\, x^6 - 252252000\, x^5 - 576576000\, x^4 -$$
$$-1037836800\, x^3 - 1383782400\, x^2 - 1383782400\, x)\, I_1(x)]\, \}$$

$$\int x^7\, e^x\, \ln x\, K_1(x)\, dx = \frac{e^x}{96621525}\, \{(-858858\, x^8 - 106722\, x^7 + 12526290\, x^6 - 72642640\, x^5 + 262741200\, x^4 -$$
$$-662547840\, x^3 + 1123808640\, x^2 - 1383782400\, x + 1383782400)\, K_0(x) + (-858858\, x^8 - 536151\, x^7 - 12004524\, x^6 +$$
$$+78190700\, x^5 - 302273600\, x^4 + 820848960\, x^3 - 1555726080\, x^2 + 1555726080\, x)\, K_1(x) +$$
$$+\ln x\, [(6441435\, x^8 - 23783760\, x^7 + 75675600\, x^6 - 201801600\, x^5 + 432432000\, x^4 - 691891200\, x^3 +$$
$$+691891200\, x^2)\, K_0(x) + (6441435\, x^8 + 27747720\, x^7 - 90810720\, x^6 + 252252000\, x^5 - 576576000\, x^4 +$$
$$+1037836800\, x^3 - 1383782400\, x^2 + 1383782400\, x)\, K_1(x)]\, \}$$

$$\int x^7\, e^{-x}\, \ln x\, K_1(x)\, dx = \frac{e^{-x}}{96621525}\, \{(858858\, x^8 - 106722\, x^7 - 12526290\, x^6 - 72642640\, x^5 - 262741200\, x^4 -$$
$$-662547840\, x^3 - 1123808640\, x^2 - 1383782400\, x - 1383782400)\, K_0(x) + (-858858\, x^8 + 536151\, x^7 - 12004524\, x^6 -$$
$$-78190700\, x^5 - 302273600\, x^4 - 820848960\, x^3 - 1555726080\, x^2 - 1555726080\, x)\, K_1(x) +$$
$$+\ln x\, [(-6441435\, x^8 - 23783760\, x^7 - 75675600\, x^6 - 201801600\, x^5 - 432432000\, x^4 - 691891200\, x^3 -$$
$$-691891200\, x^2)\, K_0(x) + (6441435\, x^8 - 27747720\, x^7 - 90810720\, x^6 - 252252000\, x^5 - 576576000\, x^4 -$$

$$-1037836800\,x^3-1383782400\,x^2-1383782400\,x)\,K_1(x)]\,\}$$

<u>**n = 8:**</u>

$$\int x^8\,e^x\,\ln x\,I_0(x)\,dx=\frac{e^x}{3989088675}\,\{(-27606150\,x^9+14784627\,x^8-500382432\,x^7+3249519840\,x^6-$$
$$-14001917440\,x^5+44566771200\,x^4-104240855040\,x^3+166972323840\,x^2-188194406400\,x+$$
$$+188194406400)\,I_0(x)+(27606150\,x^9-981552\,x^8-493929744\,x^7+3487748544\,x^6-15787083200\,x^5+$$
$$+52887833600\,x^4-132836981760\,x^3+239847444480\,x^2-239847444480\,x)\,I_1(x)+$$
$$+\ln x\,[(234652275\,x^9+876035160\,x^8-3234591360\,x^7+10291881600\,x^6-27445017600\,x^5+58810752000\,x^4-$$
$$-94097203200\,x^3+94097203200\,x^2)\,I_0(x)+(-234652275\,x^9+1001183040\,x^8-3773689920\,x^7+$$
$$+12350257920\,x^6-34306272000\,x^5+78414336000\,x^4-141145804800\,x^3+188194406400\,x^2-$$
$$-188194406400\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{3989088675}\,\{(-27606150\,x^9-14784627\,x^8-500382432\,x^7-3249519840\,x^6-$$
$$-14001917440\,x^5-44566771200\,x^4-104240855040\,x^3-166972323840\,x^2-188194406400\,x-$$
$$-188194406400)\,I_0(x)+(-27606150\,x^9-981552\,x^8+493929744\,x^7+3487748544\,x^6+15787083200\,x^5+$$
$$+52887833600\,x^4+132836981760\,x^3+239847444480\,x^2+239847444480\,x)\,I_1(x)+$$
$$+\ln x\,[(234652275\,x^9-876035160\,x^8-3234591360\,x^7-10291881600\,x^6-27445017600\,x^5-58810752000\,x^4-$$
$$-94097203200\,x^3-94097203200\,x^2)\,I_0(x)+(234652275\,x^9+1001183040\,x^8+3773689920\,x^7+$$
$$+12350257920\,x^6+34306272000\,x^5+78414336000\,x^4+141145804800\,x^3+188194406400\,x^2+$$
$$+188194406400\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^x\,\ln x\,K_0(x)\,dx=\frac{e^x}{3989088675}\,\{(-27606150\,x^9+14784627\,x^8-500382432\,x^7+3249519840\,x^6-$$
$$-14001917440\,x^5+44566771200\,x^4-104240855040\,x^3+166972323840\,x^2-188194406400\,x+$$
$$+188194406400)\,K_0(x)+(-27606150\,x^9+981552\,x^8+493929744\,x^7-3487748544\,x^6+15787083200\,x^5-$$
$$-52887833600\,x^4+132836981760\,x^3-239847444480\,x^2+239847444480\,x)\,K_1(x)+$$
$$+\ln x\,[(234652275\,x^9+876035160\,x^8-3234591360\,x^7+10291881600\,x^6-27445017600\,x^5+58810752000\,x^4-$$
$$-94097203200\,x^3+94097203200\,x^2)\,K_0(x)+(234652275\,x^9-1001183040\,x^8+3773689920\,x^7-12350257920\,x^6+$$
$$+34306272000\,x^5-78414336000\,x^4+141145804800\,x^3-188194406400\,x^2+188194406400\,x)\,K_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,K_0(x)\,dx=\frac{e^{-x}}{3989088675}\,\{-(27606150\,x^9+14784627\,x^8+500382432\,x^7+3249519840\,x^6+$$
$$+14001917440\,x^5+44566771200\,x^4+104240855040\,x^3+166972323840\,x^2+188194406400\,x+$$
$$+188194406400)\,K_0(x)+(27606150\,x^9+981552\,x^8-493929744\,x^7-3487748544\,x^6-15787083200\,x^5-$$
$$-52887833600\,x^4-132836981760\,x^3-239847444480\,x^2-239847444480\,x)\,K_1(x)+$$
$$+\ln x\,[(234652275\,x^9-876035160\,x^8-3234591360\,x^7-10291881600\,x^6-27445017600\,x^5-58810752000\,x^4-$$
$$-94097203200\,x^3-94097203200\,x^2)\,K_0(x)-(234652275\,x^9+1001183040\,x^8+3773689920\,x^7+12350257920\,x^6+$$
$$+34306272000\,x^5+78414336000\,x^4+141145804800\,x^3+188194406400\,x^2+188194406400\,x)\,K_1(x)]\,\}$$

$$\int x^8\,e^x\,\ln x\,I_1(x)\,dx=\frac{e^x}{443232075}\,\{(3067350\,x^9+327184\,x^8-56932194\,x^7+388322130\,x^6-1702592080\,x^5+$$
$$+5468744400\,x^4-12866743680\,x^3+20708177280\,x^2-23524300800\,x+23524300800)\,I_0(x)+(-3067350\,x^9-$$
$$-1860859\,x^8-55189673\,x^7+414527148\,x^6-1913825900\,x^5+6474843200\,x^4-16359577920\,x^3+$$

$+29654204160\,x^2-29654204160\,x)\,I_1(x)+\ln x\,[(-26072475\,x^9+109504395\,x^8-404323920\,x^7+1286485200\,x^6-$
$-3430627200\,x^5+7351344000\,x^4-11762150400\,x^3+11762150400\,x^2)\,I_0(x)+(26072475\,x^9+125147880\,x^8-$
$-471711240\,x^7+1543782240\,x^6-4288284000\,x^5+9801792000\,x^4-17643225600\,x^3+23524300800\,x^2-$
$-23524300800\,x)\,I_1(x)]\,\}$

$\int x^8\,e^{-x}\,\ln x\,I_1(x)\,dx = \dfrac{e^{-x}}{443232075}\,\{(-3067350\,x^9+327184\,x^8+56932194\,x^7+388322130\,x^6+1702592080\,x^5+$
$+5468744400\,x^4+12866743680\,x^3+20708177280\,x^2+23524300800\,x+23524300800)\,I_0(x)+(-3067350\,x^9+$
$+1860859\,x^8-55189673\,x^7-414527148\,x^6-1913825900\,x^5-6474843200\,x^4-16359577920\,x^3-$
$-29654204160\,x^2-29654204160\,x)\,I_1(x)+\ln x\,[(26072475\,x^9+109504395\,x^8+404323920\,x^7+1286485200\,x^6+$
$+3430627200\,x^5+7351344000\,x^4+11762150400\,x^3+11762150400\,x^2)\,I_0(x)+(26072475\,x^9-125147880\,x^8-$
$-471711240\,x^7-1543782240\,x^6-4288284000\,x^5-9801792000\,x^4-17643225600\,x^3-23524300800\,x^2-$
$-23524300800\,x)\,I_1(x)]\,\}$

$\int x^8\,e^{x}\,\ln x\,K_1(x)\,dx = \dfrac{e^{x}}{443232075}\,\{(-3067350\,x^9-327184\,x^8+56932194\,x^7-388322130\,x^6+1702592080\,x^5-$
$-5468744400\,x^4+12866743680\,x^3-20708177280\,x^2+23524300800\,x-23524300800)\,K_0(x)+(-3067350\,x^9-$
$-1860859\,x^8-55189673\,x^7+414527148\,x^6-1913825900\,x^5+6474843200\,x^4-16359577920\,x^3+$
$+29654204160\,x^2-29654204160\,x)\,K_1(x)+\ln x\,[(26072475\,x^9-109504395\,x^8+404323920\,x^7-1286485200\,x^6+$
$+3430627200\,x^5-7351344000\,x^4+11762150400\,x^3-11762150400\,x^2)\,K_0(x)+(26072475\,x^9+125147880\,x^8-$
$-471711240\,x^7+1543782240\,x^6-4288284000\,x^5+9801792000\,x^4-17643225600\,x^3+23524300800\,x^2-$
$-23524300800\,x)\,K_1(x)]\,\}$

$\int x^8\,e^{-x}\,\ln x\,K_1(x)\,dx = \dfrac{e^{-x}}{443232075}\,\{(3067350\,x^9-327184\,x^8-56932194\,x^7-388322130\,x^6-1702592080\,x^5-$
$-5468744400\,x^4-12866743680\,x^3-20708177280\,x^2-23524300800\,x-23524300800)\,K_0(x)+(-3067350\,x^9+$
$+1860859\,x^8-55189673\,x^7-414527148\,x^6-1913825900\,x^5-6474843200\,x^4-16359577920\,x^3-$
$-29654204160\,x^2-29654204160\,x)\,K_1(x)+\ln x\,[(-26072475\,x^9-109504395\,x^8-404323920\,x^7-1286485200\,x^6-$
$-3430627200\,x^5-7351344000\,x^4-11762150400\,x^3-11762150400\,x^2)\,K_0(x)+(26072475\,x^9-125147880\,x^8-$
$-471711240\,x^7-1543782240\,x^6-4288284000\,x^5-9801792000\,x^4-17643225600\,x^3-23524300800\,x^2-$
$-23524300800\,x)\,K_1(x)]\,\}$

n = 9:

$\int x^9\,e^{x}\,\ln x\,I_0(x)\,dx = \dfrac{e^{x}}{53335593025}\,\{(-295488050\,x^{10}+156946075\,x^9-6682958139\,x^8+50085741024\,x^7-$
$-253835174880\,x^6+981076078080\,x^5-2932377638400\,x^4+6569043425280\,x^3-10144737146880\,x^2+$
$+10727081164800\,x-10727081164800)\,I_0(x)+(295488050\,x^{10}-9202050\,x^9-6618605136\,x^8+53311928208\,x^7-$
$-281136719808\,x^6+1128572222400\,x^5-3537368755200\,x^4+8512679992320\,x^3-14925933711360\,x^2+$
$+14925933711360\,x)\,I_1(x)+\ln x\,[(2807136475\,x^{10}+11889048600\,x^9-49934004120\,x^8+184371707520\,x^7-$
$-586637251200\,x^6+1564366003200\,x^5-3352212864000\,x^4+5363540582400\,x^3-5363540582400\,x^2)\,I_0(x)+$
$+(-2807136475\,x^{10}+13375179675\,x^9-57067433280\,x^8+215100325440\,x^7-703964701440\,x^6+$
$+1955457504000\,x^5-4469617152000\,x^4+8045310873600\,x^3-10727081164800\,x^2+10727081164800\,x)\,I_1(x)]\,\}$

$\int x^9\,e^{-x}\,\ln x\,I_0(x)\,dx = \dfrac{e^{-x}}{53335593025}\,\{(-295488050\,x^{10}-156946075\,x^9-6682958139\,x^8-50085741024\,x^7-$

$$-253835174880\,x^6 - 981076078080\,x^5 - 2932377638400\,x^4 - 6569043425280\,x^3 - 10144737146880\,x^2 -$$
$$-10727081164800\,x - 10727081164800)\,I_0(x) + (-295488050\,x^{10} - 9202050\,x^9 + 6618605136\,x^8 + 53311928208\,x^7 +$$
$$+281136719808\,x^6 + 1128572222400\,x^5 + 3537368755200\,x^4 + 8512679992320\,x^3 + 14925933711360\,x^2 +$$
$$+14925933711360\,x)\,I_1(x) + \ln x\,[(2807136475\,x^{10} - 11889048600\,x^9 - 49934004120\,x^8 - 184371707520\,x^7 -$$
$$-586637251200\,x^6 - 1564366003200\,x^5 - 3352212864000\,x^4 - 5363540582400\,x^3 - 5363540582400\,x^2)\,I_0(x) +$$
$$+(2807136475\,x^{10} + 13375179675\,x^9 + 57067433280\,x^8 + 215100325440\,x^7 + 703964701440\,x^6 + 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 + 8045310873600\,x^3 + 10727081164800\,x^2 + 10727081164800\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^x\,\ln x\,K_0(x)\,dx = \frac{e^x}{53335593025}\,\{(-295488050\,x^{10} + 156946075\,x^9 - 6682958139\,x^8 + 50085741024\,x^7 -$$
$$-253835174880\,x^6 + 981076078080\,x^5 - 2932377638400\,x^4 + 6569043425280\,x^3 - 10144737146880\,x^2 +$$
$$+10727081164800\,x - 10727081164800)\,K_0(x) + (-295488050\,x^{10} + 9202050\,x^9 + 6618605136\,x^8 - 53311928208\,x^7 +$$
$$+281136719808\,x^6 - 1128572222400\,x^5 + 3537368755200\,x^4 - 8512679992320\,x^3 + 14925933711360\,x^2 -$$
$$-14925933711360\,x)\,K_1(x) + \ln x\,[(2807136475\,x^{10} + 11889048600\,x^9 - 49934004120\,x^8 + 184371707520\,x^7 -$$
$$-586637251200\,x^6 + 1564366003200\,x^5 - 3352212864000\,x^4 + 5363540582400\,x^3 - 5363540582400\,x^2)\,K_0(x) +$$
$$+(2807136475\,x^{10} - 13375179675\,x^9 + 57067433280\,x^8 - 215100325440\,x^7 + 703964701440\,x^6 - 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 - 8045310873600\,x^3 + 10727081164800\,x^2 - 10727081164800\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,K_0(x)\,dx = \frac{e^{-x}}{53335593025}\,\{-(295488050\,x^{10} + 156946075\,x^9 + 6682958139\,x^8 + 50085741024\,x^7 +$$
$$+253835174880\,x^6 + 981076078080\,x^5 + 2932377638400\,x^4 + 6569043425280\,x^3 + 10144737146880\,x^2 +$$
$$+10727081164800\,x + 10727081164800)\,K_0(x) + (295488050\,x^{10} + 9202050\,x^9 - 6618605136\,x^8 - 53311928208\,x^7 -$$
$$-281136719808\,x^6 - 1128572222400\,x^5 - 3537368755200\,x^4 - 8512679992320\,x^3 - 14925933711360\,x^2 -$$
$$-14925933711360\,x)\,K_1(x) + \ln x\,[(2807136475\,x^{10} - 11889048600\,x^9 - 49934004120\,x^8 - 184371707520\,x^7 -$$
$$-586637251200\,x^6 - 1564366003200\,x^5 - 3352212864000\,x^4 - 5363540582400\,x^3 - 5363540582400\,x^2)\,K_0(x) -$$
$$-(2807136475\,x^{10} + 13375179675\,x^9 + 57067433280\,x^8 + 215100325440\,x^7 + 703964701440\,x^6 + 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 + 8045310873600\,x^3 + 10727081164800\,x^2 + 10727081164800\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{32001355815}\,\{(177292830\,x^{10} + 16563690\,x^9 - 4085423914\,x^8 + 32024777664\,x^7 -$$
$$-164877988800\,x^6 + 642462822400\,x^5 - 1930087219200\,x^4 + 4339632353280\,x^3 - 6723428167680\,x^2 +$$
$$+7151387443200\,x - 7151387443200)\,I_0(x) + (-177292830\,x^{10} - 105210105\,x^9 - 3989681696\,x^8 + 33947949728\,x^7 -$$
$$-182209926528\,x^6 + 737896611200\,x^5 - 2325137561600\,x^4 + 5615525099520\,x^3 - 9871162613760\,x^2 +$$
$$+9871162613760\,x)\,I_1(x) + \ln x\,[(-1684281885\,x^{10} + 7926032400\,x^9 - 33289336080\,x^8 + 122914471680\,x^7 -$$
$$-391091500800\,x^6 + 1042910668800\,x^5 - 2234808576000\,x^4 + 3575693721600\,x^3 - 3575693721600\,x^2)\,I_0(x) +$$
$$+(1684281885\,x^{10} + 8916786450\,x^9 - 38044955520\,x^8 + 143400216960\,x^7 - 469309800960\,x^6 + 1303638336000\,x^5 -$$
$$-2979744768000\,x^4 + 5363540582400\,x^3 - 7151387443200\,x^2 + 7151387443200\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{32001355815}\,\{(-177292830\,x^{10} + 16563690\,x^9 + 4085423914\,x^8 + 32024777664\,x^7 +$$
$$+164877988800\,x^6 + 642462822400\,x^5 + 1930087219200\,x^4 + 4339632353280\,x^3 + 6723428167680\,x^2 +$$
$$+7151387443200\,x + 7151387443200)\,I_0(x) + (-177292830\,x^{10} + 105210105\,x^9 - 3989681696\,x^8 - 33947949728\,x^7 -$$
$$-182209926528\,x^6 - 737896611200\,x^5 - 2325137561600\,x^4 - 5615525099520\,x^3 - 9871162613760\,x^2 -$$
$$-9871162613760\,x)\,I_1(x) + \ln x\,[(1684281885\,x^{10} + 7926032400\,x^9 + 33289336080\,x^8 + 122914471680\,x^7 +$$
$$+391091500800\,x^6 + 1042910668800\,x^5 + 2234808576000\,x^4 + 3575693721600\,x^3 + 3575693721600\,x^2)\,I_0(x) +$$

$$+(1684281885\,x^{10}-8916786450\,x^{9}-38044955520\,x^{8}-143400216960\,x^{7}-469309800960\,x^{6}-1303638336000\,x^{5}-$$
$$-2979744768000\,x^{4}-5363540582400\,x^{3}-7151387443200\,x^{2}-7151387443200\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^{x}\,\ln x\,K_1(x)\,dx=\frac{e^{x}}{32001355815}\,\{(-177292830\,x^{10}-16563690\,x^{9}+4085423914\,x^{8}-32024777664\,x^{7}+$$
$$+164877988800\,x^{6}-642462822400\,x^{5}+1930087219200\,x^{4}-4339632353280\,x^{3}+6723428167680\,x^{2}-$$
$$-7151387443200\,x+7151387443200)\,K_0(x)+(-177292830\,x^{10}-105210105\,x^{9}-3989681696\,x^{8}+33947949728\,x^{7}-$$
$$-182209926528\,x^{6}+737896611200\,x^{5}-2325137561600\,x^{4}+5615525099520\,x^{3}-9871162613760\,x^{2}+$$
$$+9871162613760\,x)\,K_1(x)+\ln x\,[(1684281885\,x^{10}-7926032400\,x^{9}+33289336080\,x^{8}-122914471680\,x^{7}+$$
$$+391091500800\,x^{6}-1042910668800\,x^{5}+2234808576000\,x^{4}-3575693721600\,x^{3}+3575693721600\,x^{2})\,K_0(x)+$$
$$+(1684281885\,x^{10}+8916786450\,x^{9}-38044955520\,x^{8}+143400216960\,x^{7}-469309800960\,x^{6}+1303638336000\,x^{5}-$$
$$-2979744768000\,x^{4}+5363540582400\,x^{3}-7151387443200\,x^{2}+7151387443200\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{32001355815}\,\{(177292830\,x^{10}-16563690\,x^{9}-4085423914\,x^{8}-32024777664\,x^{7}-$$
$$-164877988800\,x^{6}-642462822400\,x^{5}-1930087219200\,x^{4}-4339632353280\,x^{3}-6723428167680\,x^{2}-$$
$$-7151387443200\,x-7151387443200)\,K_0(x)+(-177292830\,x^{10}+105210105\,x^{9}-3989681696\,x^{8}-33947949728\,x^{7}-$$
$$-182209926528\,x^{6}-737896611200\,x^{5}-2325137561600\,x^{4}-5615525099520\,x^{3}-9871162613760\,x^{2}-$$
$$-9871162613760\,x)\,K_1(x)+\ln x\,[(-1684281885\,x^{10}-7926032400\,x^{9}-33289336080\,x^{8}-122914471680\,x^{7}-$$
$$-391091500800\,x^{6}-1042910668800\,x^{5}-2234808576000\,x^{4}-3575693721600\,x^{3}-3575693721600\,x^{2})\,K_0(x)+$$
$$+(1684281885\,x^{10}-8916786450\,x^{9}-38044955520\,x^{8}-143400216960\,x^{7}-469309800960\,x^{6}-1303638336000\,x^{5}-$$
$$-2979744768000\,x^{4}-5363540582400\,x^{3}-7151387443200\,x^{2}-7151387443200\,x)\,K_1(x)]\,\}$$

n = 10:

$$\int x^{10}\,e^{x}\,\ln x\,I_0(x)\,dx=\frac{e^{x}}{940839860961}\,\{(-4266847442\,x^{11}+2251618941\,x^{10}-117807097980\,x^{9}+$$
$$+1000787719932\,x^{8}-5829673272192\,x^{7}+26484562500480\,x^{6}-96176811386880\,x^{5}+275819194828800\,x^{4}-$$
$$-598998804848640\,x^{3}+899357077463040\,x^{2}-901074817843200\,x+901074817843200)\,I_0(x)+$$
$$+(4266847442\,x^{11}-118195220\,x^{10}-116928608940\,x^{9}+1058156244288\,x^{8}-6371084833344\,x^{7}+$$
$$+29810373836544\,x^{6}-112008092716800\,x^{5}+336471606374400\,x^{4}-785863855042560\,x^{3}+$$
$$+1348176746004480\,x^{2}-1348176746004480\,x)\,I_1(x)+\ln x\,[(44801898141\,x^{11}+212219517510\,x^{10}-$$
$$-998680082400\,x^{9}+4194456346080\,x^{8}-15487223431680\,x^{7}+49277529100800\,x^{6}-131406744268800\,x^{5}+$$
$$+281585880576000\,x^{4}-450537408921600\,x^{3}+450537408921600\,x^{2})\,I_0(x)+$$
$$+(-44801898141\,x^{11}+235799463900\,x^{10}-1123515092700\,x^{9}+4793664395520\,x^{8}-18068427336960\,x^{7}+$$
$$+59133034920960\,x^{6}-164258430336000\,x^{5}+375447840768000\,x^{4}-675806113382400\,x^{3}+$$
$$+901074817843200\,x^{2}-901074817843200\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{940839860961}\,\{(-4266847442\,x^{11}-2251618941\,x^{10}-117807097980\,x^{9}-$$
$$-1000787719932\,x^{8}-5829673272192\,x^{7}-26484562500480\,x^{6}-96176811386880\,x^{5}-275819194828800\,x^{4}-$$
$$-598998804848640\,x^{3}-899357077463040\,x^{2}-901074817843200\,x-901074817843200)\,I_0(x)+$$
$$+(-4266847442\,x^{11}-118195220\,x^{10}+116928608940\,x^{9}+1058156244288\,x^{8}+6371084833344\,x^{7}+$$
$$+29810373836544\,x^{6}+112008092716800\,x^{5}+336471606374400\,x^{4}+785863855042560\,x^{3}+1348176746004480\,x^{2}+$$
$$+1348176746004480\,x)\,I_1(x)+\ln x\,[(44801898141\,x^{11}-212219517510\,x^{10}-998680082400\,x^{9}-4194456346080\,x^{8}-$$

$$-15487223431680\,x^7-49277529100800\,x^6-131406744268800\,x^5-281585880576000\,x^4-$$
$$-450537408921600\,x^3-450537408921600\,x^2)\,I_0(x)+(44801898141\,x^{11}+235799463900\,x^{10}+$$
$$+1123515092700\,x^9+4793664395520\,x^8+18068427336960\,x^7+59133034920960\,x^6+164258430336000\,x^5+$$
$$+375447840768000\,x^4+675806113382400\,x^3+901074817843200\,x^2+901074817843200\,x)\,I_1(x)]\,\}$$
$$\int x^{10}\,e^x\,\ln x\,K_0(x)\,dx=\frac{e^x}{940839860961}\,\{(-4266847442\,x^{11}+2251618941\,x^{10}-117807097980\,x^9+$$
$$+1000787719932\,x^8-5829673272192\,x^7+26484562500480\,x^6-96176811386880\,x^5+275819194828800\,x^4-$$
$$-598998804848640\,x^3+899357077463040\,x^2-901074817843200\,x+901074817843200)\,K_0(x)+$$
$$+(-4266847442\,x^{11}+118195220\,x^{10}+116928608940\,x^9-1058156244288\,x^8+6371084833344\,x^7-$$
$$-29810373836544\,x^6+112008092716800\,x^5-336471606374400\,x^4+785863855042560\,x^3-1348176746004480\,x^2+$$
$$+1348176746004480\,x)\,K_1(x)+\ln x\,[(44801898141\,x^{11}+212219517510\,x^{10}-998680082400\,x^9+$$
$$+4194456346080\,x^8-15487223431680\,x^7+49277529100800\,x^6-131406744268800\,x^5+281585880576000\,x^4-$$
$$-450537408921600\,x^3+450537408921600\,x^2)\,K_0(x)+(44801898141\,x^{11}-235799463900\,x^{10}+$$
$$+1123515092700\,x^9-4793664395520\,x^8+18068427336960\,x^7-59133034920960\,x^6+164258430336000\,x^5-$$
$$-375447840768000\,x^4+675806113382400\,x^3-901074817843200\,x^2+901074817843200\,x)\,K_1(x)]\,\}$$
$$\int x^{10}\,e^{-x}\,\ln x\,K_0(x)\,dx=\frac{e^{-x}}{940839860961}\,\{-(4266847442\,x^{11}+2251618941\,x^{10}+117807097980\,x^9+$$
$$+1000787719932\,x^8+5829673272192\,x^7+26484562500480\,x^6+96176811386880\,x^5+275819194828800\,x^4+$$
$$+598998804848640\,x^3+899357077463040\,x^2+901074817843200\,x+901074817843200)\,K_0(x)+$$
$$+(4266847442\,x^{11}+118195220\,x^{10}-116928608940\,x^9-1058156244288\,x^8-6371084833344\,x^7-$$
$$-29810373836544\,x^6-112008092716800\,x^5-336471606374400\,x^4-785863855042560\,x^3-$$
$$-1348176746004480\,x^2-1348176746004480\,x)\,K_1(x)+\ln x\,[(44801898141\,x^{11}-212219517510\,x^{10}-$$
$$-998680082400\,x^9-4194456346080\,x^8-15487223431680\,x^7-49277529100800\,x^6-131406744268800\,x^5-$$
$$-281585880576000\,x^4-450537408921600\,x^3-450537408921600\,x^2)\,K_0(x)-(44801898141\,x^{11}+$$
$$+235799463900\,x^{10}+1123515092700\,x^9+4793664395520\,x^8+18068427336960\,x^7+59133034920960\,x^6+$$
$$+164258430336000\,x^5+375447840768000\,x^4+675806113382400\,x^3+901074817843200\,x^2+$$
$$+901074817843200\,x)\,K_1(x)]\,\}$$
$$\int x^{10}\,e^x\,\ln x\,I_1(x)\,dx=\frac{e^x}{427654482255}\,\{(1939476110\,x^{11}+161175300\,x^{10}-54364094070\,x^9+$$
$$+481328149302\,x^8-2844440165952\,x^7+13018292481600\,x^6-47491102310400\,x^5+136629661593600\,x^4-$$
$$-297451505111040\,x^3+447630641418240\,x^2-450537408921600\,x+450537408921600)\,I_0(x)+$$
$$+(-1939476110\,x^{11}-1130913355\,x^{10}-53357417685\,x^9+507288738528\,x^8-3103413201504\,x^7+$$
$$+14636400395904\,x^6-55257417129600\,x^5+166529222092800\,x^4-389860081551360\,x^3+669992578375680\,x^2-$$
$$-669992578375680\,x)\,I_1(x)+\ln x\,[(-20364499155\,x^{11}+106109758755\,x^{10}-499340041200\,x^9+$$
$$+2097228173040\,x^8-7743611715840\,x^7+24638764550400\,x^6-65703372134400\,x^5+140792940288000\,x^4-$$
$$-225268704460800\,x^3+225268704460800\,x^2)\,I_0(x)+(20364499155\,x^{11}+117899731950\,x^{10}-561757546350\,x^9+$$
$$+2396832197760\,x^8-9034213668480\,x^7+29566517460480\,x^6-82129215168000\,x^5+187723920384000\,x^4-$$
$$-337903056691200\,x^3+450537408921600\,x^2-450537408921600\,x)\,I_1(x)]\,\}$$
$$\int x^{10}\,e^{-x}\,\ln x\,I_1(x)\,dx=\frac{e^{-x}}{427654482255}\,\{(-1939476110\,x^{11}+161175300\,x^{10}+54364094070\,x^9+$$
$$+481328149302\,x^8+2844440165952\,x^7+13018292481600\,x^6+47491102310400\,x^5+136629661593600\,x^4+$$

$$+297451505111040\,x^3+447630641418240\,x^2+450537408921600\,x+450537408921600)\,I_0(x)+$$
$$+(-1939476110\,x^{11}+1130913355\,x^{10}-53357417685\,x^9-507288738528\,x^8-3103413201504\,x^7-$$
$$-14636400395904\,x^6-55257417129600\,x^5-166529222092800\,x^4-389860081551360\,x^3-669992578375680\,x^2-$$
$$-669992578375680\,x)\,I_1(x)+\ln x\,[(20364499155\,x^{11}+106109758755\,x^{10}+499340041200\,x^9+$$
$$+2097228173040\,x^8+7743611715840\,x^7+24638764550400\,x^6+65703372134400\,x^5+140792940288000\,x^4+$$
$$+225268704460800\,x^3+225268704460800\,x^2)\,I_0(x)+(20364499155\,x^{11}-117899731950\,x^{10}-561757546350\,x^9-$$
$$-2396832197760\,x^8-9034213668480\,x^7-29566517460480\,x^6-82129215168000\,x^5-187723920384000\,x^4-$$
$$-337903056691200\,x^3-450537408921600\,x^2-450537408921600\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^x\,\ln x\,K_1(x)\,dx=\frac{e^x}{427654482255}\,\{(-1939476110\,x^{11}-161175300\,x^{10}+54364094070\,x^9-$$
$$-481328149302\,x^8+2844440165952\,x^7-13018292481600\,x^6+47491102310400\,x^5-136629661593600\,x^4+$$
$$+297451505111040\,x^3-447630641418240\,x^2+450537408921600\,x-450537408921600)\,K_0(x)+$$
$$+(-1939476110\,x^{11}-1130913355\,x^{10}-53357417685\,x^9+507288738528\,x^8-3103413201504\,x^7+$$
$$+14636400395904\,x^6-55257417129600\,x^5+166529222092800\,x^4-389860081551360\,x^3+669992578375680\,x^2-$$
$$-669992578375680\,x)\,K_1(x)+\ln x\,[+(20364499155\,x^{11}-106109758755\,x^{10}+499340041200\,x^9-$$
$$-2097228173040\,x^8+7743611715840\,x^7-24638764550400\,x^6+65703372134400\,x^5-140792940288000\,x^4+$$
$$+225268704460800\,x^3-225268704460800\,x^2)\,K_0(x)+(20364499155\,x^{11}+117899731950\,x^{10}-561757546350\,x^9+$$
$$+2396832197760\,x^8-9034213668480\,x^7+29566517460480\,x^6-82129215168000\,x^5+187723920384000\,x^4-$$
$$-337903056691200\,x^3+450537408921600\,x^2-450537408921600\,x)\,K_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{427654482255}\,\{(1939476110\,x^{11}-161175300\,x^{10}-54364094070\,x^9-$$
$$-481328149302\,x^8-2844440165952\,x^7-13018292481600\,x^6-47491102310400\,x^5-136629661593600\,x^4-$$
$$-297451505111040\,x^3-447630641418240\,x^2-450537408921600\,x-450537408921600)\,K_0(x)+$$
$$+(-1939476110\,x^{11}+1130913355\,x^{10}-53357417685\,x^9-507288738528\,x^8-3103413201504\,x^7-$$
$$-14636400395904\,x^6-55257417129600\,x^5-166529222092800\,x^4-389860081551360\,x^3-669992578375680\,x^2-$$
$$-669992578375680\,x)\,K_1(x)+\ln x\,[(-20364499155\,x^{11}-106109758755\,x^{10}-499340041200\,x^9-$$
$$-2097228173040\,x^8-7743611715840\,x^7-24638764550400\,x^6-65703372134400\,x^5-140792940288000\,x^4-$$
$$-225268704460800\,x^3-225268704460800\,x^2)\,K_0(x)+(20364499155\,x^{11}-117899731950\,x^{10}-561757546350\,x^9-$$
$$-2396832197760\,x^8-9034213668480\,x^7-29566517460480\,x^6-82129215168000\,x^5-187723920384000\,x^4-$$
$$-337903056691200\,x^3-450537408921600\,x^2-450537408921600\,x)\,K_1(x)]\,\}$$

Recurrence relations:

About the recurrence relations for the integrals $\int x^n\, e^{\pm x}\, I_\nu(x)\,dx$ and $\int x^n\, e^{\pm x}\, K_\nu(x)\,dx$ see the pages 58 and 61.

$$\int x^{n+1}\, e^x\, \ln(x)\, I_0(x)\,dx = \frac{x^{n+1}\, e^x}{2n+3}\left\{[(n+1+x)\, I_0(x) - xI_1(x)]\, \ln(x) - I_0(x)\right\} -$$

$$-\frac{(n+1)^2}{2n+3}\int x^n\, \ln x\, e^x\, I_0(x)\,dx + \frac{2}{2n+3}\int x^{n+1}\, e^x\, I_1(x)\,dx$$

$$\int x^{n+1}\, e^x\, \ln(x)\, K_0(x)\,dx = \frac{x^{n+1}\, e^x}{2n+3}\left\{[(n+1+x)\, K_0(x) + xK_1(x)]\, \ln(x) - K_0(x)\right\} -$$

$$-\frac{(n+1)^2}{2n+3}\int x^n\, \ln x\, e^x\, K_0(x)\,dx - \frac{2}{2n+3}\int x^{n+1}\, e^x\, K_1(x)\,dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, I_0(x)\,dx = \frac{x^{n+1}\, e^{-x}}{2n+3}\left\{[-(n+1-x)\, I_0(x) + xI_1(x)]\, \ln(x) + I_0(x)\right\} +$$

$$+\frac{(n+1)^2}{2n+3}\int x^n\, \ln x\, e^x\, I_0(x)\,dx - \frac{2}{2n+3}\int x^{n+1}\, e^x\, I_1(x)\,dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, K_0(x)\,dx = \frac{x^{n+1}\, e^{-x}}{2n+3}\left\{[(x-n-1)\, K_0(x) - xK_1(x)]\, \ln(x) + K_0(x)\right\} +$$

$$+\frac{(n+1)^2}{2n+3}\int x^n\, \ln x\, e^x\, K_0(x)\,dx + \frac{2}{2n+3}\int x^{n+1}\, e^x\, K_1(x)\,dx$$

The following recurrence relations for the integrals $\int x^{n+1}\, e^{\pm x}\, Z_1(x)\,dx$ refer to $\int x^n\, e^{\pm x}\, Z_0(x)\,dx$ instead of $\int x^n\, e^{\pm x}\, Z_1(x)\,dx$.

$$\int x^{n+1}\, e^x\, \ln(x)\, I_1(x)\,dx = \frac{x^{n+1}\, e^x}{2n+3}\left\{[(n+2-x)\, I_0(x) + xI_1(x)]\, \ln(x) - \frac{n+2}{n+1}\, I_0(x)\right\} -$$

$$-\frac{(n+1)(n+2)}{2n+3}\int x^n\, \ln x\, e^x\, I_0(x)\,dx + \frac{1}{n+1}\int x^{n+1}\, e^x\, I_0(x)\,dx + \frac{1}{(n+1)(2n+3)}\int x^{n+1}\, e^x\, I_1(x)\,dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, I_1(x)\,dx = \frac{x^{n+1}\, e^{-x}}{2n+3}\left\{[(n+2+x)\, I_0(x) + xI_1(x)]\, \ln(x) - \frac{n+2}{n+1}\, I_0(x)\right\} -$$

$$-\frac{(n+1)(n+2)}{2n+3}\int x^n\, \ln x\, e^x\, I_0(x)\,dx - \frac{1}{n+1}\int x^{n+1}\, e^x\, I_0(x)\,dx - \frac{1}{(n+1)(2n+3)}\int x^{n+1}\, e^x\, I_1(x)\,dx$$

$$\int x^{n+1}\, e^x\, \ln(x)\, K_1(x)\,dx = \frac{x^{n+1}\, e^x}{2n+3}\left\{[(x-n-2)\, K_0(x) + xK_1(x)]\, \ln(x) + \frac{n+2}{n+1}\, K_0(x)\right\} +$$

$$+\frac{(n+1)(n+2)}{2n+3}\int x^n\, \ln x\, e^x\, K_0(x)\,dx - \frac{1}{n+1}\int x^{n+1}\, e^x\, K_0(x)\,dx + \frac{1}{(n+1)(2n+3)}\int x^{n+1}\, e^x\, K_1(x)\,dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, K_1(x)\,dx = \frac{x^{n+1}\, e^{-x}}{2n+3}\left\{[-(n+2+x)\, K_0(x) + xK_1(x)]\, \ln(x) + \frac{n+2}{n+1}\, K_0(x)\right\} +$$

$$+\frac{(n+1)(n+2)}{2n+3}\int x^n\, \ln x\, e^x\, K_0(x)\,dx + \frac{1}{n+1}\int x^{n+1}\, e^x\, K_0(x)\,dx + \frac{1}{(n+1)(2n+3)}\int x^{n+1}\, e^x\, K_1(x)\,dx$$

1.2.13. $\int x^n\, e^{-x^2}\, J_\nu(\alpha x)\, dx$

a) The Case $\alpha = 1$, Basic Integrals:

Some improper integrals: From [13], 4.14. (34) and (35) one has (or [14], 8.2.(21); see also [7], 6.643 and 9.235)

$$\int_0^\infty e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right) = 0.78515\,05503$$

$$\int_0^\infty x\, e^{-x^2}\, J_0(x)\, dx = \frac{e^{-1/4}}{2} = 0.38940\,03915$$

$$\int_0^\infty x^2\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{16}\left[3I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.30055\,34957$$

$$\int_0^\infty x^3\, e^{-x^2}\, J_0(x)\, dx = \frac{3e^{-1/4}}{8} = 0.29205\,02937$$

$$\int_0^\infty x^4\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{64}\left[13I_0\left(\frac{1}{8}\right) + 7I_1\left(\frac{1}{8}\right)\right] = 0.32968\,09799$$

$$\int_0^\infty x^5\, e^{-x^2}\, J_0(x)\, dx = \frac{17e^{-1/4}}{32} = 0.41373\,79160$$

$$\int_0^\infty x^6\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{256}\left[87I_0\left(\frac{1}{8}\right) + 69I_1\left(\frac{1}{8}\right)\right] = 0.56005\,83093$$

$$\int_0^\infty \frac{e^{-x^2}\, J_1(x)\, dx}{x} = \frac{\sqrt{\pi}\, e^{-1/8}}{4}\left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.41706\,34325$$

$$\int_0^\infty e^{-x^2}\, J_1(x)\, dx = 1 - e^{-1/4} = 0.22119\,92169$$

$$\int_0^\infty x\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{8}\left[I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.18404\,35589$$

$$\int_0^\infty x^2\, e^{-x^2}\, J_1(x)\, dx = \frac{e^{-1/4}}{4} = 0.19470\,01958$$

$$\int_0^\infty x^3\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{32}\left[5I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.24229\,85273$$

$$\int_0^\infty x^4\, e^{-x^2}\, J_1(x)\, dx = \frac{7e^{-1/4}}{16} = 0.34072\,53426$$

$$\int_0^\infty x^5\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{128}\left[43I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.52828\,82809$$

$$\int_0^\infty x^6\, e^{-x^2}\, J_1(x)\, dx = \frac{73e^{-1/4}}{64} = 0.88831\,96432$$

Let

$$F_\nu(x) = \int_0^x e^{-t^2}\, J_\nu(t)\, dt = \nu + e^{-x^2}\, \left[P_\nu(x)\, J_0(x) + Q_\nu(x)\, J_1(x)\right]\;, \quad \nu = 0,\; 1,$$

with

$$P_\nu(x) = \sum_{k=0}^\infty a_k^{(\nu)}\, x^{2k+1-\nu} = \sum_{k=0}^\infty \frac{\alpha_k^{(\nu)}}{\gamma_k^{(\nu)}}\, x^{2k+1-\nu} \quad \text{and} \quad Q_\nu(x) = \sum_{k=0}^\infty b_k^{(\nu)}\, x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(\nu)}}{\delta_k^{(\nu)}}\, x^{2k+\nu}\,.$$

Furthermore, let

$$F_-(x) = \int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t} = e^{-x^2}\, \left[P_-(x)\, J_0(x) + Q_-(x)\, J_1(x)\right]$$

with

$$P_-(x) = \sum_{k=0}^{\infty} a_k^{(-)}\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{\alpha_k^{(-)}}{\gamma_k^{(-)}}\, x^{2k+1} \quad \text{and} \quad Q_-(x) = \sum_{k=0}^{\infty} b_k^{(-)}\, x^{2k+\nu} = \sum_{k=0}^{\infty} \frac{\beta_k^{(-)}}{\delta_k^{(-)}}\, x^{2k} \; .$$

The minimum and maximum values of $F_s(x)$ are located in the zeros $x_k^{(\nu)}$ of $J_\nu(x)$, $0 < x_k^{(\nu)} < x_{k+1}^{(\nu)}$.
Let $F_s(x_k^{(\nu)}) = \Delta_k^{(s)} + \lim_{x\to\infty} F_s(x)$.
The following table shows, that $F_s(x)$ must not be be computed for large values of x.

s	Value	$k=1$	$k=2$	$k=3$	$k=4$
0	x_k	2.4048	5.5201	8.6537	11.7915
	$\Delta_k^{(0)}$	$5.1225 \cdot 10^{-5}$	$-1.5215 \cdot 10^{-16}$	$2.6395 \cdot 10^{-36}$	$-1.6974 \cdot 10^{-64}$
1	x_k	3.8317	7.0156	10.1735	13.3237
	$\Delta_k^{(1)}$	$2.5200 \cdot 10^{-9}$	$-6.1486 \cdot 10^{-25}$	$6.6365 \cdot 10^{-49}$	$-2.4330 \cdot 10^{-81}$
-	$\Delta_k^{(-)}$	$6.2084 \cdot 10^{-10}$	$-8.5976 \cdot 10^{-26}$	$6.4624 \cdot 10^{-50}$	$-1.8160 \cdot 10^{-82}$

Remark: In any case, if $F_s(x)$ is written as

$$F_s(x) = e^{-x^2}\, \psi_s(x) + \lim_{x\to\infty} F_s(x) \; ,$$

then one has

x	1.52	2.15	2.63	3.03	3.39	3.72	4.01	4.29	4.55	4.80
$\exp(-x^2)$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}

.

The influence of $\psi_s(x)$ vanishes soon.

I) $s = 0$:
With $k \geq 1$ holds

$$a_{k+1}^{(0)} = \frac{(8k+3)a_k^{(0)} - 4a_{k-1}^{(0)}}{(2k+1)(2k+3)} \; , \quad b_{k+1}^{(0)} = \frac{(8k-1)b_k^{(0)} - 4b_{k-1}^{(0)}}{(2k+1)^2} \; .$$

$$\int_0^x e^{-t^2}\, J_0(t)\, dt =$$

$$= e^{-x^2} \left[\left(x + \frac{x^3}{3} - \frac{x^5}{45} - \frac{79}{1575}\, x^7 - \ldots \right) J_0(x) + \left(x^2 + \frac{7}{9}\, x^4 + \frac{23}{75}\, x^6 + \ldots \right) J_1(x) \right]$$

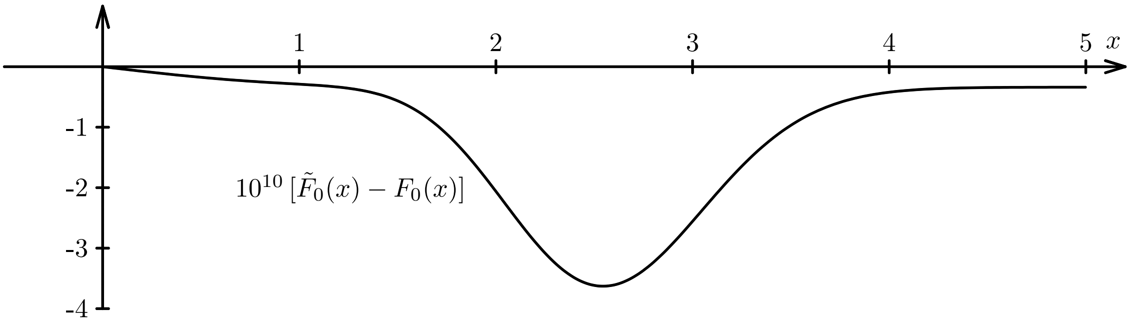
k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	3	0.33333 33333
2	-1	45	-0.02222 22222
3	-79	1575	-0.05015 87302
4	-1993	99225	-0.02008 56639
5	-7121	1403325	-0.00507 43769
6	-1354193	1404728325	-0.00096 40248
7	-40551359	273922023375	-0.00014 80398
8	-1336259641	69850115960625	-0.00001 91304
9	-48167009767	22561587455281875	-0.00000 21349
10	-1886078276353	9002073394657468125	-0.00000 02095
11	-79669949349167	4348001449619557104375	-0.00000 00183
12	-515151737265743	3571572619330350478559375	-0.00000 00014
13	-9145224759056621	888193717175574000597656 25	-0.00000 00001

k	$\beta_k^{(0)}$	$\delta_k^{(0)}$	$b_k^{(0)}$
0	0	1	0.00000 00000
1	1	1	1.00000 00000
2	7	9	0.77777 77778
3	23	75	0.3066666667
4	887	11025	0.0804535147
5	13973	893025	0.0156468184
6	85853	36018675	0.0023835691
7	5342341	18261468225	0.0002925472
8	119718871	4108830350625	0.0000291370
9	33755333	14659900880625	0.0000023026
10	5066536837	38970014695486875	0.0000001300
11	26744808373	11120208311047460625	0.0000000024
12	-19585169733827	3333467778041660446687 5	-0.0000000006
13	-594894329175841	5682047348934648488671875	-0.0000000001

One has $a_0^{(0)}, a_1^{(0)} > 0$, $a_2^{(0)}, \ldots, a_{22}^{(0)} < 0$, but $a_{23}^{(0)} > 0$. Holds $b_{11}^{(0)} \cdot b_{12}^{(0)} < 0$ and $b_{41}^{(0)} \cdot b_{42}^{(0)} < 0$.
First positive zeros of $P_0(x)$: 1.5091, 4.8104, 7.9822 .
Maxima: $P_0(1.1915) = 1.3870$, $P_0(7.9191) = 3.6414 \cdot 10^{26}$, mimimum: $P_0(4.7057) = -1.1187 \cdot 10^9$.
First positive zeros of $Q_0(x)$: 3.3521, 6.4769, 9.6124.
Maxima: $Q_0(3.2008) = 9162.0$, $Q_0(9.5602) = 9.5630 \cdot 10^{38}$, minimum: $Q_0(6.3994) = -1.4373 \cdot 10^{17}$
Approximation:

$$F_0(x) \approx \tilde{F}_0(x) = 0.7851505503 \,\mathrm{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(0)} x^{2k+1}$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$	k	$c_k^{(0)}$
0	$1.14052\,47597 \cdot 10^{-1}$	1	$-7.29834\,93536 \cdot 10^{-3}$	2	$2.05660\,25858 \cdot 10^{-4}$
3	$-3.24389\,43744 \cdot 10^{-6}$	4	$3.26550\,30992 \cdot 10^{-8}$	5	$-2.27888\,93576 \cdot 10^{-10}$



Asymptotic expansion:

$$\int_0^x e^{-t^2} J_0(t)\,dx \quad \sim \quad \frac{\sqrt{\pi}\,e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right) +$$

$$+\frac{\sqrt{2}e^{-x^2}}{\sqrt{\pi x}}\left[\left(-\frac{1}{2x} + \frac{129}{256x^3} - \frac{76203}{65536x^5} + \ldots\right)\sin\left(x + \frac{\pi}{4}\right) + \left(-\frac{3}{16x^2} + \frac{921}{2048x^4} - \frac{775773}{524288x^6} + \ldots\right)\cos\left(x + \frac{\pi}{4}\right)\right]$$

See the remark on page 138.
Let

$$\varphi_1(x) = e^{x^2}\left[\int_0^x e^{-t^2} J_0(t)\,dt - \frac{\sqrt{\pi}\,e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right)\right] , \quad \varphi_2(x) = -\frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x}\sin\left(x + \frac{\pi}{4}\right) ,$$

$$\varphi_3(x) = -\frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{1}{2x}\sin\left(x + \frac{\pi}{4}\right) + \frac{3}{16x^2}\cos\left(x + \frac{\pi}{4}\right)\right] .$$

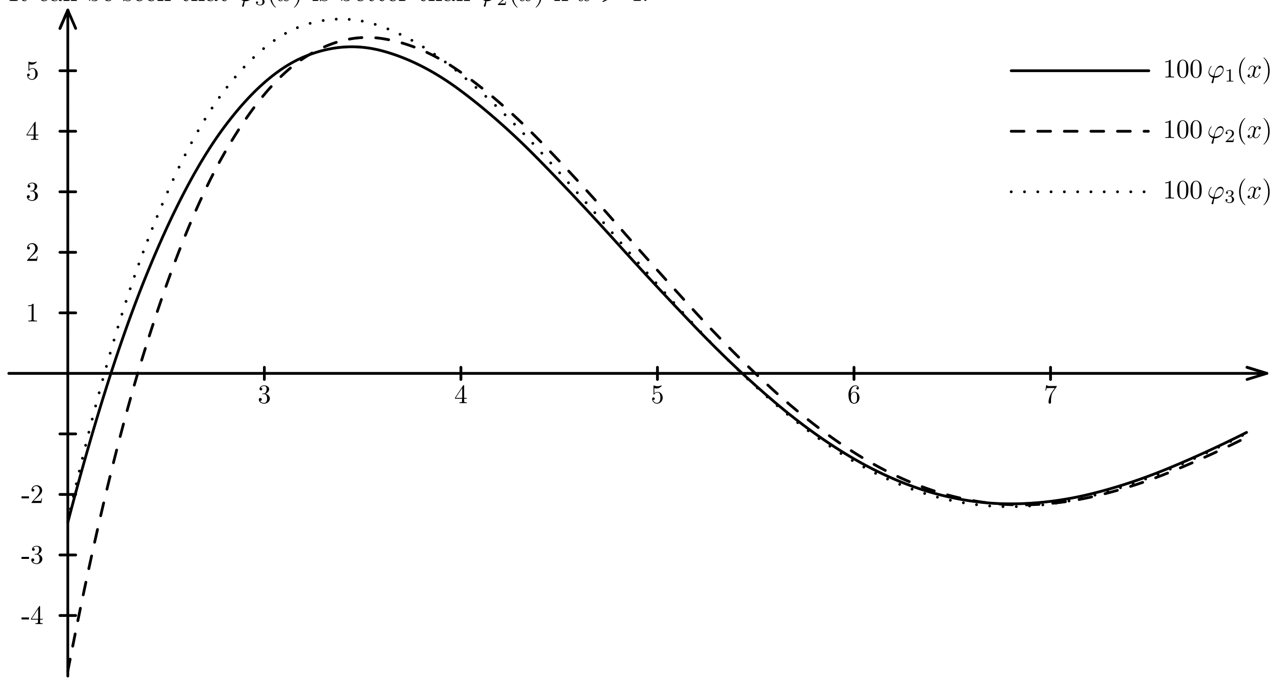
The following figure shows that $\varphi_1(x) \approx \varphi_2(x)$ if $x > 2$. From this

$$e^{x^2}\left[\int_0^x e^{-t^2} J_0(t)\,dt - \frac{\sqrt{\pi}\,e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right)\right] \approx -\frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x}\sin\left(x + \frac{\pi}{4}\right)$$

and therefore holds

$$\int_0^x e^{-t^2}\, J_0(t)\, dt \approx \frac{\sqrt{\pi}\, e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right) - \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{e^{-x^2}}{2x} \sin\left(x + \frac{\pi}{4}\right) \ .$$

It can be seen that $\varphi_3(x)$ is better than $\varphi_2(x)$ if $x > 4$.



II) $s = 1$:
With $k \geq 1$ holds

$$a_{k+1}^{(1)} = \frac{(8k-1)a_k^{(1)} - 4a_{k-1}^{(1)}}{4k(k+1)} \ , \quad b_{k+1}^{(1)} = \frac{(8k+3)b_k^{(1)} - 4b_{k-1}^{(1)}}{4(k+1)^2} \ .$$

$$\int_0^x e^{-t^2}\, J_1(t)\, dt =$$

$$= 1 + e^{-x^2} \left[\left(-1 - x^2 - \frac{3}{8}\, x^3 - \frac{13}{192}\, x^5 - \ldots \right) J_0(x) + \left(-\frac{x^3}{2} - \frac{11}{32}\, x^5 - \frac{145}{1152}\, x^7 - \ldots \right) J_1(x) \right]$$

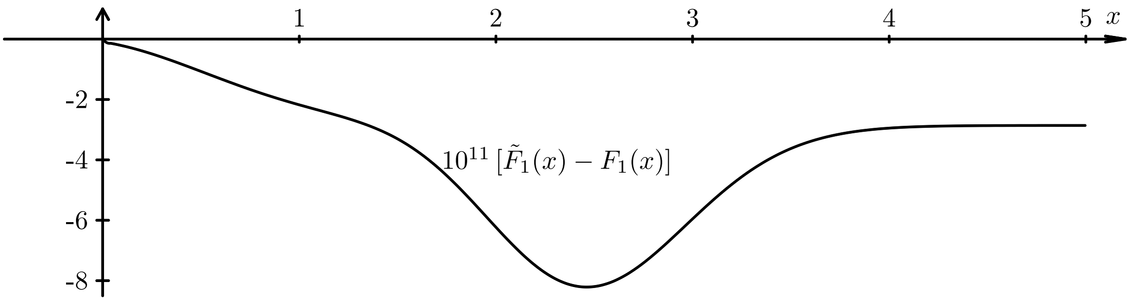
k	$\alpha_k^{(1)}$	$\gamma_k^{(1)}$	$a_k^{(1)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-3	8	-0.37500 00000
3	-13	192	-0.06770 83333
4	-11	9216	-0.00119 35764
5	431	147456	0.00292 29058
6	17513	17694720	0.00098 97303
7	88033	424673280	0.00020 72958
8	3160567	95126814720	0.00003 32248
9	17176879	3913788948480	0.00000 43888
10	4895935679	9862748150169600	0.00000 04964
11	213635978321	4339609186074624000	0.00000 00492
12	9969483318887	2291313650247401472000	0.00000 00044
13	495901729080313	1429779717754378518528000	0.00000 00003

k	$\beta_k^{(1)}$	$\delta_k^{(1)}$	$b_k^{(1)}$
0	0	1	0.00000 00000
1	-1	2	-0.50000 00000
2	-11	32	-0.34375 00000
3	-145	1152	-0.12586 80556
4	-259	8192	-0.03161 62109
5	-8893	1474560	-0.00603 09516
6	-195919	212336640	-0.00092 26811
7	-231881	1981808640	-0.00011 70047
8	-19100009	1522029035520	-0.00001 25490
9	-567362171	493137407508480	-0.00000 11505
10	-5932850387	65751654334464000	-0.00000 00902
11	-569500272763	95471402093641728000	-0.00000 00060
12	-17366529773737	54991527605937635328000	-0.00000 00003

One has $a_4^{(1)} \cdot a_5^{(1)} < 0,\ a_{27}^{(1)} \cdot a_{28}^{(1)} < 0$ and $b_{13}^{(1)} \cdot b_{14}^{(1)} < 0,\ b_{47}^{(1)} \cdot b_{48}^{(1)} < 0$.
First positive zeros of $P_1(x)$: 2.0140, 5.2565 and 8.4241.
Minima: $P_1(1.7541) = -7.0905$ and $P_1(8.3646) = -4.1348 \cdot 10^{29}$, maximum: $P_1(5.1608) = 7.8803 \cdot 10^{10}$
First positive zeros of $Q_1(x)$: 3.7880, 6.9155 and 10.0515.
Minima: $Q_1(3.6545) = -1.5968 \cdot 10^5$ and $Q_1(10.0017) = -4.3429 \cdot 10^{42}$,
maximum: $Q_1(6.8429) = 4.0769 \cdot 10^{19}$
Approximation:

$$F_1(x) \approx \tilde{F}_1(x) = 0.22119\,92169 + e^{-x^2}\left[-0.22119\,92169 + \sum_{k=1}^{6} c_k^{(1)} x^{2k}\right]$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	$2.88007\,83071 \cdot 10^{-2}$	2	$-1.22460\,84642 \cdot 10^{-3}$	3	$2.58249\,56345 \cdot 10^{-5}$
4	$-3.25444\,94146 \cdot 10^{-7}$	5	$2.72785\,19850 \cdot 10^{-9}$	6	$-1.63082\,82205 \cdot 10^{-11}$



Asymptotic expansion:

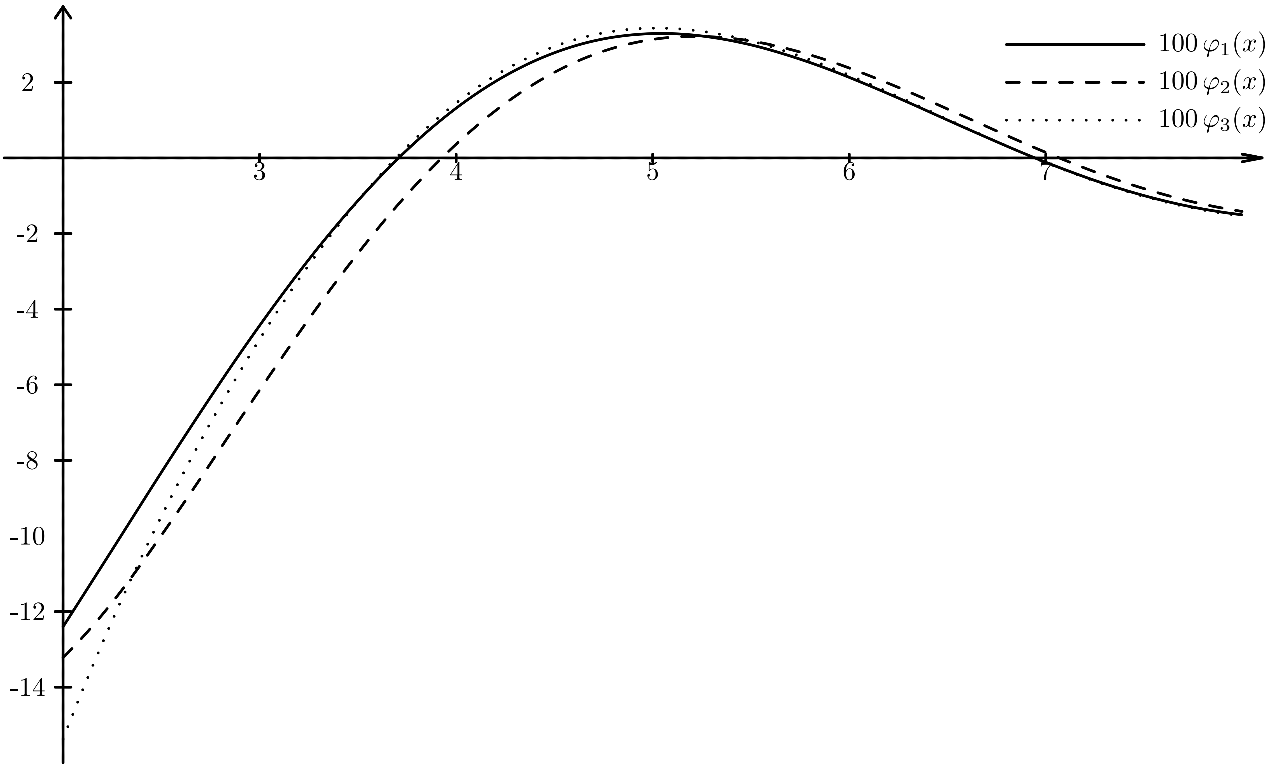
$$\int_0^x e^{-t^2}\, J_1(t)\, dt \quad \sim \quad 1 - e^{-1/4} +$$

$$+\frac{\sqrt{2}e^{-x^2}}{\sqrt{\pi x}}\left[\left(\frac{1}{2x} - \frac{137}{256x^3} + \frac{85019}{65536x^5} + \ldots\right)\cos\left(x + \frac{\pi}{4}\right) + \left(-\frac{7}{16x^2} + \frac{1773}{2048x^4} - \frac{1434089}{524288x^6} + \ldots\right)\sin\left(x + \frac{\pi}{4}\right)\right]$$

See the remark on page 138.
Let

$$\varphi_1(x) = e^{x^2}\left[\int_0^x e^{-t^2}\, J_1(t)\, dt - 1 + e^{-1/4}\right], \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x}\cos\left(x + \frac{\pi}{4}\right),$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{1}{2x}\cos\left(x + \frac{\pi}{4}\right) - \frac{7}{16x^2}\sin\left(x + \frac{\pi}{4}\right)\right].$$



III) $s = -$:

With $k \geq 1$ holds

$$a_{k+1}^{(-)} = \frac{(8k+3)a_k^{(-)} - 4a_{k-1}^{(-)}}{(2k+1)(2k+3)}\ , \quad b_{k+1}^{(-)} = \frac{(8k-1)b_k^{(-)} - 4b_{k-1}^{(-)}}{(2k+1)^2}\ .$$

$$\int_0^x \frac{e^{-t^2}\, J_0(t)\, dt}{t} =$$

$$= e^{-x^2} \left[\left(x + x^3 + \frac{7x^5}{15} + \frac{73}{525}\, x^7 - \ldots \right) J_0(x) + \left(-1 - x^2 - \frac{x^4}{3} - \frac{x^6}{25} + \ldots \right) J_1(x) \right]$$

k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	1	1.00000 00000
2	7	15	0.46666 66667
3	73	525	0.13904 76190
4	991	33075	0.02996 22071
5	2327	467775	0.00497 46139
6	307991	468242775	0.00065 77592
7	6390233	91307341125	0.00006 99860
8	136790767	23283371986875	0.00000 58750
9	2646943729	7520529151760625	0.00000 03520
10	21787108711	3000691131552489375	0.00000 00726
11	-2416192168471	1449333816539852368125	-0.00000 00017
12	-37423740194359	119052420644345015953125	-0.00000 00003

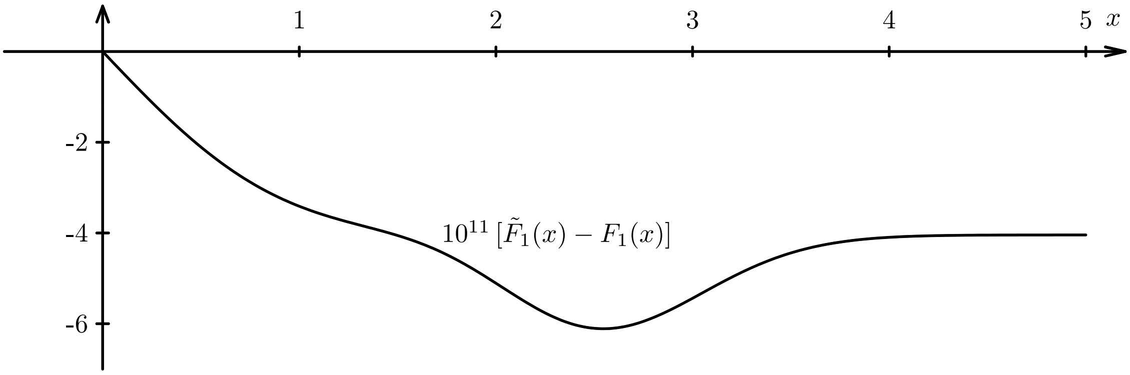
k	$\beta_k^{(-)}$	$\delta_k^{(-)}$	$b_k^{(-)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-1	3	-0.33333 33333
3	-1	25	-0.0400 00000
4	31	3675	0.00843 53741
5	1549	297675	0.00520 36617
6	16789	12006225	0.00139 83579
7	1617533	6087156075	0.00026 57289
8	54916223	1369610116875	0.00004 00962
9	24740029	4886633626875	0.00000 50628
10	7163341181	12990004898495625	0.00000 05145
11	195955925549	3706736103682486875	0.00000 00529
12	50273780536949	11111559260138868155625	0.00000 00045
13	661736445380167	1894015782978216162890625	0.00000 00003

One has $a_{10}^{(-)} \cdot a_{11}^{(-)} < 0,\ a_{41}^{(-)} \cdot a_{42}^{(-)} < 0$ and $b_3^{(-)} \cdot b_4^{(-)} < 0,\ b_{25}^{(-)} \cdot b_{26}^{(-)} < 0,\ b_{65}^{(-)} \cdot b_{66}^{(-)} < 0$.
First positive zeros of $P_-(x)$: 3.2793, 6.4739 and 9.6341.
Maxima: $P_-(3.1244) = 4757.3$ and $P_-(9.5821) = 1.2302 \cdot 10^{39}$, minimum: $P_-(6.3963) = -1.1684 \cdot 10^{17}$
First positive zeros of $Q_-(x)$: 1.8812, 4.9850 and 8.1173.
Minima: $Q_-(1.6008) = -4.7895$ and $Q_-(8.0555) = -2.7233 \cdot 10^{27}$, maximum: $Q_-(4.8841) = 5.2197 \cdot 10^9$

Approximation:

$$F_-(x) \approx \tilde{F}_-(x) = 0.41706\,34325\,\mathrm{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(-)} x^{2k+1}$$

k	$c_k^{(-)}$	k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	$2.93943\,11364 \cdot 10^{-2}$	1	$-1.23712\,57573 \cdot 10^{-3}$	2	$2.59830\,30409 \cdot 10^{-5}$
3	$-3.26773\,05781 \cdot 10^{-7}$	4	$2.73580\,96839 \cdot 10^{-9}$	5	$-1.63439\,98979 \cdot 10^{-11}$



Asymptotic expansion:

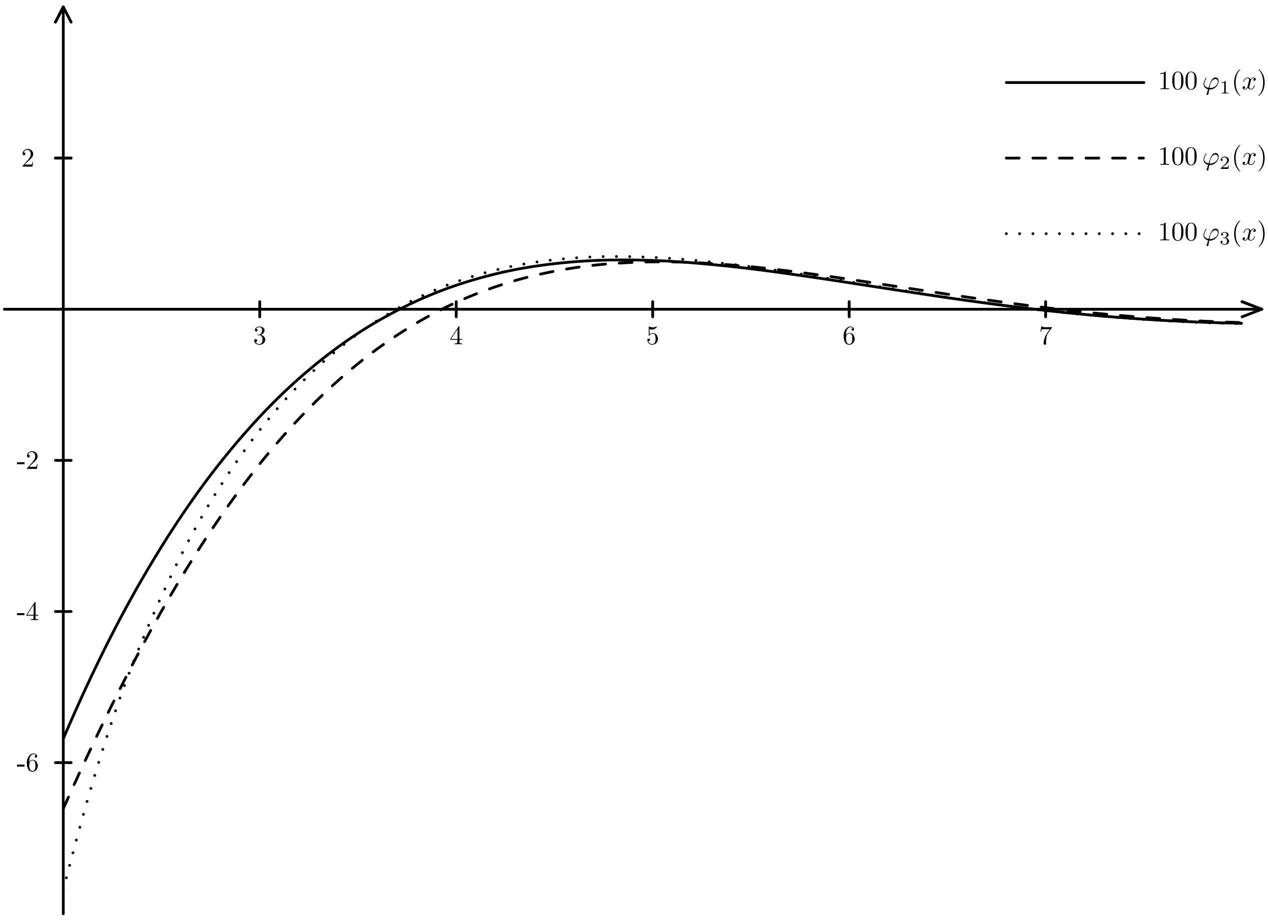
$$\int_0^x \frac{e^{-t^2}\,J_1(t)\,dt}{t} \sim \frac{\sqrt{\pi}\,e^{-1/8}}{4}\left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] +$$

$$+\sqrt{\frac{2}{\pi x}}\,e^{-x^2}\left[\left(\frac{1}{2x^2} - \frac{201}{256x^4} + \frac{150683}{65536x^6} - \dots\right)\cos\left(x+\frac{\pi}{4}\right) + \left(-\frac{7}{16x^3} + \frac{2477}{2048x^5} - \frac{2419305}{524288x^7} + \dots\right)\sin\left(x+\frac{\pi}{4}\right)\right]$$

See the remark on page 138.
Let

$$\varphi 1-(x) = e^{x^2}\left[\int_0^x \frac{e^{-t^2}\,J_1(t)\,dt}{t} - \frac{\sqrt{\pi}\,e^{-1/8}}{4}\right], \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\cdot\frac{1}{2x^2}\cos\left(x+\frac{\pi}{4}\right),$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{1}{2x^2}\cos\left(x+\frac{\pi}{4}\right) - \frac{7}{16x^3}\sin\left(x+\frac{\pi}{4}\right)\right].$$



b) Integrals ($\alpha = 1$)

Let

$$\mathcal{J}_\nu \quad = \quad \int e^{-x^2}\,J_\nu(x)\,dx\,, \qquad \nu = 0,1 \quad \text{and} \quad \mathcal{J}_- \quad = \quad \int \frac{e^{-x^2}\,J_\nu(x)\,dx}{x}\,.$$

$$\int xe^{-x^2}J_0(x)\,dx = -\frac{1}{2}\,e^{-x^2}J_0(x) - \frac{1}{2}\,\mathcal{J}_1$$

$$\int xe^{-x^2}J_1(x)\,dx = -\frac{1}{2}\,e^{-x^2}J_1(x) + \frac{1}{2}\,\mathcal{J}_0 - \frac{1}{2}\,\mathcal{J}_-$$

$$\int x^2e^{-x^2}J_0(x)\,dx = \frac{e^{-x^2}}{4}\,\left[-2xJ_0(x) + J_1(x)\right] + \frac{1}{4}\,\mathcal{J}_0 + \frac{1}{4}\,\mathcal{J}_-$$

$$\int x^2e^{-x^2}J_1(x)\,dx = \frac{e^{-x^2}}{4}\,\left[-J_0(x) - 2xJ_1(x)\right] - \frac{1}{4}\,\mathcal{J}_1$$

$$\int x^3e^{-x^2}J_0(x)\,dx = \frac{e^{-x^2}}{8}\,\left[-(4x^2+3)J_0(x) + 2xJ_1(x)\right] - \frac{3}{8}\,\mathcal{J}_1$$

$$\int x^3e^{-x^2}J_1(x)\,dx = -\frac{e^{-x^2}}{8}\,\left[2xJ_0(x) + (4x^2+1)J_1(x)\right] + \frac{3}{8}\,\mathcal{J}_0 - \frac{1}{8}\,\mathcal{J}_-$$

$$\int x^4e^{-x^2}J_0(x)\,dx = \frac{e^{-x^2}}{16}\,\left[-(8x^3+10x)J_0(x) + (4x^2+7)J_1(x)\right] + \frac{3}{16}\,\mathcal{J}_0 + \frac{7}{16}\,\mathcal{J}_-$$

$$\int x^4e^{-x^2}J_1(x)\,dx = -\frac{e^{-x^2}}{16}\,\left[(4x^2+7)J_0(x) + (8x^3+6x)J_1(x)\right] - \frac{7}{16}\,\mathcal{J}_1$$

$$\int x^5e^{-x^2}J_0(x)\,dx = \frac{e^{-x^2}}{32}\,\left[-(16x^4+28x^2+17)J_0(x) + (8x^3+22x)J_1(x)\right] - \frac{17}{32}\,\mathcal{J}_1$$

$$\int x^5 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{32}\left[(8x^3+22x)J_0(x)+(16x^4+20x^2-1)J_1(x)\right]+\frac{21}{32}\,\mathcal{J}_0+\frac{1}{32}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{64}\left[-(32x^5+72x^3+78x)J_0(x)+(16x^4+66x^2+69)J_1(x)\right]+\frac{9}{64}\,\mathcal{J}_0+\frac{69}{64}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{64}\left[(16x^4+60x^2+73)J_0(x)+(32x^5+56x^3+26x)J_1(x)\right]-\frac{73}{64}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{128}\left[-(64x^6+176x^4+276x^2+131)J_0(x)+(32x^5+152x^3+290x)J_1(x)\right]-\frac{131}{128}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{128}\left[(32xu+152x^3+298x)J_0(x)+(64x^6+144x^4+140x^2+79)J_1(x)\right]+\frac{219}{128}\,\mathcal{J}_0+\frac{79}{128}\,\mathcal{J}_1$$

$$\int x^8 e^{-x^2} J_0(x)\,dx =$$

$$= \frac{e^{-x^2}}{256}\left[-(128\,x^7+416\,x^5+856\,x^3+794\,x)J_0(x)+(64\,x^6+368\,x^4+980\,x^2+887)J_1(x)\right]-\frac{93}{256}\,\mathcal{J}_0+\frac{887}{256}\,\mathcal{J}_-$$

$$\int x^8 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{256}\left[(64\,x^6+368\,x^4+996\,x^2+1007)J_0(x)+(128\,x^7+352\,x^5+520\,x^3+22\,x)J_1(x)\right]-\frac{1007}{256}\,\mathcal{J}_1$$

$$\int x^9 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{512}\left[-(256\,x^8+960\,x^6+2448\,x^4+3420\,x^2+1089)J_0(x)\right.$$
$$\left.+(128\,x^7+864\,x^5+2952\,x^3+4662\,x)J_1(x)\right]-\frac{1089}{512}\,\mathcal{J}_1$$

$$\int x^9 e^{-x^2} J_0(x)\,dx = -\frac{e^{-x^2}}{512}\left[(128\,x^7+864\,x^5+2984\,x^3+4966\,x)J_0(x)+\right.$$
$$\left.+(256\,x^8+832\,x^6+1648\,x^4+980\,x^2-1993)J_1(x)\right]+\frac{2973}{512}\,\mathcal{J}_0+\frac{1993}{512}\,\mathcal{J}_-$$

$$\int x^{10} e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{1024}\left[-(512\,x^9+2176\,x^7+6624\,x^5+12424\,x^3+9326\,x)J_0(x)+\right.$$
$$\left.+(256\,x^8+1984\,x^6+8272\,x^4+18620\,x^2+13973)J_1(x)\right]-\frac{4647}{1025}\,\mathcal{J}_0+\frac{13973}{1024}\,\mathcal{J}_-$$

$$\int x^{10} e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{1024}\left[(256\,x^8+1984\,x^6+8336\,x^4+19356\,x^2+17201)J_0(x)+\right.$$
$$\left.+(512\,x^9+1920\,x^7+4768\,x^5+5368\,x^3-4310\,x)J_1(x)\right]-\frac{17201}{1024}\,\mathcal{J}_1$$

Recurrence relations:

$$\int x^{2n+2}\,e^{-x^2}\,J_0(x)\,dx =$$

$$= \frac{x^{2n}\,e^{-x^2}}{4}\left[J_1(x)-2xJ_0(x)\right]+\frac{4n+1}{4}\int x^{2n}\,e^{-x^2}\,J_0(x)\,dx-\frac{2n-1}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_0(x)+n\int x^{2n-1}\,e^{-x^2}\,J_0(x)\,dx-\frac{1}{2}\int x^{2n}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(x)\,dx =$$

$$= -\frac{x^{2n}\, e^{-x^2}}{2}\;[J_0(x) + 2xJ_1(x)] + \frac{n}{2}\int x^{2n-1}\, e^{-x^2}\, J_0(x)\, dx + \frac{4n-1}{4}\int x^{2n}\, e^{-x^2}\, J_1(x)\, dx$$

$$\int x^{2n+1}\, e^{-x^2}\, J_1(x)\, dx = -\frac{x^{2n}\, e^{-x^2}}{2}\, J_1(x) + \frac{1}{2}\int x^{2n}\, e^{-x^2}\, J_0(x)\, dx + \frac{2n-1}{2}\int x^{2n-1}\, e^{-x^2}\, J_1(x)\, dx$$

c) General Case $\alpha \neq 1$, Basic Integrals

Some improper integrals:

$$\int_0^\infty e^{-x^2}\, J_0(\alpha x)\, dx = \frac{\sqrt{\pi}}{2}\, e^{-\alpha^2/8}\, I_0\left(\frac{\alpha^2}{8}\right)$$

$$\int_0^\infty e^{-x^2}\, J_1(\alpha x)\, dx = \frac{1 - e^{-\alpha^2/4}}{\alpha}$$

$$\int_0^\infty \frac{e^{-x^2}\, J_1(\alpha x)\, dx}{x} = \frac{\sqrt{\pi}}{4}\, e^{-\alpha^2/8}\, \left[I_0\left(\frac{\alpha^2}{8}\right) + I_1\left(\frac{\alpha^2}{8}\right)\right]$$

Let

$$F_\nu(x;\alpha) = \int_0^x e^{-t^2}\, J_\nu(\alpha t)\, dt = \frac{\nu}{\alpha} + e^{-x^2}\;[\, P_\nu(x;\alpha)\, J_0(\alpha x) + Q_\nu(x;\alpha)\, J_1(\alpha x)]\;, \quad \nu = 0,\ 1,$$

with

$$P_\nu(x;\alpha) = \sum_{k=0}^\infty a_k^{(\nu;\alpha)}\, x^{2k+1-\nu} = \sum_{k=0}^\infty \frac{\alpha_k^{(\nu;\alpha)}}{\gamma_k^{(\nu)}}\, x^{2k+1-\nu} \quad \text{and} \quad Q_\nu(x) = \sum_{k=0}^\infty b_k^{(\nu;\alpha)}\, x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(\nu;\alpha)}}{\delta_k^{(\nu)}}\, x^{2k+\nu}\,.$$

Furthermore, let

$$F_-(x;\alpha) = \int_0^x \frac{e^{-t^2}\, J_1(\alpha t)\, dt}{t} = e^{-x^2}\;[\, P_-(x;\alpha)\, J_0(\alpha x) + Q_-(x;\alpha)\, J_1(\alpha x)]$$

with

$$P_-(x) = \sum_{k=0}^\infty a_k^{(-;\alpha)}\, x^{2k+1} = \sum_{k=0}^\infty \frac{\alpha_k^{(-;\alpha)}}{\gamma_k^{(-)}}\, x^{2k+1} \quad \text{and} \quad Q_-(x;\alpha) = \sum_{k=0}^\infty b_k^{(-;\alpha)}\, x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(-;\alpha)}}{\delta_k^{(-)}}\, x^{2k}\,.$$

<u>I) $s = 0$:</u>

$$F_0(x;\alpha) =$$

$$= e^{-x^2}\;\left[\left(x + \frac{2-\alpha^2}{3}\, x^3 + \frac{\alpha^4 - 14\alpha^2 + 12}{45}\, x^5 + \frac{-\alpha^6 + 34\alpha^4 - 232\alpha^2 + 120}{1575}\, x^7 + \ldots\right) J_0(\alpha x) + \right.$$

$$\left. + \left(\alpha x^2 + \frac{8\alpha - \alpha^3}{9}\, x^4 + \frac{\alpha^5 - 24\alpha^3 + 92\alpha}{225}\, x^6 \ldots\right) J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(0;\alpha)} = \frac{(4k+2-\alpha^2)a_k^{(0;\alpha)} - 2\alpha b_k^{(0;\alpha)}}{(2k+1)(2k+3)}\;, \quad b_{k+1}^{(0;\alpha)} = \frac{2b_k^{(0;\alpha)} + \alpha a_k^{(0;\alpha)}}{2k+1}$$

Approximation:

$$F_0(x;\alpha) = \frac{\sqrt{\pi}}{2}\, e^{-\alpha^2/8}\, I_0\left(\frac{\alpha^2}{8}\right)\, \mathrm{erf}(x) + e^{-x^2}\, \sum_{k=0}^5 c_k^{(0;\alpha)}\, x^{2k+1}$$

$$c_0^{(0;\alpha)} = -\frac{77}{1006632960}\,\alpha^{12} + \frac{21}{10485760}\,\alpha^{10} - \frac{35}{786432}\,\alpha^8 + \frac{5}{6144}\,\alpha^6 - \frac{3}{256}\,\alpha^4 + \frac{\alpha^2}{8} =$$

$$= -0.00000\,00765\,\alpha^{12} + 0.00000\,20027\,\alpha^{10} - 0.00004\,45048\,\alpha^8 + 0.00081\,38021\,\alpha^6 -$$

$$-0.01171\,87500\,\alpha^4 + 0.12500\,00000\,\alpha^2$$

$$c_1^{(0;\alpha)} = -0.00000\,00510\,\alpha^{12} + 0.00000\,13351\,\alpha^{10} - 0.00002\,96699\,\alpha^8 + 0.00054\,25347\,\alpha^6 - 0.00781\,25000\,\alpha^4$$

$$c_2^{(0;\alpha)} = -0.00000\,00204\,\alpha^{12} + 0.00000\,05341\,\alpha^{10} - 0.00001\,18679\,\alpha^{8} + 0.00021\,70139\,\alpha^{6}$$

$$c_3^{(0;\alpha)} = -0.00000\,00058\,\alpha^{12} + 0.00000\,01526\,\alpha^{10} - 0.00000\,33908\,\alpha^{8}$$

$$c_4^{(0;\alpha)} = -0.00000\,00013\,\alpha^{12} + 0.00000\,00339\,\alpha^{10}$$

$$c_5^{(0;\alpha)} = -0.00000\,00002\,\alpha^{12}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used ($\Phi(x)$ defined as on page 7):

$$F_0(x;\alpha) = \sigma^{(0)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7} \varphi_k^{(0)}(\alpha)\,x^{2k+1}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7} \psi_k^{(0)}(\alpha)\,x^{2k+2}\right] J_1(\alpha x)$$

$$\sigma^{(0)}(\alpha) = 1 + \frac{1}{\alpha^2} + \frac{9}{2\,\alpha^4} + \frac{75}{2\,\alpha^6} + \frac{3675}{8\,\alpha^8} + \frac{59535}{8\,\alpha^{10}} + \frac{2401245}{16\,\alpha^{12}} + \frac{57972915}{16\,\alpha^{14}} + \frac{13043905875}{128\,\alpha^{16}} + \frac{418854310875}{128\,\alpha^{18}} +$$

$$+\frac{30241281245175}{256\,\alpha^{20}} + \frac{1212400457192925}{256\,\alpha^{22}} + \frac{213786613951685775}{1024\,\alpha^{24}} + \frac{10278202593831046875}{1024\,\alpha^{26}} +$$

$$+\frac{1070401384414690453125}{2048\,\alpha^{28}} + \frac{60013837619516978071875}{2048\,\alpha^{30}} =$$

$$= 1 + 0.04\left(\frac{5}{\alpha}\right)^2 + 0.0072\left(\frac{5}{\alpha}\right)^4 + 0.0024\left(\frac{5}{\alpha}\right)^6 + 0.00117\,6\left(\frac{5}{\alpha}\right)^8 + 0.00076\,2048\left(\frac{5}{\alpha}\right)^{10} + 0.00061\,47187\left(\frac{5}{\alpha}\right)^{12} +$$

$$+0.00059\,36426\left(\frac{5}{\alpha}\right)^{14} + 0.00066\,78479808\left(\frac{5}{\alpha}\right)^{16} + 0.00085\,78136287\left(\frac{5}{\alpha}\right)^{-18} + 0.00123\,86829\left(\frac{5}{\alpha}\right)^{20} +$$

$$+0.00198\,63969\left(\frac{5}{\alpha}\right)^{22} + 0.00350\,26799\left(\frac{5}{\alpha}\right)^{24} + 0.00673\,5922852\left(\frac{5}{\alpha}\right)^{26} + 0.01402\,996503\left(\frac{5}{\alpha}\right)^{28} +$$

$$+0.03146\,45349\left(\frac{5}{\alpha}\right)^{30}$$

Let

$$\varphi_k^{(0)}(\alpha) = \sum_{j=0}^{7} s_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k} \quad \text{and} \quad \psi_k^{(0)}(\alpha) = \sum_{j=0}^{7} t_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k+1}$$

$s_j^{(k,0)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0200000000	-0.0163333333	0.0075600000	-0.0024200000	0.0005901587	-0.0001160714	0.0000191138
3	0	0.0098000000	-0.0105840000	0.0060984000	-0.0023370286	0.0006639286	-0.0001490873	0.0000276003
4	0	0.0063504000	-0.0085377600	0.0058893120	-0.0026291571	0.0008527794	-0.0002152821	0.0000442608
5	0	0.0051226560	-0.0082450368	0.0066254760	-0.0033770063	0.0012314134	-0.0003452341	0.0000780465
6	0	0.0049470221	-0.0092756664	0.0085100558	-0.0048763971	0.0019747393	-0.0006087629	0.0001500895
7	0	0.0055653998	-0.0119140782	0.0122885206	-0.0078199677	0.0034821237	-0.0011706978	0.0003126149

$t_j^{(k,0)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.2000000000	0.1000000000	-0.0333333333	0.0083333333	-0.0016666667	0.0002777778	-0.0000396825	0.0000049603
1	-0.0360000000	0.0333333333	-0.0163333333	0.0054000000	-0.0013444444	0.0002682540	-0.0000446429	0.0000063713
2	-0.0120000000	0.0163333333	-0.0105840000	0.0043560000	-0.0012983492	0.0003017857	-0.0000573413	0.0000092001
3	-0.0058800000	0.0105840000	-0.0085377600	0.0042066514	-0.0014606429	0.0003876270	-0.0000828008	0.0000147536
4	-0.0038102400	0.0085377600	-0.0082450368	0.0047324829	-0.0018761146	0.0005597334	-0.0001327824	0.0000260155
5	-0.0030735936	0.0082450368	-0.0092756664	0.0060786113	-0.0027091095	0.0008976088	-0.0002341396	0.0000500298
6	-0.0029682132	0.0092756664	-0.0119140782	0.0087775147	-0.0043444265	0.0015827835	-0.0004502684	0.0001042050
7	-0.0033392399	0.0119140782	-0.0172039289	0.0140759418	-0.0076606720	0.0030438144	-0.0009378448	0.0002336970

Asymptotic expansion:

$$F_0(x;\alpha) \quad \sim \quad \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right) +$$

$$+\frac{2\,e^{-x^2}}{\sqrt{\pi\alpha x}}\left[\left(\frac{1}{2x}-\frac{32\alpha^4+120\alpha^2-15}{256x^3}-\frac{2048\alpha^8+32256\alpha^6+60480\alpha^4-5040\alpha^2-4725}{65536x^5}+\ldots\right)\cos\left(x+\frac{\pi}{4}\right)+\right.$$

$$+\left(-\frac{4\alpha^2+3}{16x^2}+\frac{128\alpha^6+1120\alpha^4+420\alpha^2+105}{2048x^4}-\right.$$

$$\left.\left.\frac{8192\alpha^{10}+202752\alpha^8+887040\alpha^6+221760\alpha^4+41580\alpha^2+72765}{524288x^6}+\ldots\right)\sin\left(x+\frac{\pi}{4}\right)\right]$$

II) $s = 1$:

$$\int_0^x e^{-t^2}\, J_1(\alpha t)\, dt =$$

$$=\frac{1}{\alpha}+\frac{e^{-x^2}}{\alpha}\left[\left(-1-x^2+\frac{\alpha^2-4}{8}\,x^4-\frac{\alpha^4-20\,\alpha^2+32}{192}\,x^6+\frac{\alpha^6-44\,\alpha^4+416\,\alpha^2-384}{9216}\,x^8+\ldots\right)J_0(\alpha x)+\right.$$

$$\left.+\left(-\frac{\alpha}{2}\,x^3+\frac{\alpha^3-12\,\alpha}{32}\,x^5-\frac{\alpha^5-32\,\alpha^3+176\,\alpha}{1152}\,x^7+\frac{\alpha^7-60\,\alpha^5+928\,\alpha^3-3200\,\alpha}{73728}\,x^9+\ldots\right)J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(1;\alpha)}=\frac{2a_k^{(1;\alpha)}-\alpha b_k^{(1;\alpha)}}{2k+2}\quad,\quad b_{k+1}^{(1;\alpha)}=\frac{2\alpha a_k^{(1;\alpha)}+(4k+4-\alpha^2)b_k^{(1;\alpha)}}{(2k+2)^2}$$

Approximation:

$$F_1(x;\alpha)=c_0^{(1;\alpha)}\left(1-e^{-x^2}\right)+e^{-x^2}\sum_{k=1}^{5}c_k^{(1;\alpha)}\,x^{2k}$$

$$c_0^{(1;\alpha)}=-0.00000\,03391\,\alpha^{11}+0.00000\,81380\,\alpha^9-0.00016\,27604\,\alpha^7+0.00260\,41667\,\alpha^5-0.03125\,\alpha^3+0.25\,\alpha$$

$$c_1^{(1;\alpha)}=0.00000\,03391\,\alpha^{11}-0.00000\,81380\,\alpha^9+0.00016\,27604\,\alpha^7-0.00260\,41667\,\alpha^5+0.03125\,\alpha^3$$

$$c_2^{(1;\alpha)}=0.00000\,01695\,\alpha^{11}-0.00000\,40690\,\alpha^9+0.00008\,13802\,\alpha^7-0.00130\,20833\,\alpha^5$$

$$c_3^{(1;\alpha)}=0.00000\,00565\,\alpha^{11}-0.00000\,13563\,\alpha^9+0.00002\,71267\,\alpha^7$$

$$c_4^{(1;\alpha)}=0.00000\,00141\,\alpha^{11}-0.00000\,03391\,\alpha^9\;,\quad c_5^{(1;\alpha)}=0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used:

$$F_1(x;\alpha)=\left[\sum_{k=1}^{7}\varphi_k^{(1)}(\alpha)\,x^{2k}\right]J_0(\alpha x)+\left[\sum_{k=0}^{7}\psi_k^{(1)}(\alpha)\,x^{2k+1}\right]J_1(\alpha x)$$

Let

$$\varphi_k^{(1)}(\alpha)=\sum_{j=0}^{7}s_j^{(k,1)}\left(\frac{5}{\alpha}\right)^{2k+1}\qquad\text{and}\qquad\psi_k^{(1)}(\alpha)=\sum_{j=0}^{7}t_j^{(k,1)}\left(\frac{5}{\alpha}\right)^{2k+2}$$

$s_j^{(k,1)}$:

j	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	0.2000000000	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-.0002777778	0.0000396825
1	0.0320000000	-0.0320000000	0.0160000000	-0.0053333333	0.0013333333	-0.0002666667	0.0000444444
2	0.0102400000	-0.0153600000	0.0102400000	-0.0042666667	0.0012800000	-0.0002986667	0.0000568889
3	0.0049152000	-0.0098304000	0.0081920000	-0.0040960000	0.0014336000	-0.0003822933	0.0000819200
4	0.0031457280	-0.0078643200	0.0078643200	-0.0045875200	0.0018350080	-0.0005505024	0.0001310720
5	0.0025165824	-0.0075497472	0.0088080384	-0.0058720256	0.0026424115	-0.0008808038	0.0002306867
6	0.0024159191	-0.0084557169	0.0112742892	-0.0084557169	0.0042278584	-0.0015502148	0.0004429185
7	0.0027058294	-0.0108233176	0.0162349764	-0.0135291470	0.0074410308	-0.0029764123	0.0009212705

$t_j^{(k,1)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.0800000000	0.0800000000	-0.0400000000	0.0133333333	-0.0033333333	0.0006666667	-0.0001111111	0.0000158730
1	-0.0128000000	0.0256000000	-0.0192000000	0.0085333333	-0.0026666667	0.0006400000	-0.0001244444	0.0000203175
2	-0.0040960000	0.0122880000	-0.0122880000	0.0068266667	-0.0025600000	0.0007168000	-0.0001592889	0.0000292571
3	-0.0019660800	0.0078643200	-0.0098304000	0.0065536000	-0.0028672000	0.0009175040	-0.0002293760	0.0000468114
4	-0.0012582912	0.0062914560	-0.0094371840	0.0073400320	-0.0036700160	0.0013212058	-0.0003670016	0.0000823881
5	-0.0010066330	0.0060397978	-0.0105696461	0.0093952410	-0.0052848230	0.0021139292	-0.0006459228	0.0001581852
6	-0.0009663676	0.0067645735	-0.0135291470	0.0135291470	-0.0084557169	0.0037205154	-0.0012401718	0.0003290252
7	-0.0010823318	0.0086586541	-0.0194819717	0.0216466352	-0.0148820617	0.0071433896	-0.0025795574	0.0007370164

Asymptotic expansion:

$$F_1(x;\alpha) \sim \frac{1-e^{-\alpha^2/4}}{\alpha} + \sqrt{\frac{2}{\pi\alpha x}}\, e^{-x^2}\cdot$$

$$\cdot\left[\left(\frac{1}{2} - \frac{32\,\alpha^4+88\,\alpha^2-15}{256\,\alpha^2x^2} + \frac{2048\,\alpha^8+26112\,\alpha^6+38976\,\alpha^4-4080\,\alpha^2-4725}{65536\,\alpha^4x^4} + \ldots\right)\cos\left(x+\frac{\pi}{4}\right) -\right.$$

$$-\left(\frac{4\,\alpha^2+3}{16\,\alpha x} - \frac{128\,\alpha^6+864\,\alpha^4+324\,\alpha^2+105}{2048\,\alpha^3x^3} +\right.$$

$$\left.\left.+\frac{8192\,\alpha^{10}+169984\,\alpha^8+598272\,\alpha^6+149568\,\alpha^4+34860\,\alpha^2+72765}{524288\,\alpha^5x^5} + \ldots\right)\sin\left(x+\frac{\pi}{4}\right)\right]$$

III) $s=-$:

$$F_-(x;\alpha) =$$

$$= e^{-x^2}\left[\left(\alpha x + \frac{4\alpha-\alpha^3}{3}\,x^3 + \frac{\alpha^5-16\alpha^3+36\alpha}{45}\,x^5 + \frac{-\alpha^7+36\alpha^4-296\alpha^3+480}{1575}\,x^7 + \ldots\right) J_0(\alpha x) + \right.$$

$$\left. + \left(-1 + (\alpha^2-2)x^2 + \frac{-\alpha^4+10\alpha^2-12}{9}\,x^4 + \frac{\alpha^6-26\alpha^4+136\alpha^2-120}{225}\,x^6 \ldots\right) J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(-;\alpha)} = \frac{(4k+2-\alpha^2)a_k^{(-;\alpha)} - 2\alpha b_k^{(-;\alpha)}}{(2k+1)(2k+3)}\ , \quad b_{k+1}^{(-;\alpha)} = \frac{2b_k^{(-;\alpha)} + \alpha a_k^{(-;\alpha)}}{2k+1}$$

Approximation:

$$F_-(x;\alpha) = c_0^{(-;\alpha)}\,\mathrm{erf}(x) + e^{-x^2}\sum_{k=1}^{5} c_k^{(-;\alpha)}\,x^{2k-1}$$

$$c_0^{(-;\alpha)} = -0.00000\,01479\,\alpha^{11} + 0.00000\,39441\,\alpha^9 - 0.00009\,01517\,\alpha^7 + 0.00173\,09120\,\alpha^5 -$$

$$-0.02769\,45914\,\alpha^3 + 0.44311\,34627\,\alpha$$

$$c_1^{(-;\alpha)} = 0.00000\,016689\,\alpha^{11} - 0.00000\,44505\,\alpha^9 + 0.00010\,17253\,\alpha^7 - 0.00195\,3125\,\alpha^5 + 0.03125\,\alpha^3$$

$$c_2^{(-;\alpha)} = 0.00000\,01113\,\alpha^{11} - 0.00000\,29670\,\alpha^9 + 0.00006\,78168\,\alpha^7 - 0.00130\,20833\,\alpha^5$$

$$c_3^{(-;\alpha)} = 0.00000\,00445\,\alpha^{11} - 0.00000\,11868\,\alpha^9 + 0.00002\,71267\,\alpha^7$$

$$c_4^{(-;\alpha)} = 0.00000\,00127\,\alpha^{11} - 0.00000\,03391\,\alpha^9\ , \quad c_5^{(-;\alpha)} = 0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used ($\Phi(x)$ defined as on page 7):

$$F_-(x;\alpha) = \sigma^{(-)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7}\varphi_k^{(-)}(\alpha)\,x^{2k+1}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7}\psi_k^{(-)}(\alpha)\,x^{2k}\right] J_1(\alpha x)$$

Let

$$\varphi_k^{(0)}(\alpha) = \sum_{j=0}^{7} s_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k-1} \quad \text{and} \quad \psi_k^{(-)}(\alpha) = \sum_{j=0}^{7} t_j^{(k,-)}\left(\frac{5}{\alpha}\right)^{2k}$$

$s_j^{(k,-)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	5	0	0	0	0	0	0	0
1	0	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-0.0002777778	0.0000396825	-0.0000049603
2	0	-0.0200000000	0.0116666667	-0.0042000000	0.0011000000	-0.0002269841	0.0000386905	-0.0000056217
3	0	-0.0070000000	0.0058800000	-0.0027720000	0.0008988571	-0.0002213095	0.0000438492	-0.0000072632
4	0	-0.0035280000	0.0038808000	-0.0022651200	0.0008763857	-0.0002508175	0.0000566532	-0.0000105383
5	0	-0.0023284800	0.0031711680	-0.0022084920	0.0009932371	-0.0003240562	0.0000821986	-0.0000169666
6	0	-0.0019027008	0.0030918888	-0.0025029576	0.0012832624	-0.0004701760	0.0001323398	-0.0000300179
7	0	-0.0018551333	0.0035041406	-0.0032338212	0.0018618971	-0.0007569834	0.0002341396	-0.0000578917

$t_j^{(k,-)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0120000000	-0.0116666667	0.0058800000	-0.0019800000	0.0004993651	-0.0001005952	0.0000168651
3	0	0.0042000000	-0.0058800000	0.0038808000	-0.0016179429	0.0004868810	-0.0001140079	0.0000217897
4	0	0.0021168000	-0.0038808000	0.0031711680	-0.0015774943	0.0005517984	-0.0001472983	0.0000316148
5	0	0.0013970880	-0.0031711680	0.0030918888	-0.0017878269	0.0007129235	-0.0002137164	0.0000508999
6	0	0.0011416205	-0.0030918888	0.0035041406	-0.0023098723	0.0010343873	-0.0003440834	0.0000900537
7	0	0.0011130800	-0.0035041406	0.0045273497	-0.0033514147	0.0016653635	-0.0006087629	0.0001736750

Asymptotic expansion:

$$F_-(x;\alpha) \sim \frac{\sqrt{\pi}}{4}\, e^{-\alpha^2/8} \left[I_0\left(\frac{\alpha^2}{8}\right) + I_1\left(\frac{\alpha^2}{8}\right)\right] + \sqrt{\frac{2}{\pi\alpha x}}\, e^{-x^2} \cdot$$

$$\cdot \left[\left(\frac{1}{2x} - \frac{-32\,\alpha^4 - 152\,\alpha^2 + 15}{256\,\alpha^2 x^3} + \frac{2048\,\alpha^8 + 38400\,\alpha^6 + 86080\,\alpha^4 - 6000\,\alpha^2 - 4725}{65536\,\alpha^4 x^5} + \ldots \right) \cos\left(x + \frac{\pi}{4}\right) - \right.$$

$$- \left(\frac{4\,\alpha^2 + 3}{16\,\alpha x^2} - \frac{128\,\alpha^6 + 1376\,\alpha^4 + 516\,\alpha^2 + 105}{2048\,\alpha^3 x^4} + \right.$$

$$\left. \left. + \frac{8192\,\alpha^{10} + 235520\,\alpha^8 + 1224960\,\alpha^6 + 306240\,\alpha^4 + 48300\,\alpha^2 + 72765}{524288\,\alpha^5 x^6} + \ldots \right) \sin\left(x + \frac{\pi}{4}\right) \right]$$

d) Integrals ($\alpha \neq 1$)

$$\int x\, e^{-x^2}\, J_0(\alpha x)\, dx = -\frac{e^{-x^2}}{2}\, J_0(\alpha x) - \frac{\alpha}{2} \int e^{-x^2}\, J_1(\alpha x)\, dx$$

$$\int x\, e^{-x^2}\, J_1(\alpha x)\, dx = -\frac{e^{-x^2}}{2}\, J_1(\alpha x) - \frac{\alpha}{2} \int e^{-x^2}\, J_0(\alpha x)\, dx - \frac{1}{2} \int \frac{e^{-x^2}\, J_1(\alpha x)}{x}\, dx$$

Let

$$\int x^{2n+\nu}\, e^{-x^2}\, J_\nu(\alpha x)\, dx =$$

$$= e^{-x^2} \left[A_{n;\nu}(x;\alpha)\, J_0(\alpha x) + B_{n;\nu}(x;\alpha)\, J_1(\alpha x) \right] + P_{n;\nu} \int e^{-x^2}\, J_0(\alpha x)\, dx + Q_{n;\nu} \int \frac{e^{-x^2}\, J_1(\alpha x)\, dx}{x}$$

and

$$\int x^{2n+1-\nu}\, e^{-x^2}\, J_\nu(\alpha x)\, dx = e^{-x^2} \left[C_{n;\nu}(x;\alpha)\, J_0(\alpha x) + D_{n;\nu}(x;\alpha)\, J_1(\alpha x) \right] + R_{n;\nu} \int e^{-x^2}\, J_1(\alpha x)\, dx\,.$$

$$A_{1;\,0}(x;\alpha) = -\frac{x}{2}\,, \quad B_{1;\,0}(x;\alpha) = \frac{\alpha}{4}\,, \quad P_{1;\,0}(\alpha) = \frac{-\alpha^2 + 2}{4}\,, \quad Q_{1;\,0}(x;\alpha) = \frac{\alpha}{4}$$

$$A_{1;1}(x;\alpha) = -\frac{\alpha\, x}{8}\,, \quad B_{1;1}(x;\alpha) = \frac{-4\,x^2 + \alpha^2 - 2}{8}\,, \quad P_{1;1}(\alpha) = \frac{-\alpha^3 + 4\,\alpha}{8}\,, \quad Q_{1;1}(x;\alpha) = \frac{\alpha^2 - 2}{8}$$

$$C_{1;0}(x;\alpha) = \frac{-4\,x^2 + \alpha^2 - 4}{8}\,, \quad D_{1;0}(x;\alpha) = \frac{\alpha\, x}{4}\,, \quad R_{1;0}(\alpha) = \frac{\alpha^3 - 4\,\alpha}{8}$$

$$C_{1;1}(x;\alpha) = -\frac{\alpha}{4}\,, \quad D_{1;1}(x;\alpha) = -\frac{x}{2}\,, \quad R_{1;1}(\alpha) = -\frac{\alpha^2}{4}$$

$$A_{2;0}(x;\alpha) = \frac{-4\,x^3 + (\alpha^2 - 6)\,x}{8}\,, \quad B_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^2 - \alpha^3 + 8\,\alpha}{16}\,,$$

$$P_{2;0}(\alpha) = \frac{\alpha^4 - 10\,\alpha^2 + 12}{16}\,, \quad Q_{2;0}(x;\alpha) = \frac{-\alpha^3 + 8\,\alpha}{16}$$

$$A_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^3 + \left(\alpha^3 - 12\,\alpha\right)x}{16}\,, \quad B_{2;1}(x;\alpha) = \frac{-12 - 16\,x^4 + \left(4\,\alpha^2 - 24\right)x^2 - \alpha^4 + 14\,\alpha^2}{32}\,,$$

$$P_{2;1}(\alpha) = \frac{\alpha^5 - 16\,\alpha^3 + 36\,\alpha}{32}\,, \quad Q_{2;1}(x;\alpha) = \frac{-\alpha^4 + 14\,\alpha^2 - 12}{32}$$

$$C_{2;0}(x;\alpha) = \frac{-16\,x^4 + \left(4\,\alpha^2 - 32\right)x^2 - \alpha^4 + 16\,\alpha^2 - 32}{32}\,, \quad D_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^3 + \left(-\alpha^3 + 12\,\alpha\right)x}{16}\,,$$

$$R_{2;0}(\alpha) = \frac{-\alpha^5 + 16\,\alpha^3 - 32\,\alpha}{32}$$

$$C_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^2 + \alpha^3 - 8\,\alpha}{16}\,, \quad D_{2;1}(x;\alpha) = \frac{-4\,x^3 + \left(\alpha^2 - 4\right)x}{8}\,, \quad R_{2;1}(\alpha) = \frac{\alpha^4 - 8\,\alpha^2}{16}$$

$$A_{3;0}(x;\alpha) = \frac{-16\,x^5 + (4\,\alpha^2 - 40)x^3 - (\alpha^4 - 22\,\alpha^2 + 60)x}{32}\,,$$

$$B_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^4 + (64\,\alpha - 4\,\alpha^3)x^2 + \alpha^5 - 24\,\alpha^3 + 92\,\alpha}{64}\,,$$

$$P_{3;0}(\alpha) = \frac{-\alpha^6 + 26\,\alpha^4 - 136\,\alpha^2 + 120}{64}\,, \quad Q_{3;0}(x;\alpha) = \frac{\alpha^5 - 24\,\alpha^3 + 92\,\alpha}{64}$$

$$A_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^5 + \left(4\,\alpha^3 - 80\,\alpha\right)x^3 + \left(-\alpha^5 + 32\,\alpha^3 - 180\,\alpha\right)x}{64}\,,$$

$$B_{3;1}(x;\alpha) = \frac{-64\,x^6 + \left(16\,\alpha^2 - 160\right)x^4 + \left(-4\,\alpha^4 + 104\,\alpha^2 - 240\right)x^2 + \alpha^6 - 34\,\alpha^4 + 232\,\alpha^2 - 120}{128}\,,$$

$$P_{3;1}(\alpha) = \frac{-\alpha^7 + 36\,\alpha^5 - 296\,\alpha^3 + 480\,\alpha}{128}\,, \quad Q_{3;1}(x;\alpha) = \frac{\alpha^6 - 34\,\alpha^4 + 232\,\alpha^2 - 120}{128}$$

$$C_{3;0}(x;\alpha) = \frac{-64\,x^6 + \left(16\,\alpha^2 - 192\right)x^4 + \left(-4\,\alpha^4 + 112\,\alpha^2 - 384\right)x^2 + \alpha^6 - 36\,\alpha^4 + 288\,\alpha^2 - 384}{128}\,,$$

$$D_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^5 + \left(-4\,\alpha^3 + 80\,\alpha\right)x^3 + \left(\alpha^5 - 32\,\alpha^3 + 176\,\alpha\right)x}{54}\,,$$

$$R_{3;0}(\alpha) = \frac{\alpha^7 - 36\,\alpha^5 + 288\,\alpha^3 - 384\,\alpha}{128}$$

$$C_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^4 + \left(4\,\alpha^3 - 64\,\alpha\right)x^2 - \alpha^5 + 24\,\alpha^3 - 96\,\alpha}{64}\,,$$

$$D_{3;1}(x;\alpha) = \frac{-16\,x^5 + \left(4\,\alpha^2 - 32\right)x^3 + \left(-\alpha^4 + 20\,\alpha^2 - 32\right)x}{32}\,,$$

$$R_{3;1}(\alpha) = \frac{-\alpha^6 + 24\,\alpha^4 - 96\,\alpha^2}{64}$$

$$A_{4;0}(x;\alpha) = \frac{-64\,x^7 + \left(16\,\alpha^2 - 224\right)x^5 + \left(-4\,\alpha^4 + 136\,\alpha^2 - 560\right)x^3 + \left(\alpha^6 - 46\,\alpha^4 + 488\,\alpha^2 - 840\right)x}{128}\,,$$

$$B_{4;0}(x;\alpha) = \frac{64\,\alpha\,x^6 + \left(-16\,\alpha^3 + 384\,\alpha\right)x^4 + \left(4\,\alpha^5 - 160\,\alpha^3 + 1136\,\alpha\right)x^2 - \alpha^7 + 48\,\alpha^5 - 568\,\alpha^3 + 1408\,\alpha}{256}\,,$$

$$P_{4;0}(\alpha) = \frac{\alpha^8 - 50\,\alpha^6 + 660\,\alpha^4 - 2384\,\alpha^2 + 1680}{256}\,, \quad Q_{4;0}(x;\alpha) = \frac{-\alpha^7 + 48\,\alpha^5 - 568\,\alpha^3 + 1408\,\alpha}{256}$$

$$A_{4;1}(x;\alpha) = \frac{-64\,\alpha\,x^7 + \left(16\,\alpha^3 - 448\,\alpha\right)x^5 + \left(-4\,\alpha^5 + 192\,\alpha^3 - 1680\,\alpha\right)x^3 + \left(\alpha^7 - 60\,\alpha^5 + 936\,\alpha^3 - 3360\,\alpha\right)x}{256}\,,$$

$$B_{4;1}(x;\alpha) = \frac{1}{512}\,[-256\,x^8 + \big(64\,\alpha^2 - 896\big)\,x^6 + \big(-16\,\alpha^4 + 608\,\alpha^2 - 2240\big)\,x^4 +$$
$$+ \big(4\,\alpha^6 - 216\,\alpha^4 + 2592\,\alpha^2 - 3360\big)\,x^2 - \alpha^8 + 62\,\alpha^6 - 1044\,\alpha^4 + 4656\,\alpha^2 - 1680]\ ,$$
$$P_{4;1}(\alpha) = \frac{\alpha^9 - 64\,\alpha^7 + 1164\,\alpha^5 - 6528\,\alpha^3 + 8400\,\alpha}{512}\ ,$$
$$Q_{4;1}(x;\alpha) = \frac{-\alpha^8 + 62\,\alpha^6 - 1044\,\alpha^4 + 4656\,\alpha^2 - 1680}{512}$$
$$C_{4;0}(x;\alpha) = \frac{1}{512}\,[-256\,x^8 + \big(64\,\alpha^2 - 1024\big)\,x^6 + \big(-16\,\alpha^4 + 640\,\alpha^2 - 3072\big)\,x^4 +$$
$$+ \big(4\,\alpha^6 - 224\,\alpha^4 + 2944\,\alpha^2 - 6144\big)\,x^2 - \alpha^8 + 64\,\alpha^6 - 1152\,\alpha^4 + 6144\,\alpha^2 - 6144]\ ,$$
$$D_{4;0}(x;\alpha) = \frac{64\,\alpha\,x^7 + \big(-16\,\alpha^3 + 448\,\alpha\big)\,x^5 + \big(4\,\alpha^5 - 192\,\alpha^3 + 1664\,\alpha\big)\,x^3 + \big(-\alpha^7 + 60\,\alpha^5 - 928\,\alpha^3 + 3200\,\alpha\big)\,x}{256}\ ,$$
$$R_{4;0}(\alpha) = \frac{-\alpha^9 + 64\,\alpha^7 - 1152\,\alpha^5 + 6144\,\alpha^3 - 6144\,\alpha}{512}$$
$$C_{4;1}(x;\alpha) = \frac{-64\,\alpha\,x^6 + \big(16\,\alpha^3 - 384\,\alpha\big)\,x^4 + \big(-4\,\alpha^5 + 160\,\alpha^3 - 1152\,\alpha\big)\,x^2 + \alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha}{256}\ ,$$
$$D_{4;1}(x;\alpha) = \frac{-64\,x^7 + \big(16\,\alpha^2 - 192\big)\,x^5 + \big(-4\,\alpha^4 + 128\,\alpha^2 - 384\big)\,x^3 + \big(\alpha^6 - 44\,\alpha^4 + 416\,\alpha^2 - 384\big)\,x}{128}\ ,$$
$$R_{4;1}(\alpha) = \frac{\alpha^8 - 48\,\alpha^6 + 576\,\alpha^4 - 1536\,\alpha^2}{256}$$
$$A_{5;0}(x;\alpha) = \frac{1}{512}\,[-256\,x^9 + \big(64\,\alpha^2 - 1152\big)\,x^7 + \big(-16\,\alpha^4 + 736\,\alpha^2 - 4032\big)\,x^5 +$$
$$+ \big(4\,\alpha^6 - 264\,\alpha^4 + 4128\,\alpha^2 - 10080\big)\,x^3 + \big(-\alpha^8 + 78\,\alpha^6 - 1764\,\alpha^4 + 12144\,\alpha^2 - 15120\big)\,x]\ ,$$
$$B_{5;0}(x;\alpha) = \frac{1}{1024}\,[256\,\alpha\,x^8 + \big(-64\,\alpha^3 + 2048\,\alpha\big)\,x^6 + \big(16\,\alpha^5 - 896\,\alpha^3 + 9152\,\alpha\big)\,x^4 +$$
$$+ \big(-4\,\alpha^7 + 288\,\alpha^5 - 5472\,\alpha^3 + 23808\,\alpha\big)\,x^2 + \alpha^9 - 80\,\alpha^7 + 1908\,\alpha^5 - 14880\,\alpha^3 + 27024\,\alpha]\ ,$$
$$P_{5;0}(\alpha) = \frac{-\alpha^{10} + 82\,\alpha^8 - 2064\,\alpha^6 + 18408\,\alpha^4 - 51312\,\alpha^2 + 30240}{1024}\ ,$$
$$Q_{5;0}(x;\alpha) = \frac{\alpha^9 - 80\,\alpha^7 + 1908\,\alpha^5 - 14880\,\alpha^3 + 27024\,\alpha}{1024}$$
$$A_{5;1}(x;\alpha) = \frac{1}{1024}\,[-256\,\alpha\,x^9 + \big(64\,\alpha^3 - 2304\,\alpha\big)\,x^7 + \big(-16\,\alpha^5 + 1024\,\alpha^3 - 12096\,\alpha\big)\,x^5 +$$
$$+ \big(4\,\alpha^7 - 336\,\alpha^5 + 7584\,\alpha^3 - 40320\,\alpha\big)\,x^3 + \big(-\alpha^9 + 96\,\alpha^7 - 2844\,\alpha^5 + 28992\,\alpha^3 - 75600\,\alpha\big)\,x]\ ,$$
$$B_{5;1}(x;\alpha) = \frac{1}{2048}\,[-1024\,x^{10} + \big(256\,\alpha^2 - 4608\big)\,x^8 + \big(-64\,\alpha^4 + 3200\,\alpha^2 - 16128\big)\,x^6 +$$
$$+ \big(16\,\alpha^6 - 1184\,\alpha^4 + 20096\,\alpha^2 - 40320\big)\,x^4 + \big(-4\,\alpha^8 + 360\,\alpha^6 - 9360\,\alpha^4 + 70464\,\alpha^2 - 60480\big)\,x^2 +$$
$$+ \alpha^{10} - 98\,\alpha^8 + 3024\,\alpha^6 - 33672\,\alpha^4 + 110832\,\alpha^2 - 30240]\ ,$$
$$P_{5;1}(\alpha) = \frac{-\alpha^{11} + 100\,\alpha^9 - 3216\,\alpha^7 + 39360\,\alpha^5 - 168816\,\alpha^3 + 181440\,\alpha}{2048}\ ,$$
$$Q_{5;1}(x;\alpha) = \frac{\alpha^{10} - 98\,\alpha^8 + 3024\,\alpha^6 - 33672\,\alpha^4 + 110832\,\alpha^2 - 30240}{2048}$$
$$C_{5;0}(x;\alpha) = \frac{1}{2048}\,[-1024\,x^{10} + \big(256\,\alpha^2 - 5120\big)\,x^8 + \big(-64\,\alpha^4 + 3328\,\alpha^2 - 20480\big)\,x^6 +$$
$$+ \big(16\,\alpha^6 - 1216\,\alpha^4 + 22016\,\alpha^2 - 61440\big)\,x^4 + \big(-4\,\alpha^8 + 368\,\alpha^6 - 9984\,\alpha^4 + 83456\,\alpha^2 - 122880\big)\,x^2 +$$
$$+ \alpha^{10} - 100\,\alpha^8 + 3200\,\alpha^6 - 38400\,\alpha^4 + 153600\,\alpha^2 - 122880]\ ,$$
$$D_{5;0}(x;\alpha) = \frac{1}{1024}\,[256\,\alpha\,x^9 + \big(-64\,\alpha^3 + 2304\,\alpha\big)\,x^7 + \big(16\,\alpha^5 - 1024\,\alpha^3 + 12032\,\alpha\big)\,x^5 +$$

$$+\left(-4\,\alpha^7+336\,\alpha^5-7552\,\alpha^3+39424\,\alpha\right)x^3+\left(\alpha^9-96\,\alpha^7+2832\,\alpha^5-28416\,\alpha^3+70144\,\alpha\right)x\Big]\ ,$$

$$R_{5;0}(\alpha)=\frac{\alpha^{11}-100\,\alpha^9+3200\,\alpha^7-38400\,\alpha^5+153600\,\alpha^3-122880\,\alpha}{2048}$$

$$C_{5;1}(x;\alpha)=\frac{1}{1024}\Big[-256\,\alpha\,x^8+\left(64\,\alpha^3-2048\,\alpha\right)x^6+\left(-16\,\alpha^5+896\,\alpha^3-9216\,\alpha\right)x^4+$$
$$+\left(4\,\alpha^7-288\,\alpha^5+5504\,\alpha^3-24576\,\alpha\right)x^2-\alpha^9+80\,\alpha^7-1920\,\alpha^5+15360\,\alpha^3-30720\,\alpha\Big]\ ,$$

$$D_{5;1}(x;\alpha)=\frac{1}{512}\Big[-256\,x^9+\left(64\,\alpha^2-1024\right)x^7+\left(-16\,\alpha^4+704\,\alpha^2-3072\right)x^5+$$
$$+\left(4\,\alpha^6-256\,\alpha^4+3712\,\alpha^2-6144\right)x^3+\left(-\alpha^8+76\,\alpha^6-1632\,\alpha^4+9856\,\alpha^2-6144\right)x\Big]\ ,$$

$$R_{5;1}(\alpha)=\frac{-\alpha^{10}+80\,\alpha^8-1920\,\alpha^6+15360\,\alpha^4-30720\,\alpha^2}{1024}$$

$$A_{6;0}(x;\alpha)=\frac{1}{2048}\Big[-1024\,x^{11}+\left(256\,\alpha^2-5632\right)x^9+\left(-64\,\alpha^4+3712\,\alpha^2-25344\right)x^7+$$
$$+\left(16\,\alpha^6-1376\,\alpha^4+28288\,\alpha^2-88704\right)x^5+\left(-4\,\alpha^8+424\,\alpha^6-13392\,\alpha^4+131136\,\alpha^2-221760\right)x^3+$$
$$+\left(\alpha^{10}-118\,\alpha^8+4560\,\alpha^6-67800\,\alpha^4+342768\,\alpha^2-332640\right)x\Big]\ ,$$

$$B_{6;0}(x;\alpha)=\frac{1}{4096}\Big[1024\,\alpha\,x^{10}+\left(-256\,\alpha^3+10240\,\alpha\right)x^8+\left(64\,\alpha^5-4608\,\alpha^3+61184\,\alpha\right)x^6+$$
$$+\left(-16\,\alpha^7+1536\,\alpha^5-39808\,\alpha^3+241664\,\alpha\right)x^4+\left(4\,\alpha^9-448\,\alpha^7+15696\,\alpha^5-190848\,\alpha^3+584256\,\alpha\right)x^2-$$
$$-\alpha^{11}+120\,\alpha^9-4784\,\alpha^7+75648\,\alpha^5-438192\,\alpha^3+624768\,\alpha\Big]\ ,$$

$$P_{6;0}(\alpha)=\frac{\alpha^{12}-122\,\alpha^{10}+5020\,\alpha^8-84768\,\alpha^6+573792\,\alpha^4-1310304\,\alpha^2+665280}{4096}\ ,$$

$$Q_{6;0}(x;\alpha)=\frac{-\alpha^{11}+120\,\alpha^9-4784\,\alpha^7+75648\,\alpha^5-438192\,\alpha^3+624768\,\alpha}{4096}$$

$$A_{6;1}(x;\alpha)=\frac{1}{4096}\Big[-1024\,\alpha\,x^{11}+\left(256\,\alpha^3-11264\,\alpha\right)x^9+\left(-64\,\alpha^5+5120\,\alpha^3-76032\,\alpha\right)x^7+$$
$$+\left(16\,\alpha^7-1728\,\alpha^5+50816\,\alpha^3-354816\,\alpha\right)x^5+\left(-4\,\alpha^9+512\,\alpha^7-20784\,\alpha^5+297984\,\alpha^3-1108800\,\alpha\right)x^3+$$
$$+\left(\alpha^{11}-140\,\alpha^9+6672\,\alpha^7-130368\,\alpha^5+980592\,\alpha^3-1995840\,\alpha\right)x\Big]\ ,$$

$$B_{6;1}(x;\alpha)=\frac{1}{8192}\Big[-4096\,x^{12}+\left(1024\,\alpha^2-22528\right)x^{10}+\left(-256\,\alpha^4+15872\,\alpha^2-101376\right)x^8+$$
$$+\left(64\,\alpha^6-6016\,\alpha^4+131584\,\alpha^2-354816\right)x^6+\left(-16\,\alpha^8+1888\,\alpha^6-65856\,\alpha^4+683776\,\alpha^2-887040\right)x^4+$$
$$+\left(4\,\alpha^{10}-536\,\alpha^8+23616\,\alpha^6-396768\,\alpha^4+2134464\,\alpha^2-1330560\right)x^2-$$
$$-\alpha^{12}+142\,\alpha^{10}-6940\,\alpha^8+142176\,\alpha^6-1178976\,\alpha^4+3063072\,\alpha^2-665280\Big]\ ,$$

$$P_{6;1}(\alpha)=\frac{\alpha^{13}-144\,\alpha^{11}+7220\,\alpha^9-155520\,\alpha^7+1439712\,\alpha^5-5024256\,\alpha^3+4656960\,\alpha}{8192}\ ,$$

$$Q_{6;1}(x;\alpha)=\frac{-\alpha^{12}+142\,\alpha^{10}-6940\,\alpha^8+142176\,\alpha^6-1178976\,\alpha^4+3063072\,\alpha^2-665280}{8192}$$

$$C_{6;0}(x;\alpha)=\frac{1}{8192}\Big[-4096\,x^{12}+\left(1024\,\alpha^2-24576\right)x^{10}+\left(-256\,\alpha^4+16384\,\alpha^2-122880\right)x^8+$$
$$+\left(64\,\alpha^6-6144\,\alpha^4+141312\,\alpha^2-491520\right)x^6+\left(-16\,\alpha^8+1920\,\alpha^6-69120\,\alpha^4+774144\,\alpha^2-1474560\right)x^4+$$
$$+\left(4\,\alpha^{10}-544\,\alpha^8+24576\,\alpha^6-433152\,\alpha^4+2617344\,\alpha^2-2949120\right)x^2-$$
$$-\alpha^{12}+144\,\alpha^{10}-7200\,\alpha^8+153600\,\alpha^6-1382400\,\alpha^4+4423680\,\alpha^2-2949120\Big]\ ,$$

$$D_{6;0}(x;\alpha)=\frac{1}{4096}\Big[1024\,\alpha\,x^{11}+\left(-256\,\alpha^3+11264\,\alpha\right)x^9+\left(64\,\alpha^5-5120\,\alpha^3+75776\,\alpha\right)x^7+$$
$$+\left(-16\,\alpha^7+1728\,\alpha^5-50688\,\alpha^3+350208\,\alpha\right)x^5+\left(4\,\alpha^9-512\,\alpha^7+20736\,\alpha^5-294912\,\alpha^3+1069056\,\alpha\right)x^3+$$
$$+\left(-\alpha^{11}+140\,\alpha^9-6656\,\alpha^7+129024\,\alpha^5-949248\,\alpha^3+1806336\,\alpha\right)x\Big]\ ,$$

$$R_{6;0}(\alpha) = \frac{-\alpha^{13} + 144\,\alpha^{11} - 7200\,\alpha^9 + 153600\,\alpha^7 - 1382400\,\alpha^5 + 4423680\,\alpha^3 - 2949120\,\alpha}{8192}$$

$$C_{6;1}(x;\alpha) = \frac{1}{4096}\left[-1024\,\alpha\,x^{10} + \left(256\,\alpha^3 - 10240\,\alpha\right)x^8 + \left(-64\,\alpha^5 + 4608\,\alpha^3 - 61440\,\alpha\right)x^6 + \right.$$
$$+ \left(16\,\alpha^7 - 1536\,\alpha^5 + 39936\,\alpha^3 - 245760\,\alpha\right)x^4 + \left(-4\,\alpha^9 + 448\,\alpha^7 - 15744\,\alpha^5 + 193536\,\alpha^3 - 614400\,\alpha\right)x^2 +$$
$$\left. +\alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha\right],$$

$$D_{6;1}(x;\alpha) = \frac{1}{2048}\left[-1024\,x^{11} + \left(256\,\alpha^2 - 5120\right)x^9 + \left(-64\,\alpha^4 + 3584\,\alpha^2 - 20480\right)x^7 + \right.$$
$$+ \left(16\,\alpha^6 - 1344\,\alpha^4 + 26112\,\alpha^2 - 61440\right)x^5 + \left(-4\,\alpha^8 + 416\,\alpha^6 - 12672\,\alpha^4 + 113664\,\alpha^2 - 122880\right)x^3 +$$
$$\left. + \left(\alpha^{10} - 116\,\alpha^8 + 4352\,\alpha^6 - 61056\,\alpha^4 + 267264\,\alpha^2 - 122880\right)x\right],$$

$$R_{6;1}(\alpha) = \frac{\alpha^{12} - 120\,\alpha^{10} + 4800\,\alpha^8 - 76800\,\alpha^6 + 460800\,\alpha^4 - 737280\,\alpha^2}{4096}$$

Recurrence relations:

$$\int x^{2n+2}\,e^{-x^2}\,J_0(\alpha x)\,dx =$$
$$= \frac{x^{2n}\,e^{-x^2}}{4}\left[\alpha J_1(\alpha x) - 2xJ_0(\alpha x)\right] + \frac{4n+2-\alpha}{4}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{(2n-1)\alpha}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_0(\alpha x) + n\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(\alpha x)\,dx =$$
$$= -\frac{x^{2n+1}\,e^{-x^2}}{2}\left[\alpha J_0(\alpha x) + 2xJ_1(\alpha x)\right] + \frac{\alpha n}{2}\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{4n-\alpha^2}{4}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_1(\alpha x) + \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

e) Special Cases: $\nu = 0$

$$\int x^3\,e^{-x^2}\,J_0(2x)\,dx = \frac{x}{2}\,e^{-x^2}\left[J_1(2x) - xJ_0(2x)\right]$$
$$\int x^3\,e^{x^2}\,I_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\left[xI_0(2x) - I_1(2x)\right]$$
$$\int x^3\,e^{x^2}\,K_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\left[xK_0(2x) + K_1(2x)\right]$$

$$\int x^5\,e^{-x^2}\,J_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$
$$= \frac{x}{2}\,e^{-x^2}\left[x(\sqrt{2}-x^2)\,J_0\left((2\sqrt{2+\sqrt{2}}\,x\right) + \sqrt{2+\sqrt{2}}\,(x^2+1-\sqrt{2})\,J_1\left(2\sqrt{2+\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,e^{x^2}\,I_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$
$$= \frac{x}{2}\,e^{x^2}\left[x(\sqrt{2}+x^2)\,I_0\left((2\sqrt{2+\sqrt{2}}\,x\right) - \sqrt{2+\sqrt{2}}\,(x^2-1+\sqrt{2})\,I_1\left(2\sqrt{2+\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,e^{x^2}\,K_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$
$$= \frac{x}{2}\,e^{x^2}\left[x(\sqrt{2}+x^2)\,K_0\left((2\sqrt{2+\sqrt{2}}\,x\right) + \sqrt{2+\sqrt{2}}\,(x^2-1+\sqrt{2})\,K_1\left(2\sqrt{2+\sqrt{2}}\,x\right)\right]$$

$$\int x^5\, e^{-x^2}\, J_0\left(2\sqrt{2-\sqrt{2}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{-x^2}\, \left[-x(\sqrt{2}+x^2)\, J_0\left((2\sqrt{2-\sqrt{2}}\,x\right) + \sqrt{2-\sqrt{2}}\,(x^2+1+\sqrt{2})\, J_1\left(2\sqrt{2-\sqrt{2}}\,x\right)\right]$$

$$\int x^5\, e^{x^2}\, I_0\left(2\sqrt{2-\sqrt{2}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[x(x^2-\sqrt{2})\, I_0\left((2\sqrt{2-\sqrt{2}}\,x\right) - \sqrt{2-\sqrt{2}}\,(x^2-1-\sqrt{2})\, I_1\left(2\sqrt{2-\sqrt{2}}\,x\right)\right]$$

$$\int x^5\, e^{x^2}\, K_0\left(2\sqrt{2-\sqrt{2}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[x(x^2-\sqrt{2})\, K_0\left((2\sqrt{2-\sqrt{2}}\,x\right) + \sqrt{2-\sqrt{2}}\,(x^2-1-\sqrt{2})\, K_1\left(2\sqrt{2-\sqrt{2}}\,x\right)\right]$$

e) Special Cases: $\nu = 1$

$$\int x^4\, e^{-x^2}\, J_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}}\, e^{-x^2}\, [(1-x^2)J_1(2\sqrt{2}x) - x\sqrt{2}\, J_0(2\sqrt{2}x)]$$

$$\int x^4\, e^{x^2}\, I_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}}\, e^{x^2}\, [(1+x^2)I_1(2\sqrt{2}x) - x\sqrt{2}\, I_0(2\sqrt{2}x)]$$

$$\int x^4\, e^{x^2}\, K_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}}\, e^{x^2}\, [(1+x^2)K_1(2\sqrt{2}x) + x\sqrt{2}\, K_0(2\sqrt{2}x)]$$

$$\int x^6\, e^{-x^2}\, J_1\left(2\sqrt{3+\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{-x^2}\, \left[(-x^4+(1+\sqrt{3})x^2-\sqrt{3}+1)\, J_1\left((2\sqrt{3+\sqrt{3}}\,x\right) - \sqrt{3+\sqrt{3}}\,x\,(x^2+1-\sqrt{3})\, J_0\left(2\sqrt{3+\sqrt{3}}\,x\right)\right]$$

$$\int x^6\, e^{x^2}\, I_1\left(2\sqrt{3+\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[(x^4+(1+\sqrt{3})x^2+\sqrt{3}-1)\, I_1\left((2\sqrt{3+\sqrt{3}}\,x\right) - \sqrt{3+\sqrt{3}}\,x\,(x^2-1+\sqrt{3})\, I_0\left(2\sqrt{3+\sqrt{3}}\,x\right)\right]$$

$$\int x^6\, e^{x^2}\, K_1\left(2\sqrt{3+\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[(x^4+(1+\sqrt{3})x^2+\sqrt{3}-1)\, K_1\left((2\sqrt{3+\sqrt{3}}\,x\right) + \sqrt{3+\sqrt{3}}\,x\,(x^2-1+\sqrt{3})\, K_0\left(2\sqrt{3+\sqrt{3}}\,x\right)\right]$$

$$\int x^6\, e^{-x^2}\, J_1\left(2\sqrt{3-\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{-x^2}\, \left[(-x^4+(1-\sqrt{3})x^2+\sqrt{3}+1)\, J_1\left((2\sqrt{3-\sqrt{3}}\,x\right) - \sqrt{3-\sqrt{3}}\,x\,(x^2+1+\sqrt{3})\, J_0\left(2\sqrt{3-\sqrt{3}}\,x\right)\right]$$

$$\int x^6\, e^{x^2}\, I_1\left(2\sqrt{3-\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[(x^4+(1-\sqrt{3})x^2-\sqrt{3}-1)\, I_1\left((2\sqrt{3-\sqrt{3}}\,x\right) - \sqrt{3-\sqrt{3}}\,x\,(x^2-1-\sqrt{3})\, I_0\left(2\sqrt{3-\sqrt{3}}\,x\right)\right]$$

$$\int x^6\, e^{x^2}\, K_1\left(2\sqrt{3-\sqrt{3}}\,x\right)\, dx =$$

$$= \frac{x}{2}\, e^{x^2}\, \left[(x^4+(1-\sqrt{3})x^2-\sqrt{3}-1)\, K_1\left((2\sqrt{3-\sqrt{3}}\,x\right) + \sqrt{3-\sqrt{3}}\,x\,(x^2-1-\sqrt{3})\, K_0\left(2\sqrt{3-\sqrt{3}}\,x\right)\right]$$

1.2.12. Integrals with Orthogonal Polynomials $F_n(x)$: $\int F_n(x) \cdot Z_\nu(x)\,dx$

The used form of the orthogonal polynomials is shown in the integrals with $J_0(x)$.
$\Phi(x)$ and $\Psi(x)$ are defined on page 7.

a) Legendre Polynomials $P_n(x)$:

Weight function: $w(x) \equiv 1$, interval $-1 \le x \le 1$.
Holds $\int P_0(x)\,Z_\nu(x)\,dx = \int Z_\nu(x)\,dx$ and $\int P_1(x)\,Z_\nu(x)\,dx = \int x\,Z_\nu(x)\,dx$.
The denominator of the polynomial is written in front of the integral on the left hand side.

n = 2

$$2\int P_2(x)\cdot J_0(x)\,dx = \int (3x^2-1)\,J_0(x)\,dx = -x\,J_0(x) + 3x^2\,J_1(x) - 4\,\Phi(x)$$

$$2\int P_2(x)\cdot J_1(x)\,dx = \left(-3x^2+1\right)\,J_0(x) + 6x\,J_1(x)$$

$$2\int P_2(x)\cdot I_0(x)\,dx = -x\,I_0(x) + 3x^2\,I_1(x) + 2\,\Psi(x)$$

$$2\int P_2(x)\cdot I_1(x)\,dx = (3x^2-1)\,I_0(x) - 6x\,I_1(x)$$

n = 3

$$2\int P_3(x)\cdot J_0(x)\,dx = \int (5x^3-3x)\,J_0(x)\,dx = 10x^2\,J_0(x) + \left(5x^3-23x\right)\,J_1(x)$$

$$2\int P_3(x)\cdot J_1(x)\,dx = -5x^3\,J_0(x) + 15x^2\,J_1(x) - 18\Phi(x)$$

$$2\int P_3(x)\cdot I_0(x)\,dx = -10x^2\,I_0(x) + (5x^3+17x)\,I_1(x)$$

$$2\int P_3(x)\cdot I_1(x)\,dx = 5x^3\,I_0(x) - 15x^2\,I_1(x) - 12\,\Psi(x)$$

n = 4

$$8\int P_4(x)\cdot J_0(x)\,dx = \int \left(35\,x^4 - 30\,x^2 + 3\right)\,J_0(x)\,dx =$$
$$= \left(105\,x^3 + 3\,x\right)\,J_0(x) + \left(35\,x^4 - 345\,x^2\right)\,J_1(x) + 348\Phi(x)$$

$$8\int P_4(x)\cdot J_1(x)\,dx = \left(-35\,x^4 + 310\,x^2 - 3\right)\,J_0(x) + \left(140\,x^3 - 620\,x\right)\,J_1(x)$$

$$8\int P_4(x)\cdot I_0(x)\,dx = \left(-105\,x^3 + 3\,x\right)\,I_0(x) + \left(35\,x^4 + 285\,x^2\right)\,I_1(x) + 288\,\Psi(x)$$

$$8\int P_4(x)\cdot I_1(x)\,dx = \left(35\,x^4 + 250\,x^2 + 3\right)\,I_0(x) + \left(-140\,x^3 - 500\,x\right)\,I_1(x)$$

n = 5

$$8\int P_5(x)\cdot J_0(x)\,dx = \int \left(63\,x^5 - 70\,x^3 + 15\,x\right)\,J_0(x)\,dx =$$
$$= \left(252\,x^4 - 2156\,x^2\right)\,J_0(x) + \left(63\,x^5 - 1078\,x^3 + 4327\,x\right)\,J_1(x)$$

$$8\int P_5(x)\cdot J_1(x)\,dx = \left(-63\,x^5 + 1015\,x^3\right)\,J_0(x) + \left(315\,x^4 - 3045\,x^2\right)\,J_1(x) + 3060\,\Phi(x)$$

$$8\int P_5(x)\cdot I_0(x)\,dx = \left(-252\,x^4 - 1876\,x^2\right)\,I_0(x) + \left(63\,x^5 + 938\,x^3 + 3767\,x\right)\,I_1(x)$$

$$8\int P_5(x)\cdot I_1(x)\,dx = \left(63\,x^5 + 875\,x^3\right)\,I_0(x) + \left(-315\,x^4 - 2625\,x^2\right)\,I_1(x) - 2640\,\Psi(x)$$

n = 6

$$16\int P_6(x)\cdot J_0(x)\,dx = \int \left(231\,x^6 - 315\,x^4 + 105\,x^2 - 5\right)\,J_0(x)\,dx =$$

$$= \left(1155\,x^5 - 18270\,x^3 - 5\,x\right)\ J_0(x) + \left(231\,x^6 - 6090\,x^4 + 54915\,x^2\right)\ J_1(x) - 54920\,\Phi(x)$$

$$16\int P_6(x)\cdot J_1(x)\,dx = \left(-231\,x^6 + 5859\,x^4 - 46977\,x^2 + 5\right)\ J_0(x) + \left(1386\,x^5 - 23436\,x^3 + 93954\,x\right)\ J_1(x)$$

$$16\int P_6(x)\cdot I_0(x)\,dx = \left(-1155\,x^5 - 16380\,x^3 - 5\,x\right)\ I_0(x) + \left(231\,x^6 + 5460\,x^4 + 49245\,x^2\right)\ I_1(x) + 49240\,\Psi(x)$$

$$16\int P_6(x)\cdot I_1(x)\,dx = \left(231\,x^6 + 5229\,x^4 + 41937\,x^2 - 5\right)\ I_0(x) + \left(-1386\,x^5 - 20916\,x^3 - 83874\,x\right)\ I_1(x)$$

<u>**n = 7**</u>

$$16\int P_7(x)\cdot J_0(x)\,dx = \int \left(429\,x^7 - 693\,x^5 + 315\,x^3 - 35\,x\right)\ J_0(x)\,dx =$$

$$+ \left(2574\,x^6 - 64548\,x^4 + 517014\,x^2\right)\ J_0(x) + \left(429\,x^7 - 16137\,x^5 + 258507\,x^3 - 1034063\,x\right)\ J_1(x)$$

$$16\int P_7(x)\cdot J_1(x)\,dx =$$

$$= \left(-429\,x^7 + 15708\,x^5 - 235935\,x^3\right)\ J_0(x) + \left(3003\,x^6 - 78540\,x^4 + 707805\,x^2\right)\ J_1(x) - 707840\,\Phi(x)$$

$$16\int P_7(x)\cdot I_0(x)\,dx =$$

$$= \left(-2574\,x^6 - 59004\,x^4 - 472662\,x^2\right)\ I_0(x) + \left(429\,x^7 + 14751\,x^5 + 236331\,x^3 + 945289\,x\right)\ I_1(x)$$

$$16\int P_7(x)\cdot I_1(x)\,dx =$$

$$= \left(429\,x^7 + 14322\,x^5 + 215145\,x^3\right)\ I_0(x) + \left(-3003\,x^6 - 71610\,x^4 - 645435\,x^2\right)\ I_1(x) - 645400\,\Psi(x)$$

<u>**n = 8**</u>

$$128\int P_8(x)\cdot J_0(x)\,dx = \int \left(6435\,x^8 - 12012\,x^6 + 6930\,x^4 - 1260\,x^2 + 35\right)\ J_0(x)\,dx =$$

$$= \left(45045\,x^7 - 1636635\,x^5 + 24570315\,x^3 + 35\,x\right)\ J_0(x) +$$

$$+ \left(6435\,x^8 - 327327\,x^6 + 8190105\,x^4 - 73712205\,x^2\right)\ J_1(x) + 73712240\,\Phi(x)$$

$$128\int P_8(x)\cdot J_1(x)\,dx = \left(-6435\,x^8 + 320892\,x^6 - 7708338\,x^4 + 61667964\,x^2 - 35\right)\ J_0(x) +$$

$$+ \left(51480\,x^7 - 1925352\,x^5 + 30833352\,x^3 - 123335928\,x\right)\ J_1(x)$$

$$128\int P_8(x)\cdot I_0(x)\,dx = \left(-45045\,x^7 - 1516515\,x^5 - 22768515\,x^3 + 35\,x\right)\ I_0(x) +$$

$$+ (6435\,x^8 + 303303\,x^6 + 7589505\,x^4 + 68304285\,x^2)\,I_1(x) + 68304320\,\Psi(x)$$

$$128\int P_8(x)\cdot I_1(x)\,dx = \left(6435\,x^8 + 296868\,x^6 + 7131762\,x^4 + 57052836\,x^2 + 35\right)\ I_0(x) +$$

$$+ \left(-51480\,x^7 - 1781208\,x^5 - 28527048\,x^3 - 114105672\,x\right)\ I_1(x)$$

<u>**n = 9**</u>

$$128\int P_9(x)\cdot J_0(x)\,dx = \int \left(12155\,x^9 - 25740\,x^7 + 18018\,x^5 - 4620\,x^3 + 315\,x\right)\ J_0(x)\,dx =$$

$$= \left(97240\,x^8 - 4821960\,x^6 + 115799112\,x^4 - 926402136\,x^2\right)\ J_0(x) +$$

$$+ \left(12155\,x^9 - 803660\,x^7 + 28949778\,x^5 - 463201068\,x^3 + 1852804587\,x\right)\ J_1(x)$$

$$128\int P_9(x)\cdot J_1(x)\,dx = \left(-12155\,x^9 + 791505\,x^7 - 27720693\,x^5 + 415815015\,x^3\right)\ J_0(x) +$$

$$+ \left(109395\,x^8 - 5540535\,x^6 + 138603465\,x^4 - 1247445045\,x^2\right)\ J_1(x) + 1247445360\,\Phi(x)$$

$$128\int P_9(x)\cdot I_0(x)\,dx = \left(-97240\,x^8 - 4513080\,x^6 - 108385992\,x^4 - 867078696\,x^2\right)\ I_0(x) +$$

$$+\left(12155\,x^9+752180\,x^7+27096498\,x^5+433539348\,x^3+1734157707\,x\right)\;I_1(x)$$

$$128\int P_9(x)\cdot I_1(x)\,dx=\left(12155\,x^9+740025\,x^7+25918893\,x^5+388778775\,x^3\right)\;I_0(x)+$$

$$+\left(-109395\,x^8-5180175\,x^6-129594465\,x^4-1166336325\,x^2\right)\;I_1(x)-1166336640\,\Psi(x)$$

n = 10

$$256\int P_{10}(x)\cdot J_0(x)\,dx=\int\left(46189\,x^{10}-109395\,x^8+90090\,x^6-30030\,x^4+3465\,x^2-63\right)\;J_0(x)\,dx=$$

$$=\left(415701\,x^9-26954928\,x^7+943872930\,x^5-14158184040\,x^3-63\,x\right)\;J_0(x)+$$

$$+\left(46189\,x^{10}-3850704\,x^8+188774586\,x^6-4719394680\,x^4+42474555585\,x^2\right)\;J_1(x)-42474555648\,\Phi(x)$$

$$256\int P_{10}(x)\cdot J_1(x)\,dx=$$

$$=\left(-46189\,x^{10}+3804515\,x^8-182706810\,x^6+4384993470\,x^4-35079951225\,x^2+63\right)\;J_0(x)+$$

$$+\left(461890\,x^9-30436120\,x^7+1096240860\,x^5-17539973880\,x^3+70159902450\,x\right)\;J_1(x)$$

$$256\int P_{10}(x)\cdot I_0(x)\,dx=\left(-415701\,x^9-25423398\,x^7-890269380\,x^5-13353950610\,x^3-63\,x\right)\;I_0(x)+$$

$$+\left(46189\,x^{10}+3631914\,x^8+178053876\,x^6+4451316870\,x^4+40061855295\,x^2\right)\;I_1(x)+40061855232\,\Psi(x)$$

$$256\int P_{10}(x)\cdot I_1(x)\,dx=\left(46189\,x^{10}+3585725\,x^8+172204890\,x^6+4132887330\,x^4+33063102105\,x^2-63\right)\;I_0(x)+$$

$$+(-461890\,x^9-28685800\,x^7-1033229340\,x^5-16531549320\,x^3-66126204210\,x)\,I_1(x)$$

n = 11

$$256\int P_{11}(x)\cdot J_0(x)\,dx=$$

$$=\int\left(88179\,x^{11}-230945\,x^9+218790\,x^7-90090\,x^5+15015\,x^3-693\,x\right)\;J_0(x)\,dx=$$

$$=\left(881790\,x^{10}-72390760\,x^8+3476069220\,x^6-83426021640\,x^4+667408203150\,x^2\right)\;J_0(x)+$$

$$+\left(88179\,x^{11}-9048845\,x^9+579344870\,x^7-20856505410\,x^5+333704101575\,x^3-1334816406993\,x\right)\;J_1(x)$$

$$256\int P_{11}(x)\cdot J_1(x)\,dx=$$

$$=\left(-88179\,x^{11}+8960666\,x^9-564740748\,x^7+19766016270\,x^5-296490259065\,x^3\right)\;J_0(x)+$$

$$+\left(969969\,x^{10}-80645994\,x^8+3953185236\,x^6-98830081350\,x^4+889470777195\,x^2\right)\;J_1(x)-$$

$$-889470777888\,\Phi(x)$$

$$256\int P_{11}(x)\cdot I_0(x)\,dx=$$

$$=\left(-881790\,x^{10}-68695640\,x^8-3298703460\,x^6-79168522680\,x^4-633348211470\,x^2\right)\;I_0(x)+$$

$$+\left(88179\,x^{11}+8586955\,x^9+549783910\,x^7+19792130670\,x^5+316674105735\,x^3+1266696422247\,x\right)\;I_1(x)$$

$$256\int P_{11}(x)\cdot I_1(x)\,dx=$$

$$=\left(88179\,x^{11}+8498776\,x^9+535641678\,x^7+18747368640\,x^5+281210544615\,x^3\right)\;I_0(x)+$$

$$+\left(-969969\,x^{10}-76488984\,x^8-3749491746\,x^6-93736843200\,x^4-843631633845\,x^2\right)\;I_1(x)-$$

$$-843631633152\,\Psi(x)$$

n = 12

$$1024\int P_{12}(x)\cdot J_0(x)\,dx=$$

$$= \int \left(676039\,x^{12} - 1939938\,x^{10} + 2078505\,x^8 - 1021020\,x^6 + 225225\,x^4 - 18018\,x^2 + 231\right)\,J_0(x)\,dx =$$
$$= \left(7436429\,x^{11} - 753665913\,x^9 + 47495502054\,x^7 - 1662347676990\,x^5 + 24935215830525\,x^3 + 231\,x\right)\,J_0(x) +$$
$$+ (676039\,x^{12} - 83740657\,x^{10} + 6785071722\,x^8 - 332469535398\,x^6 + 8311738610175\,x^4 -$$
$$-74805647509593\,x^2)\,J_1(x) + 74805647509824\,\Phi(x)$$

$$1024 \int P_{12}(x) \cdot J_1(x)\,dx = (-676039\,x^{12} + 83064618\,x^{10} - 6647247945\,x^8 + 319068922380\,x^6 -$$
$$-7657654362345\,x^4 + 61261234916778\,x^2 - 231)\,J_0(x) + (8112468\,x^{11} - 830646180\,x^9 + 53177983560\,x^7 -$$
$$-1914413534280\,x^5 + 30630617449380\,x^3 - 122522469833556\,x)\,J_1(x)$$

$$1024 \int P_{12}(x) \cdot I_0(x)\,dx = (-7436429\,x^{11} - 718747029\,x^9 - 45295612362\,x^7 - 1585341327570\,x^5 -$$
$$-23780120589225\,x^3 + 231\,x)\,I_0(x) + (676039\,x^{12} + 79860781\,x^{10} + 6470801766\,x^8 + 317068265514\,x^6 +$$
$$+7926706863075\,x^4 + 71340361749657\,x^2)\,I_1(x) + 71340361749888\,\Psi(x)$$

$$1024 \int P_{12}(x) \cdot I_1(x)\,dx = (676039\,x^{12} + 79184742\,x^{10} + 6336857865\,x^8 + 304168156500\,x^6 + 7300035981225\,x^4 +$$
$$+58400287831782\,x^2 + 231)\,I_0(x) + (-8112468\,x^{11} - 791847420\,x^9 - 50694862920\,x^7 - 1825008939000\,x^5 -$$
$$-29200143924900\,x^3 - 116800575663564\,x)\,I_1(x)$$

<u>**n = 13**</u>

$$1024 \int P_{13}(x) \cdot J_0(x)\,dx =$$
$$= \int \left(1300075\,x^{13} - 4056234\,x^{11} + 4849845\,x^9 - 2771340\,x^7 + 765765\,x^5 - 90090\,x^3 + 3003\,x\right)\,J_0(x)\,dx =$$
$$= (15600900\,x^{12} - 1912670340\,x^{10} + 153052425960\,x^8 - 7346533074120\,x^6 + 176316796841940\,x^4 -$$
$$-1410534374915700\,x^2)\,J_0(x) + (1300075\,x^{13} - 191267034\,x^{11} + 19131553245\,x^9 - 1224422179020\,x^7 +$$
$$+44079199210485\,x^5 - 705267187457850\,x^3 + 2821068749834403\,x)\,J_1(x)$$

$$1024 \int P_{13}(x) \cdot J_1(x)\,dx = (-1300075\,x^{13} + 189966959\,x^{11} - 18811578786\,x^9 + 1185132234858\,x^7 -$$
$$-41479628985795\,x^5 + 622194434877015\,x^3)\,J_0(x) + (16900975\,x^{12} - 2089636549\,x^{10} + 169304209074\,x^8 -$$
$$-8295925644006\,x^6 + 207398144928975\,x^4 - 1866583304631045\,x^2)\,J_1(x) + 1866583304634048\,\Phi(x)$$

$$\int P_{13}(x) \cdot I_0(x)\,dx = (-15600900\,x^{12} - 1831545660\,x^{10} - 146562451560\,x^8 - 7034981046840\,x^6 -$$
$$-168839548187220\,x^4 - 1350716385317580\,x^2)\,I_0(x) + (1300075\,x^{13} + 183154566\,x^{11} + 18320306445\,x^9 +$$
$$+1172496841140\,x^7 + 42209887046805\,x^5 + 675358192658790\,x^3 + 2701432770638163\,x)\,I_1(x)$$

$$1024 \int P_{13}(x) \cdot I_1(x)\,dx = (1300075\,x^{13} + 181854491\,x^{11} + 18008444454\,x^9 + 1134529229262\,x^7 +$$
$$+39708523789935\,x^5 + 595627856758935\,x^3)\,I_0(x) + (-16900975\,x^{12} - 2000399401\,x^{10} - 162076000086\,x^8 -$$
$$-7941704604834\,x^6 - 198542618949675\,x^4 - 1786883570276805\,x^2)\,I_1(x) - 1786883570279808\,\Psi(x)$$

<u>**n = 14**</u>

$$2048 \int P_{14}(x) \cdot J_0(x)\,dx = \int (5014575\,x^{14} - 16900975\,x^{12} + 22309287\,x^{10} - 14549535\,x^8 + 4849845\,x^6 -$$
$$-765765\,x^4 + 45045\,x^2 - 429)\,J_0(x)\,dx = (65189475\,x^{13} - 9508005650\,x^{11} + 941493342933\,x^9 -$$
$$-59314182451524\,x^7 + 2075996410052565\,x^5 - 31139946153085770\,x^3 - 429\,x)\,J_0(x) +$$
$$+(5014575\,x^{14} - 864364150\,x^{12} + 104610371437\,x^{10} - 8473454635932\,x^8 + 415199282010513\,x^6 -$$

$$-10379982051028590\,x^4+93419838459302355\,x^2)\,J_1(x)-93419838459302784\,\Phi(x)$$

$$2048\int P_{14}(x)\cdot J_1(x)\,dx=(-5014575\,x^{14}+859349575\,x^{12}-103144258287\,x^{10}+8251555212495\,x^8-$$

$$-396074655049605\,x^6+9505791721956285\,x^4-76046333775695325\,x^2+429)\,J_0(x)+$$

$$+(70204050\,x^{13}-10312194900\,x^{11}+1031442582870\,x^9-66012441699960\,x^7+2376447930297630\,x^5-$$

$$-38023166887825140\,x^3+152092667551390650\,x)\,J_1(x)$$

$$2048\int P_{14}(x)\cdot I_0(x)\,dx=(-65189475\,x^{13}-9136184200\,x^{11}-904683019383\,x^9-56994928374384\,x^7-$$

$$-1994822517352665\,x^5-29922337757992680\,x^3-429\,x)\,I_0(x)+(5014575\,x^{14}+830562200\,x^{12}+$$

$$+100520335487\,x^{10}+8142132624912\,x^8+398964503470533\,x^6+9974112585997560\,x^4+$$

$$+89767013274023085\,x^2)\,I_1(x)+89767013274022656\,\Psi(x)$$

$$2048\int P_{14}(x)\cdot I_1(x)\,dx=(5014575\,x^{14}+825547625\,x^{12}+99088024287\,x^{10}+7927027393425\,x^8+$$

$$+380497319734245\,x^6+9131935672856115\,x^4+73055485382893965\,x^2-429)\,I_0(x)+$$

$$+(-70204050\,x^{13}--9906571500\,x^{11}-990880242870\,x^9-63416219147400\,x^7-2282983918405470\,x^5-$$

$$-36527742691424460\,x^3-146110970765787930\,x)\,I_1(x)$$

n = 15

$$2048\int P_{15}(x)\cdot J_0(x)\,dx=\int(9694845\,x^{15}-35102025\,x^{13}+50702925\,x^{11}-37182145\,x^9++14549535\,x^7-$$

$$-2909907\,x^5+255255\,x^3-6435\,x)\,J_0(x)\,dx=(135727830\,x^{14}-23223499740\,x^{12}+2787326998050\,x^{10}-$$

$$-222986457301160\,x^8+10703350037752890\,x^6-256880400917708988\,x^4+2055043207342182414\,x^2)\,J_0(x)+$$

$$+(9694845\,x^{15}-1935291645\,x^{13}+278732699805\,x^{11}-27873307162645\,x^9+1783891672958815\,x^7-$$

$$-64220100229427247\,x^5+1027521603671091207\,x^3-4110086414684371263\,x)\,J_1(x)$$

$$2048\int P_{15}(x)\cdot J_1(x)\,dx=(-9694845\,x^{15}+1925596800\,x^{13}-275411045325\,x^{11}+27265730669320\,x^9-$$

$$-1717741046716695\,x^7+60120936637994232\,x^5-901814049570168735\,x^3)\,J_0(x)+(145422675\,x^{14}-$$

$$-25032758400\,x^{12}+3029521498575\,x^{10}-245391576023880\,x^8+12024187327016865\,x^6-$$

$$-300604683189971160\,x^4+2705442148710506205\,x^2)\,J_1(x)-2705442148710512640\,\Phi(x)$$

$$\int P_{15}(x)\cdot I_0(x)\,dx=(-135727830\,x^{14}-22381051140\,x^{12}-2686233166050\,x^{10}-214898355826840\,x^8-$$

$$-10315121166985530\,x^6-247562907996013092\,x^4-1980503263968615246\,x^2)\,I_0(x)+$$

$$+(9694845\,x^{15}+1865087595\,x^{13}+268623316605\,x^{11}+26862294478355\,x^9+1719186861164255\,x^7+$$

$$-61890726999003273\,x^5+990251631984307623\,x^3+3961006527937224057\,x)\,I_1(x)$$

$$2048\int P_{15}(x)\cdot I_1(x)\,dx=(9694845\,x^{15}+1855392750\,x^{13}+265371866175\,x^{11}+26271777569180\,x^9+$$

$$+1655122001407875\,x^7+57929270046365718\,x^5+868939050695741025\,x^3)\,I_0(x)+$$

$$+(-145422675\,x^{14}-24120105750\,x^{12}-2919090527925\,x^{10}-236445998122620\,x^8-11585854009855125\,x^6-$$

$$-28964635023182859O\,x^4-2606817152087223075\,x^2)\,I_1(x)-2606817152087216640\,\Psi(x)$$

b) Chebyshev Polynomials of the first kind $T_n(x)$:

Weight function: $w(x) = (1-x^2)^{-1/2}$, interval $-1 \le x \le 1$.

Holds $\int T_0(x)\, Z_\nu(x)\, dx = \int Z_\nu(x)\, dx$ and $\int T_1(x)\, Z_\nu(x)\, dx = \int x\, Z_\nu(x)\, dx$.

n = 2

$$\int T_2(x) \cdot J_0(x)\, dx = \int (2\,x^2-1)\, J_0(x)\, dx = -xJ_0(x) + 2x^2\, J_1(x) - 3\,\Phi(x)$$

$$\int T_2(x) \cdot J_1(x)\, dx = (-2\,x^2+1)\, J_0(x) + 4x\, J_1(x)$$

$$\int T_2(x) \cdot I_0(x)\, dx = -x\, I_0(x) + 2x^2\, I_1(x) + \Psi(x)$$

$$\int T_2(x) \cdot I_1(x)\, dx = (2x^2-1)\, I_0(x) - 4\, I_1(x)$$

n = 3

$$\int T_3(x) \cdot J_0(x)\, dx = \int (4\,x^3-3\,x)\, J_0(x)\, dx = 8\,x^2\, J_0(x) + (4\,x^3-19\,x)\, J_1(x)$$

$$\int T_3(x) \cdot J_1(x)\, dx = -4x^3\, J_0(x) + 12x^2\, J_1(x) - 15\,\Phi(x)$$

$$\int T_3(x) \cdot I_0(x)\, dx = -8\,x^2\, I_0(x) + (4\,x^3+13\,x)\, I_1(x)$$

$$\int T_3(x) \cdot I_1(x)\, dx = 4x^3\, I_0(x) - 12x^2\, I_1(x) - 9\,\Psi(x)$$

n = 4

$$\int T_4(x) \cdot J_0(x)\, dx = \int (8\,x^4-8\,x^2+1)\, J_0(x)\, dx = (24\,x^3+x)\, J_0(x) + (8\,x^4-80\,x^2)\, J_1(x) + 81\,\Phi(x)$$

$$\int T_4(x) \cdot J_1(x)\, dx = (-8\,x^4+72\,x^2-1)\, J_0(x) + (32\,x^3-144\,x)\, J_1(x)$$

$$\int T_4(x) \cdot I_0(x)\, dx = (-24x^3+x)\, I_0(x) + (8\,x^4+64\,x^2)\, I_1(x) + 65\,\Psi(x)$$

$$\int T_4(x) \cdot I_1(x)\, dx = (8\,x^4+56\,x^2+1)\, I_0(x) + (-32\,x^3-112\,x)\, I_1(x)$$

n = 5

$$\int T_5(x) \cdot J_0(x)\, dx = \int (16\,x^5-20\,x^3+5\,x)\, J_0(x)\, dx =$$

$$= (64\,x^4-552\,x^2)\, J_0(x) + (16\,x^5-276\,x^3+1109\,x)\, J_1(x)$$

$$\int T_5(x) \cdot J_1(x)\, dx = (-16\,x^5+260\,x^3)\, J_0(x) + (80\,x^4-780\,x^2)\, J_1(x) + 785\,\Phi(x)$$

$$\int T_5(x) \cdot I_0(x)\, dx = (-64\,x^4-472\,x^2)\, I_0(x) + (16\,x^5+236\,x^3+949\,x)\, I_1(x)$$

$$\int T_5(x) \cdot I_1(x)\, dx = (16\,x^5+220\,x^3)\, I_0(x) + (-80\,x^4-660\,x^2)\, I_1(x) - 665\,\Psi(x)$$

n = 6

$$\int T_6(x) \cdot J_0(x)\, dx = \int (32\,x^6-48\,x^4+18\,x^2-1)\, J_0(x)\, dx =$$

$$= (160\,x^5-2544\,x^3-x)\, J_0(x) + (32\,x^6-848\,x^4+7650\,x^2)\, J_1(x) - 7651\,\Phi(x)$$

$$\int T_6(x) \cdot J_1(x)\, dx = (-32\,x^6+816\,x^4-6546\,x^2+1)\, J_0(x) + (192\,x^5-3264\,x^3+13092\,x)\, J_1(x)$$

$$\int T_6(x) \cdot I_0(x)\, dx = (-160\,x^5-2256\,x^3-x)\, I_0(x) + (32\,x^6+752\,x^4+6786\,x^2)\, I_1(x) + 6785\,\Psi(x)$$

$$\int T_6(x)\cdot I_1(x)\,dx = (32\,x^6+720\,x^4+5778\,x^2-1)\,I_0(x)+(-192\,x^5-2880\,x^3-11556\,x)\,I_1(x)$$

<u>**n = 7**</u>

$$\int T_7(x)\cdot J_0(x)\,dx = \int (64\,x^7-112\,x^5+56\,x^3-7\,x)\,J_0(x)\,dx =$$

$$= (384\,x^6-9664\,x^4+77424\,x^2)\,J_0(x)+(64\,x^7-2416\,x^5+38712\,x^3-154855\,x)\,J_1(x)$$

$$\int T_7(x)\cdot J_1(x)\,dx = (-64\,x^7+2352\,x^5-35336\,x^3)\,J_0(x)+(448\,x^6-11760\,x^4+106008\,x^2)\,J_1(x)-106015\,\Phi(x)$$

$$\int T_7(x)\cdot I_0(x)\,dx =$$

$$= (-384\,x^6-8768\,x^4-70256\,x^2)\,I_0(x)+(64\,x^7+2192\,x^5+35128\,x^3+140505\,x)\,I_1(x)$$

$$\int T_7(x)\cdot I_1(x)\,dx = (64\,x^7+2128\,x^5+31976\,x^3)\,I_0(x)+(-448\,x^6-10640\,x^4-95928\,x^2)\,I_1(x)-95921\,\Psi(x)$$

<u>**n = 8**</u>

$$\int T_8(x)\cdot J_0(x)\,dx = \int (128\,x^8-256\,x^6+160\,x^4-32\,x^2+1)\,J_0(x)\,dx =$$

$$= (896\,x^7-32640\,x^5+490080\,x^3+x)J_0(x)+(128\,x^8-6528\,x^6+163360\,x^4-1470272\,x^2)J_1(x)+1470273\,\Phi(x)$$

$$\int T_8(x)\cdot J_1(x)\,dx = (-128\,x^8+6400\,x^6-153760\,x^4+1230112\,x^2-1)\,J_0(x)+$$

$$+(1024\,x^7-38400\,x^5+615040\,x^3-2460224\,x)\,J_1(x)$$

$$\int T_8(x)\cdot I_0(x)\,dx = (-896\,x^7-30080\,x^5-451680\,x^3+x)\,I_0(x)+$$

$$+(128\,x^8+6016\,x^6+150560\,x^4+1355008\,x^2)\,I_1(x)+1355009\,\Psi(x)$$

$$\int T_8(x)\cdot I_1(x)\,dx = (128\,x^8+5888\,x^6+141472\,x^4+1131744\,x^2+1)\,I_0(x)+$$

$$+(-1024\,x^7-35328\,x^5-565888\,x^3-2263488\,x)\,I_1(x)$$

<u>**n = 9**</u>

$$\int T_9(x)\cdot J_0(x)\,dx = \int (256\,x^9-576\,x^7+432\,x^5-120\,x^3+9\,x)\,J_0(x)\,dx =$$

$$= (2048\,x^8-101760\,x^6+2443968\,x^4-19551984\,x^2)\,J_0(x)+$$

$$+(256\,x^9-16960\,x^7+610992\,x^5-9775992\,x^3+39103977)\,J_1(x)$$

$$\int T_9(x)\cdot J_1(x)\,dx = (-256\,x^9+16704\,x^7-585072\,x^5+8776200\,x^3)\,J_0(x)+$$

$$+(2304\,x^8-116928\,x^6+2925360\,x^4-26328600\,x^2)\,J_1(x)+26328609\,\Phi(x)$$

$$\int T_9(x)\cdot I_0(x)\,dx = (-2048\,x^8-94848\,x^6-2278080\,x^4-18224400\,x^2)\,I_0(x)+$$

$$+(256\,x^9+15808\,x^7+569520\,x^5+9112200\,x^3+36448809\,x)\,I_1(x)$$

$$\int T_9(x)\cdot I_1(x)\,dx = (256\,x^9+15552\,x^7+544752\,x^5+8171160\,x^3)\,I_0(x)+$$

$$+(-2304\,x^8-108864\,x^6-2723760\,x^4-24513480\,x^2)\,I_1(x)-24513489\,\Psi(x)$$

<u>**n = 10**</u>

$$\int T_{10}(x)\cdot J_0(x)\,dx = \int (512\,x^{10}-1280\,x^8+1120\,x^6-400\,x^4+50\,x^2-1)\,J_0(x)\,dx =$$

$$= (4608\,x^9-299264\,x^7+10479840\,x^5-157198800\,x^3-x)\,J_0(x)+$$

$$+(512\,x^{10}-42752\,x^8+2095968\,x^6-52399600\,x^4+471596450\,x^2)\,J_1(x)-471596451\,\Phi(x)$$

$$\int T_{10}(x)\cdot J_1(x)\,dx = (-512\,x^{10}+42240\,x^8-2028640\,x^6+48687760\,x^4-389502130\,x^2+1)\,J_0(x)+$$

$$+(5120\,x^9-337920\,x^7+12171840\,x^5-194751040\,x^3+779004260\,x)\,J_1(x)$$

$$\int T_{10}(x)\cdot I_0(x)\,dx = (-4608\,x^9-281344\,x^7-9852640\,x^5-147788400\,x^3-x)\,I_0(x)+$$

$$+(512\,x^{10}+40192\,x^8+1970528\,x^6+49262800\,x^4+443365250\,x^2)\,I_1(x)+443365249\,\Psi(x)$$

$$\int T_{10}(x)\cdot I_1(x)\,dx = (512\,x^{10}+39680\,x^8+1905760\,x^6+45737840\,x^4+365902770\,x^2-1)\,I_0(x)+$$

$$+(-5120\,x^9-317440\,x^7-11434560\,x^5-182951360\,x^3-731805540\,x)\,I_1(x)$$

$\underline{\mathbf{n=11}}$

$$\int T_{11}(x)\cdot J_0(x)\,dx = \int(1024\,x^{11}-2816\,x^9+2816\,x^7-1232\,x^5+220\,x^3-11\,x)\,J_0(x)\,dx=$$

$$=(10240\,x^{10}-841728\,x^8+40419840\,x^6-970081088\,x^4+7760649144\,x^2)\,J_0(x)+$$

$$+(1024\,x^{11}-105216\,x^9+6736640\,x^7-242520272\,x^5+3880324572\,x^3-15521298299\,x)\,J_1(x)$$

$$\int T_{11}(x)\cdot J_1(x)\,dx = (-1024\,x^{11}+104192\,x^9-6566912\,x^7+229843152\,x^5-3447647500\,x^3)\,J_0(x)+$$

$$+(11264\,x^{10}-937728\,x^8+45968384\,x^6-1149215760\,x^4+10342942500\,x^2)\,J_1(x)-10342942511\,\Phi(x)$$

$$\int T_{11}(x)\cdot I_0(x)\,dx = (-10240\,x^{10}-796672\,x^8-38257152\,x^6-918166720\,x^4-7345334200\,x^2)\,I_0(x)+$$

$$+(1024\,x^{11}+99584\,x^9+6376192\,x^7+229541680\,x^5+3672667100\,x^3+14690668389\,x)\,I_1(x)$$

$$\int T_{11}(x)\cdot I_1(x)\,dx = (1024\,x^{11}+98560\,x^9+6212096\,x^7+217422128\,x^5+3261332140\,x^3)\,I_0(x)+$$

$$+(-11264\,x^{10}-887040\,x^8-43484672\,x^6-1087110640\,x^4-9783996420\,x^2)\,I_1(x)-9783996409\,\Psi(x)$$

$\underline{\mathbf{n=12}}$

$$\int T_{12}(x)\cdot J_0(x)\,dx = \int(2048\,x^{12}-6144\,x^{10}+6912\,x^8-3584\,x^6+840\,x^4-72\,x^2+1)\,J_0(x)\,dx=$$

$$=(22528\,x^{11}-2285568\,x^9+144039168\,x^7-5041388800\,x^5+75620834520\,x^3+x)\,J_0(x)+$$

$$+(2048\,x^{12}-253952\,x^{10}+20577024\,x^8-1008277760\,x^6+25206944840\,x^4-226862503632\,x^2)\,J_1(x)+$$

$$+226862503633\,\Phi(x)$$

$$\int T_{12}(x)\cdot J_1(x)\,dx = (-2048\,x^{12}+251904\,x^{10}-20159232\,x^8+967646720\,x^6-23223522120\,x^4+$$

$$+185788177032\,x^2-1)\,J_0(x)+(24576\,x^{11}-2519040\,x^9+161273856\,x^7-5805880320\,x^5+$$

$$+92894088480\,x^3-371576354064\,x)\,J_1(x)$$

$$\int T_{12}(x)\cdot I_0(x)\,dx = (-22528\,x^{11}-2174976\,x^9-137071872\,x^7-4797497600\,x^5-71962466520\,x^3+x)\,I_0(x)+$$

$$+(2048\,x^{12}+241664\,x^{10}+19581696\,x^8+959499520\,x^6+23987488840\,x^4+215887399488\,x^2)\,I_1(x)+$$

$$+215887399489\,\Psi(x)$$

$$\int T_{12}(x)\cdot I_1(x)\,dx =$$

$$=(2048\,x^{12}+239616\,x^{10}+19176192\,x^8+920453632\,x^6+22090888008\,x^4+176727103992\,x^2+1)\,I_0(x)+$$

$$+(-24576\,x^{11}-2396160\,x^9-153409536\,x^7-5522721792\,x^5-88363552032\,x^3-353454207984\,x)\,I_1(x)$$

n = 13

$$\int T_{13}(x)\cdot J_0(x)\,dx =$$

$$= \int (4096\,x^{13} - 13312\,x^{11} + 16640\,x^9 - 9984\,x^7 + 2912\,x^5 - 364\,x^3 + 13\,x)\,J_0(x)\,dx =$$

$$= (49152\,x^{12} - 6031360\,x^{10} + 482641920\,x^8 - 23166872064\,x^6 + 556004941184\,x^4 - 4448039530200\,x^2)\,J_0(x) +$$
$$+ (4096\,x^{13} - 603136\,x^{11} + 60330240\,x^9 - 3861145344\,x^7 + 139001235296\,x^5 - 2224019765100\,x^3 +$$
$$+ 8896079060413\,x)\,J_1(x)$$

$$\int T_{13}(x)\cdot J_1(x)\,dx = (-4096\,x^{13} + 599040\,x^{11} - 59321600\,x^9 + 3737270784\,x^7 - 130804480352\,x^5 +$$
$$+ 1962067205644\,x^3)\,J_0(x) + (53248\,x^{12} - 6589440\,x^{10} + 533894400\,x^8 - 26160895488\,x^6 + 654022401760\,x^4 -$$
$$- 5886201616932\,x^2)\,J_1(x) + 5886201616945\,\Phi(x)$$

$$\int T_{13}(x)\cdot I_0(x)\,dx = (-49152\,x^{12} - 5765120\,x^{10} - 461342720\,x^8 - 22144390656\,x^6 -$$
$$- 531465387392\,x^4 - 4251723098408\,x^2)\,I_0(x) + (4096\,x^{13} + 576512\,x^{11} + 57667840\,x^9 +$$
$$+ 3690731776\,x^7 + 132866346848\,x^5 + 2125861549204\,x^3 + 8503446196829\,x)\,I_1(x)$$

$$\int T_{13}(x)\cdot I_1(x)\,dx = (4096\,x^{13} + 572416\,x^{11} + 56685824\,x^9 + 3571196928\,x^7 + 124991895392\,x^5 +$$
$$+ 1874878430516\,x^3)\,I_0(x) + (-53248\,x^{12} - 6296576\,x^{10} - 510172416\,x^8 - 24998378496\,x^6 -$$
$$- 624959476960\,x^4 - 5624635291548\,x^2)\,I_1(x) - 5624635291561\,\Psi(x)$$

n = 14

$$\int T_{14}(x)\cdot J_0(x)\,dx = \int (8192\,x^{14} - 28672\,x^{12} + 39424\,x^{10} - 26880\,x^8 + 9408\,x^6 - 1568\,x^4 + 98\,x^2 - 1)\,J_0(x)\,dx =$$

$$= (106496\,x^{13} - 15544320\,x^{11} + 1539242496\,x^9 - 96972465408\,x^7 + 3394036336320\,x^5 - 50910545049504\,x^3 -$$
$$- x)\,J_0(x) + (8192\,x^{14} - 1413120\,x^{12} + 171026944\,x^{10} - 13853209344\,x^8 + 678807267264\,x^6 - 16970181683168\,x^4 +$$
$$+ 152731635148610\,x^2)\,J_1(x) - 152731635148611\,\Phi(x)$$

$$\int T_{14}(x)\cdot J_1(x)\,dx = (-8192\,x^{14} + 1404928\,x^{12} - 168630784\,x^{10} + 13490489600\,x^8 - 647543510208\,x^6 +$$
$$+ 15541044246560\,x^4 - 124328353972578\,x^2 + 1)\,J_0(x) + (114688\,x^{13} - 16859136\,x^{11} + 1686307840\,x^9 -$$
$$- 107923916800\,x^7 + 3885261061248\,x^5 - 62164176986240\,x^3 + 248656707945156\,x)\,J_1(x)$$

$$\int T_{14}(x)\cdot I_0(x)\,dx = (-106496\,x^{13} - 14913536\,x^{11} - 1476794880\,x^9 - 93037889280\,x^7 - 3256326171840\,x^5 -$$
$$- 48844892572896\,x^3 - x)\,I_0(x) + (8192\,x^{14} + 1355776\,x^{12} + 164088320\,x^{10} + 13291127040\,x^8 + 651265234368\,x^6 +$$
$$+ 16281630857632\,x^4 + 146534677718786\,x^2)\,I_1(x) + 146534677718785\,\Psi(x)$$

$$\int T_{14}(x)\cdot I_1(x)\,dx = (8192\,x^{14} + 1347584\,x^{12} + 161749504\,x^{10} + 12939933440\,x^8 + 621116814528\,x^6 +$$
$$+ 14906803547104\,x^4 + 119254428376930\,x^2 - 1)\,I_0(x) + (-114688\,x^{13} - 16171008\,x^{11} - 1617495040\,x^9 -$$
$$- 103519467520\,x^7 - 3726700887168\,x^5 - 59627214188416\,x^3 - 238508856753860\,x)\,I_1(x)$$

n = 15

$$\int T_{15}(x)\cdot J_0(x)\,dx =$$

$$= \int (16384\,x^{15} - 61440\,x^{13} + 92160\,x^{11} - 70400\,x^9 + 28800\,x^7 - 6048\,x^5 + 560\,x^3 - 15\,x)\,J_0(x)\,dx =$$

$$= (229376\,x^{14} - 39272448\,x^{12} + 4713615360\,x^{10} - 377089792000\,x^8 + 18100310188800\,x^6 -$$

$$-434407444555392\,x^4 + 3475259556444256\,x^2)\,J_0(x) + (16384\,x^{15} - 3272704\,x^{13} + 471361536\,x^{11} -$$
$$-47136224000\,x^9 + 3016718364800\,x^7 - 108601861138848\,x^5 + 1737629778222128\,x^3 - 6950519112888527\,x)\,J_1(x)$$

$$\int T_{15}(x)\cdot J_1(x)\,dx = (-16384\,x^{15} + 3256320\,x^{13} - 465745920\,x^{11} + 46108916480\,x^9 - 2904861767040\,x^7 +$$
$$+101670161852448\,x^5 - 1525052427787280\,x^3)\,J_0(x) + (245760\,x^{14} - 42332160\,x^{12} + 5123205120\,x^{10} -$$
$$-414980248320\,x^8 + 20334032369280\,x^6 - 508350809262240\,x^4 + 4575157283361840\,x^2)\,J_1(x) -$$
$$-4575157283361855\,\Phi(x)$$

$$\int T_{15}(x)\cdot I_0(x)\,dx = (-229376\,x^{14} - 37797888\,x^{12} - 4536668160\,x^{10} - 362932889600\,x^8 -$$
$$-17420778873600\,x^6 - 418098692942208\,x^4 - 3344789543538784\,x^2)\,I_0(x) +$$
$$+(16384\,x^{15} + 3149824\,x^{13} + 453666816\,x^{11} + 45366611200\,x^9 + 2903463145600\,x^7 +$$
$$+104524673235552\,x^5 + 1672394771769392\,x^3 + 6689579087077553\,x)\,I_1(x)$$

$$\int T_{15}(x)\cdot I_1(x)\,dx = (16384\,x^{15} + 3133440\,x^{13} + 448174080\,x^{11} + 44369163520\,x^9 + 2795257330560\,x^7 +$$
$$+97834006563552\,x^5 + 1467510098453840\,x^3)\,I_0(x) + (-245760\,x^{14} - 40734720\,x^{12} - 4929914880\,x^{10} -$$
$$-399322471680\,x^8 - 19566801313920\,x^6 - 489170032817760\,x^4 - 4402530295361520\,x^2)\,I_1(x) -$$
$$-4402530295361505\,\Psi(x)$$

c) Chebyshev Polynomials of the second kind $U_n(x)$:

Weight function: $w(x) = (1-x^2)^{1/2}$, interval $-1 \le x \le 1$.

Holds $\int U_0(x)\, Z_\nu(x)\, dx = \int Z_\nu(x)\, dx$ and $\int U_1(x)\, Z_\nu(x)\, dx = \int 2x\, Z_\nu(x)\, dx$.

n = 2

$$\int U_2(x) \cdot J_0(x)\, dx = \int (4x^2-1)\, J_0(x)\, dx = -x\, J_0(x) + 4x^2\, J_1(x) - 5\,\Phi(x)$$

$$\int U_2(x) \cdot J_1(x)\, dx = (-4x^2+1)\, J_0(x) + 8x\, J_1(x)$$

$$\int U_2(x) \cdot I_0(x)\, dx = -x\, I_0(x) + 4x^2\, I_1(x) + 3\,\Psi(x)$$

$$\int U_2(x) \cdot I_1(x)\, dx = (4x^2-1)\, I_0(x) - 8x\, I_1(x)$$

n = 3

$$\int U_3(x) \cdot J_0(x)\, dx = \int (8\,x^3 - 4\,x)\, J_0(x)\, dx = 16x^2\, J_0(x) + (8x^3 - 36x)\, J_1(x)$$

$$\int U_3(x) \cdot J_1(x)\, dx = -8x^3\, J_0(x) + 24x^2\, J_1(x) - 28\,\Phi(x)$$

$$\int U_3(x) \cdot I_0(x)\, dx = -16x^2\, I_0(x) + (8x^3 + 28x)\, I_1(x)$$

$$\int U_3(x) \cdot I_1(x)\, dx = 8x^3\, I_0(x) - 24x^2\, I_1(x) - 20\,\Psi(x)$$

n = 4

$$\int U_4(x) \cdot J_0(x)\, dx = \int (16\,x^4 - 12\,x^2 + 1)\, J_0(x)\, dx = (48\,x^3 + x)\, J_0(x) + (16\,x^4 - 156\,x^2)\, J_1(x) + 157\,\Phi(x)$$

$$\int U_4(x) \cdot J_1(x)\, dx = (-16\,x^4 + 140\,x^2 - 1)\, J_0(x) + (64\,x^3 - 280\,x)\, J_1(x)$$

$$\int U_4(x) \cdot I_0(x)\, dx = (-48\,x^3 + x)\, I_0(x) + (16\,x^4 + 132\,x^2)\, I_1(x) + 133\,\Psi(x)$$

$$\int U_4(x) \cdot I_1(x)\, dx = (16\,x^4 + 116\,x^2 + 1)\, I_0(x) + (-64\,x^3 - 232\,x)\, I_1(x)$$

n = 5

$$\int U_5(x) \cdot J_0(x)\, dx = \int (32\,x^5 - 32\,x^3 + 6\,x)\, J_0(x)\, dx = (128\,x^4 - 1088\,x^2)\, J_0(x) + (32\,x^5 - 544\,x^3 + 2182\,x)\, J_1(x)$$

$$\int U_5(x) \cdot J_1(x)\, dx = (-32\,x^5 + 512\,x^3)\, J_0(x) + (160\,x^4 - 1536\,x^2)\, J_1(x) + 1542\,\Phi(x)$$

$$\int U_5(x) \cdot I_0(x)\, dx = (-128\,x^4 - 960\,x^2)\, I_0(x) + (32\,x^5 + 480\,x^3 + 1926\,x)\, I_1(x)$$

$$\int U_5(x) \cdot I_1(x)\, dx = (32\,x^5 + 448\,x^3)\, I_0(x) + (-160\,x^4 - 1344\,x^2)\, I_1(x) - 1350\,\Psi(x)$$

n = 6

$$\int U_6(x) \cdot J_0(x)\, dx = \int (64\,x^6 - 80\,x^4 + 24\,x^2 - 1)\, J_0(x)\, dx =$$

$$= (320\,x^5 - 5040\,x^3 - x)\, J_0(x) + (64\,x^6 - 1680\,x^4 + 15144\,x^2)\, J_1(x) - 15145\,\Phi(x)$$

$$\int U_6(x) \cdot J_1(x)\, dx = (-64\,x^6 + 1616\,x^4 - 12952\,x^2 + 1)\, J_0(x) + (384\,x^5 - 6464\,x^3 + 25904\,x)\, J_1(x)$$

$$\int U_6(x) \cdot I_0(x)\, dx = (-320\,x^5 - 4560\,x^3 - x)\, I_0(x) + (64\,x^6 + 1520\,x^4 + 13704\,x^2)\, I_1(x) + 13703\,\Psi(x)$$

$$\int U_6(x)\cdot I_1(x)\,dx = (64\,x^6+1456\,x^4+11672\,x^2-1)\,I_0(x)+(-384\,x^5-5824\,x^3-23344\,x)\,I_1(x)$$

n = 7

$$\int U_7(x)\cdot J_0(x)\,dx = \int (128\,x^7-192\,x^5+80\,x^3-8\,x)\,J_0(x)\,dx =$$

$$= (768\,x^6-19200\,x^4+153760\,x^2)\,J_0(x)+(128\,x^7-4800\,x^5+76880\,x^3-307528\,x)\,J_1(x)$$

$$\int U_7(x)\cdot J_1(x)\,dx = (-128\,x^7+4672\,x^5-70160\,x^3)\,J_0(x)+(896\,x^6-23360\,x^4+210480\,x^2)\,J_1(x)-210488\,\Phi(x)$$

$$\int U_7(x)\cdot I_0(x)\,dx = (-768\,x^6-17664\,x^4-141472\,x^2)\,I_0(x)+(128\,x^7+4416\,x^5+70736\,x^3+282936\,x)\,I_1(x)$$

$$\int U_7(x)\cdot I_1(x)\,dx = (128\,x^7+4288\,x^5+64400\,x^3)\,I_0(x)+(-896\,x^6-21440\,x^4-193200\,x^2)\,I_1(x)-193192\,\Psi(x)$$

n = 8

$$\int U_8(x)\cdot J_0(x)\,dx = \int (256\,x^8-448\,x^6+240\,x^4-40\,x^2+1)\,J_0(x)\,dx =$$

$$= (1792\,x^7-64960\,x^5+975120\,x^3+x)\,J_0(x)+(256\,x^8-12992\,x^6+325040\,x^4-2925400\,x^2)\,J_1(x)+2925401\,\Phi(x)$$

$$\int U_8(x)\cdot J_1(x)\,dx = (-256\,x^8+12736\,x^6-305904\,x^4+2447272\,x^2-1)\,J_0(x)+$$

$$+(2048\,x^7-76416\,x^5+1223616\,x^3-4894544\,x)\,J_1(x)$$

$$\int U_8(x)\cdot I_0(x)\,dx = (-1792\,x^7-60480\,x^5-907920\,x^3+x)\,I_0(x)+$$

$$+(256\,x^8+12096\,x^6+302640\,x^4+2723720\,x^2)\,I_1(x)+2723721\,\Psi(x)$$

$$\int U_8(x)\cdot I_1(x)\,dx = (256\,x^8+11840\,x^6+284400\,x^4+2275160\,x^2+1)\,I_0(x)+$$

$$+(-2048\,x^7-71040\,x^5-1137600\,x^3-4550320\,x)\,I_1(x)$$

n = 9

$$\int U_9(x)\cdot J_0(x)\,dx = \int (512\,x^9-1024\,x^7+672\,x^5-160\,x^3+10\,x)\,J_0(x)\,dx =$$

$$= (4096\,x^8-202752\,x^6+4868736\,x^4-38950208\,x^2)\,J_0(x)+$$

$$+(512\,x^9-33792\,x^7+1217184\,x^5-19475104\,x^3+77900426\,x)\,J_1(x)$$

$$\int U_9(x)\cdot J_1(x)\,dx = (-512\,x^9+33280\,x^7-1165472\,x^5+17482240\,x^3)\,J_0(x)+$$

$$+(4608\,x^8-232960\,x^6+5827360\,x^4-52446720\,x^2)\,J_1(x)+52446730\Phi(x)$$

$$\int U_9(x)\cdot I_0(x)\,dx = (-4096\,x^8-190464\,x^6-4573824\,x^4-36590272\,x^2)\,I_0(x)+$$

$$+(512\,x^9+31744\,x^7+1143456\,x^5+18295136\,x^3+73180554\,x)\,I_1(x)$$

$$\int U_9(x)\cdot I_1(x)\,dx = (512\,x^9+31232\,x^7+1093792\,x^5+16406720\,x^3)\,I_0(x)+$$

$$+(-4608\,x^8-218624\,x^6-5468960\,x^4-49220160\,x^2)\,I_1(x)-49220170\,\Psi(x)$$

n = 10

$$\int U_{10}(x)\cdot J_0(x)\,dx = \int (1024\,x^{10}-2304\,x^8+1792\,x^6-560\,x^4+60\,x^2-1)\,J_0(x)\,dx =$$

$$= (9216\,x^9-596736\,x^7+20894720\,x^5-313422480\,x^3-x)\,J_0(x)+$$

$$+(1024\,x^{10}-85248\,x^8+4178944\,x^6-104474160\,x^4+940267500\,x^2)\,J_1(x)-940267501\,\Phi(x)$$

$$\int U_{10}(x)\cdot J_1(x)\,dx = (-1024\,x^{10}+84224\,x^8-4044544\,x^6+97069616\,x^4-776556988\,x^2+1)\,J_0(x)+$$

$$+(10240\,x^9 - 673792\,x^7 + 24267264\,x^5 - 388278464\,x^3 + 1553113976\,x)\,J_1(x)$$

$$\int U_{10}(x)\cdot I_0(x)\,dx = (-9216\,x^9 - 564480\,x^7 - 19765760\,x^5 - 296484720\,x^3 - x)\,I_0(x)+$$

$$+(1024\,x^{10} + 80640\,x^8 + 3953152\,x^6 + 98828240\,x^4 + 889454220\,x^2)\,I_1(x) + 889454219\,\Psi(x)$$

$$\int U_{10}(x)\cdot I_1(x)\,dx = (1024\,x^{10} + 79616\,x^8 + 3823360\,x^6 + 91760080\,x^4 + 734080700\,x^2 - 1)\,I_0(x)+$$

$$+(-10240\,x^9 - 636928\,x^7 - 22940160\,x^5 - 367040320\,x^3 - 1468161400\,x)\,I_1(x)$$

d) Laguerre Polynomials $L_n(x)$:

Weight function: $w(x) = e^{-x}$, interval $0 \leq x < \infty$

Holds $\int L_0(x)\, Z_\nu(x)\, dx = \int Z_\nu(x)\, dx$.

n = 1

$$\int L_1(x) \cdot J_0(x)\, dx = \int (-x+1)\, J_0(x)\, dx = x\, J_0(x) - x\, J_1(x) + \Phi(x)$$

$$\int L_1(x) \cdot J_1(x)\, dx = -J_0(x) - \Phi(x)$$

$$\int L_1(x) \cdot I_0(x)\, dx = x\, I_0(x) - x\, I_1(x) + \Psi(x)$$

$$\int L_1(x) \cdot I_1(x)\, dx = I_0(x) + \Psi(x)$$

n = 2

$$\int L_2(x) \cdot J_0(x)\, dx = \int (x^2 - 4\,x + 2)\, J_0(x)\, dx = 2x\, J_0(x) + (x^2 - 4x)\, J_1(x) + \Phi(x)$$

$$\int L_2(x) \cdot J_1(x)\, dx = (-x^2 - 2)\, J_0(x) + 2x\, J_1(x) - 4\,\Phi(x)$$

$$\int L_2(x) \cdot I_0(x)\, dx = 2x\, I_0(x) + (x^2 - 4x)\, I_1(x) + 3\,\Psi(x)$$

$$\int L_2(x) \cdot I_1(x)\, dx = (x^2 + 2)\, I_0(x) - 2x\, I_1(x) + 4\,\Psi(x)$$

n = 3

$$\int L_3(x) \cdot J_0(x)\, dx = \int (-x^3 + 9\,x^2 - 18\,x + 6)\, J_0(x)\, dx = (-2\,x^2 + 6\,x)\, J_0(x) + (-x^3 + 9\,x^2 - 14\,x)\, J_1(x) - 3\,\Phi(x)$$

$$\int L_3(x) \cdot J_1(x)\, dx = (x^3 - 9\,x^2 - 6)\, J_0(x) + (-3\,x^2 + 18\,x)\, J_1(x) - 15\,\Phi(x)$$

$$\int L_3(x) \cdot I_0(x)\, dx = (2\,x^2 + 6\,x)\, I_0(x) + (-x^3 + 9\,x^2 - 22\,x)\, I_1(x) + 15\,\Psi(x)$$

$$\int L_3(x) \cdot I_1(x)\, dx = (-x^3 + 9\,x^2 + 6)\, I_0(x) + (3\,x^2 - 18\,x)\, I_1(x) + 21\,\Psi(x)$$

n = 4

$$\int L_4(x) \cdot J_0(x)\, dx = \int (x^4 - 16\,x^3 + 72\,x^2 - 96\,x + 24)\, J_0(x)\, dx =$$

$$= (3\,x^3 - 32\,x^2 + 24\,x)\, J_0(x) + (x^4 - 16\,x^3 + 63\,x^2 - 32\,x)\, J_1(x) - 39\,\Phi(x)$$

$$\int L_4(x) \cdot J_1(x)\, dx = (-x^4 + 16\,x^3 - 64\,x^2 - 24)\, J_0(x) + (4\,x^3 - 48\,x^2 + 128\,x)\, J_1(x) - 48\,\Phi(x)$$

$$\int L_4(x) \cdot I_0(x)\, dx = (-3\,x^3 + 32\,x^2 + 24\,x)\, I_0(x) + (x^4 - 16\,x^3 + 81\,x^2 - 160\,x)\, I_1(x) + 105\,\Psi(x)$$

$$\int L_4(x) \cdot I_1(x)\, dx = (x^4 - 16\,x^3 + 80\,x^2 + 24)\, I_0(x) + (-4\,x^3 + 48\,x^2 - 160\,x)\, I_1(x) + 144\,\Psi(x)$$

n = 5

$$\int L_5(x) \cdot J_0(x)\, dx = \int (-x^5 + 25\,x^4 - 200\,x^3 + 600\,x^2 - 600\,x + 120)\, J_0(x)\, dx =$$

$$= (-4\,x^4 + 75\,x^3 - 368\,x^2 + 120\,x)\, J_0(x) + (-x^5 + 25\,x^4 - 184\,x^3 + 375\,x^2 + 136\,x)\, J_1(x) - 255\,\Phi(x)$$

$$\int L_5(x) \cdot J_1(x)\, dx = (x^5 - 25\,x^4 + 185\,x^3 - 400\,x^2 - 120)\, J_0(x) + (-5\,x^4 + 100\,x^3 - 555\,x^2 + 800\,x)\, J_1(x) - 45\,\Phi(x)$$

$$\int L_5(x) \cdot I_0(x)\, dx = (4\,x^4 - 75\,x^3 + 432\,x^2 + 120\,x)\, I_0(x) + (-x^5 + 25\,x^4 - 216\,x^3 + 825\,x^2 - 1464\,x)\, I_1(x) + 945\,\Psi(x)$$

$$\int L_5(x)\cdot I_1(x)\,dx = (-x^5+25\,x^4-215\,x^3+800\,x^2+120)\,I_0(x)+(5\,x^4-100\,x^3+645\,x^2-1600\,x)\,I_1(x)+1245\,\Psi(x)$$

n = 6

$$\int L_6(x)\cdot J_0(x)\,dx = \int (x^6-36\,x^5+450\,x^4-2400\,x^3+5400\,x^2-4320\,x+720)\,J_0(x)\,dx =$$

$$= (5\,x^5-144\,x^4+1275\,x^3-3648\,x^2+720\,x)\,J_0(x)+$$
$$+(x^6-36\,x^5+425\,x^4-1824\,x^3+1575\,x^2+2976\,x)\,J_1(x)-855\,\Phi(x)$$

$$\int L_6(x)\cdot J_1(x)\,dx = (-x^6+36\,x^5-426\,x^4+1860\,x^3-1992\,x^2-720)\,J_0(x)+$$
$$+(6\,x^5-180\,x^4+1704\,x^3-5580\,x^2+3984\,x)\,J_1(x)+1260\,\Phi(x)$$

$$\int L_6(x)\cdot I_0(x)\,dx = (-5\,x^5+144\,x^4-1425\,x^3+5952\,x^2+720\,x)\,I_0(x)+$$
$$+(x^6-36\,x^5+475\,x^4-2976\,x^3+9675\,x^2-16224\,x)\,I_1(x)+10395\,\Psi(x)$$

$$\int L_6(x)\cdot I_1(x)\,dx = (x^6-36\,x^5+474\,x^4-2940\,x^3+9192\,x^2+720)\,I_0(x)+$$
$$+(-6\,x^5+180\,x^4-1896\,x^3+8820\,x^2-18384\,x)\,I_1(x)+13140\,\Psi(x)$$

n = 7

$$\int L_7(x)\cdot J_0(x)\,dx = \int (-x^7+49\,x^6-882\,x^5+7350\,x^4-29400\,x^3+52920\,x^2-35280\,x+5040)\,J_0(x)\,dx =$$

$$= (-6\,x^6+245\,x^5-3384\,x^4+18375\,x^3-31728\,x^2+5040\,x)\,J_0(x)+$$
$$+(-x^7+49\,x^6-846\,x^5+6125\,x^4-15864\,x^3-2205\,x^2+28176\,x)\,J_1(x)\ +7245\Phi(x)$$

$$\int L_7(x)\cdot J_1(x)\,dx = (x^7-49\,x^6+847\,x^5-6174\,x^4+16695\,x^3-3528\,x^2-5040)\,J_0(x)+$$
$$+(-7\,x^6+294\,x^5-4235\,x^4+24696\,x^3-50085\,x^2+7056\,x)\,J_1(x)+14805\,\Phi(x)$$

$$\int L_7(x)\cdot I_0(x)\,dx = (6\,x^6-245\,x^5+3672\,x^4-25725\,x^3+88176\,x^2+5040\,x)\,I_0(x)+$$
$$+(-x^7+49\,x^6-918\,x^5+8575\,x^4-44088\,x^3+130095\,x^2-211632\,x)\,I_1(x)+135135\,\Psi(x)$$

$$\int L_7(x)\cdot I_1(x)\,dx = (-x^7+49\,x^6-917\,x^5+8526\,x^4-43155\,x^3+121128\,x^2+5040)\,I_0(x)+$$
$$+(7\,x^6-294\,x^5+4585\,x^4-34104\,x^3+129465\,x^2-242256\,x)\,I_1(x)+164745\,\Psi(x)$$

n = 8

$$\int L_8(x)\cdot J_0(x)\,dx =$$

$$= \int (x^8-64\,x^7+1568\,x^6-18816\,x^5+117600\,x^4-376320\,x^3+564480\,x^2-322560\,x+40320)\,J_0(x)\,dx =$$

$$= (7\,x^7-384\,x^6+7595\,x^5-66048\,x^4+238875\,x^3-224256\,x^2+40320\,x)\,J_0(x)+$$
$$+(x^8-64\,x^7+1519\,x^6-16512\,x^5+79625\,x^4-112128\,x^3-152145\,x^2+125952\,x)\,J_1(x)+192465\,\Phi(x)$$

$$\int L_8(x)\cdot J_1(x)\,dx = (-x^8+64\,x^7-1520\,x^6+16576\,x^5-81120\,x^4+127680\,x^3+84480\,x^2-40320)\,J_0(x)+$$
$$+(8\,x^7-448\,x^6+9120\,x^5-82880\,x^4+324480\,x^3-383040\,x^2-168960\,x)\,J_1(x)+60480\,\Phi(x)$$

$$\int L_8(x)\cdot I_0(x)\,dx = (-7\,x^7+384\,x^6-8085\,x^5+84480\,x^4-474075\,x^3+1428480\,x^2+40320\,x)\,I_0(x)+$$
$$+(x^8-64\,x^7+1617\,x^6-21120\,x^5+158025\,x^4-714240\,x^3+1986705\,x^2-3179520\,x)\,I_1(x)+2027025\,\Psi(x)$$

$$\int L_8(x)\cdot I_1(x)\,dx=(x^8-64\,x^7+1616\,x^6-21056\,x^5+156384\,x^4-692160\,x^3+1815552\,x^2+40320)\,I_0(x)+$$
$$+(-8\,x^7+448\,x^6-9696\,x^5+105280\,x^4-625536\,x^3+2076480\,x^2-3631104\,x)\,I_1(x)+2399040\,\Psi(x)$$

n = 9

$$\int L_9(x)\cdot J_0(x)\,dx=\int(-x^9+81\,x^8-2592\,x^7+42336\,x^6-381024\,x^5+1905120\,x^4-5080320\,x^3+$$
$$+6531840\,x^2-3265920\,x+362880)\,J_0(x)\,dx=(-8\,x^8+567\,x^7-15168\,x^6+191835\,x^5-1160064\,x^4+$$
$$+2837835\,x^3-880128\,x^2+362880\,x)\,J_0(x)+(-x^9+81\,x^8-2528\,x^7+38367\,x^6-290016\,x^5+$$
$$+945945\,x^4-440064\,x^3-1981665\,x^2-1505664\,x)\,J_1(x)+2344545\,\Phi(x)$$

$$\int L_9(x)\cdot J_1(x)\,dx=$$
$$=(x^9-81\,x^8+2529\,x^7-38448\,x^6+292509\,x^5-982368\,x^4+692685\,x^3+1327104\,x^2-362880)\,J_0(x)+$$
$$+(-9\,x^8+648\,x^7-17703\,x^6+230688\,x^5-1462545\,x^4+3929472\,x^3-2078055\,x^2-2654208\,x)\,J_1(x)-$$
$$-1187865\,\Phi(x)$$

$$\int L_9(x)\cdot I_0(x)\,dx=$$
$$=(8\,x^8-567\,x^7+15936\,x^6-231525\,x^5+1906560\,x^4-9188235\,x^3+25413120\,x^2+362880\,x)\,I_0(x)+$$
$$+(-x^9+81\,x^8-2656\,x^7+46305\,x^6-476640\,x^5+3062745\,x^4-12706560\,x^3+34096545\,x^2-54092160\,x)\,I_1(x)+$$
$$+34459425\,\Psi(x)$$

$$\int L_9(x)\cdot I_1(x)\,dx=(-x^9+81\,x^8-2655\,x^7+46224\,x^6-473949\,x^5+3014496\,x^4-12189555\,x^3+$$
$$+30647808\,x^2+362880)\,I_0(x)+$$
$$+(9\,x^8-648\,x^7+18585\,x^6-277344\,x^5+2369745\,x^4-12057984\,x^3+36568665\,x^2-61295616\,x)\,I_1(x)+39834585\,\Psi(x)$$

n = 10

$$\int L_{10}(x)\cdot J_0(x)\,dx=\int(x^{10}-100\,x^9+4050\,x^8-86400\,x^7+1058400\,x^6-7620480\,x^5+31752000\,x^4-$$
$$-72576000\,x^3+81648000\,x^2-36288000\,x+3628800)\,J_0(x)\,dx=(9\,x^9-800\,x^8+27783\,x^7-480000\,x^6+$$
$$+4319595\,x^5-18961920\,x^4+30462075\,x^3+6543360\,x^2+3628800\,x)\,J_0(x)+(x^{10}-100\,x^9+3969\,x^8-80000\,x^7+$$
$$+863919\,x^6-4740480\,x^5+10154025\,x^4+3271680\,x^3-9738225\,x^2-49374720\,x)\,J_1(x)+13367025\,\Phi(x)$$

$$\int L_{10}(x)\cdot J_1(x)\,dx=(-x^{10}+100\,x^9-3970\,x^8+80100\,x^7-867840\,x^6+4816980\,x^5-10923840\,x^4+$$
$$+321300\,x^3+5742720\,x^2-3628800)\,J_0(x)+(10\,x^9-900\,x^8+31760\,x^7-560700\,x^6+5207040\,x^5-24084900\,x^4+$$
$$+43695360\,x^3-963900\,x^2-11485440\,x)\,J_1(x)-35324100\,\Phi(x)$$

$$\int L_{10}(x)\cdot I_0(x)\,dx=(-9\,x^9+800\,x^8-28917\,x^7+556800\,x^6-6304095\,x^5+43845120\,x^4-189817425\,x^3+$$
$$+495912960\,x^2+3628800\,x)\,I_0(x)+(x^{10}-100\,x^9+4131\,x^8-92800\,x^7+1260819\,x^6-10961280\,x^5+$$
$$+63272475\,x^4-247956480\,x^3+651100275\,x^2-1028113920\,x)\,I_1(x)+654729075\,\Psi(x)$$

$$\int L_{10}(x)\cdot I_1(x)\,dx=(x^{10}-100\,x^9+4130\,x^8-92700\,x^7+1256640\,x^6-10864980\,x^5+61911360\,x^4-$$
$$-235550700\,x^3+576938880\,x^2+3628800)\,I_0(x)+(-10\,x^9+900\,x^8-33040\,x^7+648900\,x^6-7539840\,x^5+$$
$$+54324900\,x^4-247645440\,x^3+706652100\,x^2-1153877760\,x)\,I_1(x)+742940100\,\Psi(x)$$

n = 11

$$\int L_{11}(x)\cdot J_0(x)\,dx = \int(-x^{11}+121\,x^{10}-6050\,x^9+163350\,x^8-2613600\,x^7+25613280\,x^6-153679680\,x^5+$$
$$+548856000\,x^4-1097712000\,x^3+1097712000\,x^2-439084800\,x+39916800)\,J_0(x)\,dx =$$
$$=(-10\,x^{10}+1089\,x^9-47600\,x^8+1074843\,x^7-13396800\,x^6+90446895\,x^5-293195520\,x^4+$$
$$+289864575\,x^3+150140160\,x^2+39916800\,x)\,J_0(x)+$$
$$+(-x^{11}+121\,x^{10}-5950\,x^9+153549\,x^8-2232800\,x^7+18089379\,x^6-73298880\,x^5+$$
$$+96621525\,x^4+75070080\,x^3+228118275\,x^2-739365120\,x)\,J_1(x)-188201475\,\Phi(x)$$

$$\int L_{11}(x)\cdot J_1(x)\,dx = (x^{11}-121\,x^{10}+5951\,x^9-153670\,x^8+2238687\,x^7-18237120\,x^6+75325635\,x^5-$$
$$-111165120\,x^4-32172525\,x^3-208391040\,x^2-39916800)\,J_0(x)+(-11\,x^{10}+1210\,x^9-53559\,x^8+1229360\,x^7-$$
$$-15670809\,x^6+109422720\,x^5-376628175\,x^4+444660480\,x^3+96517575\,x^2+416782080\,x)\,J_1(x)-535602375\,\Phi(x)$$

$$\int L_{11}(x)\cdot I_0(x)\,dx = (10\,x^{10}-1089\,x^9+49200\,x^8-1212057\,x^7+18043200\,x^6-170488395\,x^5+$$
$$+1047755520\,x^4-4203893925\,x^3+10577468160\,x^2+39916800\,x)\,I_0(x)+(-x^{11}+121\,x^{10}-6150\,x^9+173151\,x^8-$$
$$-3007200\,x^7+34097679\,x^6-261938880\,x^5+1401297975\,x^4-5288734080\,x^3+13709393775\,x^2-$$
$$-21594021120\,x)\,I_1(x)+13749310575\,\Psi(x)$$

$$\int L_{11}(x)\cdot I_1(x)\,dx = (-x^{11}+121\,x^{10}-6149\,x^9+173030\,x^8-3000987\,x^7+33918720\,x^6-$$
$$-258714225\,x^5+1362905280\,x^4-4978425375\,x^3+12000954240\,x^2+39916800)\,I_0(x)+$$
$$+(11\,x^{10}-1210\,x^9+55341\,x^8-1384240\,x^7+21006909\,x^6-203512320\,x^5+1293571125\,x^4-$$
$$-5451621120\,x^3+14935276125\,x^2-24001908480\,x)\,I_1(x)+15374360925\,\Psi(x)$$

n = 12

$$\int L_{12}(x)\cdot J_0(x)\,dx = \int(x^{12}-144\,x^{11}+8712\,x^{10}-290400\,x^9+5880600\,x^8-75271680\,x^7+614718720\,x^6-$$
$$-3161410560\,x^5+9879408000\,x^4-17563392000\,x^3+15807052800\,x^2-5748019200\,x+479001600)\,J_0(x)\,dx =$$
$$=(11\,x^{11}-1440\,x^{10}+77319\,x^9-2208000\,x^8+36293103\,x^7-345646080\,x^6+1803334995\,x^5-4350136320\,x^4+$$
$$+2588199075\,x^3-325693440\,x^2+479001600\,x)\,J_0(x)+$$
$$+(x^{12}-144\,x^{11}+8591\,x^{10}-276000\,x^9+5184729\,x^8-57607680\,x^7+360666999\,x^6-1087534080\,x^5+$$
$$+862733025\,x^4-162846720\,x^3+8042455575\,x^2-5096632320\,x)\,J_1(x)-7563453975\,\Phi(x)$$

$$\int L_{12}(x)\cdot J_1(x)\,dx = (-x^{12}+144\,x^{11}-8592\,x^{10}+276144\,x^9-5193240\,x^8+57874608\,x^7-365443200\,x^6+$$
$$+1135799280\,x^5-1108771200\,x^4+526402800\,x^3-6936883200\,x^2-479001600)\,J_0(x)+$$
$$+(12\,x^{11}-1584\,x^{10}+85920\,x^9-2485296\,x^8+41545920\,x^7-405122256\,x^6+2192659200\,x^5-5678996400\,x^4+$$
$$+4435084800\,x^3-1579208400\,x^2+13873766400\,x)\,J_1(x)-4168810800\,\Phi(x)$$

$$\int L_{12}(x)\cdot I_0(x)\,dx = (-11\,x^{11}+1440\,x^{10}-79497\,x^9+2438400\,x^8-46172511\,x^7+568673280\,x^6-$$
$$-4689631485\,x^5+26293800960\,x^4-99982696275\,x^3+245477191680\,x^2+479001600\,x)\,I_0(x)+$$
$$+(x^{12}-144\,x^{11}+8833\,x^{10}-304800\,x^9+6596073\,x^8-94778880\,x^7+937926297\,x^6-6573450240\,x^5+$$
$$+33327565425\,x^4-122738595840\,x^3+315755141625\,x^2-496702402560\,x)\,I_1(x)+316234143225\,\Psi(x)$$

$$\int L_{12}(x)\cdot I_1(x)\,dx = (x^{12}-144\,x^{11}+8832\,x^{10}-304656\,x^9+6587160\,x^8-94465008\,x^7+930902400\,x^6-$$
$$-6467685840\,x^5+32221065600\,x^4-114578679600\,x^3+273575577600\,x^2+479001600)\,I_0(x)+$$
$$+(-12\,x^{11}+1584\,x^{10}-88320\,x^9+2741904\,x^8-52697280\,x^7+661255056\,x^6-5585414400\,x^5+$$
$$+32338429200\,x^4-128884262400\,x^3+343736038800\,x^2-547151155200\,x)\,I_1(x)+349484058000\,\Psi(x)$$

e) Hermite Polynomials $H_n(x)$:

Weight function: $w(x) = e^{-x^2}$, interval $-\infty < x < \infty$.

Holds $\int H_0(x)\, Z_\nu(x)\, dx = \int Z_\nu(x)\, dx$ and $\int H_1(x)\, Z_\nu(x)\, dx = \int 2x\, Z_\nu(x)\, dx$.

n = 2

$$\int H_2(x) \cdot J_0(x)\, dx = \int (4\,x^2 - 2)\, J_0(x)\, dx = -2x\, J_0(x) + 4x^2\, J_1(x) - 6\,\Phi(x)$$

$$\int H_2(x) \cdot J_1(x)\, dx = (-4\,x^2 + 2)\, J_0(x) + 8x\, J_1(x)$$

$$\int H_2(x) \cdot I_0(x)\, dx = -2x\, I_0(x) + 4x^2\, I_1(x) + 2\,\Psi(x)$$

$$\int H_2(x) \cdot I_1(x)\, dx = (4x^2 - 2)\, I_0(x) - 8x\, I_1(x)$$

n = 3

$$\int H_3(x) \cdot J_0(x)\, dx = \int (8\,x^3 - 12\,x)\, J_0(x)\, dx = 16x^2\, J_0(x) + (8x^3 - 44x)\, J_1(x)$$

$$\int H_3(x) \cdot J_1(x)\, dx = -8x^3\, J_0(x) + 24x^2\, J_1(x) - 36\,\Phi(x)$$

$$\int H_3(x) \cdot I_0(x)\, dx = -16x^2\, I_0(x) + (8x^3 + 20x)\, I_1(x)$$

$$\int H_3(x) \cdot I_1(x)\, dx = 8x^3\, I_0(x) - 24x^2\, I_1(x) - 12\,\Psi(x)$$

n = 4

$$\int H_4(x) \cdot J_0(x)\, dx = \int (16\,x^4 - 48\,x^2 + 12)\, J_0(x)\, dx = (48\,x^3 + 12\,x)\, J_0(x) + (16\,x^4 - 192\,x^2)\, J_1(x) + 204\,\Phi(x)$$

$$\int H_4(x) \cdot J_1(x)\, dx = (-16\,x^4 + 176\,x^2 - 12)\, J_0(x) + (64\,x^3 - 352\,x)\, J_1(x)$$

$$\int H_4(x) \cdot I_0(x)\, dx = (-48\,x^3 + 12\,x)\, I_0(x) + (16\,x^4 + 96\,x^2)\, I_1(x) + 108\,\Psi(x)$$

$$\int H_4(x) \cdot I_1(x)\, dx = (16\,x^4 + 80\,x^2 + 12)\, I_0(x) + (-64\,x^3 - 160\,x)\, I_1(x)$$

n = 5

$$\int H_5(x) \cdot J_0(x)\, dx = \int (32\,x^5 - 160\,x^3 + 120\,x)\, J_0(x)\, dx =$$

$$= (128\,x^4 - 1344\,x^2)\, J_0(x) + (32\,x^5 - 672\,x^3 + 2808\,x)\, J_1(x)$$

$$\int H_5(x) \cdot J_1(x)\, dx = (-32\,x^5 + 640\,x^3)\, J_0(x) + (160\,x^4 - 1920\,x^2)\, J_1(x) + 2040\,\Phi(x)$$

$$\int H_5(x) \cdot I_0(x)\, dx = (-128\,x^4 - 704\,x^2)\, I_0(x) + (32\,x^5 + 352\,x^3 + 1528\,x)\, I_1(x)$$

$$\int H_5(x) \cdot I_1(x)\, dx = (32\,x^5 + 320\,x^3)\, I_0(x) + (-160\,x^4 - 960\,x^2)\, I_1(x) - 1080\,\Psi(x)$$

n = 6

$$\int H_6(x) \cdot J_0(x)\, dx = \int (64\,x^6 - 480\,x^4 + 720\,x^2 - 120)\, J_0(x)\, dx =$$

$$= (320\,x^5 - 6240\,x^3 - 120\,x)\, J_0(x) + (64\,x^6 - 2080\,x^4 + 19440\,x^2)\, J_1(x) - 19560\,\Phi(x)$$

$$\int H_6(x) \cdot J_1(x)\, dx = (-64\,x^6 + 2016\,x^4 - 16848\,x^2 + 120)\, J_0(x) + (384\,x^5 - 8064\,x^3 + 33696\,x)\, J_1(x)$$

$$\int H_6(x) \cdot I_0(x)\, dx = (-320\,x^5 - 3360\,x^3 - 120\,x)\, I_0(x) + (64\,x^6 + 1120\,x^4 + 10800\,x^2)\, I_1(x) + 10680\,\Psi(x)$$

$$\int H_6(x)\cdot I_1(x)\,dx = (64\,x^6+1056\,x^4+9168\,x^2-120)\,I_0(x)+(-384\,x^5-4224\,x^3-18336\,x)\,I_1(x)$$

n = 7

$$\int H_7(x)\cdot J_0(x)\,dx = \int(128\,x^7-1344\,x^5+3360\,x^3-1680\,x)\,J_0(x)\,dx =$$

$$=(768\,x^6-23808\,x^4+197184\,x^2)\,J_0(x)+(128\,x^7-5952\,x^5+98592\,x^3-396048\,x)\,J_1(x)$$

$$\int H_7(x)\cdot J_1(x)\,dx = (-128\,x^7+5824\,x^5-90720\,x^3)\,J_0(x)+(896\,x^6-29120\,x^4+272160\,x^2)\,J_1(x)-273840\,\Phi(x)$$

$$\int H_7(x)\cdot I_0(x)\,dx = (-768\,x^6-13056\,x^4-111168\,x^2)\,I_0(x)+(128\,x^7+3264\,x^5+55584\,x^3+220656\,x)\,I_1(x)$$

$$\int H_7(x)\cdot I_1(x)\,dx = (128\,x^7+3136\,x^5+50400\,x^3)\,I_0(x)+(-896\,x^6-15680\,x^4-151200\,x^2)\,I_1(x)-149520\,\Psi(x)$$

n = 8

$$\int H_8(x)\cdot J_0(x)\,dx = \int(256\,x^8-3584\,x^6+13440\,x^4-13440\,x^2+1680)\,J_0(x)\,dx =$$

$$=(1792\,x^7-80640\,x^5+1249920\,x^3+1680\,x)\,J_0(x)+(256\,x^8-16128\,x^6+416640\,x^4-3763200\,x^2)\,J_1(x)+3764880\,\Phi(x)$$

$$\int H_8(x)\cdot J_1(x)\,dx = (-256\,x^8+15872\,x^6-394368\,x^4+3168384\,x^2-1680)\,J_0(x)+$$

$$+(2048\,x^7-95232\,x^5+1577472\,x^3-6336768\,x)\,J_1(x)$$

$$\int H_8(x)\cdot I_0(x)\,dx = (-1792\,x^7-44800\,x^5-712320\,x^3+1680\,x)\,I_0(x)+$$

$$+(256\,x^8+8960\,x^6+237440\,x^4+2123520\,x^2)\,I_1(x)+2125200\,\Psi(x)$$

$$\int H_8(x)\cdot I_1(x)\,dx = (256\,x^8+8704\,x^6+222336\,x^4+1765248\,x^2+1680)\,I_0(x)+$$

$$+(-2048\,x^7-52224\,x^5-889344\,x^3-3530496\,x)\,I_1(x)$$

n = 9

$$\int H_9(x)\cdot J_0(x)\,dx = \int(512\,x^9-9216\,x^7+48384\,x^5-80640\,x^3+30240\,x)\,J_0(x)\,dx =$$

$$=(4096\,x^8-251904\,x^6+6239232\,x^4-50075136\,x^2)\,J_0(x)+$$

$$+(512\,x^9-41984\,x^7+1559808\,x^5-25037568\,x^3+100180512\,x)\,J_1(x)$$

$$\int H_9(x)\cdot J_1(x)\,dx = (-512\,x^9+41472\,x^7-1499904\,x^5+22579200\,x^3)\,J_0(x)+$$

$$+(4608\,x^8-290304\,x^6+7499520\,x^4-67737600\,x^2)\,J_1(x)+67767840\,\Phi(x)$$

$$\int H_9(x)\cdot I_0(x)\,dx = (-4096\,x^8-141312\,x^6-3585024\,x^4-28518912\,x^2)\,I_0(x)+$$

$$+(512\,x^9+23552\,x^7+896256\,x^5+14259456\,x^3+57068064)\,I_1(x)$$

$$\int H_9(x)\cdot I_1(x)\,dx = (512\,x^9+23040\,x^7+854784\,x^5+12741120\,x^3)\,I_0(x)+$$

$$+(-4608\,x^8-161280\,x^6-4273920\,x^4-38223360\,x)\,I_1(x)-38253600\,\Psi(x)$$

n = 10

$$\int H_{10}(x)\cdot J_0(x)\,dx = \int(1024\,x^{10}-23040\,x^8+161280\,x^6-403200\,x^4+302400\,x^2-30240)\,J_0(x)\,dx =$$

$$=(9216\,x^9-741888\,x^7+26772480\,x^5-402796800\,x^3-30240\,x)\,J_0(x)+$$

$$+(1024\,x^{10}-105984\,x^8+5354496\,x^6-134265600\,x^4+1208692800\,x^2)\,J_1(x)-1208723040\,\Phi(x)$$

$$\int H_{10}(x)\cdot J_1(x)\,dx = (-1024\,x^{10}+104960\,x^8-5199360\,x^6+125187840\,x^4-1001805120\,x^2+30240)\,J_0(x)+$$
$$+(10240\,x^9-839680\,x^7+31196160\,x^5-500751360\,x^3+2003610240\,x)\,J_1(x)$$
$$\int H_{10}(x)\cdot I_0(x)\,dx = (-9216\,x^9-419328\,x^7-15482880\,x^5-231033600\,x^3-30240\,x)\,I_0(x)+$$
$$+(1024\,x^{10}+59904\,x^8+3096576\,x^6+77011200\,x^4+693403200\,x^2)\,I_1(x)+693372960\,\Psi(x)$$
$$\int H_{10}(x)\cdot I_1(x)\,dx = (1024\,x^{10}+58880\,x^8+2987520\,x^6+71297280\,x^4+570680640\,x^2-30240)\,I_0(x)+$$
$$+(-10240\,x^9-471040\,x^7-17925120\,x^5-285189120\,x^3-1141361280\,x)\,I_1(x)$$

n = 11

$$\int H_{11}(x)\cdot J_0(x)\,dx = \int(2048\,x^{11}-56320\,x^9+506880\,x^7-1774080\,x^5+2217600\,x^3-665280\,x)\,J_0(x)\,dx =$$
$$= (20480\,x^{10}-2088960\,x^8+103311360\,x^6-2486568960\,x^4+19896986880\,x^2)\,J_0(x)+$$
$$+(2048\,x^{11}-261120\,x^9+17218560\,x^7-621642240\,x^5+9948493440\,x^3-39794639040\,x)\,J_1(x)$$
$$\int H_{11}(x)\cdot J_1(x)\,dx = (-2048\,x^{11}+259072\,x^9-16828416\,x^7+590768640\,x^5-8863747200\,x^3)\,J_0(x)+$$
$$+(22528\,x^{10}-2331648\,x^8+117798912\,x^6-2953843200\,x^4+26591241600\,x^2)\,J_1(x)-26591906880\,\Phi(x)$$
$$\int H_{11}(x)\cdot I_0(x)\,dx = (-20480\,x^{10}-1187840\,x^8-60057600\,x^6-1434286080\,x^4-11478723840\,x^2)\,I_0(x)+$$
$$+(2048\,x^{11}+148480\,x^9+10009600\,x^7+358571520\,x^5+5739361920\,x^3+22956782400\,x)\,I_1(x)$$
$$\int H_{11}(x)\cdot I_1(x)\,dx = (2048\,x^{11}+146432\,x^9+9732096\,x^7+338849280\,x^5+5084956800\,x^3)\,I_0(x)+$$
$$-(-22528\,x^{10}-1317888\,x^8-68124672\,x^6-1694246400\,x^4-15254870400\,x^2)\,I_1(x)-15254205120\,\Psi(x)$$

n = 12

$$\int H_{12}(x)\cdot J_0(x)\,dx =$$
$$= \int(4096\,x^{12}-135168\,x^{10}+1520640\,x^8-7096320\,x^6+13305600\,x^4-7983360\,x^2+665280)\,J_0(x)\,dx =$$
$$= (45056\,x^{11}-5677056\,x^9+368299008\,x^7-12925946880\,x^5+193929120000\,x^3+665280\,x)\,J_0(x)+$$
$$+(4096\,x^{12}-630784\,x^{10}+52614144\,x^8-2585189376\,x^6+64643040000\,x^4-581795343360\,x^2)\,J_1(x)+$$
$$+581796008640\,\Phi(x)$$
$$\int H_{12}(x)\cdot J_1(x)\,dx =$$
$$= (-4096\,x^{12}+626688\,x^{10}-51655680\,x^8+2486568960\,x^6-59690960640\,x^4+477535668480\,x^2-665280)\,J_0(x)+$$
$$+(49152\,x^{11}-6266880\,x^9+413245440\,x^7-14919413760\,x^5+238763842560\,x^3-955071336960\,x)\,J_1(x)$$
$$\int H_{12}(x)\cdot I_0(x)\,dx =$$
$$+(-45056\,x^{11}-3244032\,x^9-215018496\,x^7-7490165760\,x^5-112392403200\,x^3+665280\,x)\,I_0(x)+$$
$$+(4096\,x^{12}+360448\,x^{10}+30716928\,x^8+1498033152\,x^6+37464134400\,x^4+337169226240\,x^2)\,I_1(x)+$$
$$+337169891520\,\Psi(x)$$
$$\int H_{12}(x)\cdot I_1(x)\,dx =$$
$$= (4096\,x^{12}+356352\,x^{10}+30028800\,x^8+1434286080\,x^6+34436171520\,x^4+275481388800\,x^2+665280)\,I_0(x)+$$
$$+(-49152\,x^{11}-3563520\,x^9-240230400\,x^7-8605716480\,x^5-137744686080\,x^3-550962777600\,x)\,I_1(x)$$

n = 13

$$\int H_{13}(x)\cdot J_0(x)\,dx =$$

$$= \int (8192\,x^{13} - 319488\,x^{11} + 4392960\,x^9 - 26357760\,x^7 + 69189120\,x^5 - 69189120\,x^3 + 17297280\,x)\,J_0(x)\,dx =$$

$$= (98304\,x^{12} - 14991360\,x^{10} + 1234452480\,x^8 - 59411865600\,x^6 + 1426161530880\,x^4 - 11409430625280\,x^2)\,J_0(x) +$$
$$+ (8192\,x^{13} - 1499136\,x^{11} + 154306560\,x^9 - 9901977600\,x^7 + 356540382720\,x^5 - 5704715312640\,x^3 +$$
$$+ 22818878547840\,x)\,J_1(x)$$

$$\int H_{13}(x)\cdot J_1(x)\,dx =$$

$$= (-8192\,x^{13} + 1490944\,x^{11} - 151996416\,x^9 + 9602131968\,x^7 - 336143808000\,x^5 + 5042226309120\,x^3)\,J_0(x) +$$
$$+ (106496\,x^{12} - 16400384\,x^{10} + 1367967744\,x^8 - 67214923776\,x^6 + 1680719040000\,x^4 - 15126678927360\,x^2)\,J_1(x) +$$
$$+ 15126696224640\,\Phi(x)$$

$$\int H_{13}(x)\cdot I_0(x)\,dx =$$

$$= (-98304\,x^{12} - 8601600\,x^{10} - 723271680\,x^8 - 34558894080\,x^6 - 829690214400\,x^4 - 6637383336960\,x^2)\,I_0(x) +$$
$$+ (8192\,x^{13} + 860160\,x^{11} + 90408960\,x^9 + 5759815680\,x^7 + 207422553600\,x^5 + 3318691668480\,x^3 +$$
$$+ 13274783971200\,x)\,I_1(x)$$

$$\int H_{13}(x)\cdot I_1(x)\,dx =$$

$$= (8192\,x^{13} + 851968\,x^{11} + 88737792\,x^9 + 5564123136\,x^7 + 194813498880\,x^5 + 2922133294080\,x^3)\,I_0(x) +$$
$$+ (-106496\,x^{12} - 9371648\,x^{10} - 798640128\,x^8 - 38948861952\,x^6 - 974067494400\,x^4 - 8766399882240\,x^2)\,I_1(x) -$$
$$- 8766417179520\,\Psi(x)$$

n = 14

$$\int H_{14}(x)\cdot J_0(x)\,dx = \int (16384\,x^{14} - 745472\,x^{12} + 12300288\,x^{10} - 92252160\,x^8 + 322882560\,x^6 - 484323840\,x^4 +$$
$$+ 242161920\,x^2 - 17297280)\,J_0(x)\,dx = (212992\,x^{13} - 38658048\,x^{11} + 3937849344\,x^9 - 248730273792\,x^7 +$$
$$+ 8707173995520\,x^5 - 130609062904320\,x^3 - 17297280\,x)\,J_0(x) + (16384\,x^{14} - 3514368\,x^{12} + 437538816\,x^{10} -$$
$$- 35532896256\,x^8 + 1741434799104\,x^6 - 43536354301440\,x^4 + 391827430874880\,x^2)\,J_1(x) - 391827448172160\,\Phi(x)$$

$$\int H_{14}(x)\cdot J_1(x)\,dx = (-16384\,x^{14} + 3497984\,x^{12} - 432058368\,x^{10} + 34656921600\,x^8 - 1663855119360\,x^6 +$$
$$+ 39933007188480\,x^4 - 319464299669760\,x^2 + 17297280)\,J_0(x) + (229376\,x^{13} - 41975808\,x^{11} + 4320583680\,x^9 -$$
$$- 277255372800\,x^7 + 9983130716160\,x^5 - 159732028753920\,x^3 + 638928599339520\,x)\,J_1(x)$$

$$\int H_{14}(x)\cdot I_0(x)\,dx = (-212992\,x^{13} - 22257664\,x^{11} - 2314211328\,x^9 - 145149548544\,x^7 - 5081848611840\,x^5 -$$
$$- 76226276206080\,x^3 - 17297280\,x)\,I_0(x) + (16384\,x^{14} + 2023424\,x^{12} + 257134592\,x^{10} + 20735649792\,x^8 +$$
$$+ 1016369722368\,x^6 + 25408758735360\,x^4 + 228679070780160\,x^2)\,I_1(x) + 228679053482880\,\Psi(x)$$

$$\int H_{14}(x)\cdot I_1(x)\,dx = (16384\,x^{14} + 2007040\,x^{12} + 253145088\,x^{10} + 20159354880\,x^8 + 967971916800\,x^6 +$$
$$+ 23230841679360\,x^4 + 185846975596800\,x^2 - 17297280)\,I_0(x) + (-229376\,x^{13} - 24084480\,x^{11} - 2531450880\,x^9 -$$
$$- 161274839040\,x^7 - 5807831500800\,x^5 - 92923366717440\,x^3 - 371693951193600\,x)\,I_1(x)$$

n = 15

$$\int H_{15}(x)\cdot J_0(x)\,dx=\int(32768\,x^{15}-1720320\,x^{13}+33546240\,x^{11}-307507200\,x^{9}+1383782400\,x^{7}-$$
$$-2905943040\,x^{5}+2421619200\,x^{3}-518918400\,x)\,J_0(x)\,dx=$$
$$=(458752\,x^{14}-97714176\,x^{12}+12061163520\,x^{10}-967353139200\,x^{8}+46441253376000\,x^{6}-$$
$$-1114601704796160\,x^{4}+8916818481607680\,x^{2})\,J_0(x)+$$
$$+(32768\,x^{15}-8142848\,x^{13}+1206116352\,x^{11}-120919142400\,x^{9}+7740208896000\,x^{7}-$$
$$-278650426199040\,x^{5}+4458409240803840\,x^{3}-17833637482133760\,x)\,J_1(x)$$

$$\int H_{15}(x)\cdot J_1(x)\,dx=(-32768\,x^{15}+8110080\,x^{13}-1193287680\,x^{11}+118442987520\,x^{9}-7463291996160\,x^{7}+$$
$$+261218125808640\,x^{5}-3918274308748800\,x^{3})\,J_0(x)+(491520\,x^{14}-105431040\,x^{12}+13126164480\,x^{10}-$$
$$-1065986887680\,x^{8}+52243043973120\,x^{6}-1306090629043200\,x^{4}+11754822926246400\,x^{2})\,J_1(x)-$$
$$-11754823445164800\,\Phi(x)$$

$$\int H_{15}(x)\cdot I_0(x)\,dx=(-458752\,x^{14}-56426496\,x^{12}-7106641920\,x^{10}-566071296000\,x^{8}-27179724902400\,x^{6}-$$
$$-652301773885440\,x^{4}-5218419034321920\,x^{2})\,I_0(x)+(32768\,x^{15}+4702208\,x^{13}+710664192\,x^{11}+$$
$$+70758912000\,x^{9}+4529954150400\,x^{7}+163075443471360\,x^{5}+2609209517160960\,x^{3}+10436837549725440\,x)\,I_1(x)$$

$$\int H_{15}(x)\cdot I_1(x)\,dx=(32768\,x^{15}+4669440\,x^{13}+701276160\,x^{11}+69118832640\,x^{9}+4355870238720\,x^{7}+$$
$$+152452552412160\,x^{5}+2286790707801600\,x^{3})\,I_0(x)+(-491520\,x^{14}-60702720\,x^{12}-7714037760\,x^{10}-$$
$$-622069493760\,x^{8}-30491091671040\,x^{6}-762262762060800\,x^{4}-6860372123404800\,x^{2})\,I_1(x)-$$
$$-6860371604486400\,\Psi(x)$$

1.3.2. Integrals of the type $\int x^n \operatorname{Ei}(x) \cdot Z_\nu(x)\, dx$

About $\operatorname{Ei}(x)$ see [1], 5.1., or [7], 8.2. In [4], page 657, is no reference to the fact, that the integral should be used as principal value.

$$\int x \operatorname{Ei}(x)\, I_0(x)\, dx = x\operatorname{Ei}(x) I_1(x) + e^x \left[(x-1)I_0(x) - xI_1(x)\right]$$

$$\int x \operatorname{Ei}(x)\, K_0(x)\, dx = -x\operatorname{Ei}(x) K_1(x) + e^x \left[(x-1)K_0(x) + xK_1(x)\right]$$

$$\int x^2 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)[x^2 I_0(x) - 2xI_1(x)] + \frac{e^x}{3}\left[(-x^2-6\,x+6)I_0(x) + (x^2+5\,x)I_1(x)\right]$$

$$\int x^2 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)[x^2 K_0(x) + 2xK_1(x)] + \frac{e^x}{3}\left[(x^2+6\,x-6)K_0(x) + (x^2+5\,x)K_1(x)\right]$$

$$\int x^3 \operatorname{Ei}(x)\, I_0(x)\, dx = \operatorname{Ei}(x)[-2x^2 I_0(x) + (x^3+4\,x)I_1(x)] +$$
$$+\frac{e^x}{15}\left[(3\,x^3+7\,x^2+60\,x-60)I_0(x) - (3\,x^3+16\,x^2+44\,x)I_1(x)\right]$$

$$\int x^3 \operatorname{Ei}(x)\, K_0(x)\, dx = -\operatorname{Ei}(x)[2x^2 K_0(x) + (x^3+4x)K_1(x)] +$$
$$+\frac{e^x}{15}\left[(3\,x^3+7\,x^2+60\,x-60)K_0(x) + (3\,x^3+16\,x^2+44\,x)K_1(x)\right]$$

$$\int x^4 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)\left[(x^4+8\,x^2)I_0(x) - (16\,x+4\,x^3)I_1(x)\right] +$$
$$+\frac{e^x}{105}\left[-(15\,x^4+102\,x^3+178\,x^2+1680\,x-1680)I_0(x) + (15\,x^4+57\,x^3+484\,x^2+1196\,x)I_1(x)\right]$$

$$\int x^4 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)\left[(x^4+8\,x^2)K_0(x) + (4\,x^3+16\,x)K_1(x)\right] +$$
$$+\frac{e^x}{105}\left[(15\,x^4+102\,x^3+178\,x^2+1680\,x-1680)K_0(x) + (15\,x^4+57\,x^3+484\,x^2+1196\,x)K_1(x)\right]$$

$$\int x^5 \operatorname{Ei}(x)\, I_0(x)\, dx = \operatorname{Ei}(x)\left[-(4\,x^4+32\,x^2)I_0(x) + (x^5+16\,x^3+64\,x)I_1(x)\right] +$$
$$+\frac{e^x}{45}\left[(5\,x^5+15\,x^4+192\,x^3+288\,x^2+2880\,x-2880)\,I_0(x) - (5\,x^5+40\,x^4+72\,x^3+864\,x^2+2016\,x)I_1(x)\right]$$

$$\int x^5 \operatorname{Ei}(x)\, K_0(x)\, dx = -\operatorname{Ei}(x)\left[(4\,x^4+32\,x^2)K_0(x) + (x^5+16\,x^3+64\,x)K_1(x)\right] +$$
$$+\frac{e^x}{45}\left[(5\,x^5+15\,x^4+192\,x^3+288\,x^2+2880\,x-2880)K_0(x) + (5\,x^5+40\,x^4+72\,x^3+864\,x^2+2016\,x)K_1(x)\right]$$

$$\int x^6 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)\left[(x^6+24\,x^4+192\,x^2)I_0(x) - (384\,x+96\,x^3+6\,x^5)I_1(x)\right] +$$
$$+\frac{e^x}{3465}\left[-(315\,x^6+3010\,x^5+5430\,x^4+91104\,x^3+130656\,x^2+1330560\,x-1330560)I_0(x) + \right.$$
$$\left. +(315\,x^6+1435\,x^5+20480\,x^4+29664\,x^3+403968\,x^2+926592\,x)I_1(x)\right]$$

$$\int x^6 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)\left[(x^6+24\,x^4+192\,x^2)K_0(x) + (6\,x^5+96\,x^3+384\,x)K_1(x)\right] +$$

$$+\frac{e^x}{3465}\,\big[(315\,x^6+3010\,x^5+5430\,x^4+91104\,x^3+130656\,x^2+1330560\,x-1330560)K_0(x)+$$
$$+(315\,x^6+1435\,x^5+20480\,x^4+29664\,x^3+403968\,x^2+926592\,x)K_1(x)\big]$$

$$\int x^7\,\mathrm{Ei}(x)\,I_0(x)\,dx=\mathrm{Ei}(x)\,[-(6\,x^6+144\,x^4+1152\,x^2)I_0(x)+(x^7+36\,x^5+576\,x^3+2304\,x)I_1(x)]+$$
$$+\frac{e^x}{15015}\,\big[(1155\,x^7+4515\,x^6+88060\,x^5+120180\,x^4+2402304\,x^3+3363456\,x^2+34594560\,x-34594560)I_0(x)+$$
$$-(1155\,x^7+12600\,x^6+25060\,x^5+560480\,x^4+720864\,x^3+10570368\,x^2+24024192\,x)I_1(x)\big]$$
$$\int x^7\,\mathrm{Ei}(x)\,K_0(x)\,dx=-\,\mathrm{Ei}(x)\,[(6\,x^6+144\,x^4+1152\,x^2)K_0(x)+(x^7+36\,x^5+576\,x^3+2304\,x)K_1(x)]+$$
$$+\frac{e^x}{15015}\,\big[(1155\,x^7+4515\,x^6+88060\,x^5+120180\,x^4+2402304\,x^3+3363456\,x^2+34594560\,x-34594560)K_0(x)+$$
$$+\,(1155\,x^7+12600\,x^6+25060\,x^5+560480\,x^4+720864\,x^3+10570368\,x^2+24024192\,x)K_1(x)\big]$$

$$\int x^8\,\mathrm{Ei}(x)\,I_1(x)\,dx=\mathrm{Ei}(x)[(x^8+48\,x^6+1152\,x^4+9216\,x^2)I_0(x)-(8\,x^7+288\,x^5+4608\,x^3+18432\,x)I_1(x)]+$$
$$+\frac{e^x}{15015}\,\big[-(1001\,x^8+12474\,x^7+25830\,x^6+731920\,x^5+902640\,x^4+19312512\,x^3+26813568\,x^2+$$
$$+276756480\,x-276756480)I_0(x)+(1001\,x^8+5467\,x^7+113148\,x^6+166180\,x^5+4562240\,x^4+5625792\,x^3+$$
$$+84751104\,x^2+192005376\,x)I_1(x)\big]$$
$$\int x^8\,\mathrm{Ei}(x)\,K_1(x)\,dx=-\,\mathrm{Ei}(x)[(x^8+48\,x^6+1152\,x^4+9216\,x^2)K_0(x)+(8\,x^7+288\,x^5+4608\,x^3+18432\,x)K_1(x)]+$$
$$+\frac{e^x}{15015}\,\big[(1001\,x^8+12474\,x^7+25830\,x^6+731920\,x^5+902640\,x^4+19312512\,x^3+26813568\,x^2+$$
$$+276756480\,x-276756480)K_0(x)+(1001\,x^8+5467\,x^7+113148\,x^6+166180\,x^5+4562240\,x^4+5625792\,x^3+$$
$$+84751104\,x^2+192005376\,x)K_1(x)\big]$$

$$\int x^9\,\mathrm{Ei}(x)\,I_0(x)\,dx=$$
$$=\mathrm{Ei}(x)\,[-(8\,x^8+384\,x^6+9216\,x^4+73728\,x^2)I_0(x)+(x^9+64\,x^7+2304\,x^5+36864\,x^3+147456\,x)I_1(x)]+$$
$$+\frac{e^x}{23205}\,\big[(1365\,x^9+6643\,x^8+175392\,x^7+252000\,x^6+9228800\,x^5+10775040\,x^4+239388672\,x^3+$$
$$+330897408\,x^2+3421716480\,x-3421716480)I_0(x)-(1365\,x^9+18928\,x^8+42896\,x^7+1479744\,x^6+$$
$$+1830080\,x^5+56919040\,x^4+68631552\,x^3+1049063424\,x^2+2372653056\,x)I_1(x)\big]$$
$$\int x^9\,\mathrm{Ei}(x)\,K_0(x)\,dx=$$
$$=-\,\mathrm{Ei}(x)\,[(8\,x^8+384\,x^6+9216\,x^4+73728\,x^2)K_0(x)+(x^9+64\,x^7+2304\,x^5+36864\,x^3+147456\,x)K_1(x)]+$$
$$+\frac{e^x}{23205}\,\big[(1365\,x^9+6643\,x^8+175392\,x^7+252000\,x^6+9228800\,x^5+10775040\,x^4+239388672\,x^3+$$
$$+330897408\,x^2+3421716480\,x-3421716480)K_0(x)+(1365\,x^9+18928\,x^8+42896\,x^7+1479744\,x^6+$$
$$+1830080\,x^5+56919040\,x^4+68631552\,x^3+1049063424\,x^2+2372653056\,x)K_1(x)\big]$$

1.3.3. Integrals of the type $\int x^n \operatorname{Si}(x)\cdot J_\nu(x)\,dx$ and $\int x^n \operatorname{Ci}(x)\cdot J_\nu(x)\,dx$

Let

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t\,dt}{t} \quad \text{and} \quad \operatorname{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt\,.$$

About C see page 113.

(In [4], p. 656, the function $\operatorname{ci}(x)$ is defined by some integral which fails to converge.)

$$\int x\operatorname{Si}(x)\,J_0(x)\,dx = x\operatorname{Si}(x)\,J_1(x) + \sin x\,J_0(x) - \sin x\,J_1(x) - \cos x\,J_0(x)$$

$$\int x\operatorname{Ci}(x)\,J_0(x)\,dx = x\operatorname{Ci}(x)\,J_1(x) + x\sin x\,J_0(x) + \cos x\,J_0(x) - x\cos x\,J_1(x)$$

$$\int x^2\operatorname{Si}(x)\,J_1(x)\,dx = \frac{1}{3}\,\big[-3\,x^2\operatorname{Si}(x)\,J_0(x) + 6\,x\operatorname{Si}(x)\,J_1(x) + \left(x^2+6\right)\sin x\,J_0(x) - 5\,x\sin x\,J_1(x) -$$
$$-6\,x\cos x\,J_0(x) - x^2\cos x\,J_1(x)\big]$$

$$\int x^2\operatorname{Ci}(x)\,J_1(x)\,dx = \frac{1}{3}\,\big[-3\,x^2\operatorname{Ci}(x)\,J_0(x) + 6\,x\operatorname{Ci}(x)\,J_0(x) + 6\,x\sin x\,J_0(x) + x^2\sin x\,J_1(x) +$$
$$+\left(x^2+6\right)\cos x\,J_0(x) - 5\,x\cos x\,J_1(x)\big]$$

$$\int x^3\operatorname{Si}(x)\,J_0(x)\,dx = \frac{1}{15}\,\big[30\,x^2\operatorname{Si}(x)\,J_0(x) + \left(15\,x^3-60\,x\right)\operatorname{Si}(x)\,J_1(x) + \left(-7\,x^2-60\right)\sin x\,J_0(x) +$$
$$+\left(-3\,x^3+44\,x\right)\sin x\,J_1(x) + \left(-3\,x^3+60\,x\right)\cos x\,J_0(x) + 16\,x^2\cos x\,J_1(x)\big]$$

$$\int x^3\operatorname{Ci}(x)\,J_0(x)\,dx = \frac{1}{15}\,\big[30\,x^2\operatorname{Ci}(x)\,J_0(x) + \left(15\,x^3-60\,x\right)\operatorname{Ci}(x)\,J_0(x) + \left(3\,x^3-60\,x\right)\sin x\,J_0(x) -$$
$$-16\,x^2\sin x\,J_1(x) + \left(-7\,x^2-60\right)\cos x\,J_0(x) + \left(-3\,x^3+44\,x\right)\cos x\,J_1(x)\big]$$

$$\int x^4\operatorname{Si}(x)\,J_1(x)\,dx = \frac{1}{105}\,\big[\left(-105\,x^4+840\,x^2\right)\operatorname{Si}(x)\,J_0(x) + \left(420\,x^3-1680\,x\right)\operatorname{Si}(x)\,J_1(x) +$$
$$+\left(15\,x^4-178\,x^2-1680\right)\sin x\,J_0(x) + \left(-57\,x^3+1196\,x\right)\sin x\,J_1(x) + \left(-102\,x^3+1680\,x\right)\cos x\,J_0(x) +$$
$$+\left(-15\,x^4+484\,x^2\right)\cos x\,J_1(x)\big]$$

$$\int x^4\operatorname{Ci}(x)\,J_1(x)\,dx = \frac{1}{105}\,\big[\left(-105\,x^4+840\,x^2\right)\operatorname{Ci}(x)\,J_0(x) + \left(420\,x^3-1680\,x\right)\operatorname{Ci}(x)\,J_0(x) +$$
$$+\left(102\,x^3-1680\,x\right)\sin x\,J_0(x) + \left(15\,x^4-484\,x^2\right)\sin x\,J_1(x) + \left(15\,x^4-178\,x^2-1680\right)\cos x\,J_0(x) +$$
$$+\left(-57\,x^3+1196\,x\right)\cos x\,J_1(x)\big]$$

$$\int x^5\operatorname{Si}(x)\,J_0(x)\,dx = \frac{1}{45}\,\big[\left(180\,x^4-1440\,x^2\right)\operatorname{Si}(x)\,J_0(x) + \left(45\,x^5-720\,x^3+2880\,x\right)\operatorname{Si}(x)\,J_1(x) +$$
$$+\left(-15\,x^4+288\,x^2+2880\right)\sin x\,J_0(x) + \left(-5\,x^5+72\,x^3-2016\,x\right)\sin x\,J_1(x) +$$
$$+\left(-5\,x^5+192\,x^3-2880\,x\right)\cos x\,J_0(x) + \left(40\,x^4-864\,x^2\right)\cos x\,J_1(x)\big]$$

$$\int x^5\,\mathrm{Ci}(x)\,J_0(x)\,dx = \frac{1}{45}\,\big[\big(180\,x^4-1440\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x)+\big(45\,x^5-720\,x^3+2880\,x\big)\,\mathrm{Ci}(x)\,J_0(x)+$$
$$+\big(5\,x^5-192\,x^3+2880\,x\big)\sin x\,J_0(x)+\big(-40\,x^4+864\,x^2\big)\sin x\,J_1(x)+$$
$$+\big(-15\,x^4+288\,x^2+2880\big)\cos x\,J_0(x)+\big(-5\,x^5+72\,x^3-2016\,x\big)\cos x\,J_1(x)\big]$$

$$\int x^6\,\mathrm{Si}(x)\,J_1(x)\,dx = \frac{1}{3465}\,\big[\big(-3465\,x^6+83160\,x^4-665280\,x^2\big)\,\mathrm{Si}(x)\,J_0(x)+$$
$$+\big(20790\,x^5-332640\,x^3+1330560\,x\big)\,\mathrm{Si}(x)\,J_1(x)+$$
$$+\big(315\,x^6-5430\,x^4+130656\,x^2+1330560\big)\sin x\,J_0(x)+\big(-1435\,x^5+29664\,x^3-926592\,x\big)\sin x\,J_1(x)+$$
$$+\big(-3010\,x^5+91104\,x^3-1330560\,x\big)\cos x\,J_0(x)+\big(-315\,x^6+20480\,x^4-403968\,x^2\big)\cos x\,J_1(x)\big]$$

$$\int x^6\,\mathrm{Ci}(x)\,J_1(x)\,dx = \frac{1}{3465}\,\big[\big(-3465\,x^6+83160\,x^4-665280\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x)+$$
$$+\big(20790\,x^5-332640\,x^3+1330560\,x\big)\,\mathrm{Ci}(x)\,J_0(x)+\big(3010\,x^5-91104\,x^3+1330560\,x\big)\sin x\,J_0(x)+$$
$$+\big(315\,x^6-20480\,x^4+403968\,x^2\big)\sin x\,J_1(x)+\big(315\,x^6-5430\,x^4+130656\,x^2+1330560\big)\cos x\,J_0(x)+$$
$$+\big(-1435\,x^5+29664\,x^3-926592\,x\big)\cos x\,J_1(x)\big]$$

$$\int x^7\,\mathrm{Si}(x)\,J_0(x)\,dx = \frac{1}{15015}\,\big[\big(90090\,x^6-2162160\,x^4+17297280\,x^2\big)\,\mathrm{Si}(x)\,J_0(x)+$$
$$+\big(15015\,x^7-540540\,x^5+8648640\,x^3-34594560\,x\big)\,\mathrm{Si}(x)\,J_1(x)+$$
$$+\big(-4515\,x^6+120180\,x^4-3363456\,x^2-34594560\big)\sin x\,J_0(x)+$$
$$+\big(-1155\,x^7+25060\,x^5-720864\,x^3+24024192\,x\big)\sin x\,J_1(x)+$$
$$+\big(-1155\,x^7+88060\,x^5-2402304\,x^3+34594560\,x\big)\cos x\,J_0(x)+$$
$$+\big(12600\,x^6-560480\,x^4+10570368\,x^2\big)\cos x\,J_1(x)\big]$$

$$\int x^7\,\mathrm{Ci}(x)\,J_0(x)\,dx = \frac{1}{15015}\,\big[\big(90090\,x^6-2162160\,x^4+17297280\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x)+$$
$$+\big(15015\,x^7-540540\,x^5+8648640\,x^3-34594560\,x\big)\,\mathrm{Ci}(x)\,J_0(x)+$$
$$+\big(1155\,x^7-88060\,x^5+2402304\,x^3-34594560\,x\big)\sin x\,J_0(x)+$$
$$+\big(-12600\,x^6+560480\,x^4-10570368\,x^2\big)\sin x\,J_1(x)+$$
$$+\big(-4515\,x^6+120180\,x^4-3363456\,x^2-34594560\big)\cos x\,J_0(x)+$$
$$+\big(-1155\,x^7+25060\,x^5-720864\,x^3+24024192\,x\big)\cos x\,J_1(x)\big]$$

$$\int x^8\,\mathrm{Si}(x)\,J_1(x)\,dx = \frac{1}{15015}\,\big[\big(-15015\,x^8+720720\,x^6-17297280\,x^4+138378240\,x^2\big)\,\mathrm{Si}(x)\,J_0(x)+$$
$$+\big(120120\,x^7-4324320\,x^5+69189120\,x^3-276756480\,x\big)\,\mathrm{Si}(x)\,J_1(x)+$$
$$+\big(1001\,x^8-25830\,x^6+902640\,x^4-26813568\,x^2-276756480\big)\sin x\,J_0(x)+$$
$$+\big(-5467\,x^7+166180\,x^5-5625792\,x^3+192005376\,x\big)\sin x\,J_1(x)+$$
$$+\big(-12474\,x^7+731920\,x^5-19312512\,x^3+276756480\,x\big)\cos x\,J_0(x)+$$
$$+\big(-1001\,x^8+113148\,x^6-4562240\,x^4+84751104\,x^2\big)\cos x\,J_1(x)\big]$$

$$\int x^8\,\mathrm{Ci}(x)\,J_1(x)\,dx = \frac{1}{15015}\,\big[\big(-15015\,x^8 + 720720\,x^6 - 17297280\,x^4 + 138378240\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x) +$$
$$+\big(120120\,x^7 - 4324320\,x^5 + 69189120\,x^3 - 276756480\,x\big)\,\mathrm{Ci}(x)\,J_0(x) +$$
$$+\big(12474\,x^7 - 731920\,x^5 + 19312512\,x^3 - 276756480\,x\big)\sin x\,J_0(x) +$$
$$+\big(1001\,x^8 - 113148\,x^6 + 4562240\,x^4 - 84751104\,x^2\big)\sin x\,J_1(x) +$$
$$+\big(1001\,x^8 - 25830\,x^6 + 902640\,x^4 - 26813568\,x^2 - 276756480\big)\cos x\,J_0(x) +$$
$$+\big(-5467\,x^7 + 166180\,x^5 - 5625792\,x^3 + 192005376\,x\big)\cos x\,J_1(x)\big]$$

$$\int x^9\,\mathrm{Si}(x)\,J_0(x)\,dx = \frac{1}{23205}\,\big[\big(185640\,x^8 - 8910720\,x^6 + 213857280\,x^4 - 1710858240\,x^2\big)\,\mathrm{Si}(x)\,J_0(x) +$$
$$+\big(23205\,x^9 - 1485120\,x^7 + 53464320\,x^5 - 855429120\,x^3 + 3421716480\,x\big)\,\mathrm{Si}(x)\,J_1(x) +$$
$$+\big(-6643\,x^8 + 252000\,x^6 - 10775040\,x^4 + 330897408\,x^2 + 3421716480\big)\sin x\,J_0(x) +$$
$$+\big(-1365\,x^9 + 42896\,x^7 - 1830080\,x^5 + 68631552\,x^3 - 2372653056\,x\big)\sin x\,J_1(x) +$$
$$+\big(-1365\,x^9 + 175392\,x^7 - 9228800\,x^5 + 239388672\,x^3 - 3421716480\,x\big)\cos x\,J_0(x) +$$
$$+\big(18928\,x^8 - 1479744\,x^6 + 56919040\,x^4 - 1049063424\,x^2\big)\cos x\,J_1(x)\big]$$

$$\int x^9\,\mathrm{Ci}(x)\,J_0(x)\,dx = \frac{1}{23205}\,\big[\big(185640\,x^8 - 8910720\,x^6 + 213857280\,x^4 - 1710858240\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x) +$$
$$+\big(23205\,x^9 - 1485120\,x^7 + 53464320\,x^5 - 855429120\,x^3 + 3421716480\,x\big)\,\mathrm{Ci}(x)\,J_0(x) +$$
$$+\big(1365\,x^9 - 175392\,x^7 + 9228800\,x^5 - 239388672\,x^3 + 3421716480\,x\big)\sin x\,J_0(x) +$$
$$+\big(-18928\,x^8 + 1479744\,x^6 - 56919040\,x^4 + 1049063424\,x^2\big)\sin x\,J_1(x) +$$
$$+\big(-6643\,x^8 + 252000\,x^6 - 10775040\,x^4 + 330897408\,x^2 + 3421716480\big)\cos x\,J_0(x) +$$
$$+\big(-1365\,x^9 + 42896\,x^7 - 1830080\,x^5 + 68631552\,x^3 - 2372653056\,x\big)\cos x\,J_1(x)\big]$$

1.3.4. $\int x^n \operatorname{erf}(x)\, J_\nu(\alpha x)\, dx$

a) The Case $\alpha = 1$

About the basic integrals

$$F_0(x) = \int_0^x e^{-t^2}\, J_0(t)\, dt \quad \text{and} \quad F_-(x) = \int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t}$$

see page 137 and following.

$$\int \operatorname{erf}(x)\, J_1(x)\, dx = -\operatorname{erf}(x)\, J_0(x) + \frac{2}{\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx$$

$$\int x \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{\sqrt{\pi}}\, J_1(x) + x \operatorname{erf}(x)\, J_1(x) - \frac{1}{\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{1}{\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^2 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{2\sqrt{\pi}} \left[-2x\, J_0(x) + 5 J_1(x)\right] + \operatorname{erf}(x) \left[-x^2\, J_0(x) + 2x\, J_1(x)\right] -$$

$$-\frac{3}{2\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{5}{2\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^3 \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{4\sqrt{\pi}} \left[10x\, J_0(x) + (4x^2 - 19)\, J_1(x)\right] + \operatorname{erf}(x) \left[2x^2\, J_0(x) + (x^3 - 4x)\, J_1(x)\right] +$$

$$+\frac{9}{4\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{19}{4\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^4 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{8\sqrt{\pi}} \left[(-8\,x^3 + 70\,x)\, J_0(x) + (36\,x^2 - 145)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x) \left[(-x^4 + 8\,x^2)\, J_0(x) + (4\,x^3 - 16\,x)\, J_1(x)\right] + \frac{75}{8\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{145}{8\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^5 \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{16\sqrt{\pi}} \left[(72\,x^3 - 538\,x)\, J_0(x) + (16\,x^4 - 268\,x^2 + 1159)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x) \left[(4\,x^4 - 32\,x^2)\, J_0(x) + (x^5 - 16\,x^3 + 64\,x)\, J_1(x)\right] - \frac{621}{16\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{1159}{16\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^6 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{32\sqrt{\pi}} \left[(-32\,x^5 + 792\,x^3 - 6534\,x)\, J_0(x) + (208\,x^4 - 3156\,x^2 + 13977)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x) \left[(-x^6 + 24\,x^4 - 192\,x^2)\, J_0(x) + (6\,x^5 - 96\,x^3 + 384\,x)\, J_1(x)\right] -$$

$$-\frac{7443}{32\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{13977}{32\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^7 \operatorname{erf}(x)\, J_0(x)\, dx =$$

$$= \frac{e^{-x^2}}{64\sqrt{\pi}} \left[(416\,x^5 - 9352\,x^3 + 78706\,x)\, J_0(x) + (64\,x^6 - 2352\,x^4 + 38012\,x^2 - 167803)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x) \left[(6\,x^6 - 144\,x^4 + 1152\,x^2)\, J_0(x) + (x^7 - 36\,x^5 + 576\,x^3 - 2304\,x)\, J_1(x)\right] +$$

$$+\frac{89097}{64\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{167803}{64\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^8 \operatorname{erf}(x)\, J_1(x)\, dx =$$

$$= \frac{e^{-x^2}}{128\sqrt{\pi}} \left[(-128\,x^7 + 6240\,x^5 - 150488\,x^3 + 1258502\,x)\, J_0(x) + (1088\,x^6 - 37264\,x^4 + 609172\,x^2 - 2683961)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x)\ \left[(-x^8+48\,x^6-1152\,x^4+9216\,x^2)\,J_0(x)+(8\,x^7-288\,x^5+4608\,x^3-18432\,x)\,J_1(x)\right]+$$

$$+\frac{1425459}{128\sqrt{\pi}}\int e^{-x^2}\,J_0(x)\,dx-\frac{2683961}{128\sqrt{\pi}}\int\frac{e^{-x^2}\,J_1(x)\,dx}{x}$$

$$\int x^9\operatorname{erf}(x)\,J_0(x)\,dx=\frac{e^{-x^2}}{256\sqrt{\pi}}\ \left[(2176\,x^7-98976\,x^5+2410792\,x^3-20131066\,x)\,J_0(x)+\right.$$

$$\left.+\ (256\,x^8-16576\,x^6+597872\,x^4-9745772\,x^2+42941383)\,J_1(x)\right]+$$

$$+\operatorname{erf}(x)\ \left[(8\,x^8-384\,x^6+9216\,x^4-73728\,x^2)\,J_0(x)+(x^9-64\,x^7+2304\,x^5-36864\,x^3+147456\,x)\,J_1(x)\right]-$$

$$-\frac{22810317}{256\sqrt{\pi}}\int e^{-x^2}\,J_0(x)\,dx+\frac{42941383}{\sqrt{256\pi}}\int\frac{e^{-x^2}\,J_1(x)\,dx}{x}$$

Recurrence relations:

$$\int x^{2n+2}\operatorname{erf}(x)\,J_1(x)\,dx=\frac{4n+1}{2}\int x^{2n}\operatorname{erf}(x)\,J_1(x)\,dx+\frac{4n+5}{2}\int x^{2n+1}\operatorname{erf}(x)\,J_0(x)\,dx-$$

$$-n(2n+1)\int x^{2n-1}\operatorname{erf}(x)\,J_0(x)\,dx-\frac{x^{2n+1}\,e^{-x^2}}{\sqrt{\pi}}\,J_0(x)+\frac{x^{2n}}{2}\,\left[(2n+1-2x^2)\,J_0(x)-x\,J_1(x)\right]\,\operatorname{erf}(x)$$

$$\int x^{2n+1}\operatorname{erf}(x)\,J_0(x)\,dx=-\frac{16n^2+3}{8n+2}\int x^{2n}\operatorname{erf}(x)\,J_1(x)\,dx+\frac{16n^3-36n^2+18n-1}{8n+2}\int x^{2n-2}\operatorname{erf}(x)\,J_1(x)\,dx+$$

$$+\frac{(4n-1)(2n-1)(n-1)}{4n+1}\int x^{2n-3}\operatorname{erf}(x)\,J_0(x)\,dx+\frac{x^{2n-1}\,e^{-x^2}}{(4n+1)\sqrt{\pi}}\,[(4n-1)\,J_0(x)+(4n+1)x\,J_1(x)]+$$

$$+\frac{x^{2n-2}}{8n+2}\,\{[(4n-3)x^2-(2n-1)(4n-1)]\,J_0(x)+[(8n+2)x^3-2n(4n-3)x]\,J_1(x)\}\operatorname{erf}(x)$$

b) The General Case

About the basic integrals

$$F_0(x)=\int_0^x e^{-t^2}\,J_0(\alpha t)\,dt\quad\text{and}\quad F_-(x)=\int_0^x\frac{e^{-t^2}\,J_1(\alpha t)\,dt}{t}$$

see page 146 and following.

$$\int\operatorname{erf}(x)\,J_1(\alpha x)\,dx=-\frac{\operatorname{erf}(x)\,J_0(\alpha x)}{\alpha}+\frac{2}{\alpha\sqrt{\pi}}\int e^{-x^2}\,J_0(\alpha x)\,dx$$

$$\int x\operatorname{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{\sqrt{\pi}\,\alpha}\,J_1(\alpha x)+\frac{x\operatorname{erf}(x)}{\alpha}\,J_1(\alpha x)-\frac{1}{\sqrt{\pi}}\int e^{-x^2}\,J_0(\alpha x)\,dx+\frac{1}{\sqrt{\pi}\,\alpha}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^2\operatorname{erf}(x)\,J_1(\alpha x)\,dx=\frac{e^{-x^2}}{2\sqrt{\pi}\,\alpha^2}\,[-2\alpha x\,J_0(\alpha x)+(\alpha^2+4)\,J_1(\alpha x)]+\frac{\operatorname{erf}(x)}{\alpha^2}\,[-\alpha x^2\,J_0(\alpha x)+2x\,J_1(\alpha x)]-$$

$$-\frac{\alpha^2+2}{2\sqrt{\pi}\,\alpha}\int e^{-x^2}\,J_0(\alpha x)\,dx+\frac{\alpha^2+4}{2\sqrt{\pi}\,\alpha^2}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^3\operatorname{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^3}\,\{2\alpha(\alpha^2+4)x\,J_0(\alpha x)+[4\alpha^2x^2-(\alpha^4+2\,\alpha^2+16)]\,J_1(\alpha x)\}+$$

$$+\frac{\operatorname{erf}(x)}{\alpha^3}\,[2\alpha x^2\,J_0(\alpha x)+(\alpha^2x^3-4x)\,J_1(\alpha x)]+\frac{\alpha^4+8}{4\sqrt{\pi}\,\alpha^2}\int e^{-x^2}\,J_0(\alpha x)\,dx-\frac{\alpha^4+2\,\alpha^2+16}{4\sqrt{\pi}\,\alpha^3}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^4\operatorname{erf}(x)\,J_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^4}\,\{[-8\alpha^3x^3+2(\alpha^5+2\,\alpha^3+32\alpha)x]\,J_0(\alpha x)+[(4\,\alpha^4+32\,\alpha^2)x^2-\alpha^6-16\,\alpha^2-128]\,J_1(\alpha x)\}+$$

$$+\frac{\text{erf}(x)}{\alpha^4}\left[(-\alpha^3x^4+8\,\alpha\,x^2)\,J_0(\alpha x)+(4\,\alpha^2x^3-16\,x)\,J_1(\alpha x)\right]+$$

$$+\frac{\alpha^6-2\,\alpha^4+12\,\alpha^2+64}{8\sqrt{\pi}\,\alpha^3}\int e^{-x^2}\,J_0(\alpha x)\,dx-\frac{\alpha^6+16\,\alpha^2+128}{8\sqrt{\pi}\,\alpha^4}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^5\text{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{16\sqrt{\pi}\,\alpha^5}\,\{[(8\,\alpha^5+64\,\alpha^3)x^3-(2\,\alpha^7-8\,\alpha^5+32\,\alpha^3+512\,\alpha)x]\,J_0(\alpha x)+$$

$$+[16\alpha^4x^4-(4\,\alpha^6+8\,\alpha^4+256\,\alpha^2)x^2+\alpha^8-6\,\alpha^6+12\,\alpha^4+128\,\alpha^2+1024]\,J_1(\alpha x)\}+$$

$$+\frac{\text{erf}(x)}{\alpha^5}\left[(4\,\alpha^3x^4-32\,\alpha\,x^2)\,J_0(\alpha x)+(\alpha^4x^5-16\,\alpha^2x^3+64\,x)\,J_1(\alpha x)\right]-$$

$$-\frac{\alpha^8-8\,\alpha^6+20\,\alpha^4+96\,\alpha^2+512}{16\sqrt{\pi}\,\alpha^4}\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^8-6\,\alpha^6+12\,\alpha^4+128\,\alpha^2+1024}{16\sqrt{\pi}\,\alpha^5}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^6\,\text{erf}(x)\,J_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{32\sqrt{\pi}\,\alpha^6}\,\{[-32\alpha^5x^5+(8\,\alpha^7+16\,\alpha^5+768\,\alpha^3)x^3-(2\,\alpha^9-20\,\alpha^7+24\,\alpha^5+384\,\alpha^3+6144\,\alpha)x]\,J_0(\alpha x)+$$

$$+[(16\,\alpha^6+192\,\alpha^4)x^4-(4\,\alpha^8-16\,\alpha^6+96\,\alpha^4+3072\,\alpha^2)x^2+\alpha^{10}-12\,\alpha^8+20\,\alpha^6+144\,\alpha^4+1536\,\alpha^2+12288]\,J_1(\alpha x)\}+$$

$$+\frac{\text{erf}(x)}{\alpha^6}\left[-(\alpha^5x^6-24\,\alpha^3x^4+192\,\alpha\,x^2)\,J_0(\alpha x)+(6\,\alpha^4x^5-96\,\alpha^2x^3+384\,x)\,J_1(\alpha x)\right]-$$

$$-\frac{\alpha^{10}-14\,\alpha^8+40\,\alpha^6+120\,\alpha^4+1152\,\alpha^2+6144}{32\sqrt{\pi}\,\alpha^5}\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^{10}-12\,\alpha^8+20\,\alpha^6+144\,\alpha^4+1536\,\alpha^2+12288}{32\sqrt{\pi}\,\alpha^6}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^7\,\text{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{64\sqrt{\pi}\,\alpha^7}\,\{[(32\,\alpha^7+384\,\alpha^5)x^5-(8\,\alpha^9-64\,\alpha^7+192\,\alpha^5+9216\,\alpha^3)x^3+$$

$$+(2\,\alpha^{11}-40\,\alpha^9+120\,\alpha^7+288\,\alpha^5+4608\,\alpha^3+73728\,\alpha)x]\,J_0(\alpha x)+[64\alpha^6x^6-(16\,\alpha^8+32\,\alpha^6+2304\,\alpha^4)x^4+$$

$$+(4\,\alpha^{10}-56\,\alpha^8+48\,\alpha^6+1152\,\alpha^4+36864\,\alpha^2)x^2-\alpha^{12}+22\,\alpha^{10}-88\,\alpha^8-120\,\alpha^6-1728\,\alpha^4-18432\,\alpha^2-147456]\,J_1(\alpha x)\}+$$

$$+\frac{\text{erf}(x)}{\alpha^7}\left[(6\,\alpha^5x^6-144\,\alpha^3x^4+1152\,\alpha\,x^2)\,J_0(\alpha x)+(\alpha^6x^7-36\,\alpha^4x^5+576\,\alpha^2x^3-2304\,x)\,J_1(\alpha x)\right]+$$

$$+\frac{\alpha^{12}-24\,\alpha^{10}+128\,\alpha^8+1440\,\alpha^4+13824\,\alpha^2+73728}{64\sqrt{\pi}\,\alpha^6}\int e^{-x^2}\,J_0(\alpha x)\,dx-$$

$$-\frac{\alpha^{12}-22\,\alpha^{10}+88\,\alpha^8+120\,\alpha^6+1728\,\alpha^4+18432\,\alpha^2+147456}{64\sqrt{\pi}\,\alpha^7}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^8\,\text{erf}(x)\,J_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{128\sqrt{\pi}\,\alpha^8}\,\{[-128\alpha^7x^7+(32\,\alpha^9+64\,\alpha^7+6144\,\alpha^5)x^5-(8\,\alpha^{11}-144\,\alpha^9+96\,\alpha^7+3072\,\alpha^5+147456\,\alpha^3)x^3+$$

$$+(2\,\alpha^{13}-60\,\alpha^{11}+336\,\alpha^9+240\,\alpha^7+4608\,\alpha^5+73728\,\alpha^3+1179648\,\alpha)x]\,J_0(\alpha x)+$$

$$+[(64\,\alpha^8+1024\,\alpha^6)x^6-(16\,\alpha^{10}-128\,\alpha^8+512\,\alpha^6+36864\,\alpha^4)x^4+$$

$$+(4\,\alpha^{12}-96\,\alpha^{10}+240\,\alpha^8+768\,\alpha^6+18432\,\alpha^4+589824\,\alpha^2)x^2-$$

$$-\alpha^{14}+32\,\alpha^{12}-216\,\alpha^{10}-1920\,\alpha^6-27648\,\alpha^4-294912\,\alpha^2-2359296]\,J_1(\alpha x)\}+$$

$$+\frac{\text{erf}(x)}{\alpha^8}\left[-(\alpha^7x^8-48\,\alpha^5x^6+1152\,\alpha^3x^4-9216\,\alpha\,x^2)\,J_0(\alpha x)+(8\,\alpha^6x^7-288\,\alpha^4x^5+4608\,\alpha^2x^3-18432\,x)\,J_1(\alpha x)\right]+$$

$$+\frac{\alpha^{14}-34\,\alpha^{12}+276\,\alpha^{10}-336\,\alpha^8+1680\,\alpha^6+23040\,\alpha^4+221184\,\alpha^2+1179648}{128\sqrt{\pi}\,\alpha^7}\int e^{-x^2}\,J_0(\alpha x)\,dx-$$

$$-\frac{\alpha^{14}-32\,\alpha^{12}+216\,\alpha^{10}+1920\,\alpha^6+27648\,\alpha^4+294912\,\alpha^2+2359296}{128\sqrt{\pi}\,\alpha^8}\int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^9\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{256\sqrt{\pi}\,\alpha^9}\,\{[(128\,\alpha^9+2048\,\alpha^7)x^7-(32\,\alpha^{11}-384\,\alpha^9+1024\,\alpha^7+98304\,\alpha^5)x^5+$$

$$+(8\,\alpha^{13}-256\,\alpha^{11}+1056\,\alpha^9+1536\,\alpha^7+49152\,\alpha^5+2359296\,\alpha^3)x^3-$$

$$-(2\,\alpha^{15}-88\,\alpha^{13}+912\,\alpha^{11}-1344\,\alpha^9+3840\,\alpha^7+73728\,\alpha^5+1179648\,\alpha^3+18874368\,\alpha)x]\,J_0(\alpha x)+$$

$$+[256\alpha^8x^8-(64\,\alpha^{10}+128\,\alpha^8+16384\,\alpha^6)x^6+(16\,\alpha^{12}-352\,\alpha^{10}+192\,\alpha^8+8192\,\alpha^6+589824\,\alpha^4)x^4-$$

$$-(4\,\alpha^{14}-152\,\alpha^{12}+1056\,\alpha^{10}+480\,\alpha^8+12288\,\alpha^6+294912\,\alpha^4+9437184\,\alpha^2)x^2+$$

$$+\alpha^{16}-46\,\alpha^{14}+532\,\alpha^{12}-1200\,\alpha^{10}+1680\,\alpha^8+30720\,\alpha^6+442368\,\alpha^4+4718592\,\alpha^2+37748736]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^9}\,[(8\,\alpha^7x^8-384\,\alpha^5x^6+9216\,\alpha^3x^4-73728\,\alpha\,x^2)\,J_0(\alpha x)+$$

$$+(\alpha^8x^9-64\,\alpha^6x^7+2304\,\alpha^4x^5-36864\,\alpha^2x^3+147456\,x)\,J_1(\alpha x)]-$$

$$-\frac{\alpha^{16}-48\,\alpha^{14}+620\,\alpha^{12}-2112\,\alpha^{10}+3024\,\alpha^8+26880\,\alpha^6+368640\,\alpha^4+3538944\,\alpha^2+18874368}{256\sqrt{\pi}\,\alpha^8}\cdot$$

$$\cdot\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^{16}-46\,\alpha^{14}+532\,\alpha^{12}-1200\,\alpha^{10}+1680\,\alpha^8+30720\,\alpha^6+442368\,\alpha^4+4718592\,\alpha^2+37748736}{256\sqrt{\pi}\,\alpha^9}\cdot$$

$$\cdot\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

<u>Recurrence relations:</u>

$$\int x^{2n+2}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx=\frac{\alpha^2+4n+4}{2\alpha}\int x^{2n+1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx-\frac{n(2n+1)}{\alpha}\int x^{2n-1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx+$$

$$+\frac{4n+1}{2}\int x^{2n}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx-\frac{x^{2n+1}\,e^{-x^2}}{\alpha\sqrt{\pi}}\,J_0(\alpha x)+\frac{x^{2n}}{2\alpha}\,\left[(2n+1-2x^2)\,J_0(\alpha x)-\alpha x\,J_1(\alpha x)\right]\,\mathrm{erf}(x)$$

$$\int x^{2n+1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx=-\frac{\alpha^4+2\,\alpha^2+16\,n^2}{2\,(4\,n+\alpha^2)\,\alpha}\int x^{2n}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx+$$

$$+\frac{16n^3-12(\alpha^2+2)n^2+(10\alpha+8)n-\alpha^2}{2\alpha(4n+\alpha^2)}\int x^{2n-2}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx+$$

$$+\frac{(4n-1)(2n-1)(n-1)}{4n+\alpha^2}\int x^{2n-3}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx+$$

$$+\frac{x^{2n-1}\,e^{-x^2}}{\alpha(4n+\alpha^2)\sqrt{\pi}}\,\left[(4n-1)\alpha\,J_0(\alpha x)+(4n+\alpha^2)x\,J_1(\alpha x)\right]+$$

$$+\frac{x^{2n-2}\,\mathrm{erf}(x)}{2\alpha(4n+\alpha^2)}\,\left\{[\alpha(4n-\alpha^2-2)x^2-\alpha(4n-1)(2n-1)]\,J_0(\alpha x)+[2(4n+\alpha^2)x^3+2n(2-4n+\alpha^2)]\,J_1(\alpha x)\right\}$$

2. Products of two Bessel functions

2.1. Bessel Functions with the the same Argument x :

See also [10], 3. .

2.1.1. Integrals of the type $\int x^{2n+1} Z_\nu^2(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p=1,2$.

$$\int \frac{J_1^2(x)}{x}\,dx = -\frac{1}{2}\left[J_0^2(x)+J_1^2(x)\right]$$

$$\int \frac{I_1^2(x)}{x}\,dx = \frac{1}{2}\left[I_0^2(x)-I_1^2(x)\right]$$

$$\int \frac{K_1^2(x)}{x}\,dx = \frac{1}{2}\left[K_0^2(x)-K_1^2(x)\right]$$

$$\int xJ_0^2(x)\,dx = \frac{x^2}{2}\left[J_0^2(x)+J_1^2(x)\right]$$

$$\int xJ_1^2(x)\,dx = \frac{x}{2}\left[xJ_0^2(x)+xJ_1^2(x)-2J_0(x)\cdot J_1(x)\right]$$

$$\int xI_0^2(x)\,dx = \frac{x^2}{2}\left[I_0^2(x)-I_1^2(x)\right]$$

$$\int xI_1^2(x)\,dx = \frac{x}{2}\left[xI_1^2(x)-xI_0^2(x)+2I_0(x)\cdot I_1(x)\right] \qquad *E*$$

$$\int xK_0^2(x)\,dx = \frac{x^2}{2}\left[K_0^2(x)-K_1^2(x)\right]$$

$$\int xK_1^2(x)\,dx = \frac{x}{2}\left[xK_1^2(x)-xK_0^2(x)-2K_0(x)\cdot K_1(x)\right] \qquad *E*$$

$$\int x^3J_0^2(x)\,dx = \frac{x^4}{6}J_0^2(x)+\frac{x^3}{3}J_0(x)J_1(x)+\left(\frac{x^4}{6}-\frac{x^2}{3}\right)J_1^2(x)$$

$$\int x^3J_1^2(x)\,dx = \frac{x^4}{6}J_0^2(x)-\frac{2x^3}{3}J_0(x)J_1(x)+\left(\frac{x^4}{6}+\frac{2x^2}{3}\right)J_1^2(x)$$

$$\int x^3I_0^2(x)\,dx = \frac{x^4}{6}I_0^2(x)+\frac{x^3}{3}I_0(x)I_1(x)-\left(\frac{x^4}{6}+\frac{x^2}{3}\right)I_1^2(x)$$

$$\int x^3I_1^2(x)\,dx = -\frac{x^4}{6}I_0^2(x)+\frac{2x^3}{3}I_0(x)I_1(x)+\left(\frac{x^4}{6}-\frac{2x^2}{3}\right)I_1^2(x)$$

$$\int x^3K_0^2(x)\,dx = \frac{x^4}{6}K_0^2(x)-\frac{x^3}{3}K_0(x)K_1(x)-\left(\frac{x^4}{6}+\frac{x^2}{3}\right)K_1^2(x)$$

$$\int x^3K_1^2(x)\,dx = -\frac{x^4}{6}K_0^2(x)-\frac{2x^3}{3}K_0(x)K_1(x)+\left(\frac{x^4}{6}-\frac{2x^2}{3}\right)K_1^2(x)$$

$$\int x^5J_0^2(x)\,dx = \left(\frac{x^6}{10}+\frac{4x^4}{15}\right)J_0^2(x)+\left(\frac{2x^5}{5}-\frac{16x^3}{15}\right)J_0(x)J_1(x)+\left(\frac{x^6}{10}-\frac{8x^4}{15}+\frac{16x^2}{15}\right)J_1^2(x)$$

$$\int x^5J_1^2(x)\,dx = \left(\frac{x^6}{10}-\frac{2x^4}{5}\right)J_0^2(x)+\left(-\frac{3x^5}{5}+\frac{8x^3}{5}\right)J_0(x)J_1(x)+\left(\frac{x^6}{10}+\frac{4x^4}{5}-\frac{8x^2}{5}\right)J_1^2(x)$$

$$\int x^5I_0^2(x)\,dx = \left(\frac{x^6}{10}-\frac{4x^4}{15}\right)I_0^2(x)+\left(\frac{2x^5}{5}+\frac{16x^3}{15}\right)I_0(x)I_1(x)-\left(\frac{x^6}{10}+\frac{8x^4}{15}+\frac{16x^2}{15}\right)I_1^2(x)$$

$$\int x^5I_1^2(x)\,dx = -\left(\frac{x^6}{10}+\frac{2x^4}{5}\right)I_0^2(x)+\left(\frac{3x^5}{5}+\frac{8x^3}{5}\right)I_0(x)I_1(x)+\left(\frac{x^6}{10}-\frac{4x^4}{5}-\frac{8x^2}{5}\right)I_1^2(x)$$

$$\int x^5 K_0^2(x)\,dx = \left(\frac{x^6}{10} - \frac{4x^4}{15}\right) K_0^2(x) - \left(\frac{2x^5}{5} + \frac{16x^3}{15}\right) K_0(x)K_1(x) - \left(\frac{x^6}{10} + \frac{8x^4}{15} + \frac{16x^2}{15}\right) K_1^2(x)$$

$$\int x^5 K_1^2(x)\,dx = -\left(\frac{x^6}{10} + \frac{2x^4}{5}\right) K_0^2(x) - \left(\frac{3x^5}{5} + \frac{8x^3}{5}\right) K_0(x)K_1(x) + \left(\frac{x^6}{10} - \frac{4x^4}{5} - \frac{8x^2}{5}\right) K_1^2(x)$$

$$\begin{aligned}\int x^7 J_0^2(x)\,dx &= \left(\frac{x^8}{14} + \frac{18x^6}{35} - \frac{72x^4}{35}\right) J_0^2(x) + \left(\frac{3x^7}{7} - \frac{108x^5}{35} + \frac{288x^3}{35}\right) J_0(x)J_1(x) + \\ &+ \left(\frac{x^8}{14} - \frac{27x^6}{35} + \frac{144x^4}{35} - \frac{288x^2}{35}\right) J_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^7 J_1^2(x)\,dx &= \left(\frac{x^8}{14} - \frac{24x^6}{35} + \frac{96x^4}{35}\right) J_0^2(x) + \left(-\frac{4x^7}{7} + \frac{144x^5}{35} - \frac{384x^3}{35}\right) J_0(x)J_1(x) + \\ &+ \left(\frac{x^8}{14} + \frac{36x^6}{35} - \frac{192x^4}{35} + \frac{384x^2}{35}\right) J_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^7 I_0^2(x)\,dx &= \left(\frac{x^8}{14} - \frac{18x^6}{35} - \frac{72x^4}{35}\right) I_0^2(x) + \left(\frac{3x^7}{7} + \frac{108x^5}{35} + \frac{288x^3}{35}\right) I_0(x)I_1(x) - \\ &- \left(\frac{x^8}{14} + \frac{27x^6}{35} + \frac{144x^4}{35} + \frac{288x^2}{35}\right) I_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^7 I_1^2(x)\,dx &= -\left(\frac{x^8}{14} + \frac{24x^6}{35} + \frac{96x^4}{35}\right) I_0^2(x) + \left(\frac{4x^7}{7} + \frac{144x^5}{35} + \frac{384x^3}{35}\right) I_0(x)I_1(x) + \\ &+ \left(\frac{x^8}{14} - \frac{36x^6}{35} - \frac{192x^4}{35} - \frac{384x^2}{35}\right) I_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^7 K_0^2(x)\,dx &= \left(\frac{x^8}{14} - \frac{18x^6}{35} - \frac{72x^4}{35}\right) K_0^2(x) - \left(\frac{3x^7}{7} + \frac{108x^5}{35} + \frac{288x^3}{35}\right) K_0(x)K_1(x) - \\ &- \left(\frac{x^8}{14} + \frac{27x^6}{35} + \frac{144x^4}{35} + \frac{288x^2}{35}\right) K_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^7 K_1^2(x)\,dx &= -\left(\frac{x^8}{14} + \frac{24x^6}{35} + \frac{96x^4}{35}\right) K_0^2(x) - \left(\frac{4x^7}{7} + \frac{144x^5}{35} + \frac{384x^3}{35}\right) K_0(x)K_1(x) + \\ &+ \left(\frac{x^8}{14} - \frac{36x^6}{35} - \frac{192x^4}{35} - \frac{384x^2}{35}\right) K_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^9 J_0^2(x)\,dx &= \left(\frac{x^{10}}{18} + \frac{16x^8}{21} - \frac{256x^6}{35} + \frac{1024x^4}{35}\right) J_0^2(x) + \\ &+ \left(\frac{4x^9}{9} - \frac{128x^7}{21} + \frac{1536x^5}{35} - \frac{4096x^3}{35}\right) J_0(x)J_1(x) + \\ &+ \left(\frac{x^{10}}{18} - \frac{64x^8}{63} + \frac{384x^6}{35} - \frac{2048x^4}{35} + \frac{4096x^2}{35}\right) J_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^9 J_1^2(x)\,dx &= \left(\frac{x^{10}}{18} - \frac{20x^8}{21} + \frac{64x^6}{7} - \frac{256x^4}{7}\right) J_0^2(x) + \\ &+ \left(-\frac{5x^9}{9} + \frac{160x^7}{21} - \frac{384x^5}{7} + \frac{1024x^3}{7}\right) J_0(x)J_1(x) + \\ &+ \left(\frac{x^{10}}{18} + \frac{80x^8}{63} - \frac{96x^6}{7} + \frac{512x^4}{7} - \frac{1024x^2}{7}\right) J_1^2(x)\end{aligned}$$

$$\begin{aligned}\int x^9 I_0^2(x)\,dx &= \left(\frac{x^{10}}{18} - \frac{16x^8}{21} - \frac{256x^6}{35} - \frac{1024x^4}{35}\right) I_0^2(x) + \\ &+ \left(\frac{4x^9}{9} + \frac{128x^7}{21} + \frac{1536x^5}{35} + \frac{4096x^3}{35}\right) I_0(x)I_1(x) -\end{aligned}$$

$$-\left(\frac{x^{10}}{18}+\frac{64x^8}{63}+\frac{384x^6}{35}+\frac{2048x^4}{35}+\frac{4096x^2}{35}\right)I_1^2(x)$$

$$\int x^9 I_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18}+\frac{20x^8}{21}+\frac{64x^6}{7}+\frac{256x^4}{7}\right)I_0^2(x)+$$
$$+\left(\frac{5x^9}{9}+\frac{160x^7}{21}+\frac{384x^5}{7}+\frac{1024x^3}{7}\right)I_0(x)I_1(x)+$$
$$+\left(\frac{x^{10}}{18}-\frac{80x^8}{63}-\frac{96x^6}{7}-\frac{512x^4}{7}-\frac{1024x^2}{7}\right)I_1^2(x)$$

$$\int x^9 K_0^2(x)\,dx \;=\; \left(\frac{x^{10}}{18}-\frac{16x^8}{21}-\frac{256x^6}{35}-\frac{1024x^4}{35}\right)K_0^2(x)-$$
$$-\left(\frac{4x^9}{9}+\frac{128x^7}{21}+\frac{1536x^5}{35}+\frac{4096x^3}{35}\right)K_0(x)K_1(x)-$$
$$-\left(\frac{x^{10}}{18}+\frac{64x^8}{63}+\frac{384x^6}{35}+\frac{2048x^4}{35}+\frac{4096x^2}{35}\right)K_1^2(x)$$

$$\int x^9 K_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18}+\frac{20x^8}{21}+\frac{64x^6}{7}+\frac{256x^4}{7}\right)K_0^2(x)-$$
$$-\left(\frac{5x^9}{9}+\frac{160x^7}{21}+\frac{384x^5}{7}+\frac{1024x^3}{7}\right)K_0(x)K_1(x)+$$
$$+\left(\frac{x^{10}}{18}-\frac{80x^8}{63}-\frac{96x^6}{7}-\frac{512x^4}{7}-\frac{1024x^2}{7}\right)K_1^2(x)$$

Let

$$\int x^m\cdot J_0^2(x)\,dx = A_m(x)\cdot J_0^2(x)+B_m(x)\cdot J_0(x)\cdot J_1(x)+C_m(x)\cdot J_1^2(x)\;,$$
$$\int x^m\cdot J_1^2(x)\,dx = D_m(x)\cdot J_0^2(x)+E_m(x)\cdot J_0(x)\cdot J_1(x)+F_m(x)\cdot J_1^2(x)\;,$$
$$\int x^m\cdot I_0^2(x)\,dx = A_m^*(x)\cdot I_0^2(x)+B_m^*(x)\cdot I_0(x)\cdot I_1(x)+C_m^*(x)\cdot I_1^2(x)\;,$$
$$\int x^m\cdot I_1^2(x)\,dx = D_m^*(x)\cdot I_0^2(x)+E_m^*(x)\cdot I_0(x)\cdot I_1(x)+F_m^*(x)\cdot I_1^2(x)$$

and

$$\int x^m\cdot K_0^2(x)\,dx = A_m^*(x)\cdot K_0^2(x)-B_m^*(x)\cdot K_0(x)\cdot K_1(x)+C_m^*(x)\cdot K_1^2(x)\;,$$
$$\int x^m\cdot K_1^2(x)\,dx = D_m^*(x)\cdot K_0^2(x)-E_m^*(x)\cdot K_0(x)\cdot K_1(x)+F_m^*(x)\cdot K_1^2(x)\;,$$

then holds

$$A_{11}=\frac{1}{22}\,x^{12}+\frac{100}{99}\,x^{10}-\frac{4000}{231}\,x^8+\frac{12800}{77}\,x^6-\frac{51200}{77}\,x^4$$
$$B_{11}=\frac{5}{11}\,x^{11}-\frac{1000}{99}\,x^9+\frac{32000}{231}\,x^7-\frac{76800}{77}\,x^5+\frac{204800}{77}\,x^3$$
$$C_{11}=\frac{1}{22}\,x^{12}-\frac{125}{99}\,x^{10}+\frac{16000}{693}\,x^8-\frac{19200}{77}\,x^6+\frac{102400}{77}\,x^4-\frac{204800}{77}\,x^2$$
$$D_{11}=\frac{1}{22}\,x^{12}-\frac{40}{33}\,x^{10}+\frac{1600}{77}\,x^8-\frac{15360}{77}\,x^6+\frac{61440}{77}\,x^4$$
$$E_{11}=-\frac{6}{11}\,x^{11}+\frac{400}{33}\,x^9-\frac{12800}{77}\,x^7+\frac{92160}{77}\,x^5-\frac{245760}{77}\,x^3$$
$$F_{11}=\frac{1}{22}\,x^{12}+\frac{50}{33}\,x^{10}-\frac{6400}{231}\,x^8+\frac{23040}{77}\,x^6-\frac{122880}{77}\,x^4+\frac{245760}{77}\,x^2$$
$$A_{11}^*=\frac{1}{22}\,x^{12}-\frac{100}{99}\,x^{10}-\frac{4000}{231}\,x^8-\frac{12800}{77}\,x^6-\frac{51200}{77}\,x^4$$

$$B_{11}^* = \frac{5}{11}\,x^{11} + \frac{1000}{99}\,x^9 + \frac{32000}{231}\,x^7 + \frac{76800}{77}\,x^5 + \frac{204800}{77}\,x^3$$
$$C_{11}^* = -\frac{1}{22}\,x^{12} - \frac{125}{99}\,x^{10} - \frac{16000}{693}\,x^8 - \frac{19200}{77}\,x^6 - \frac{102400}{77}\,x^4 - \frac{204800}{77}\,x^2$$
$$D_{11}^* = -\frac{1}{22}\,x^{12} - \frac{40}{33}\,x^{10} - \frac{1600}{77}\,x^8 - \frac{15360}{77}\,x^6 - \frac{61440}{77}\,x^4$$
$$E_{11}^* = \frac{6}{11}\,x^{11} + \frac{400}{33}\,x^9 + \frac{12800}{77}\,x^7 + \frac{92160}{77}\,x^5 + \frac{245760}{77}\,x^3$$
$$F_{11}^* = \frac{1}{22}\,x^{12} - \frac{50}{33}\,x^{10} - \frac{6400}{231}\,x^8 - \frac{23040}{77}\,x^6 - \frac{122880}{77}\,x^4 - \frac{245760}{77}\,x^2$$

$$A_{13} = \frac{1}{26}\,x^{14} + \frac{180}{143}\,x^{12} - \frac{4800}{143}\,x^{10} + \frac{576000}{1001}\,x^8 - \frac{5529600}{1001}\,x^6 + \frac{22118400}{1001}\,x^4$$
$$B_{13} = \frac{6}{13}\,x^{13} - \frac{2160}{143}\,x^{11} + \frac{48000}{143}\,x^9 - \frac{4608000}{1001}\,x^7 + \frac{33177600}{1001}\,x^5 - \frac{88473600}{1001}\,x^3$$
$$C_{13} = \frac{1}{26}\,x^{14} - \frac{216}{143}\,x^{12} + \frac{6000}{143}\,x^{10} - \frac{768000}{1001}\,x^8 + \frac{8294400}{1001}\,x^6 - \frac{44236800}{1001}\,x^4 + \frac{88473600}{1001}\,x^2$$
$$D_{13} = \frac{1}{26}\,x^{14} - \frac{210}{143}\,x^{12} + \frac{5600}{143}\,x^{10} - \frac{96000}{143}\,x^8 + \frac{921600}{143}\,x^6 - \frac{3686400}{143}\,x^4$$
$$E_{13} = -\frac{7}{13}\,x^{13} + \frac{2520}{143}\,x^{11} - \frac{56000}{143}\,x^9 + \frac{768000}{143}\,x^7 - \frac{5529600}{143}\,x^5 + \frac{14745600}{143}\,x^3$$
$$F_{13} = \frac{1}{26}\,x^{14} + \frac{252}{143}\,x^{12} - \frac{7000}{143}\,x^{10} + \frac{128000}{143}\,x^8 - \frac{1382400}{143}\,x^6 + \frac{7372800}{143}\,x^4 - \frac{14745600}{143}\,x^2$$
$$A_{13}^* = \frac{1}{26}\,x^{14} - \frac{180}{143}\,x^{12} - \frac{4800}{143}\,x^{10} - \frac{576000}{1001}\,x^8 - \frac{5529600}{1001}\,x^6 - \frac{22118400}{1001}\,x^4$$
$$B_{13}^* = \frac{6}{13}\,x^{13} + \frac{2160}{143}\,x^{11} + \frac{48000}{143}\,x^9 + \frac{4608000}{1001}\,x^7 + \frac{33177600}{1001}\,x^5 + \frac{88473600}{1001}\,x^3$$
$$C_{13}^* = -\frac{1}{26}\,x^{14} - \frac{216}{143}\,x^{12} - \frac{6000}{143}\,x^{10} - \frac{768000}{1001}\,x^8 - \frac{8294400}{1001}\,x^6 - \frac{44236800}{1001}\,x^4 - \frac{88473600}{1001}\,x^2$$
$$D_{13}^* = -\frac{1}{26}\,x^{14} - \frac{210}{143}\,x^{12} - \frac{5600}{143}\,x^{10} - \frac{96000}{143}\,x^8 - \frac{921600}{143}\,x^6 - \frac{3686400}{143}\,x^4$$
$$E_{13}^* = \frac{7}{13}\,x^{13} + \frac{2520}{143}\,x^{11} + \frac{56000}{143}\,x^9 + \frac{768000}{143}\,x^7 + \frac{5529600}{143}\,x^5 + \frac{14745600}{143}\,x^3$$
$$F_{13}^* = \frac{1}{26}\,x^{14} - \frac{252}{143}\,x^{12} - \frac{7000}{143}\,x^{10} - \frac{128000}{143}\,x^8 - \frac{1382400}{143}\,x^6 - \frac{7372800}{143}\,x^4 - \frac{14745600}{143}\,x^2$$

$$A_{15} = \frac{1}{30}\,x^{16} + \frac{98}{65}\,x^{14} - \frac{8232}{143}\,x^{12} + \frac{219520}{143}\,x^{10} - \frac{3763200}{143}\,x^8 + \frac{36126720}{143}\,x^6 - \frac{144506880}{143}\,x^4$$
$$B_{15} = \frac{7}{15}\,x^{15} - \frac{1372}{65}\,x^{13} + \frac{98784}{143}\,x^{11} - \frac{2195200}{143}\,x^9 + \frac{30105600}{143}\,x^7 - \frac{216760320}{143}\,x^5 + \frac{578027520}{143}\,x^3$$
$$C_{15} = \frac{x^{16}}{30} - \frac{343}{195}\,x^{14} + \frac{49392}{715}\,x^{12} - \frac{274400}{143}\,x^{10} + \frac{5017600}{143}\,x^8 - \frac{54190080}{143}\,x^6 + \frac{289013760}{143}\,x^4 - \frac{578027520}{143}\,x^2$$
$$D_{15} = \frac{x^{16}}{30} - \frac{112}{65}\,x^{14} + \frac{9408}{143}\,x^{12} - \frac{250880}{143}\,x^{10} + \frac{4300800}{143}\,x^8 - \frac{41287680}{143}\,x^6 + \frac{165150720}{143}\,x^4$$
$$E_{15} = -\frac{8}{15}\,x^{15} + \frac{1568}{65}\,x^{13} - \frac{112896}{143}\,x^{11} + \frac{2508800}{143}\,x^9 - \frac{34406400}{143}\,x^7 + \frac{247726080}{143}\,x^5 - \frac{660602880}{143}\,x^3$$
$$F_{15} = \frac{x^{16}}{30} + \frac{392}{195}\,x^{14} - \frac{56448}{715}\,x^{12} + \frac{313600}{143}\,x^{10} - \frac{5734400}{143}\,x^8 + \frac{61931520}{143}\,x^6 - \frac{330301440}{143}\,x^4 + \frac{660602880}{143}\,x^2$$
$$A_{15}^* = \frac{x^{16}}{30} - \frac{98}{65}\,x^{14} - \frac{8232}{143}\,x^{12} - \frac{219520}{143}\,x^{10} - \frac{3763200}{143}\,x^8 - \frac{36126720}{143}\,x^6 - \frac{144506880}{143}\,x^4$$

$$B^*_{15} = \frac{7}{15}\,x^{15} + \frac{1372}{65}\,x^{13} + \frac{98784}{143}\,x^{11} + \frac{2195200}{143}\,x^{9} + \frac{30105600}{143}\,x^{7} + \frac{216760320}{143}\,x^{5} + \frac{578027520}{143}\,x^{3}$$

$$C^*_{15} = -\frac{x^{16}}{30} - \frac{343}{195}\,x^{14} - \frac{49392}{715}\,x^{12} - \frac{274400}{143}\,x^{10} - \frac{5017600}{143}\,x^{8} - \frac{54190080}{143}\,x^{6} - \frac{289013760}{143}\,x^{4} - \frac{578027520}{143}\,x^{2}$$

$$D^*_{15} = -\frac{x^{16}}{30} - \frac{112}{65}\,x^{14} - \frac{9408}{143}\,x^{12} - \frac{250880}{143}\,x^{10} - \frac{4300800}{143}\,x^{8} - \frac{41287680}{143}\,x^{6} - \frac{165150720}{143}\,x^{4}$$

$$E^*_{15} = \frac{8}{15}\,x^{15} + \frac{1568}{65}\,x^{13} + \frac{112896}{143}\,x^{11} + \frac{2508800}{143}\,x^{9} + \frac{34406400}{143}\,x^{7} + \frac{247726080}{143}\,x^{5} + \frac{660602880}{143}\,x^{3}$$

$$F^*_{15} = \frac{x^{16}}{30} - \frac{392}{195}\,x^{14} - \frac{56448}{715}\,x^{12} - \frac{313600}{143}\,x^{10} - \frac{5734400}{143}\,x^{8} - \frac{61931520}{143}\,x^{6} - \frac{330301440}{143}\,x^{4} - \frac{660602880}{143}\,x^{2}$$

Recurrence Formulas:

$$\int x^{2n+1} J_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2+2n^2)J_0^2(x) + x^2 J_1^2(x) + 2nx\,J_0(x)J_1(x)\right\} - \frac{2n^3}{2n+1}\int x^{2n-1}J_0^2(x)\,dx$$

$$\int x^{2n+1} J_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{x^2\,J_0^2(x) + \left[x^2+2n(n+1)\right]\,J_1^2(x) - 2(n+1)xJ_0(x)J_1(x)\right\} - \frac{2n(n^2-1)}{2n+1}\int x^{2n-1}J_1^2(x)\,dx$$

$$\int x^{2n+1} I_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)I_0^2(x) - x^2 I_1^2(x) + 2nx\,I_0(x)I_1(x)\right\} + \frac{2n^3}{2n+1}\int x^{2n-1}I_0^2(x)\,dx$$

$$\int x^{2n+1} I_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{-x^2\,I_0^2(x) + \left[x^2-2n(n+1)\right]\,I_1^2(x) + 2(n+1)xI_0(x)I_1(x)\right\} + \frac{2n(n^2-1)}{2n+1}\int x^{2n-1}I_1^2(x)\,dx$$

$$\int x^{2n+1} K_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)K_0^2(x) - x^2 K_1^2(x) - 2nx\,K_0(x)K_1(x)\right\} + \frac{2n^3}{2n+1}\int x^{2n-1}K_0^2(x)\,dx$$

$$\int x^{2n+1} K_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{-x^2\,K_0^2(x) + \left[x^2-2n(n+1)\right]\,K_1^2(x) - 2(n+1)xK_0(x)K_1(x)\right\} + \frac{2n(n^2-1)}{2n+1}\int x^{2n-1}K_1^2(x)\,dx$$

2.1.2. Integrals of the type $\int x^{-2n} Z_\nu^2(x)\,dx$

See also [4], 1.8.3.
Concerning the case $x^{+2n} Z_\nu^2(x)$ see 2.1.3., p. 195 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p=1,2$.

$$\int \frac{J_0^2(x)}{x^2}\,dx = -\left(2x+\frac{1}{x}\right) J_0^2(x) + 2\,J_0(x)\,J_1(x) - 2x\,J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^2}\,dx = \left(2x-\frac{1}{x}\right) I_0^2(x) - 2\,I_0(x)\,I_1(x) - 2x\,I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^2}\,dx = \left(2x-\frac{1}{x}\right) K_0^2(x) + 2\,K_0(x)\,K_1(x) - 2x\,K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,J_0^2(x) - \frac{2}{3}\,J_0(x)\,J_1(x) + \left(\frac{2x}{3}-\frac{1}{3x}\right) J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,I_0^2(x) - \frac{2}{3}\,I_0(x)\,I_1(x) - \left(\frac{2x}{3}+\frac{1}{3x}\right) I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,K_0^2(x) + \frac{2}{3}\,K_0(x)\,K_1(x) - \left(\frac{2x}{3}+\frac{1}{3x}\right) K_1^2(x)$$

$$\int \frac{J_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4+6\,x^2-9)\,J_0^2(x) + \frac{1}{27x^2}\,(-16\,x^2+6)\,J_0(x)\,J_1(x) + \frac{1}{27x}\,(16\,x^2-2)\,J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4-6\,x^2-9)\,I_0^2(x) - \frac{1}{27x^2}\,(16\,x^2+6)\,I_0(x)\,I_1(x) - \frac{1}{27x}\,(16\,x^2+2)\,I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4-6\,x^2-9)\,K_0^2(x) + \frac{1}{27x^2}\,(16\,x^2+6)\,K_0(x)\,K_1(x) - \frac{1}{27x}\,(16\,x^2+2)\,K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^4}\,dx = \frac{-16\,x^2-6}{45x}\,J_0^2(x) + \frac{16\,x^2-6}{45x^2}\,J_0(x)\,J_1(x) + \frac{-16\,x^4+2\,x^2-9}{45x^3}\,J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^4}\,dx = \frac{16\,x^2-6}{45x}\,I_0^2(x) - \frac{16\,x^2+6}{45x^2}\,I_0(x)\,I_1(x) - \frac{16\,x^4+2\,x^2+9}{45x^3}\,I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^4}\,dx = \frac{16\,x^2-6}{45x}\,K_0^2(x) + \frac{16\,x^2+6}{45x^2}\,K_0(x)\,K_1(x) - \frac{16\,x^4+2\,x^2+9}{45x^3}\,K_1^2(x)$$

$$\begin{aligned}\int \frac{J_0^2(x)}{x^6}\,dx = {} & \frac{-256\,x^6-96\,x^4+90\,x^2-675}{3375x^5}\,J_0^2(x) + \frac{256\,x^4-96\,x^2+270}{3375x^4}\,J_0(x)\,J_1(x) + \\ & + \frac{-256\,x^4+32\,x^2-54}{3375x^3}\,J_1^2(x)\end{aligned}$$

$$\begin{aligned}\int \frac{I_0^2(x)}{x^6}\,dx = {} & \frac{256\,x^6-96\,x^4-90\,x^2-675}{3375x^5}\,I_0^2(x) - \frac{256\,x^4+96\,x^2+270}{3375x^4}\,I_0(x)\,I_1(x) - \\ & - \frac{256\,x^4+32\,x^2+54}{3375x^3}\,I_1^2(x)\end{aligned}$$

$$\begin{aligned}\int \frac{K_0^2(x)}{x^6}\,dx = {} & \frac{256\,x^6-96\,x^4-90\,x^2-675}{3375x^5}\,K_0^2(x) + \frac{256\,x^4+96\,x^2+270}{3375x^4}\,K_0(x)\,K_1(x) - \\ & - \frac{256\,x^4+32\,x^2+54}{3375x^3}\,K_1^2(x)\end{aligned}$$

$$\begin{aligned}\int \frac{J_1^2(x)}{x^6}\,dx = {} & \frac{1}{4725x^5}\,\big[x^2(256\,x^4+96\,x^2-90)\,J_0^2(x) + x(-256\,x^4+96\,x^2-270)\,J_0(x)\,J_1(x) + \\ & +(256\,x^6-32\,x^4+54\,x^2-675)\,J_1^2(x)\big]\end{aligned}$$

$$\int \frac{I_1^2(x)}{x^6}\,dx = \frac{1}{4725x^5}\,\big[x^2(256\,x^4-96\,x^2-90)\,I_0^2(x) - x(256\,x^4+96\,x^2+270)\,I_0(x)\,I_1(x) -$$

$$-(256\,x^6+32\,x^4+54\,x^2+675)\,I_1^2(x)\big]$$

$$\int\frac{K_1^2(x)}{x^6}\,dx \;=\; \frac{1}{4725x^5}\,\big[\,x^2(256\,x^4-96\,x^2-90)\,K_0^2(x)+x(256\,x^4+96\,x^2+270)\,K_0(x)\,K_1(x)-$$
$$-(256\,x^6+32\,x^4+54\,x^2+675)\,K_1^2(x)\big]$$

$$\int\frac{J_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8+768\,x^6-720\,x^4+3150\,x^2-55125)\,J_0^2(x)+$$
$$+x(-2048\,x^6+768\,x^4-2160\,x^2+15750)\,J_0(x)\,J_1(x)+x^2\,(2048\,x^6-256\,x^4+432\,x^2-2250)\,J_1^2(x)\big]$$

$$\int\frac{I_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8-768\,x^6-720\,x^4-3150\,x^2-55125)\,I_0^2(x)-$$
$$-x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,I_0(x)\,I_1(x)-x^2\,(2048\,x^6+256\,x^4+432\,x^2+2250)\,I_1^2(x)\big]$$

$$\int\frac{K_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8-768\,x^6-720\,x^4-3150\,x^2-55125)\,K_0^2(x)+$$
$$+x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,K_0(x)\,K_1(x)-x^2\,(2048\,x^6+256\,x^4+432\,x^2+2250)\,K_1^2(x)\big]$$

$$\int\frac{J_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(-2048\,x^6-768\,x^4+720\,x^2-3150)\,J_0^2(x)+$$
$$+x(2048\,x^6-768\,x^4+2160\,x^2-15750)\,J_0(x)\,J_1(x)+(-2048\,x^8+256\,x^6-432\,x^4+2250\,x^2-55125)\,J_1^2(x)\big]$$

$$\int\frac{I_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(2048\,x^6-768\,x^4-720\,x^2-3150)\,I_0^2(x)-$$
$$-x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,I_0(x)\,I_1(x)-(2048\,x^8+256\,x^6+432\,x^4+2250\,x^2+55125)\,I_1^2(x)\big]$$

$$\int\frac{K_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(2048\,x^6-768\,x^4-720\,x^2-3150)\,K_0^2(x)+$$
$$+x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,K_0(x)\,K_1(x)-(2048\,x^8+256\,x^6+432\,x^4+2250\,x^2+55125)\,K_1^2(x)\big]$$

$$\int\frac{J_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(-65536\,x^{10}-24576\,x^8+23040\,x^6-100800\,x^4+992250\,x^2-31255875)\,J_0^2(x)+$$
$$+x(65536\,x^8-24576\,x^6+69120\,x^4-504000\,x^2+6945750)\,J_0(x)\,J_1(x)+$$
$$+x^2\,(-65536\,x^8+8192\,x^6-13824\,x^4+72000\,x^2-771750)\,J_1^2(x)\big]$$

$$\int\frac{I_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(65536\,x^{10}-24576\,x^8-23040\,x^6-100800\,x^4-992250\,x^2-31255875)\,I_0^2(x)-$$
$$-x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,I_0(x)\,I_1(x)-$$
$$-x^2\,(65536\,x^8+8192\,x^6+13824\,x^4+72000\,x^2+771750)\,I_1^2(x)\big]$$

$$\int\frac{K_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(65536\,x^{10}-24576\,x^8-23040\,x^6-100800\,x^4-992250\,x^2-31255875)\,K_0^2(x)+$$
$$+x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,K_0(x)\,K_1(x)-$$
$$-x^2\,(65536\,x^8+8192\,x^6+13824\,x^4+72000\,x^2+771750)\,K_1^2(x)\big]$$

$$\int\frac{J_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\,\big[x^2(65536\,x^8+24576\,x^6-23040\,x^4+100800\,x^2-992250)\,J_0^2(x)+$$
$$+x(-65536\,x^8+24576\,x^6-69120\,x^4+504000\,x^2-6945750)\,J_0(x)\,J_1(x)+$$
$$+(65536\,x^{10}-8192\,x^8+13824\,x^6-72000\,x^4+771750\,x^2-31255875)\,J_1^2(x)\big]$$

$$\int\frac{I_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\,\big[x^2(65536\,x^8-24576\,x^6-23040\,x^4-100800\,x^2-992250)\,I_0^2(x)-$$
$$-x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,I_0(x)\,I_1(x)-$$
$$-(65536\,x^{10}+8192\,x^8+13824\,x^6+72000\,x^4+771750\,x^2-31255875)\,I_1^2(x)\big]$$

$$\int \frac{K_1^2(x)}{x^{10}}\,dx = \frac{1}{343814625x^9}\left[x^2(65536\,x^8-24576\,x^6-23040\,x^4-100800\,x^2-992250)\,K_0^2(x)+\right.$$
$$+x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,K_0(x)\,K_1(x)-$$
$$\left.-(65536\,x^{10}+8192\,x^8+13824\,x^6+72000\,x^4+771750\,x^2-31255875)\,K_1^2(x)\right]$$

Recurrence formulas:

$$\int \frac{J_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{-[2x^2+(2n+1)^2]\,J_0^2(x)+(4n+2)xJ_0(x)J_1(x)-2x^2\,J_1^2(x)}{x^{2n+1}}-8n\int\frac{J_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{[2x^2-(2n+1)^2]\,I_0^2(x)-(4n+2)xI_0(x)I_1(x)-2x^2\,I_1^2(x)}{x^{2n+1}}+8n\int\frac{I_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{[2x^2-(2n+1)^2]\,K_0^2(x)+(4n+2)xK_0(x)K_1(x)-2x^2\,K_1^2(x)}{x^{2n+1}}+8n\int\frac{K_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{J_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,J_0^2(x)+(4n-2)xJ_0(x)J_1(x)+(4n^2-1+2x^2)\,J_1^2(x)}{x^{2n+1}}+8n\int\frac{J_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,I_0^2(x)+(4n-2)xI_0(x)I_1(x)+(4n^2-1-2x^2)\,I_1^2(x)}{x^{2n+1}}-8n\int\frac{I_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,K_0^2(x)-(4n-2)xK_0(x)K_1(x)+(4n^2-1-2x^2)\,K_1^2(x)}{x^{2n+1}}-8n\int\frac{K_1^2(x)}{x^{2n}}\,dx\right]$$

2.1.3. Integrals of the type $\int x^{2n}\, Z_\nu^2(x)\, dx$

a) The functions $\Theta(x)$ and $\Omega(x)$:

From HANKEL's asymptotic expansion of $J_\nu(x)$ and $Y_\nu(x)$ (see [1], 9.2, or [5], XIII. A. 4) and such of $\mathbf{H}_\nu(x)$ follows, that no finite representations of the integrals $\int Z_\nu^2(x)\, dx$ by functions of the type

$$A(x)\, J_0^2(x) + B(x)\, J_0(x)\, J_1(x) + C(x)\, J_1^2(x) + [D(x)\, J_0(x) + E(x)\, J_1(x)]\, \Phi(x) + F(x)\, \Phi^2(x)$$

with

$$A(x) \quad = \quad \sum_{i=-m}^{n} a_i\, x^i\ , \ \ldots\ ,$$

can be expected. Indeed, one has

$$\lim_{x\to+\infty} \frac{1}{\ln x} \int_0^x J_\nu^2(t)\, dt \quad = \quad \frac{1}{\pi}$$

and this contradicts the upper statement.
At least should be given some other representations or approximations.

$$\Theta(x) = \int_0^x J_0^2(t)\, dt = 2\sum_{k=0}^{\infty} \frac{(-1)^k \cdot (2k)!}{(2k+1)\cdot (k!)^4} \cdot \left(\frac{x}{2}\right)^{2k+1}\ ,$$

$$\Omega(x) = \int_0^x I_0^2(t)\, dt = 2\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+1)\cdot (k!)^4} \cdot \left(\frac{x}{2}\right)^{2k+1}\ .$$

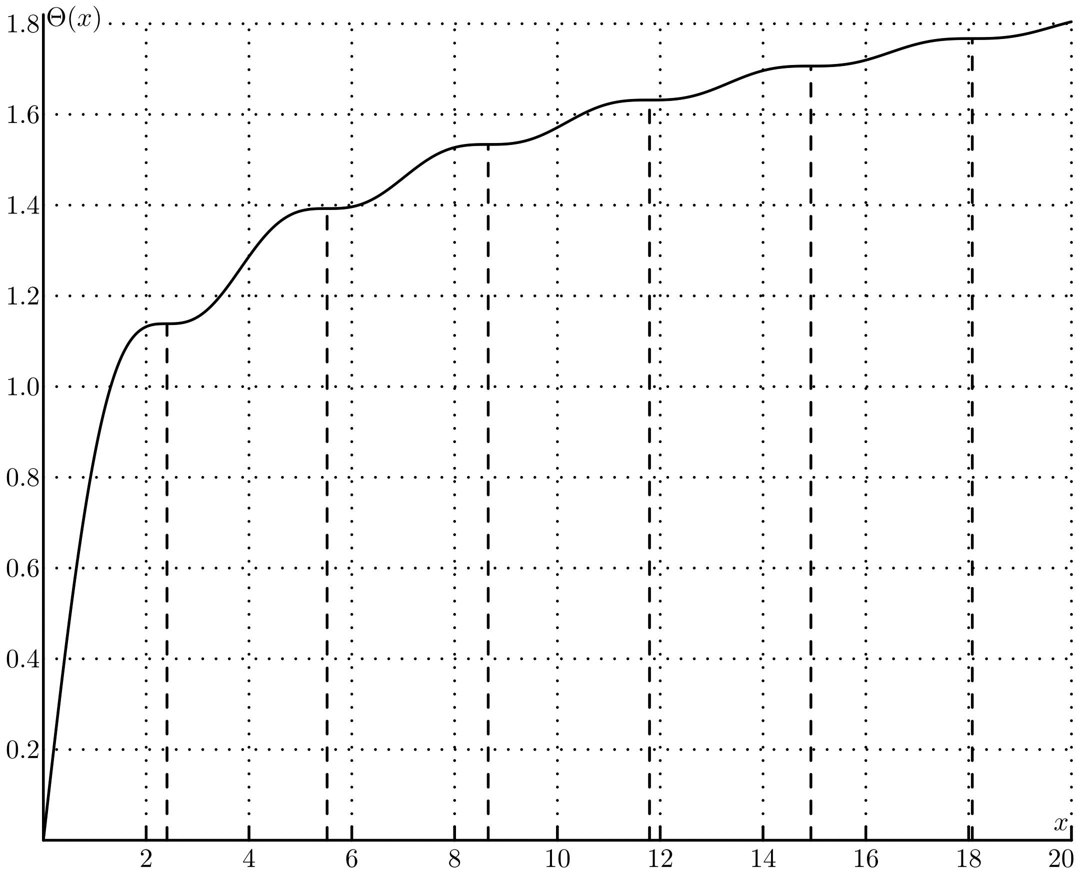


FIGURE 6 : *Function* $\Theta(x)$

The dashed lines are located in the zeros of $J_0(x)$.

If $\Theta(x)$ is computed by its series expansion with floating point numbers with n decimal digits, then the rounding error is (roughly spoken) about $10^{-n} \cdot \Omega(x)$. The computation of $\Omega(x)$ does not cause problems.

x	$\Theta(x)$	$\Omega(x)$	x	$\Theta(x)$	$\Omega(x)$
1	0.850 894 480	1.186 711 080	11	1.623 448 675	27 934 437.937
2	1.132 017 958	4.122 544 686	12	1.631 897 146	187 937 123.616
3	1.153 502 059	16.143 998 37	13	1.653 795 366	1 274 682 776.62
4	1.286 956 020	77.509 947 74	14	1.696 509 451	8 704 524 383.83
5	1.386 983 380	425.031 292 0	15	1.706 616 878	59 786 647 515.3
6	1.396 339 284	2 509.864 255	16	1.719 735 792	412 698 941 831.
7	1.460 064 224	15 483.965 76	17	1.755 251 443	2 861 234 688 170
8	1.527 171 173	98 307.748 55	18	1.767 226 854	19 912 983 676 244
9	1.534 810 723	637 083.688 6	19	1.774 861 457	139 056 981 172 080
10	1.571 266 461	4 193 041.1057	20	1.804 335 251	974 012 122 207 867

Differential equations:

$$2x\Theta''' \cdot \Theta' - 2\Theta'' \cdot \Theta' - x\Theta''^2 + 4x\Theta'^2 = 0$$

$$2x\Omega''' \cdot \Omega' - 2\Omega'' \cdot \Omega' - x\Omega''^2 - 4x\Omega'^2 = 0$$

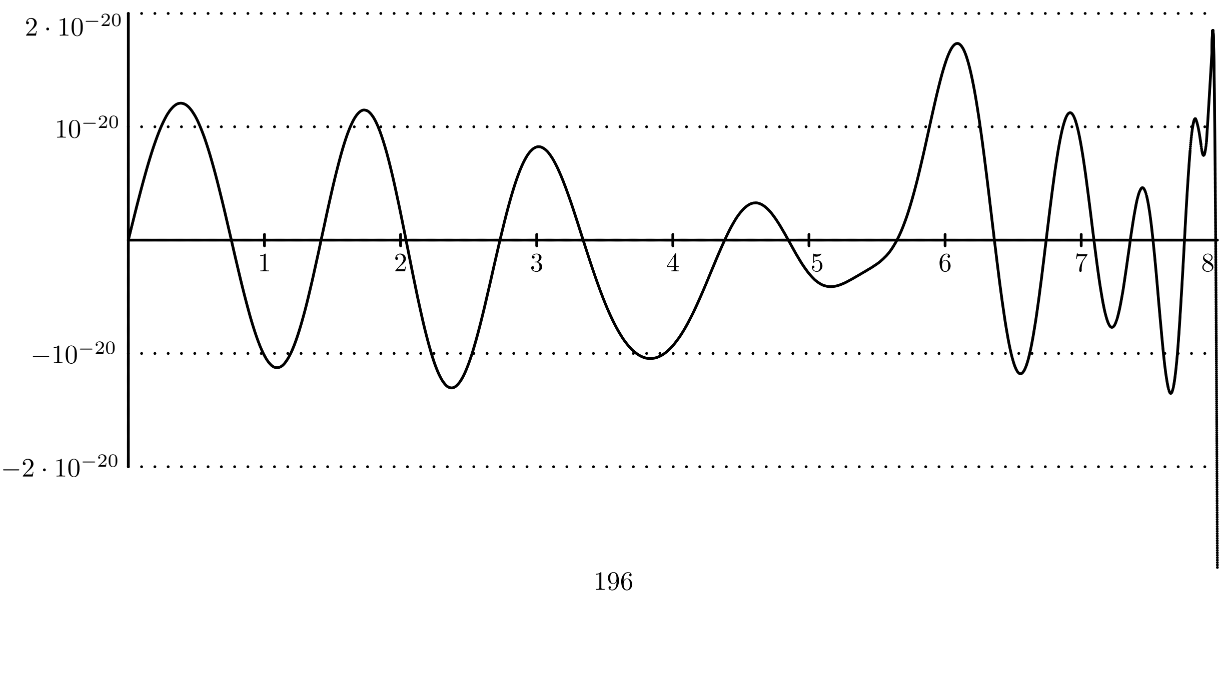
Approximation by Chebyshev polynomials:

From [2], table 9.1., follows, that in the case $-8 \le x \le 8$ holds

$$\Theta(x) \quad \approx \quad \sum_{k=0}^{23} c_k \, T_{2k+1}\left(\frac{x}{8}\right) . \qquad (*)$$

k	c_k	k	c_k
0	1.80296 30053 02073 59034	12	0.00000 59437 33032 29013
1	-0.41322 52443 66465 65056	13	-0.00000 05921 36076 87261
2	0.21926 25129 41565 79685	14	0.00000 00501 50462 51669
3	-0.12660 62713 07010 86382	15	-0.00000 00036 59712 13919
4	0.08920 60707 21441 83736	16	0.00000 00002 32718 06621
5	-0.08107 02495 61597 55273	17	-0.00000 00000 13019 50604
6	0.05544 00433 79678 61623	18	0.00000 00000 00646 15991
7	-0.02523 73073 64048 13366	19	-0.00000 00000 00028 65553
8	0.00802 84592 74213 97781	20	0.00000 00000 00001 14279
9	-0.00188 87924 36267 70784	21	-0.00000 00000 00000 04122
10	0.00034 34505 49931 43439	22	0.00000 00000 00000 00135
11	-0.00004 99025 73931 36611	23	-0.00000 00000 00000 00004

This approximation differs from $\Theta(x)$ as shown in the following figure:



Asymptotic series of $\Theta(x)$ for $x \to +\infty$:

$$\Theta(x) \quad \sim \quad \frac{1}{\pi} \left[\ln 8x + \mathbf{C} + \mathcal{A}(x)\cos 2x + \mathcal{B}(x)\sin 2x + \mathcal{C}(x)\right]$$

with Euler's constant $\mathbf{C}$ = 0.577 215 664 901 533 ... and

$$\mathcal{A}(x) = -\frac{1}{2x} + \frac{29}{64x^3} - \frac{6747}{4096x^5} + \frac{1796265}{131072x^7} - \frac{3447866835}{16777216x^9} + \frac{2611501938675}{536870912x^{11}} - \frac{5739627264576975}{34359738368x^{13}} +$$
$$+\frac{8634220069330080225}{1099511627776x^{15}} - \frac{1363263923927901083838 75}{281474976710656x^{17}} + \frac{341752571613441977621007375}{9007199254740992x^{19}} - \ldots ,$$
$$\mathcal{B}(x) = -\frac{3}{8x^2} + \frac{195}{256x^4} - \frac{71505}{16384x^6} + \frac{26103735}{524288x^8} - \frac{63761381145}{67108864x^{10}} + \frac{58671892003725}{2147483648x^{12}} - \frac{151798966421827725}{137438953472x^{14}} +$$
$$+\frac{262762002151603329375}{4398046511104x^{16}} - \frac{469243026363058463378 3625}{1125899906842624x^{18}} + \frac{131268801014295816003488606 25}{36028797018963968x^{20}} - \ldots ,$$
$$\mathcal{C}(x) = \frac{1}{16x^2} - \frac{27}{512x^4} + \frac{375}{2048x^6} - \frac{385875}{262144x^8} + \frac{11252115}{524288x^{10}} - \frac{8320313925}{16777216x^{12}} -$$
$$+\frac{1119167124075}{67108864x^{14}} - \frac{26440323306271875}{34359738368x^{16}} + \frac{1603719856835971875}{34359738368x^{18}} - \frac{3959969219293655192625}{1099511627776x^{20}} + \ldots .$$

The asymptotic series

$$\mathcal{A}(x) = \sum_{k=1}^{\infty} \frac{a_k}{x^{2k-1}} , \qquad \mathcal{B}(x) = \sum_{k=1}^{\infty} \frac{b_k}{x^{2k}} , \qquad \mathcal{C}(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^{2k}}$$

begin with

k	a_k	b_k	c_k
1	-0.500 000 000 000 000	-0.375 000 000 000 000	0.062 500 000 000 000
2	0.453 125 000 000 000	0.761 718 750 000 000	-0.052 734 375 000 000
3	-1.647 216 796 875 000	-4.364 318 847 656 250	0.183 105 468 750 000
4	13.704 414 367 675 78	49.788 923 263 549 80	-1.471 996 307 373 047
5	-205.508 877 933 025 4	-950.118 618 384 003 6	21.461 706 161 499 02
6	4 864.301 418 280 229	27 321.228 759 235 24	-495.929 355 919 361 1
7	-167 045.138 793 094 5	-1 104 482.845 562 072	16 676.889 718 696 48
8	7 852 777.406 997 193	59 745 162.196 032 50	-769 514.686 726 961 4
9	-484 328 639.035 390 4	-4 167 715 296.104 454	46 674 390.813 451 37
10	37 942 157 373.010 10	364 344 113 252.523 3	-3 601 571 024.131 458

Let x_k denote the k-th positive zero of $J_0(x)$, then holds

$$\Theta(x_k) \;\sim\; \frac{1}{\pi}\left[\ln x_k + \frac{5}{2^4\cdot x_k^2} - \frac{331}{2^9\cdot x_k^4} + \frac{7\,987}{2^{11}\cdot x_k^6} - \frac{753\,375}{2^{14}\cdot x_k^8} + \frac{246\,293\,295}{2^{18}\cdot x_k^{10}} - \ldots\right] =$$
$$= \frac{1}{\pi}\left[\ln x_k + \frac{0.312\,500}{x_k^2} - \frac{0.646\,484}{x_k^4} + \frac{3.899\,902}{x_k^6} - \frac{45.982\,36}{x_k^8} + \frac{939.5344}{x_k^{10}} - \ldots\right] .$$

Simple approximation: $\Theta(x) \approx (\ln 8x + \mathbf{C})/\pi$:

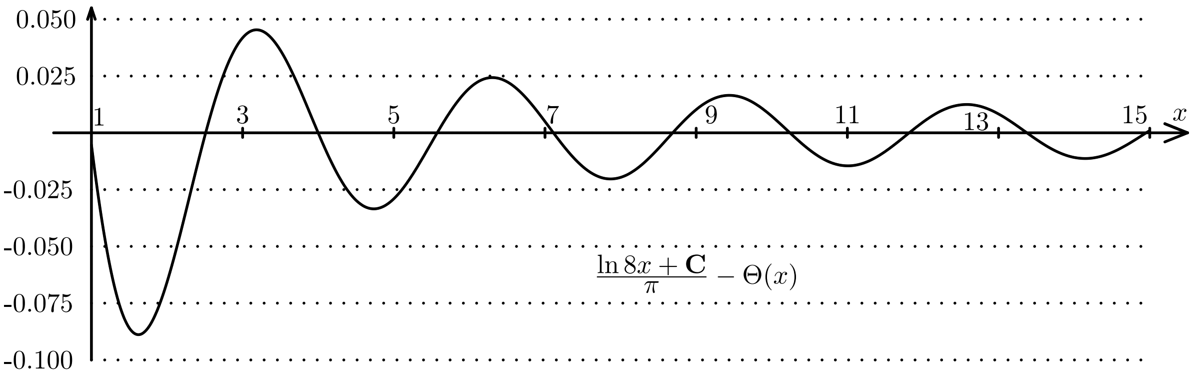


FIGURE 7

Let

$$\Delta_n(x) = \frac{1}{\pi}\left[\ln 8x + \mathbf{C} + \sum_{k=1}^{n} \frac{a_k x\,\cos 2x + b_k\,\sin 2x + c_k}{x^{2k}}\right] - \Theta(x)$$

with $\Delta_0(x) = (\ln 8x + \mathbf{C})/\pi$.
In the following table are given some first consecutive maxima and minima $\Delta^*_{n,k}$ of the differences $\Delta_n(x)$

$$\Delta^*_{n,k} \;=\; \Delta_n(x_{n,k})\;.$$

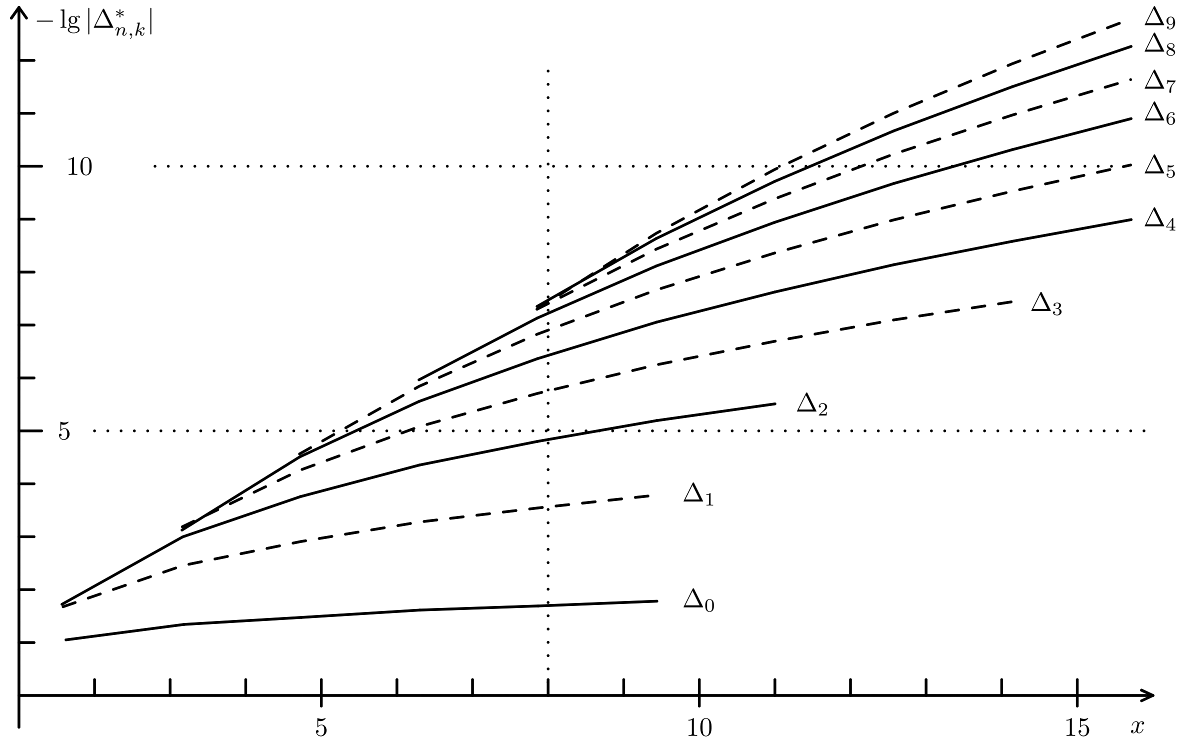
Values	$x_{n,1},\ \Delta^*_{n,1}$	$x_{n,2},\ \Delta^*_{n,2}$	$x_{n,3},\ \Delta^*_{n,3}$	$x_{n,4},\ \Delta^*_{n,4}$	$x_{n,5},\ \Delta^*_{n,5}$	$x_{n,6},\ \Delta^*_{n,6}$	$x_{n,7},\ \Delta^*_{n,7}$
x	1.6216	3.1847	4.7356	6.3043	7.8688	9.4386	11.0064
$\Delta_0(x)$	-8.886E-02	4.536E-02	-3.347E-02	2.432E-02	-2.031E-02	1.651E-02	-1.454E-02
x	1.5839	3.1735	4.7253	6.2993	7.8633	9.4353	11.0027
$\Delta_1(x)$	2.104E-02	-3.483E-03	1.237E-03	-5.267E-04	2.863E-04	-1.638E-04	1.066E-04
x	1.5694	3.1664	4.7195	6.2969	7.8602	9.4338	11.0007
$\Delta_2(x)$	-1.894E-02	1.007E-03	-1.756E-04	4.430E-05	-1.583E-05	6.405E-06	-3.092E-06
x	1.5650	3.1612	4.7160	6.2952	7.8583	9.4330	10.9995
$\Delta_3(x)$	3.978E-02	-6.484E-04	5.541E-05	-8.322E-06	1.965E-06	-5.644E-07	2.029E-07
x	1.5642	3.1574	4.7138	6.2939	7.8569	9.4324	10.9986
$\Delta_4(x)$	-1.578E-01	7.486E-04	-3.105E-05	2.773E-06	-4.335E-07	8.855E-08	-2.376E-08
x	1.5644	3.1545	4.7124	6.2928	7.8559	9.4319	10.9979
$\Delta_5(x)$	1.041E+00	-1.376E-03	2.730E-05	-1.444E-06	1.494E-07	-2.172E-08	4.352E-09
x	1.5649	3.1523	4.7115	6.2918	7.8551	9.4314	10.9973
$\Delta_6(x)$	-1.045E+01	3.722E-03	-3.486E-05	1.085E-06	-7.420E-08	7.673E-09	-1.149E-09
x	1.5655	3.1507	4.7110	6.2909	7.8545	9.4310	10.9968
$\Delta_7(x)$	1.493E+02	-1.403E-02	6.123E-05	-1.115E-06	5.025E-08	-3.692E-09	4.132E-10
x	1.5660	3.1494	4.7107	6.2902	7.8540	9.4305	10.9964
$\Delta_8(x)$	-2.891E+03	7.055E-02	-1.421E-04	1.506E-06	-4.457E-08	2.324E-09	-1.942E-10
x	1.5664	3.1483	4.7105	6.2895	7.8537	9.4302	10.9961
$\Delta_9(x)$	7.303E+04	-4.582E-01	4.224E-04	-2.590E-06	5.022E-08	-1.854E-09	1.156E-10

If $x > 8$, then $|\Delta_n(x)|$ is restricted by $|\Delta_n(x)| \le |\Delta^*_{n,5}|$. More accurate:

n	0	1	2	3	4
$\lvert\Delta_n(x)\rvert\ <$	1.9625E-02	2.7573E-04	1.5193E-05	1.8792E-06	4.1271E-07
n	5	6	7	8	9
$\lvert\Delta_n(x)\rvert\ <$	1.4150E-07	6.9856E-08	4.6990E-08	4.1368E-08	4.6224E-08

Therefore the formula (*) from page 196 may be continued to $x > 8$ by an asymptotic formula ($n = 7$, 8 or 9) with a uniformly bounded absolute error less then 0.5E-07.

The following figure shows the values of $-\lg|\Delta^*_{n,k}|$ from the preceding table, connected by a polygonal line. It gives the intervals where some special asymptotic formula is preferable.



The differences $\Delta_n(x)$ are shown in the following figures:

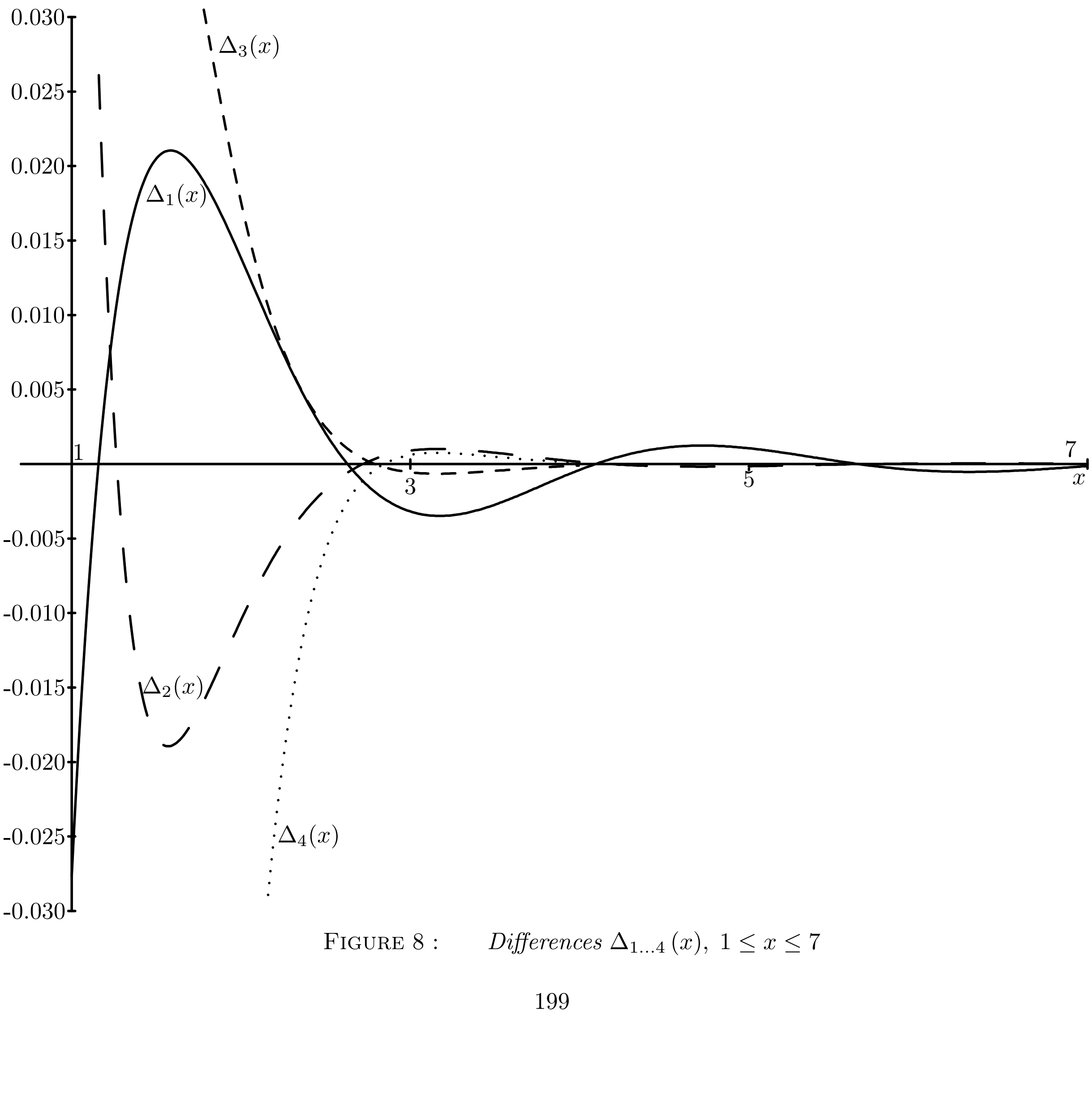


FIGURE 8 : *Differences* $\Delta_{1\ldots4}(x),\ 1 \le x \le 7$

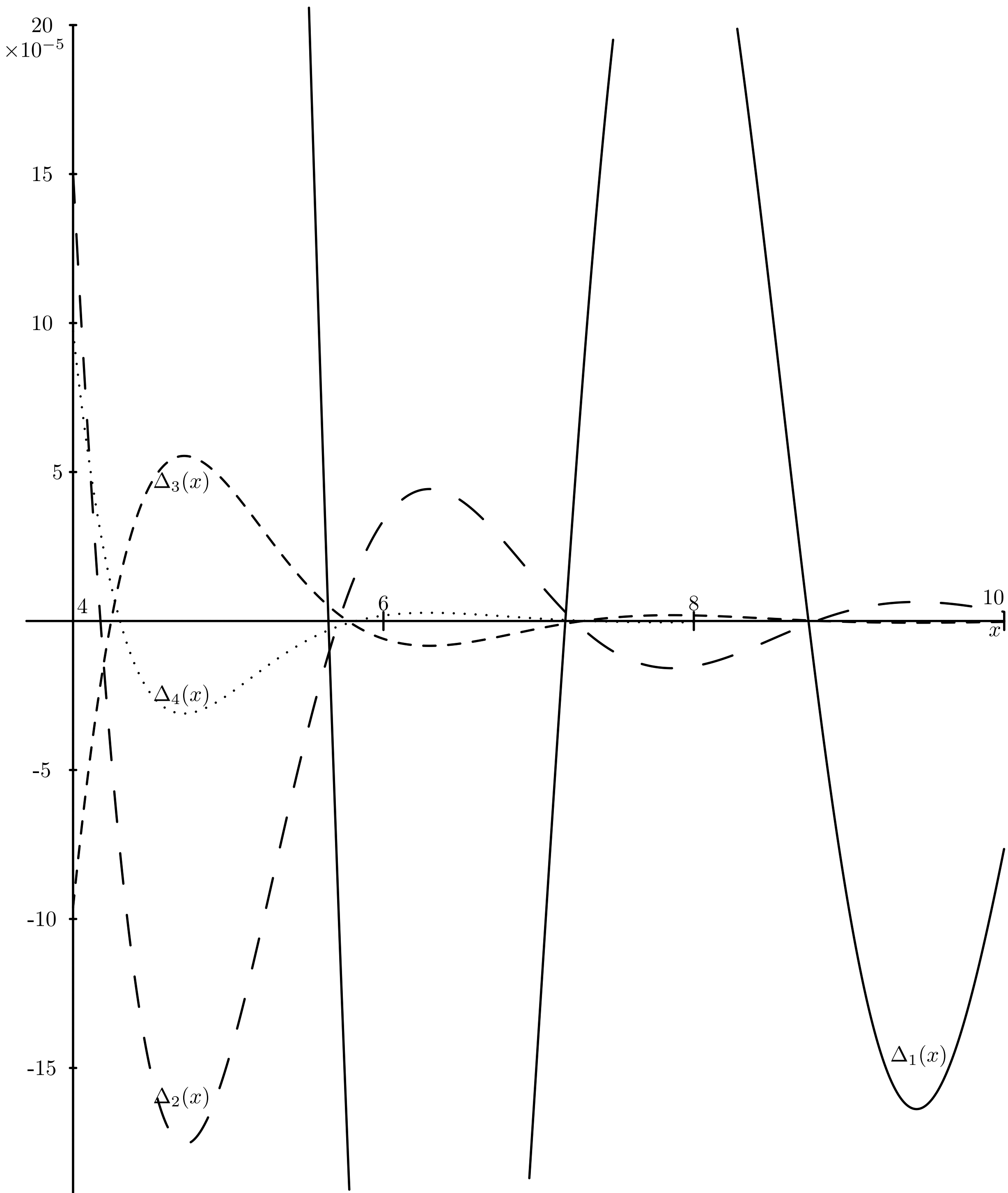


FIGURE 9 : *Differences* $\Delta_{1\ldots4}(x),\ 4 \le x \le 10$

Asymptotic behaviour of $\Omega(x)$ for $x \to \infty$:

$$\Omega(x) \sim$$

$$\sim \frac{e^{2x}}{4\pi x}\left[1+\frac{3}{4x}+\frac{29}{32x^2}+\frac{195}{128x^3}+\frac{6\,747}{2\,048x^4}+\frac{71\,505}{8\,192x^5}+\frac{1\,796\,265}{65\,536x^6}+\frac{26\,103\,735}{262\,144x^7}+\frac{430\,983\,354}{1\,048\,576x^8}+\ldots\right]$$

$$=\frac{e^{2x}}{4\pi x}\left[1+\frac{0.75}{x}+\frac{0.90625}{x^2}+\frac{1.5234}{x^3}+\frac{3.2944}{x^4}+\frac{8.7286}{x^5}+\frac{27.409}{x^6}+\frac{99.578}{x^7}+\frac{411.02}{x^8}+\ldots\right]$$

b) Integrals:

Holds

$$\int J_0^2(x)\,dx = x[J_0^2(x)+J_1^2(x)] + \int J_1^2(x)\,dx\ ,$$

$$\int I_0^2(x)\,dx = x[I_0^2(x)-I_1^2(x)] - \int I_1^2(x)\,dx\ .$$

These formulas express every integral by the other one. Therefore the next integrals are given with $\int J_0^2(x)\,dx$ or $\int I_0^2(x)\,dx$. The last integrals are represented by the functions $\Theta(x)$ respectively $\Omega(x)$, see page 195 .

$$\int x^2\,J_0^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3+x)\,J_0^2(x)+2x^2 J_0(x)J_1(x)+2x^3\,J_1^2(x)\right] - \frac{1}{8}\int J_0^2(x)\,dx$$

$$\int x^2\,I_0^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3-x)\,I_0^2(x)+2x^2 I_0(x)I_1(x)-2x^3\,I_1^2(x)\right] + \frac{1}{8}\int I_0^2(x)\,dx$$

$$\int x^2\,J_1^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3-3x)\,J_0^2(x)-6x^2 J_0(x)J_1(x)+2x^3\,J_1^2(x)\right] + \frac{3}{8}\int J_0^2(x)\,dx$$

$$\int x^2\,I_1^2(x)\,dx = \frac{1}{8}\,\left[(-2\,x^3-3x)\,I_0^2(x)+6x^2 I_0(x)I_1(x)+2x^3\,I_1^2(x)\right] + \frac{3}{8}\int I_0^2(x)\,dx$$

$$\int x^4\,J_0^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5+18\,x^3-27\,x)\,J_0^2(x)+(48\,x^4-54\,x^2)J_0(x)J_1(x)+(16\,x^5-54\,x^3)\,J_1^2(x)\right]+\frac{27}{128}\int J_0^2(x)\,dx$$

$$\int x^4\,I_0^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5-18\,x^3-27\,x)\,I_0^2(x)+(48\,x^4+54\,x^2)I_0(x)I_1(x)+(-16\,x^5-54\,x^3)\,I_1^2(x)\right]+\frac{27}{128}\int I_0^2(x)\,dx$$

$$\int x^4\,J_1^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5-30\,x^3+45\,x)\,J_0^2(x)+(-80\,x^4+90\,x^2)J_0(x)J_1(x)+(16\,x^5+90\,x^3)\,J_1^2(x)\right] -$$

$$-\frac{45}{128}\int J_0^2(x)\,dx$$

$$\int x^4\,I_1^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(-16\,x^5-30\,x^3-45\,x)\,I_0^2(x)+(80\,x^4+90\,x^2)I_0(x)I_1(x)+(16\,x^5-90\,x^3)\,I_1^2(x)\right] +$$

$$+\frac{45}{128}\int I_0^2(x)\,dx$$

With

$$\int x^{2n}\,J_0^2(x)\,dx = \frac{1}{\beta_n}\left[A_n(x)\,J_0^2(x)+B_n(x)J_0(x)J_1(x)+C_n(x)\,J_1^2(x)+\gamma_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\,I_0^2(x)\,dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\,I_0^2(x)+B_n^*(x)I_0(x)I_1(x)+C_n^*(x)\,I_1^2(x)+\gamma_n^*\int I_0^2(x)\,dx\right]$$

$$\int x^{2n}\,J_1^2(x)\,dx = \frac{1}{\xi_n}\left[P_n(x)\,J_0^2(x)+Q_n(x)J_0(x)J_1(x)+R_n(x)\,J_1^2(x)+\varrho_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\,I_1^2(x)\,dx = \frac{1}{\xi_n^*}\left[P_n^*(x)\,I_0^2(x)+Q_n^*(x)I_0(x)I_1(x)+R_n^*(x)\,I_1^2(x)+\varrho_n^*\int I_0^2(x)\,dx\right]$$

holds

$$\beta_3 = 3072\,, \quad \gamma_3 = -3375$$
$$A_3(x) = 256\,x^7 + 1200\,x^5 - 2250\,x^3 + 3375\,x$$
$$B_3(x) = 1280\,x^6 - 6000\,x^4 + 6750\,x^2\,, \quad C_3(x) = 256\,x^7 - 2000\,x^5 + 6750\,x^3$$

$$\beta_3^* = 3072\,, \quad \gamma_3^* = 3375$$
$$A_3^*(x) = 256\,x^7 - 1200\,x^5 - 2250\,x^3 - 3375\,x$$
$$B_3^*(x) = 1280\,x^6 + 6000\,x^4 + 6750\,x^2\,, \quad C_3^*(x) = -256\,x^7 - 2000\,x^5 - 6750\,x^3$$

$$\xi_3 = 3072\,, \quad \varrho_3 = 4725$$
$$P_3(x) = 256\,x^7 - 1680\,x^5 + 3150\,x^3 - 4725\,x$$
$$Q_3(x) = -1792\,x^6 + 8400\,x^4 - 9450\,x^2\,, \quad R_3(x) = 256\,x^7 + 2800\,x^5 - 9450\,x^3$$

$$\xi_3^* = 3072\,, \quad \varrho_3^* = 4725$$
$$P_3^*(x) = -256\,x^7 - 1680\,x^5 - 3150\,x^3 - 4725\,x$$
$$Q_3^*(x) = 1792\,x^6 + 8400\,x^4 + 9450\,x^2\,, \quad R_3^*(x) = 256\,x^7 - 2800\,x^5 - 9450\,x^3$$

$$\beta_4 = 98304\,, \quad \gamma_4 = 1157625$$
$$A_4(x) = 6144\,x^9 + 62720\,x^7 - 411600\,x^5 + 771750\,x^3 - 1157625\,x$$
$$B_4(x) = 43008\,x^8 - 439040\,x^6 + 2058000\,x^4 - 2315250\,x^2$$
$$C_4(x) = 6144\,x^9 - 87808\,x^7 + 686000\,x^5 - 2315250\,x^3$$

$$\beta_4^* = 98304\,, \quad \gamma_4^* = 1157625$$
$$A_4^*(x) = 6144\,x^9 - 62720\,x^7 - 411600\,x^5 - 771750\,x^3 - 1157625\,x$$
$$B_4^*(x) = 43008\,x^8 + 439040\,x^6 + 2058000\,x^4 + 2315250\,x^2$$
$$C_4^*(x) = -6144\,x^9 - 87808\,x^7 - 686000\,x^5 - 2315250\,x^3$$

$$\xi_4 = 32768\,, \quad \varrho_4 = -496125$$
$$P_4(x) = 2048\,x^9 - 26880\,x^7 + 176400\,x^5 - 330750\,x^3 + 496125\,x$$
$$Q_4(x) = -18432\,x^8 + 188160\,x^6 - 882000\,x^4 + 992250\,x^2$$
$$R_4(x) = 2048\,x^9 + 37632\,x^7 - 294000\,x^5 + 992250\,x^3$$

$$\xi_4^* = 32768\,, \quad \varrho_4^* = 496125$$
$$P_4^*(x) = -2048\,x^9 - 26880\,x^7 - 176400\,x^5 - 330750\,x^3 - 496125\,x$$
$$Q_4^*(x) = 18432\,x^8 + 188160\,x^6 + 882000\,x^4 + 992250\,x^2$$
$$R_4^*(x) = 2048\,x^9 - 37632\,x^7 - 294000\,x^5 - 992250\,x^3$$

$$\beta_5 = 1310720\,, \quad \gamma_5 = -281302875$$
$$A_5(x) = 65536\,x^{11} + 1161216\,x^9 - 15240960\,x^7 + 100018800\,x^5 - 187535250\,x^3 + 281302875\,x$$
$$B_5(x) = 589824\,x^{10} - 10450944\,x^8 + 106686720\,x^6 - 500094000\,x^4 + 562605750\,x^2$$
$$C_5(x) = 65536\,x^{11} - 1492992\,x^9 + 21337344\,x^7 - 166698000\,x^5 + 562605750\,x^3$$

$$\beta_5^* = 1310720\,, \quad \gamma_5^* = 281302875^*$$

$$A_5(x)^* = 65536\,x^{11} - 1161216\,x^9 - 15240960\,x^7 - 100018800\,x^5 - 187535250\,x^3 - 281302875\,x$$
$$B_5(x)^* = 589824\,x^{10} + 10450944\,x^8 + 106686720\,x^6 + 500094000\,x^4 + 562605750\,x^2$$
$$C_5(x)^* = -65536\,x^{11} - 1492992\,x^9 - 21337344\,x^7 - 166698000\,x^5 - 562605750\,x^3$$

$$\xi_5 = 1310720\,, \quad \varrho_5 = 343814625$$
$$P_5(x) = 65536\,x^{11} - 1419264\,x^9 + 18627840\,x^7 - 122245200\,x^5 + 229209750\,x^3 - 343814625\,x$$
$$Q_5(x) = -720896\,x^{10} + 12773376\,x^8 - 130394880\,x^6 + 611226000\,x^4 - 687629250\,x^2$$
$$R_5(x) = 65536\,x^{11} + 1824768\,x^9 - 26078976\,x^7 + 203742000\,x^5 - 687629250\,x^3$$

$$\xi_5^* = 1310720\,, \quad \varrho_5^* = 343814625$$
$$P_5^*(x) = -65536\,x^{11} - 1419264\,x^9 - 18627840\,x^7 - 122245200\,x^5 - 229209750\,x^3 - 343814625\,x$$
$$Q_5^*(x) = 720896\,x^{10} + 12773376\,x^8 + 130394880\,x^6 + 611226000\,x^4 + 687629250\,x^2$$
$$R_5^*(x) = 65536\,x^{11} - 1824768\,x^9 - 26078976\,x^7 - 203742000\,x^5 - 687629250\,x^3$$

$$\beta_6 = 62914560\,, \quad \gamma_6 = 374414126625$$
$$A_6(x) = 2621440\,x^{13} + 71368704\,x^{11} - 1545578496\,x^9 + 20285717760\,x^7 - 133125022800\,x^5 + 249609417750\,x^3 - \\ -374414126625\,x$$
$$B_6(x) = 28835840\,x^{12} - 785055744\,x^{10} + 13910206464\,x^8 - 142000024320\,x^6 + 665625114000\,x^4 - 748828253250\,x^2$$
$$C_6(x) = 2621440\,x^{13} - 87228416\,x^{11} + 1987172352\,x^9 - 28400004864\,x^7 + 221875038000\,x^5 - 748828253250\,x^3$$

$$\beta_6^* = 62914560\,, \quad \gamma_6^* = 374414126625$$
$$A_6^*(x) = 2621440\,x^{13} - 71368704\,x^{11} - 1545578496\,x^9 - 20285717760\,x^7 - 133125022800\,x^5 - 249609417750\,x^3 - \\ -374414126625\,x$$
$$B_6^*(x) = 28835840\,x^{12} + 785055744\,x^{10} + 13910206464\,x^8 + 142000024320\,x^6 + 665625114000\,x^4 + 748828253250\,x^2$$
$$C_6^*(x) = -2621440\,x^{13} - 87228416\,x^{11} - 1987172352\,x^9 - 28400004864\,x^7 - 221875038000\,x^5 - 748828253250\,x^3$$

$$\xi_6 = 62914560\,, \quad \varrho_6 = -442489422375$$
$$P_6(x) = 2621440\,x^{13} - 84344832\,x^{11} + 1826592768\,x^9 - 23974030080\,x^7 + 157329572400\,x^5 - \\ -294992948250\,x^3 + 442489422375\,x$$
$$Q_6(x) = -34078720\,x^{12} + 927793152\,x^{10} - 16439334912\,x^8 + 167818210560\,x^6 - \\ -786647862000\,x^4 + 884978844750\,x^2$$
$$R_6(x) = 2621440\,x^{13} + 103088128\,x^{11} - 2348476416\,x^9 + 33563642112\,x^7 - \\ -262215954000\,x^5 + 884978844750\,x^3$$

$$\xi_6^* = 62914560\,, \quad \varrho_6^* = 442489422375$$
$$P_6^*(x) = -2621440\,x^{13} - 84344832\,x^{11} - 1826592768\,x^9 - 23974030080\,x^7 - 157329572400\,x^5 - \\ -294992948250\,x^3 - 442489422375\,x$$
$$Q_6^*(x) = 34078720\,x^{12} + 927793152\,x^{10} + 16439334912\,x^8 + 167818210560\,x^6 + \\ +786647862000\,x^4 + 884978844750\,x^2$$
$$R_6^*(x) = 2621440\,x^{13} - 103088128\,x^{11} - 2348476416\,x^9 - 33563642112\,x^7 - \\ -262215954000\,x^5 - 884978844750\,x^3$$

$$\beta_7 = 3523215360\,, \quad \gamma_7 = -822587836195125$$

$$A_7(x) = 125829120\,x^{15} + 4873256960\,x^{13} - 156797042688\,x^{11} + 3395635955712\,x^9 - 44567721918720\,x^7 + $$
$$+292475675091600\,x^5 - 548391890796750\,x^3 + 822587836195125\,x$$

$$B_7(x) = 1635778560\,x^{14} - 63352340480\,x^{12} + 1724767469568\,x^{10} - 30560723601408\,x^8 + $$
$$+311974053431040\,x^6 - 1462378375458000\,x^4 + 1645175672390250\,x^2$$

$$C_7(x) = 125829120\,x^{15} - 5759303680\,x^{13} + 191640829952\,x^{11} - 4365817657344\,x^9 + 62394810686208\,x^7 - $$
$$-487459458486000\,x^5 + 1645175672390250\,x^3$$

$$\beta_7^* = 3523215360\,, \quad \gamma_7^* = 822587836195125$$

$$A_7^*(x) = 125829120\,x^{15} - 4873256960\,x^{13} - 156797042688\,x^{11} - 3395635955712\,x^9 - 44567721918720\,x^7 - $$
$$-292475675091600\,x^5 - 548391890796750\,x^3 - 822587836195125\,x$$

$$B_7^*(x) = 1635778560\,x^{14} + 63352340480\,x^{12} + 1724767469568\,x^{10} + 30560723601408\,x^8 + $$
$$+311974053431040\,x^6 + 1462378375458000\,x^4 + 1645175672390250\,x^2$$

$$C_7^*(x) = -125829120\,x^{15} - 5759303680\,x^{13} - 191640829952\,x^{11} - 4365817657344\,x^9 - 62394810686208\,x^7 - $$
$$-487459458486000\,x^5 - 1645175672390250\,x^3$$

$$\xi_7 = 234881024\,, \quad \varrho_7 = 63275987399625$$

$$P_7(x) = 8388608\,x^{15} - 374865920\,x^{13} + 12061310976\,x^{11} - 261202765824\,x^9 + 3428286301440\,x^7 - $$
$$-22498128853200\,x^5 + 42183991599750\,x^3 - 63275987399625\,x$$

$$Q_7(x) = -125829120\,x^{14} + 4873256960\,x^{12} - 132674420736\,x^{10} + 2350824892416\,x^8 - 23998004110080\,x^6 + $$
$$+112490644266000\,x^4 - 126551974799250\,x^2$$

$$R_7(x) = 8388608\,x^{15} + 443023360\,x^{13} - 14741602304\,x^{11} + 335832127488\,x^9 - 4799600822016\,x^7 + $$
$$+37496881422000\,x^5 - 126551974799250\,x^3$$

$$\xi_7^* = 234881024\,, \quad \varrho_7^* = 63275987399625$$

$$P_7^*(x) = -8388608\,x^{15} - 374865920\,x^{13} - 12061310976\,x^{11} - 261202765824\,x^9 - 3428286301440\,x^7 - $$
$$-22498128853200\,x^5 - 42183991599750\,x^3 - 63275987399625\,x$$

$$Q_7^*(x) = 125829120\,x^{14} + 4873256960\,x^{12} + 132674420736\,x^{10} + 2350824892416\,x^8 + 23998004110080\,x^6 + $$
$$+112490644266000\,x^4 + 126551974799250\,x^2$$

$$R_7^*(x) = 8388608\,x^{15} - 443023360\,x^{13} - 14741602304\,x^{11} - 335832127488\,x^9 - 4799600822016\,x^7 - $$
$$-37496881422000\,x^5 - 126551974799250\,x^3$$

$$\beta_8 = 15032385536\,, \quad \gamma_8 = 185082263143903125$$

$$A_8(x) = 469762048\,x^{17} + 24536678400\,x^{15} - 1096482816000\,x^{13} + 35279334604800\,x^{11} - $$
$$-764018090035200\,x^9 + 10027737431712000\,x^7 - 65807026895610000\,x^5 + 123388175429268750\,x^3 - $$
$$-185082263143903125\,x$$

$$B_8(x) = 7046430720\,x^{16} - 368050176000\,x^{14} + 14254276608000\,x^{12} - 388072680652800\,x^{10} + $$
$$+6876162810316800\,x^8 - 70194162021984000\,x^6 + 329035134478050000\,x^4 - 370164526287806250\,x^2$$

$$C_8(x) = 469762048\,x^{17} - 28311552000\,x^{15} + 1295843328000\,x^{13} - 43119186739200\,x^{11} + $$
$$+982308972902400\,x^9 - 14038832404396800\,x^7 + 109678378159350000\,x^5 - 370164526287806250\,x^3$$

$$\beta_8^* = 15032385536\ , \quad \gamma_8^* = 185082263143903125$$

$$A_8^*(x) = 469762048\,x^{17} - 24536678400\,x^{15} - 1096482816000\,x^{13} - 35279334604800\,x^{11} - $$
$$-764018090035200\,x^9 - 10027737431712000\,x^7 - 65807026895610000\,x^5 - 123388175429268750\,x^3 - $$
$$-185082263143903125\,x$$

$$B_8^*(x) = 7046430720\,x^{16} + 368050176000\,x^{14} + 14254276608000\,x^{12} + 388072680652800\,x^{10} + $$
$$+6876162810316800\,x^8 + 70194162021984000\,x^6 + 329035134478050000\,x^4 + 370164526287806250\,x^2$$

$$C_8^*(x) = -469762048\,x^{17} - 28311552000\,x^{15} - 1295843328000\,x^{13} - 43119186739200\,x^{11} - $$
$$-982308972902400\,x^9 - 14038832404396800\,x^7 - 109678378159350000\,x^5 - 370164526287806250\,x^3$$

$$\xi_8 = 15032385536\ , \quad \varrho_8 = -209759898229756875$$

$$P_8(x) = 469762048\,x^{17} - 27808235520\,x^{15} + 1242680524800\,x^{13} - 39983245885440\,x^{11} + 865887168706560\,x^9 - $$
$$-11364769089273600\,x^7 + 74581297148358000\,x^5 - 139839932153171250\,x^3 + 209759898229756875\,x$$

$$Q_8(x) = -7985954816\,x^{16} + 417123532800\,x^{14} - 16154846822400\,x^{12} + 439815704739840\,x^{10} - $$
$$-7792984518359040\,x^8 + 79553383624915200\,x^6 - 372906485741790000\,x^4 + 419519796459513750\,x^2$$

$$R_8(x) = 469762048\,x^{17} + 32086425600\,x^{15} - 1468622438400\,x^{13} + 48868411637760\,x^{11} - $$
$$-1113283502622720\,x^9 + 15910676724983040\,x^7 - 124302161913930000\,x^5 + 419519796459513750\,x^3$$

$$\xi_8^* = 15032385536\ , \quad \varrho_8^* = 209759898229756875$$

$$P_8^*(x) = -469762048\,x^{17} - 27808235520\,x^{15} - 1242680524800\,x^{13} - 39983245885440\,x^{11} - 865887168706560\,x^9 - $$
$$-11364769089273600\,x^7 - 74581297148358000\,x^5 - 139839932153171250\,x^3 - 209759898229756875\,x$$

$$Q_8^*(x) = 7985954816\,x^{16} + 417123532800\,x^{14} + 16154846822400\,x^{12} + 439815704739840\,x^{10} + $$
$$+7792984518359040\,x^8 + 79553383624915200\,x^6 + 372906485741790000\,x^4 + 419519796459513750\,x^2$$

$$R_8^*(x) = 469762048\,x^{17} - 32086425600\,x^{15} - 1468622438400\,x^{13} - 48868411637760\,x^{11} - $$
$$-1113283502622720\,x^9 - 15910676724983040\,x^7 - 124302161913930000\,x^5 - 419519796459513750\,x^3$$

2.1.4. Integrals of the type $\int x^{2n} Z_0(x) Z_1(x)\,dx$

In the following formulas both $J_0(x)$ and $J_1(x)$ together may be substituted by $Y_0(x)$ and $Y_1(x)$ respectively or $H_0^{(p)}(x),\ H_1^{(p)}(x),\ p=1,2.$

$$\int J_0(x)J_1(x)\,dx = -\frac{1}{2}J_0^2(x)$$

$$\int I_0(x)I_1(x)\,dx = \frac{1}{2}I_0^2(x)$$

$$\int x^2\cdot J_0(x)J_1(x)\,dx = \frac{x^2}{2}J_1^2(x)$$

$$\int x^2\cdot I_0(x)I_1(x)\,dx = \frac{x^2}{2}I_1^2(x)$$

$$\int x^4\cdot J_0(x)J_1(x)\,dx = -\frac{x^4}{6}J_0^2(x)+\frac{2x^3}{3}J_0(x)J_1(x)+\left(\frac{x^4}{3}-\frac{2x^2}{3}\right)J_1^2(x)$$

$$\int x^4\cdot I_0(x)I_1(x)\,dx = \frac{x^4}{6}I_0^2(x)-\frac{2x^3}{3}I_0(x)I_1(x)+\left(\frac{x^4}{3}+\frac{2x^2}{3}\right)I_1^2(x)$$

$$\int x^6\cdot J_0(x)J_1(x)\,dx = \left(-\frac{x^6}{5}+\frac{4x^4}{5}\right)J_0^2(x)+\left(\frac{6x^5}{5}-\frac{16x^3}{5}\right)J_0(x)J_1(x)+\left(\frac{3x^6}{10}-\frac{8x^4}{5}+\frac{16x^2}{5}\right)J_1^2(x)$$

$$\int x^6\cdot I_0(x)I_1(x)\,dx = \left(\frac{x^6}{5}+\frac{4x^4}{5}\right)I_0^2(x)-\left(\frac{6x^5}{5}+\frac{16x^3}{5}\right)I_0(x)I_1(x)+\left(\frac{3x^6}{10}+\frac{8x^4}{5}+\frac{16x^2}{5}\right)I_1^2(x)$$

$$\int x^8\cdot J_0(x)J_1(x)\,dx = \left(-\frac{3x^8}{14}+\frac{72x^6}{35}-\frac{288x^4}{35}\right)J_0^2(x)+\left(\frac{12x^7}{7}-\frac{432x^5}{35}+\frac{1152x^3}{35}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{2x^8}{7}-\frac{108x^6}{35}+\frac{576x^4}{35}-\frac{1152x^2}{35}\right)J_1^2(x)$$

$$\int x^8\cdot I_0(x)I_1(x)\,dx = \left(\frac{3x^8}{14}+\frac{72x^6}{35}+\frac{288x^4}{35}\right)I_0^2(x)-\left(\frac{12x^7}{7}+\frac{432x^5}{35}+\frac{1152x^3}{35}\right)I_0(x)I_1(x)+$$
$$+\left(\frac{2x^8}{7}+\frac{108x^6}{35}+\frac{576x^4}{35}+\frac{1152x^2}{35}\right)I_1^2(x)$$

$$\int x^{10}\cdot J_0(x)J_1(x)\,dx = \left(-\frac{2x^{10}}{9}+\frac{80x^8}{21}-\frac{256x^6}{7}+\frac{1024x^4}{7}\right)J_0^2(x)+$$
$$+\left(\frac{20x^9}{9}-\frac{640x^7}{21}+\frac{1536x^5}{7}-\frac{4096x^3}{7}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{5x^{10}}{18}-\frac{320x^8}{63}+\frac{384x^6}{7}-\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)J_1^2(x)$$

$$\int x^{10}\cdot I_0(x)I_1(x)\,dx = \left(\frac{2x^{10}}{9}+\frac{80x^8}{21}+\frac{256x^6}{7}+\frac{1024x^4}{7}\right)I_0^2(x)-$$
$$-\left(\frac{20x^9}{9}+\frac{640x^7}{21}+\frac{1536x^5}{7}+\frac{4096x^3}{7}\right)I_0(x)I_1(x)+$$
$$+\left(\frac{5x^{10}}{18}+\frac{320x^8}{63}+\frac{384x^6}{7}+\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)I_1^2(x)$$

Let

$$\int x^m J_0(x)J_1(x)\,dx = P_m(x)J_0^2(x)+Q_m(x)J_0(x)J_1(x)+R_m(x)J_1^2(x)$$

$$\int x^m I_0(x)I_1(x)\,dx = P_m^*(x)I_0^2(x)+Q_m^*(x)I_0(x)I_1(x)+R_m^*(x)I_1^2(x)$$

then holds

$$P_{12} = -\frac{5}{22}\,x^{12}+\frac{200}{33}\,x^{10}-\frac{8000}{77}\,x^8+\frac{76800}{77}\,x^6-\frac{307200}{77}\,x^4$$

$$Q_{12} = \frac{30}{11}\,x^{11} - \frac{2000}{33}\,x^{9} + \frac{64000}{77}\,x^{7} - \frac{460800}{77}\,x^{5} + \frac{1228800}{77}\,x^{3}$$
$$R_{12} = \frac{3}{11}\,x^{12} - \frac{250}{33}\,x^{10} + \frac{32000}{231}\,x^{8} - \frac{115200}{77}\,x^{6} + \frac{614400}{77}\,x^{4} - \frac{1228800}{77}\,x^{2}$$
$$P^*_{12} = \frac{5}{22}\,x^{12} + \frac{200}{33}\,x^{10} + \frac{8000}{77}\,x^{8} + \frac{76800}{77}\,x^{6} + \frac{307200}{77}\,x^{4}$$
$$Q^*_{12} = -\frac{30}{11}\,x^{11} - \frac{2000}{33}\,x^{9} - \frac{64000}{77}\,x^{7} - \frac{460800}{77}\,x^{5} - \frac{1228800}{77}\,x^{3}$$
$$R^*_{12} = \frac{3}{11}\,x^{12} + \frac{250}{33}\,x^{10} + \frac{32000}{231}\,x^{8} + \frac{115200}{77}\,x^{6} + \frac{614400}{77}\,x^{4} + \frac{1228800}{77}\,x^{2}$$

$$P_{14} = -\frac{3}{13}\,x^{14} + \frac{1260}{143}\,x^{12} - \frac{33600}{143}\,x^{10} + \frac{576000}{143}\,x^{8} - \frac{5529600}{143}\,x^{6} + \frac{22118400}{143}\,x^{4}$$
$$Q_{14} = \frac{42}{13}\,x^{13} - \frac{15120}{143}\,x^{11} + \frac{336000}{143}\,x^{9} - \frac{4608000}{143}\,x^{7} + \frac{33177600}{143}\,x^{5} - \frac{88473600}{143}\,x^{3}$$
$$R_{14} = \frac{7}{26}\,x^{14} - \frac{1512}{143}\,x^{12} + \frac{42000}{143}\,x^{10} - \frac{768000}{143}\,x^{8} + \frac{8294400}{143}\,x^{6} - \frac{44236800}{143}\,x^{4} + \frac{88473600}{143}\,x^{2}$$
$$P^*_{14} = \frac{3}{13}\,x^{14} + \frac{1260}{143}\,x^{12} + \frac{33600}{143}\,x^{10} + \frac{576000}{143}\,x^{8} + \frac{5529600}{143}\,x^{6} + \frac{22118400}{143}\,x^{4}$$
$$Q^*_{14} = -\frac{42}{13}\,x^{13} - \frac{15120}{143}\,x^{11} - \frac{336000}{143}\,x^{9} - \frac{4608000}{143}\,x^{7} - \frac{33177600}{143}\,x^{5} - \frac{88473600}{143}\,x^{3}$$
$$R^*_{14} = \frac{7}{26}\,x^{14} + \frac{1512}{143}\,x^{12} + \frac{42000}{143}\,x^{10} + \frac{768000}{143}\,x^{8} + \frac{8294400}{143}\,x^{6} + \frac{44236800}{143}\,x^{4} + \frac{88473600}{143}\,x^{2}$$

$$P_{16} = -\frac{7}{30}\,x^{16} + \frac{784}{65}\,x^{14} - \frac{65856}{143}\,x^{12} + \frac{1756160}{143}\,x^{10} - \frac{30105600}{143}\,x^{8} + \frac{289013760}{143}\,x^{6} - \frac{1156055040}{143}\,x^{4}$$
$$Q_{16} = \frac{56}{15}\,x^{15} - \frac{10976}{65}\,x^{13} + \frac{790272}{143}\,x^{11} - \frac{17561600}{143}\,x^{9} + \frac{240844800}{143}\,x^{7} - \frac{1734082560}{143}\,x^{5} + \frac{4624220160}{143}\,x^{3}$$
$$R_{16} =$$
$$\frac{4}{15}\,x^{16} - \frac{2744}{195}\,x^{14} + \frac{395136}{715}\,x^{12} - \frac{2195200}{143}\,x^{10} + \frac{40140800}{143}\,x^{8} - \frac{433520640}{143}\,x^{6} + \frac{2312110080}{143}\,x^{4} - \frac{4624220160}{143}\,x^{2}$$
$$P^*_{16} = \frac{7}{30}\,x^{16} + \frac{784}{65}\,x^{14} + \frac{65856}{143}\,x^{12} + \frac{1756160}{143}\,x^{10} + \frac{30105600}{143}\,x^{8} + \frac{289013760}{143}\,x^{6} + \frac{1156055040}{143}\,x^{4}$$
$$Q^*_{16} = -\frac{56}{15}\,x^{15} - \frac{10976}{65}\,x^{13} - \frac{790272}{143}\,x^{11} - \frac{17561600}{143}\,x^{9} - \frac{240844800}{143}\,x^{7} - \frac{1734082560}{143}\,x^{5} - \frac{4624220160}{143}\,x^{3}$$
$$R^*_{16} =$$
$$\frac{4}{15}\,x^{16} + \frac{2744}{195}\,x^{14} + \frac{395136}{715}\,x^{12} + \frac{2195200}{143}\,x^{10} + \frac{40140800}{143}\,x^{8} + \frac{433520640}{143}\,x^{6} + \frac{2312110080}{143}\,x^{4} + \frac{4624220160}{143}\,x^{2}$$

Recurrence Formulas:

$$\int x^{2n+2} J_0(x) J_1(x)\,dx =$$
$$= \frac{x^{2n+1}}{4n+2}\left[-nxJ_0^2(x) + 2n(n+1)J_0(x)J_1(x) + (n+1)xJ_1^2(x)\right] - \frac{2n^2(n+1)}{2n+1}\int x^{2n} J_0(x) J_1(x)\,dx$$
$$\int x^{2n+2} I_0(x) I_1(x)\,dx =$$
$$= \frac{x^{2n+1}}{4n+2}\left[nxI_0^2(x) - 2n(n+1)I_0(x)I_1(x) + (n+1)xI_1^2(x)\right] + \frac{2n^2(n+1)}{2n+1}\int x^{2n} I_0(x) I_1(x)\,dx$$

2.1.5. Integrals of the type $\int x^{2n+1} Z_0(x) Z_1(x)\,dx$

The integrals $\int J_0^2(x)\,dx$ and $\int I_0^2(x)\,dx$ may be defined as the functions $\Theta(x)$ and $\Omega(x)$ in 2.1.3. , page 196 .

$$*E* \qquad \int x\,J_0(x)J_1(x)\,dx = -\frac{x}{2}\,J_0^2(x) + \frac{1}{2}\int J_0^2(x)\,dx$$

$$*E* \qquad \int x\,I_0(x)I_1(x)\,dx = \frac{x}{2}\,I_0^2(x) - \frac{1}{2}\int I_0^2(x)\,dx$$

$$\int x^3\,J_0(x)J_1(x)\,dx = \frac{1}{16}\left[(-2\,x^3+3\,x)\,J_0^2(x) + 6\,x^2 J_0(x)J_1(x) + 6\,x^3\,J_1^2(x)\right] - \frac{3}{16}\int J_0^2(x)\,dx$$

$$\int x^3\,I_0(x)I_1(x)\,dx = \frac{1}{16}\left[(2\,x^3+3\,x)\,I_0^2(x) - 6x^2 I_0(x)I_1(x) + x^3\,I_1^2(x)\right] - \frac{3}{16}\int I_0^2(x)\,dx$$

With

$$\int x^{2n+1}\,J_0(x)\cdot J_1(x)\,dx = \frac{1}{\beta_n}\left[A_n(x)\,J_0^2(x) + B_n(x)J_0(x)J_1(x) + C_n(x)\,J_1^2(x) + \gamma_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n+1}\,I_0(x)\cdot I_1(x)\,dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\,I_0^2(x) + I_n^*(x)I_0(x)I_1(x) + C_n^*(x)\,I_1^2(x) + \gamma_n^*\int I_0^2(x)\,dx\right]$$

holds

$$\beta_2 = 256\,, \qquad \gamma_2 = 135$$
$$A_2(x) = -48\,x^5 + 90\,x^3 - 135\,x\,, \quad B_2(x) = 240\,x^4 - 270\,x^2\,, \quad C_2(x) = 80\,x^5 - 270\,x^3$$

$$\beta_2^* = 256\,, \qquad \gamma_2^* = -135$$
$$A_2^*(x) = 48\,x^5 + 90\,x^3 + 135\,x\,, \quad B_2^*(x) = -240\,x^4 - 270\,x^2\,, \quad C_2^*(x) = 80\,x^5 + 270\,x^3$$

$$\beta_3 = 30720\,, \qquad \gamma_3 = -118125$$
$$A_3(x) = -6400\,x^7 + 42000\,x^5 - 78750\,x^3 + 118125\,x$$
$$B_3(x) = 44800\,x^6 - 210000\,x^4 + 236250\,x^2\,, \quad C_3(x) = 8960\,x^7 - 70000\,x^5 + 236250\,x^3$$

$$\beta_3^* = 30720\,, \qquad \gamma_3^* = -118125$$
$$A_3^*(x) = 6400\,x^7 + 42000\,x^5 + 78750\,x^3 + 118125\,x$$
$$B_3^*(x) = -44800\,x^6 - 210000\,x^4 - 236250\,x^2\,, \quad C_3^*(x) = 8960\,x^7 + 70000\,x^5 + 236250\,x^3$$

$$\beta_4 = 983040\,, \qquad \gamma_4 = 52093125$$
$$A_4(x) = -215040\,x^9 + 2822400\,x^7 - 18522000\,x^5 + 34728750\,x^3 - 52093125\,x$$
$$B_4(x) = 1935360\,x^8 - 19756800\,x^6 + 92610000\,x^4 - 104186250\,x^2$$
$$C_4(x) = 276480\,x^9 - 3951360\,x^7 + 30870000\,x^5 - 104186250\,x^3$$

$$\beta_4^* = 983040\,, \qquad \gamma_4^* = -52093125$$
$$A_4^*(x) = 215040\,x^9 + 2822400\,x^7 + 18522000\,x^5 + 34728750\,x^3 + 52093125\,x$$
$$B_4^*(x) = -1935360\,x^8 - 19756800\,x^6 - 92610000\,x^4 - 104186250\,x^2$$
$$C_4^*(x) = 276480\,x^9 + 3951360\,x^7 + 30870000\,x^5 + 104186250\,x^3$$

$$\beta_5 = 7864320\,, \qquad \gamma_5 = -9282994875$$
$$A_5(x) = -1769472\,x^{11} + 38320128\,x^9 - 502951680\,x^7 + 3300620400\,x^5 - 6188663250\,x^3 +$$

$$+9282994875\,x$$

$$B_5(x) = 19464192\,x^{10} - 344881152\,x^8 + 3520661760\,x^6 - 16503102000\,x^4 + 18565989750\,x^2$$

$$C_5(x) = 2162688\,x^{11} - 49268736\,x^9 + 704132352\,x^7 - 5501034000\,x^5 + 18565989750\,x^3$$

$$\beta_5^* = 7864320\;, \qquad \gamma_5^* = -9282994875$$

$$A_5^*(x) = 1769472\,x^{11} + 38320128\,x^9 + 502951680\,x^7 + 3300620400\,x^5 + 6188663250\,x^3 +$$
$$+9282994875\,x$$

$$B_5^*(x) = -19464192\,x^{10} - 344881152\,x^8 - 3520661760\,x^6 - 16503102000\,x^4 - 18565989750\,x^2$$

$$C_5^*(x) = 2162688\,x^{11} + 49268736\,x^9 + 704132352\,x^7 + 5501034000\,x^5 + 18565989750\,x^3$$

$$\beta_6 = 125829120\;, \qquad \gamma_6 = 4867383646125$$

$$A_6(x) = -28835840\,x^{13} + 927793152\,x^{11} - 20092520448\,x^9 + 263714330880\,x^7 - 1730625296400\,x^5 +$$
$$+3244922430750\,x^3 - 4867383646125\,x$$

$$B_6(x) = 374865920\,x^{12} - 10205724672\,x^{10} + 180832684032\,x^8 - 1846000316160\,x^6 +$$
$$+8653126482000\,x^4 - 9734767292250\,x^2$$

$$C_6(x) = 34078720\,x^{13} - 1133969408\,x^{11} + 25833240576\,x^9 - 369200063232\,x^7 + 2884375494000\,x^5 -$$
$$-9734767292250\,x^3$$

$$\beta_6^* = 125829120\;, \qquad \gamma_6^* = -4867383646125$$

$$A_6^*(x) = 28835840\,x^{13} + 927793152\,x^{11} + 20092520448\,x^9 + 263714330880\,x^7 + 1730625296400\,x^5 +$$
$$+3244922430750\,x^3 + 4867383646125\,x$$

$$B_6^*(x) = -374865920\,x^{12} - 10205724672\,x^{10} - 180832684032\,x^8 - 1846000316160\,x^6 -$$
$$-8653126482000\,x^4 - 9734767292250\,x^2$$

$$C_6^*(x) = 34078720\,x^{13} + 1133969408\,x^{11} + 25833240576\,x^9 + 369200063232\,x^7 + 2884375494000\,x^5 +$$
$$+9734767292250\,x^3$$

$$\beta_7 = 7046430720\;, \qquad \gamma_7 = -12338817542926875$$

$$A_7(x) = -1635778560\,x^{15} + 73098854400\,x^{13} - 2351955640320\,x^{11} + 50934539335680\,x^9 -$$
$$-668515828780800\,x^7 + 4387135126374000\,x^5 - 8225878361951250\,x^3 + 12338817542926875\,x$$

$$B_7(x) = 24536678400\,x^{14} - 950285107200\,x^{12} + 25871512043520\,x^{10} - 458410854021120\,x^8 +$$
$$+4679610801465600\,x^6 - 21935675631870000\,x^4 + 24677635085853750\,x^2$$

$$C_7(x) = 1887436800\,x^{15} - 86389555200\,x^{13} + 2874612449280\,x^{11} - 65487264860160\,x^9 +$$
$$+935922160293120\,x^7 - 7311891877290000\,x^5 + 24677635085853750\,x^3$$

$$\beta_7^* = 7046430720\;, \qquad \gamma_7^* = -12338817542926875$$

$$A_7^*(x) = 1635778560\,x^{15} + 73098854400\,x^{13} + 2351955640320\,x^{11} + 50934539335680\,x^9 +$$
$$+668515828780800\,x^7 + 4387135126374000\,x^5 + 8225878361951250\,x^3 + 12338817542926875\,x$$

$$B_7^*(x) = -24536678400\,x^{14} - 950285107200\,x^{12} - 25871512043520\,x^{10} - 458410854021120\,x^8 -$$
$$-4679610801465600\,x^6 - 21935675631870000\,x^4 - 24677635085853750\,x^2$$

$$C_7^*(x) = 1887436800\,x^{15} + 86389555200\,x^{13} + 2874612449280\,x^{11} + 65487264860160\,x^9 +$$
$$+935922160293120\,x^7 + 7311891877290000\,x^5 + 24677635085853750\,x^3$$

$$\beta_8 = 450971566080\ , \qquad \gamma_8 = 47195977101695296875$$

$$A_8(x) = -105696460800\,x^{17} + 6256852992000\,x^{15} - 279603118080000\,x^{13} + 8996230324224000\,x^{11} - $$
$$-194824612958976000\,x^9 + 2557073045086560000\,x^7 - 16780791858380550000\,x^5 + $$
$$+31463984734463531250\,x^3 - 47195977101695296875\,x$$

$$B_8(x) = 1796839833600\,x^{16} - 93852794880000\,x^{14} + 3634840535040000\,x^{12} - 98958533566464000\,x^{10} + $$
$$+1753421516630784000\,x^8 - 17899511315605920000\,x^6 + 83903959291902750000\,x^4 - $$
$$-94391954203390593750\,x^2$$

$$C_8(x) = 119789322240\,x^{17} - 7219445760000\,x^{15} + 330440048640000\,x^{13} - 10995392618496000\,x^{11} + $$
$$+250488788090112000\,x^9 - 3579902263121184000\,x^7 + 27967986430634250000\,x^5 - $$
$$-94391954203390593750\,x^3$$

$$\beta_8^* = 450971566080\ , \qquad \gamma_8^* = -47195977101695296875$$

$$A_8^*(x) = 105696460800\,x^{17} + 6256852992000\,x^{15} + 279603118080000\,x^{13} + 8996230324224000\,x^{11} + $$
$$+194824612958976000\,x^9 + 2557073045086560000\,x^7 + 16780791858380550000\,x^5 + $$
$$+31463984734463531250\,x^3 + 47195977101695296875\,x$$

$$B_8^*(x) = -1796839833600\,x^{16} - 93852794880000\,x^{14} - 3634840535040000\,x^{12} - 98958533566464000\,x^{10} - $$
$$-1753421516630784000\,x^8 - 17899511315605920000\,x^6 - 83903959291902750000\,x^4 - $$
$$-94391954203390593750\,x^2$$

$$C_8^*(x) = 119789322240\,x^{17} + 7219445760000\,x^{15} + 330440048640000\,x^{13} + 10995392618496000\,x^{11} + $$
$$+250488788090112000\,x^9 + 3579902263121184000\,x^7 + 27967986430634250000\,x^5 + $$
$$+94391954203390593750\,x^3$$

2.1.6. Integrals of the type $\int x^{-(2n+1)} Z_0(x) Z_1(x)\, dx$

See also [4], 1.8.3..

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x} = x[J_0^2(x) + J_1^2(x)] - J_0(x)\, J_1(x)$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x} = x[I_0^2(x) - I_1^2(x)] - I_0(x)\, I_1(x)$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x} = x[-K_0^2(x) + K_1^2(x)] - K_0(x)\, K_1(x)$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x} = -\, x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - I_0(x)\, K_1(x)$$

$$\int \frac{K_0(x)\, I_1(x)\, dx}{x} = x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - K_0(x)\, I_1(x)$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(-8\,x^2-3)\, J_0^2(x) + (8\,x^2-3)\, J_0(x)\, J_1(x) + x(-8\,x^2+1)\; J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(8\,x^2-3)\, I_0^2(x) - (8\,x^2+3)\, I_0(x)\, J_1(x) - x(8\,x^2+1)\; I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(-8\,x^2+3)\, K_0^2(x) + (-8\,x^2-3)\, K_0(x)\, K_1(x) + x(8\,x^2+1)\; K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2}\left[(-8\,x^3+3\,x)\, I_0(x)K_0(x) + (-4\,x^2-3)\, I_0(x)K_1(x) + 4\,x^2\, I_1(x)K_0(x) - (8\,x^3+x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2}\left[(8\,x^3-3\,x)\, I_0(x)K_0(x) + 4\,x^2\, I_0(x)K_1(x) - (4\,x^2+3)\, I_1(x)K_0(x) + (8\,x^3+x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(128\,x^4+48\,x^2-45)\, J_0^2(x) + (-128\,x^4+48\,x^2-135)\, J_0(x)\, J_1(x) + \right.$$
$$\left. + \,x(128\,x^4-16\,x^2+27)\; J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(128\,x^4-48\,x^2-45)\, I_0^2(x) - (128\,x^4+48\,x^2+135)\, I_0(x)\, I_1(x) - \right.$$
$$\left. -\, x(128\,x^4+16\,x^2+27)\; I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(-128\,x^4+48\,x^2+45)\, K_0^2(x) + (-128\,x^4-48\,x^2-135)\, K_0(x)\, K_1(x) + \right.$$
$$\left. +\, x(128\,x^4+16\,x^2+27)\; K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[(-128\,x^5+48\,x^3+45\,x)\, I_0(x)K_0(x) + (-64\,x^4-24\,x^2-135)\, I_0(x)K_1(x) + \right.$$
$$\left. + (64\,x^4+24\,x^2)\, I_1(x)K_0(x) + (-128\,x^5-16\,x^3-27\,x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^5} = \frac{1}{675x^4}\left[(128\,x^5-48\,x^3-45\,x)\, I_0(x)K_0(x) + (64\,x^4+24\,x^2)\, I_0(x)K_1(x) + \right.$$
$$\left. + (-64\,x^4-24\,x^2-135)\, I_1(x)K_0(x) + (128\,x^5+16\,x^3+27\,x)\, I_1(x)K_1(x)\right]$$

The integrals for $K_0(x)K_1(x)$ may be found in a simple way from such for $I_0(x)I_1(x)$. The same holds for $I_1(x)K_0(x)$, which is similar to $I_0(x)K_1(x)$.

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^7} = \frac{1}{55\,125 x^6} \left[x(-1\,024\, x^6 - 384\, x^4 + 360\, x^2 - 1\,575)\, J_0^2(x) + \right.$$

$$\left. +(1\,024\, x^6 - 384\, x^4 + 1\,080\, x^2 - 7\,875)\, J_0(x)\, J_1(x) + x\,(-1\,024\, x^6 + 128\, x^4 - 216\, x^2 + 1\,125)\; J_1^2(x) \right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^7} = \frac{1}{55\,125 x^6} \left[x(1\,024\, x^6 - 384\, x^4 - 360\, x^2 - 1\,575)\, I_0^2(x) - \right.$$

$$\left. -(1\,024\, x^6 + 384\, x^4 + 1\,080\, x^2 + 7\,875)\, I_0(x)\, I_1(x) - x\,(1\,024\, x^6 + 128\, x^4 + 216\, x^2 + 1\,125)\; I_1^2(x) \right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^7} = \frac{1}{55125 x^6} \left[(-1024\, x^7 + 384\, x^5 + 360\, x^3 + 1575\, x)\, I_0(x) K_0(x) - \right.$$

$$-(512\, x^6 + 192\, x^4 + 540\, x^2 + 7875)\, I_0(x) K_1(x) + (512\, x^6 + 192\, x^4 + 540\, x^2)\, I_1(x) K_0(x) -$$

$$\left. -(1024\, x^7 + 128\, x^5 + 216\, x^3 + 1125\, x)\, I_1(x) K_1(x) \right]$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^9} = \frac{1}{31\,255\,875 x^8} \left[x\,(32\,768\, x^8 + 12\,288\, x^6 - 11\,520\, x^4 + 50\,400\, x^2 - 496\,125)\, J_0^2(x) + \right.$$

$$+(-32\,768\, x^8 + 12\,288\, x^6 - 34\,560\, x^4 + 252\,000\, x^2 - 3\,472\,875)\, J_0(x)\, J_1(x) +$$

$$\left. +x\,(32\,768\, x^8 - 4\,096\, x^6 + 6\,912\, x^4 - 36\,000\, x^2 + 385\,875)\; J_1^2(x) \right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^9} = \frac{1}{31\,255\,875 x^8} \left[x\,(32\,768\, x^8 - 12\,288\, x^6 - 11\,520\, x^4 - 50\,400\, x^2 - 496\,125)\, I_0^2(x) - \right.$$

$$-(32\,768\, x^8 + 12\,288\, x^6 + 34\,560\, x^4 + 252\,000\, x^2 + 3\,472\,875)\, I_0(x)\, I_1(x) -$$

$$\left. -x\,(32\,768\, x^8 + 4\,096\, x^6 + 6\,912\, x^4 + 36\,000\, x^2 + 385\,875)\; I_1^2(x) \right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^9} = \frac{1}{31255875 x^8} \left[(-32768\, x^9 + 12288\, x^7 + 11520\, x^5 + 50400\, x^3 + 496125\, x)\, I_0(x) K_0(x) - \right.$$

$$-(16384\, x^8 + 6144\, x^6 + 17280\, x^4 + 126000\, x^2 + 3472875)\, I_0(x) K_1(x) +$$

$$+(16384\, x^8 + 6144\, x^6 + 17280\, x^4 + 126000\, x^2)\, I_1(x) K_0(x) -$$

$$\left. -(32768\, x^9 + 4096\, x^7 + 6912\, x^5 + 36000\, x^3 + 385875\, x)\, I_1(x) K_1(x) \right]$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^{11}} = \frac{1}{3\,403\,7647\,875 x^{10}} \left[x\,(-1\,310\,720\, x^{10} - \right.$$

$$-491\,520\, x^8 + 460\,800\, x^6 - 2\,016\,000\, x^4 + 19\,845\,000\, x^2 - 343\,814\,625)\, J_0^2(x) +$$

$$+(1\,310\,720\, x^{10} - 491\,520\, x^8 + 1\,382\,400\, x^6 - 10\,080\,000\, x^4 + 138\,915\,000\, x^2 - 3\,094\,331\,625)\, J_0(x)\, J_1(x) +$$

$$\left. +x\,(-1\,310\,720\, x^{10} + 163\,840\, x^8 - 276\,480\, x^6 + 1\,440\,000\, x^4 - 15\,435\,000\, x^2 + 281\,302\,875)\, J_1^2(x) \right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^{11}} = \frac{1}{34\,037\,647\,875 x^{10}} \left[x\,(1\,310\,720\, x^{10} - \right.$$

$$-491\,520\, x^8 - 460\,800\, x^6 - 2\,016\,000\, x^4 - 19\,845\,000\, x^2 - 343\,814\,625)\, I_0^2(x) -$$

$$-(1\,310\,720\, x^{10} + 491\,520\, x^8 + 1\,382\,400\, x^6 + 10\,080\,000\, x^4 + 138\,915\,000\, x^2 + 3\,094\,331\,625)\, I_0(x)\, I_1(x) -$$

$$\left. -x\,(1\,310\,720\, x^{10} + 163\,840\, x^8 + 276\,480\, x^6 + 1\,440\,000\, x^4 + 15\,435\,000\, x^2 + 281\,302\,875)\; I_1^2(x) \right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^{11}} = \frac{1}{6807529575 x^{10}} \cdot$$

$$\cdot \left[(-262144\, x^{11} + 98304\, x^9 + 92160\, x^7 + 403200\, x^5 + 3969000\, x^3 + 68762925\, x)\, I_0(x) K_0(x) - \right.$$

$$-(131072\, x^{10} + 49152\, x^8 + 138240\, x^6 + 1008000\, x^4 + 13891500\, x^2 + 618866325)\, I_0(x) K_1(x) +$$

$$+(131072\,x^{10}+49152\,x^8+138240\,x^6+1008000\,x^4+13891500\,x^2)\,I_1(x)K_0(x)-$$
$$-(262144\,x^{11}+32768\,x^9+55296\,x^7+288000\,x^5+3087000\,x^3+56260575\,x)\,I_1(x)K_1(x)\big]$$

$$\int\frac{J_0(x)\,J_1(x)\,dx}{x^{13}} = \frac{1}{4\,218\,399\,159\,975x^{12}}\,\big[x\,(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-1\,474\,560\,x^8+$$
$$+6\,451\,200\,x^6-63\,504\,000\,x^4+1\,100\,206\,800\,x^2-29\,499\,294\,825)\,J_0^2(x)+$$
$$+(-4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-4\,423\,680\,x^8+32\,256\,000\,x^6-$$
$$-444\,528\,000\,x^4+9\,901\,861\,200\,x^2-324\,492\,243\,075)\,J_0(x)\,J_1(x)+$$
$$+x\,(4\,194\,304\,x^{12}-524\,288\,x^{10}+884\,736\,x^8-4\,608\,000\,x^6+$$
$$+49\,392\,000\,x^4-900\,169\,200\,x^2+24\,960\,941\,775)\;J_1^2(x)\big]$$

$$\int\frac{I_0(x)\,I_1(x)\,dx}{x^{13}} = \frac{1}{4\,218\,399\,159\,975x^{12}}\,\big[x\,(4\,194\,304\,x^{12}-1\,572\,864\,x^{10}-1\,474\,560\,x^8-$$
$$-6\,451\,200\,x^6-63\,504\,000\,x^4-1\,100\,206\,800\,x^2-29\,499\,294\,825)\,I_0^2(x)-$$
$$-(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}+4\,423\,680\,x^8+32\,256\,000\,x^6+$$
$$+444\,528\,000\,x^4+9\,901\,861\,200\,x^2+324\,492\,243\,075)\,I_0(x)\,I_1(x)-$$
$$-x\,(4\,194\,304\,x^{12}+524\,288\,x^{10}+884\,736\,x^8+4\,608\,000\,x^6+$$
$$+49\,392\,000\,x^4+900\,169\,200\,x^2+24\,960\,941\,775)\;I_1^2(x)\big]$$

$$\int\frac{I_0(x)\,K_1(x)\,dx}{x^{13}} = \frac{1}{4218399159975x^{12}}\,\cdot$$
$$\cdot\,\big[(-4194304\,x^{13}+1572864\,x^{11}+1474560\,x^9+6451200\,x^7+63504000\,x^5+1100206800\,x^3+$$
$$+29499294825\,x)\,I_0(x)K_0(x)-(2097152\,x^{12}+786432\,x^{10}+2211840\,x^8+16128000\,x^6+222264000\,x^4+$$
$$*4950930600\,x^2+324492243075)\,I_0(x)K_1(x)+(2097152\,x^{12}+786432\,x^{10}+2211840\,x^8+16128000\,x^6+$$
$$+222264000\,x^4+4950930600\,x^2)\,I_1(x)K_0(x)-(4194304\,x^{13}+524288\,x^{11}+884736\,x^9+4608000\,x^7+$$
$$+49392000\,x^5+900169200\,x^3+24960941775\,x)\,I_1(x)K_1(x)\big]$$

<u>Recurrence Relations:</u>

$$\int\frac{J_0(x)\,J_1(x)\,dx}{x^{2n+1}} =$$
$$=\frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,J_0^2(x)}{2n-1}-J_0(x)\,J_1(x)+\frac{x\,J_1^2(x)}{2n+1}\right]-\frac{8n}{(2n+1)^2(2n-1)}\int\frac{J_0(x)\,J_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{I_0(x)\,I_1(x)\,dx}{x^{2n+1}} =$$
$$=-\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_0^2(x)}{2n-1}+I_0(x)\,I_1(x)+\frac{x\,I_1^2(x)}{2n+1}\right]+\frac{8n}{(2n+1)^2(2n-1)}\int\frac{I_0(x)\,I_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{K_0(x)\,K_1(x)\,dx}{x^{2n+1}} =$$
$$=\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,K_0^2(x)}{2n-1}-K_0(x)\,K_1(x)+\frac{x\,K_1^2(x)}{2n+1}\right]+\frac{8n}{(2n+1)^2(2n-1)}\int\frac{K_0(x)\,K_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{I_0(x)\,K_1(x)\,dx}{x^{2n+1}}=\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_0(x)\,K_0(x)}{2n-1}-I_0(x)\,K_1(x)-\frac{x\,I_1(x)\,K_1(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^2(2n-1)}\int\frac{I_0(x)\,K_1(x)-I_1(x)\,K_0(x)}{x^{2n-1}}\,dx$$

$$\int\frac{I_1(x)\,K_0(x)\,dx}{x^{2n+1}}=\frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,I_0(x)\,K_0(x)}{2n-1}-I_1(x)\,K_0(x)+\frac{x\,I_1(x)\,K_1(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^2(2n-1)}\int\frac{I_1(x)\,K_0(x)-I_0(x)\,K_1(x)}{x^{2n-1}}\,dx$$

2.1.7. Integrals of the type $\int x^{2n+1} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_1(x) \\ K_\nu(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x \cdot J_0(x) \cdot I_0(x)\,dx = \frac{x}{2}\,[J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x)]$$

$$\int x \cdot J_0(x) \cdot K_0(x)\,dx = \frac{x}{2}\,[-J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x)]$$

$$\int x^3 \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[x^3 J_0(x) \cdot I_1(x) + x^3 J_1(x) \cdot I_0(x) - 2x^2 J_1(x) \cdot I_1(x)\right]$$

$$\int x^3 \cdot J_0(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[-x^3 J_0(x) \cdot K_1(x) + x^3 J_1(x) \cdot K_0(x) + 2x^2 J_1(x) \cdot K_1(x)\right]$$

$$\int x^5 \cdot J_0(x) \cdot I_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[8\,x^2 J_0(x) \cdot I_0(x) + (x^5 - 4\,x^3 - 8\,x) J_0(x) \cdot I_1(x) + (x^5 + 4\,x^3 - 8\,x) J_1(x) \cdot I_0(x) - 4\,x^4 J_1(x) \cdot I_1(x)\right]$$

$$\int x^5 \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[8\,x^2 J_0(x) \cdot K_0(x) - (x^5 - 4\,x^3 - 8\,x) J_0(x) \cdot K_1(x) + (x^5 + 4\,x^3 - 8\,x) J_1(x) \cdot K_0(x) + 4\,x^4 J_1(x) \cdot I_K(x)\right]$$

$$\int x^7 \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[48\,x^4 J_0(x) \cdot I_0(x) + (x^7 - 12\,x^5 - 96\,x^3) J_0(x) \cdot I_1(x) + \right.$$
$$\left. + (x^7 + 12\,x^5 - 96\,x^3) J_1(x) \cdot I_0(x) + (-6\,x^6 + 192\,x^2) J_1(x) \cdot I_1(x)\right]$$

$$\int x^7 \cdot J_0(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[48\,x^4 J_0(x) \cdot K_0(x) - (x^7 - 12\,x^5 - 96\,x^3) J_0(x) \cdot K_1(x) + \right.$$
$$\left. + (x^7 + 12\,x^5 - 96\,x^3) J_1(x) \cdot K_0(x) + (6\,x^6 - 192\,x^2) J_1(x) \cdot K_1(x)\right]$$

$$\int x^9 \cdot J_0(x) \cdot I_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[(144\,x^6 - 3456\,x^2) J_0(x) \cdot I_0(x) + (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x) J_0(x) \cdot I_1(x) + \right.$$
$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x) J_1(x) \cdot I_0(x) + (-8\,x^8 + 1728\,x^4) J_1(x) \cdot I_1(x)\right]$$

$$\int x^9 \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[(144\,x^6 - 3456\,x^2) J_0(x) \cdot K_0(x) - (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x) J_0(x) \cdot K_1(x) + \right.$$
$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x) J_1(x) \cdot K_0(x) + (8\,x^8 - 1728\,x^4) J_1(x) \cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x)]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,[P_n(x) J_0(x) \cdot K_0(x) - Q_n(x) J_0(x) \cdot K_1(x) + R_n(x) J_1(x) \cdot K_0(x) - S_n(x) J_1(x) \cdot K_1(x)]\ .$$

$$P_{11}(x) = 320\,x^8 - 61440\,x^4$$
$$Q_{11}(x) = x^{11} - 40\,x^9 - 1280\,x^7 + 15360\,x^5 + 122880\,x^3$$

$$R_{11}(x) = x^{11} + 40\,x^9 - 1280\,x^7 - 15360\,x^5 + 122880\,x^3$$
$$S_{11}(x) = -10\,x^{10} + 7680\,x^6 - 245760\,x^2$$

$$P_{13}(x) = 600\,x^{10} - 432000\,x^6 + 10368000\,x^2$$
$$Q_{13}(x) = x^{13} - 60\,x^{11} - 3000\,x^9 + 72000\,x^7 + 1296000\,x^5 - 5184000\,x^3 - 10368000\,x$$
$$R_{13}(x) = x^{13} + 60\,x^{11} - 3000\,x^9 - 72000\,x^7 + 1296000\,x^5 + 5184000\,x^3 - 10368000\,x$$
$$S_{13}(x) = -12\,x^{12} + 24000\,x^8 - 5184000\,x^4$$

$$P_{15}(x) = 1008\,x^{12} - 1935360\,x^8 + 371589120\,x^4$$
$$Q_{15}(x) = x^{15} - 84\,x^{13} - 6048\,x^{11} + 241920\,x^9 + 7741440\,x^7 - 92897280\,x^5 - 743178240\,x^3$$
$$R_{15}(x) = x^{15} + 84\,x^{13} - 6048\,x^{11} - 241920\,x^9 + 7741440\,x^7 + 92897280\,x^5 - 743178240\,x^3$$
$$S_{15}(x) = -14\,x^{14} + 60480\,x^{10} - 46448640\,x^6 + 1486356480\,x^2$$

b) $\nu = 1$:

$$\int x \cdot J_1(x) \cdot I_1(x)\,dx = \frac{x}{2}\,\left[-J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x)\right]$$
$$\int x \cdot J_1(x) \cdot K_1(x)\,dx = -\,\frac{x}{2}\,\left[J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x)\right]$$
$$\int x^3 \cdot J_1(x) \cdot I_1(x)\,dx = \frac{1}{2}\,\left[2\,x^2 J_0(x) \cdot I_0(x) - (x^3 + 2\,x) J_0(x) \cdot I_1(x) + (x^3 - 2\,x) J_1(x) \cdot I_0(x)\right]$$
$$\int x^3 \cdot J_1(x) \cdot K_1(x)\,dx = \frac{1}{2}\,\left[-2\,x^2 J_0(x) \cdot K_0(x) - (x^3 + 2\,x) J_0(x) \cdot K_1(x) - (x^3 - 2\,x) J_1(x) \cdot K_0(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_0(x)\,dx =$$
$$= \frac{1}{2}\,\left[-P_n(x) J_0(x) \cdot K_0(x) + Q_n(x) J_0(x) \cdot K_1(x) - R_n(x) J_1(x) \cdot K_0(x) + S_n(x) J_1(x) \cdot K_1(x)\right]\ .$$

$$P_5(x) = 4\,x^4\ , \quad Q_5(x) = -x^5 - 8\,x^3\ , \quad R_5(x) = x^5 - 8\,x^3\ , \quad S_5(x) = 16\,x^2$$

$$P_7(x) = 6\,x^6 - 144\,x^2\ , \quad Q_7(x) = -x^7 - 18\,x^5 + 72\,x^3 + 144\,x$$
$$R_7(x) = x^7 - 18\,x^5 - 72\,x^3 + 144\,x\ , \quad S_7(x) = 72\,x^4$$

$$P_9(x) = 8\,x^8 - 1536\,x^4\ , \quad Q_9(x) = -x^9 - 32\,x^7 + 384\,x^5 + 3072\,x^3$$
$$R_9(x) = x^9 - 32\,x^7 - 384\,x^5 + 3072\,x^3\ , \quad S_9(x) = 192\,x^6 - 6144\,x^2$$

$$P_{11}(x) = 10\,x^{10} - 7200\,x^6 + 172800\,x^2$$
$$Q_{11}(x) = -x^{11} - 50\,x^9 + 1200\,x^7 + 21600\,x^5 - 86400\,x^3 - 172800\,x$$
$$R_{11}(x) = x^{11} - 50\,x^9 - 1200\,x^7 + 21600\,x^5 + 86400\,x^3 - 172800\,x$$
$$S_{11}(x) = 400\,x^8 - 86400\,x^4$$

$$P_{13}(x) = 12\,x^{12} - 23040\,x^{8} + 4423680\,x^{4}$$
$$Q_{13}(x) = -x^{13} - 72\,x^{11} + 2880\,x^{9} + 92160\,x^{7} - 1105920\,x^{5} - 8847360\,x^{3}$$
$$R_{13}(x) = x^{13} - 72\,x^{11} - 2880\,x^{9} + 92160\,x^{7} + 1105920\,x^{5} - 8847360\,x^{3}$$
$$S_{13}(x) = 720\,x^{10} - 552960\,x^{6} + 17694720\,x^{2}$$

$$P_{15}(x) = 14\,x^{14} - 58800\,x^{10} + 42336000\,x^{6} - 1016064000\,x^{2}$$
$$Q_{15}(x) = -x^{15} - 98\,x^{13} + 5880\,x^{11} + 294000\,x^{9} - 7056000\,x^{7} - 127008000\,x^{5} + 508032000\,x^{3} + 1016064000\,x$$
$$R_{15}(x) = x^{15} - 98\,x^{13} - 5880\,x^{11} + 294000\,x^{9} + 7056000\,x^{7} - 127008000\,x^{5} - 508032000\,x^{3} + 1016064000\,x$$
$$S_{15}(x) = 1176\,x^{12} - 2352000\,x^{8} + 508032000\,x^{4}$$

c) Recurrence relations:

$$\int x^{2n+1}\,J_0(x)\,I_0(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) - 2n\,J_1(x)\,I_1(x)\right] + 2n(n-1)\int x^{2n-1}\,J_1(x)\,I_1(x)\,dx$$
$$\int x^{2n+1}\,J_1(x)\,I_1(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[-x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) + 2n\,J_0(x)\,I_0(x)\right] - 2n^2\int x^{2n-1}\,J_0(x)\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,I_0(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[-x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_1(x)\,K_1(x)\right] - 2n(n-1)\int x^{2n-1}\,J_1(x)\,K_1(x)\,dx$$
$$\int x^{2n+1}\,J_1(x)\,K_1(x)\,dx =$$
$$= -\,\frac{x^{2n}}{2}\,\left[x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_0(x)\,K_0(x)\right] + 2n^2\int x^{2n-1}\,J_0(x)\,K_0(x)\,dx$$

2.1.8. Integrals of the type $\int x^{2n} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_{1-\nu}(x) \\ K_{1-\nu}(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x^2 \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2}\left[x^2 J_0(x) \cdot I_0(x) - x J_0(x) \cdot I_1(x) - x J_1(x) \cdot I_0(x) + x^2 J_1(x) \cdot I_1(x)\right]$$

$$\int x^2 \cdot J_0(x) \cdot K_1(x)\, dx = \frac{1}{2}\left[-x^2 J_0(x) \cdot K_0(x) - x J_0(x) \cdot K_1(x) + x J_1(x) \cdot K_0(x) + x^2 J_1(x) \cdot K_1(x)\right]$$

$$\int x^4 \cdot J_0(x) \cdot I_1(x)\, dx =$$

$$= \frac{1}{2}\left[(x^4 - 2\,x^2) J_0(x) \cdot I_0(x) + (-x^3 + 2\,x) J_0(x) \cdot I_1(x) + (-3\,x^3 + 2\,x) J_1(x) \cdot I_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot I_1(x)\right]$$

$$\int x^4 \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2}\left[-(x^4 - 2\,x^2) J_0(x) \cdot K_0(x) - (x^3 - 2\,x) J_0(x) \cdot K_1(x) + (3\,x^3 - 2\,x) J_1(x) \cdot K_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2}\left[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2}\left[-P_n(x) J_0(x) \cdot K_0(x) + Q_n(x) J_0(x) \cdot K_1(x) - R_n(x) J_1(x) \cdot K_0(x) + S_n(x) J_1(x) \cdot K_1(x)\right] .$$

$$P_6(x) = x^6 - 8\,x^4 - 24\,x^2 , \quad Q_6(x) = -x^5 + 28\,x^3 + 24\,x$$
$$R_6(x) = -5\,x^5 + 4\,x^3 + 24\,x , \quad S_6(x) = x^6 + 12\,x^4 - 32\,x^2$$

$$P_8(x) = x^8 - 18\,x^6 - 192\,x^4 + 432\,x^2 , \quad Q_8(x) = -x^7 + 102\,x^5 + 168\,x^3 - 432\,x$$
$$R_8(x) = -7\,x^7 + 6\,x^5 + 600\,x^3 - 432\,x , \quad S_8(x) = x^8 + 24\,x^6 - 216\,x^4 - 768\,x^2$$

$$P_{10}(x) = x^{10} - 32\,x^8 - 720\,x^6 + 6144\,x^4 + 17280\,x^2$$
$$Q_{10}(x) = -x^9 + 248\,x^7 + 624\,x^5 - 20928\,x^3 - 17280\,x$$
$$R_{10}(x) = -9\,x^9 + 8\,x^7 + 3696\,x^5 - 3648\,x^3 - 17280\,x$$
$$S_{10}(x) = x^{10} + 40\,x^8 - 768\,x^6 - 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = x^{12} - 50\,x^{10} - 1920\,x^8 + 36000\,x^6 + 368640\,x^4 - 864000\,x^2$$
$$Q_{12}(x) = -x^{11} + 490\,x^9 + 1680\,x^7 - 200160\,x^5 - 305280\,x^3 + 864000\,x$$
$$R_{12}(x) = -11\,x^{11} + 10\,x^9 + 13680\,x^7 - 15840\,x^5 - 1169280\,x^3 + 864000\,x$$
$$S_{12}(x) = x^{12} + 60\,x^{10} - 2000\,x^8 - 46080\,x^6 + 432000\,x^4 + 1474560\,x^2$$

$$P_{14}(x) = x^{14} - 72\,x^{12} - 4200\,x^{10} + 138240\,x^8 + 3024000\,x^6 - 26542080\,x^4 - 72576000\,x^2$$
$$Q_{14}(x) = -x^{13} + 852\,x^{11} + 3720\,x^9 - 1056960\,x^7 - 2436480\,x^5 + 89372160\,x^3 + 72576000\,x$$
$$R_{14}(x) = -13\,x^{13} + 12\,x^{11} + 38280\,x^9 - 48960\,x^7 - 15707520\,x^5 + 16796160\,x^3 + 72576000\,x$$
$$S_{14}(x) = x^{14} + 84\,x^{12} - 4320\,x^{10} - 168000\,x^8 + 3317760\,x^6 + 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= x^{16} - 98\,x^{14} - 8064\,x^{12} + 411600\,x^{10} + 15482880\,x^8 - 296352000\,x^6 - 2972712960\,x^4 + 7112448000\,x^2$$
$$Q_{16}(x) =$$
$$= -x^{15} + 1358\,x^{13} + 7224\,x^{11} - 3993360\,x^9 - 12539520\,x^7 + 1632234240\,x^5 + 2389201920\,x^3 - 7112448000\,x$$
$$R_{16}(x) =$$
$$= -15\,x^{15} + 14\,x^{13} + 89544\,x^{11} - 122640\,x^9 - 111323520\,x^7 + 145877760\,x^5 + 9501649920\,x^3 - 7112448000\,x$$
$$S_{16}(x) =$$
$$= x^{16} + 112\,x^{14} - 8232\,x^{12} - 483840\,x^{10} + 16464000\,x^8 + 371589120\,x^6 - 3556224000\,x^4 - 11890851840\,x^2$$

a) $\nu = 1$:

$$\int x^2 \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[-x^2 J_0(x)\cdot I_0(x) + xJ_0(x)\cdot I_1(x) + xJ_1(x)\cdot I_0(x) + x^2 J_1(x)\cdot I_1(x)\right]$$
$$\int x^2 \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[-x^2 J_0(x)\cdot K_0(x) - xJ_0(x)\cdot K_1(x) + xJ_1(x)\cdot K_0(x) - x^2 J_1(x)\cdot K_1(x)\right]$$
$$\int x^4 \cdot J_1(x) \cdot I_0(x)\,dx =$$
$$= \frac{1}{2}\,\left[(-x^4 - 2\,x^2)J_0(x)\cdot I_0(x) + (3\,x^3 + 2\,x)J_0(x)\cdot I_1(x) + (x^3 + 2\,x)J_1(x)\cdot I_0(x) + (x^4 - 4\,x^2)J_1(x)\cdot I_1(x)\right]$$
$$\int x^4 \cdot J_1(x) \cdot K_0(x)\,dx =$$
$$= \frac{1}{2}\,\left[(-x^4 - 2\,x^2)J_0(x)\cdot K_0(x) - (3\,x^3 + 2\,x)J_0(x)\cdot K_1(x) + (x^3 + 2\,x)J_1(x)\cdot K_0(x) - (x^4 - 4\,x^2)J_1(x)\cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[P_n(x)J_0(x)\cdot I_0(x) + Q_n(x)J_0(x)\cdot I_1(x) + R_n(x)J_1(x)\cdot I_0(x) + S_n(x)J_1(x)\cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[P_n(x)J_0(x)\cdot K_0(x) - Q_n(x)J_0(x)\cdot K_1(x) + R_n(x)J_1(x)\cdot K_0(x) - S_n(x)J_1(x)\cdot K_1(x)\right]\,.$$

$$P_6(x) = -x^6 - 8\,x^4 + 24\,x^2\,, \quad Q_6(x) = 5\,x^5 + 4\,x^3 - 24\,x$$
$$R_6(x) = x^5 + 28\,x^3 - 24\,x\,, \quad S_6(x) = x^6 - 12\,x^4 - 32\,x^2$$

$$P_8(x) = -x^8 - 18\,x^6 + 192\,x^4 + 432\,x^2\,, \quad Q_8(x) = 7\,x^7 + 6\,x^5 - 600\,x^3 - 432\,x$$
$$R_8(x) = x^7 + 102\,x^5 - 168\,x^3 - 432\,x\,, \quad S_8(x) = x^8 - 24\,x^6 - 216\,x^4 + 768\,x^2$$

$$P_{10}(x) = -x^{10} - 32\,x^8 + 720\,x^6 + 6144\,x^4 - 17280\,x^2$$
$$Q_{10}(x) = 9\,x^9 + 8\,x^7 - 3696\,x^5 - 3648\,x^3 + 17280\,x$$
$$R_{10}(x) = x^9 + 248\,x^7 - 624\,x^5 - 20928\,x^3 + 17280\,x$$
$$S_{10}(x) = x^{10} - 40\,x^8 - 768\,x^6 + 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = -x^{12} - 50\,x^{10} + 1920\,x^8 + 36000\,x^6 - 368640\,x^4 - 864000\,x^2$$
$$Q_{12}(x) = 11\,x^{11} + 10\,x^9 - 13680\,x^7 - 15840\,x^5 + 1169280\,x^3 + 864000\,x$$
$$R_{12}(x) = x^{11} + 490\,x^9 - 1680\,x^7 - 200160\,x^5 + 305280\,x^3 + 864000\,x$$

$$S_{12}(x) = x^{12} - 60\,x^{10} - 2000\,x^8 + 46080\,x^6 + 432000\,x^4 - 1474560\,x^2$$

$$P_{14}(x) = -x^{14} - 72\,x^{12} + 4200\,x^{10} + 138240\,x^8 - 3024000\,x^6 - 26542080\,x^4 + 72576000\,x^2$$
$$Q_{14}(x) = 13\,x^{13} + 12\,x^{11} - 38280\,x^9 - 48960\,x^7 + 15707520\,x^5 + 16796160\,x^3 - 72576000\,x$$
$$R_{14}(x) = x^{13} + 852\,x^{11} - 3720\,x^9 - 1056960\,x^7 + 2436480\,x^5 + 89372160\,x^3 - 72576000\,x$$
$$S_{14}(x) = x^{14} - 84\,x^{12} - 4320\,x^{10} + 168000\,x^8 + 3317760\,x^6 - 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= -x^{16} - 98\,x^{14} + 8064\,x^{12} + 411600\,x^{10} - 15482880\,x^8 - 296352000\,x^6 + 2972712960\,x^4 + 7112448000\,x^2$$
$$Q_{16}(x) =$$
$$= 15\,x^{15} + 14\,x^{13} - 89544\,x^{11} - 122640\,x^9 + 111323520\,x^7 + 145877760\,x^5 - 9501649920\,x^3 - 7112448000\,x$$
$$R_{16}(x) =$$
$$= x^{15} + 1358\,x^{13} - 7224\,x^{11} - 3993360\,x^9 + 12539520\,x^7 + 1632234240\,x^5 - 2389201920\,x^3 - 7112448000\,x$$
$$S_{16}(x) =$$
$$= x^{16} - 112\,x^{14} - 8232\,x^{12} + 483840\,x^{10} + 16464000\,x^8 - 371589120\,x^6 - 3556224000\,x^4 + 11890851840\,x^2$$

c) Recurrence Relations:

$$\int x^{2n+2}\,J_0(x)\,I_1(x)\,dx = \frac{x^{2n+1}}{2}\,\left[x\,J_0(x)\,I_0(x) - J_0(x)\,I_1(x) - (2n+1)\,J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)\right] +$$
$$+ n\int x^{2n}\,J_0(x)\,I_1(x)\,dx + n(2n+1)\int x^{2n}\,J_1(x)\,I_0(x)\,dx$$

$$\int x^{2n+2}\,J_1(x)\,I_0(x)\,dx = \frac{x^{2n+1}}{2}\,\left[-x\,J_0(x)\,I_0(x) + (2n+1)\,J_0(x)\,I_1(x) + J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)\right] -$$
$$- n(2n+1)\int x^{2n}\,J_0(x)\,I_1(x)\,dx - n\int x^{2n}\,J_1(x)\,I_0(x)\,dx$$

$$\int x^{2n+2}\,J_0(x)\,K_1(x)\,dx = \frac{x^{2n+1}}{2}\,\left[-x\,J_0(x)\,K_0(x) - J_0(x)\,K_1(x) + (2n+1)\,J_1(x)\,K_0(x) + x\,J_1(x)\,K_1(x)\right] +$$
$$+ n\int x^{2n}\,J_0(x)\,K_1(x)\,dx - n(2n+1)\int x^{2n}\,J_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,J_1(x)\,K_0(x)\,dx = \frac{x^{2n+1}}{2}\,\left[-x\,J_0(x)\,K_0(x) - (2n+1)\,J_0(x)\,K_1(x) + J_1(x)\,K_0(x) - x\,J_1(x)\,K_1(x)\right] +$$
$$+ n(2n+1)\int x^{2n}\,J_0(x)\,K_1(x)\,dx - n\int x^{2n}\,J_1(x)\,K_0(x)\,dx$$

2.1.9. Integrals of the type $\int x^{2n[+1]} J_\mu(x)\, Y_\nu\, dx$, :

a) $\int x^{2n+1}\, J_0(x)Y_0(x)\, dx$:

$$\int x\, J_0(x)Y_0(x)\, dx = \frac{x^2}{2}\, [J_0(x)Y_0(x) + J_1(x)Y_1(x)]$$

$$\int x^3\, J_0(x)Y_0(x)\, dx =$$

$$= \frac{x^4}{6} J_0(x)Y_0(x) + \frac{x^3}{6}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \left(\frac{x^4}{6} - \frac{x^2}{3}\right)\, J_1(x)Y_1(x)$$

$$\int x^5\, J_0(x)Y_0(x)\, dx = \left(\frac{x^6}{10} + \frac{4}{15}\, x^4\right) J_0(x)Y_0(x) +$$

$$+\left(\frac{x^5}{5} - \frac{8}{15}\, x^3\right)[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \left(\frac{x^6}{10} - \frac{8}{15}\, x^4 + \frac{16}{15}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^7\, J_0(x)Y_0(x)\, dx = \left(\frac{x^8}{14} + \frac{18}{35}\, x^6 - \frac{72}{35}\, x^4\right) J_0(x)Y_0(x) +$$

$$+\left(\frac{3}{14}\, x^7 - \frac{54}{35}\, x^5 + \frac{144}{35}\, x^3\right)[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\left(\frac{x^8}{14} - \frac{27}{35}\, x^6 + \frac{144}{35}\, x^4 - \frac{288}{35}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^9\, J_0(x)Y_0(x)\, dx = \left(\frac{x^{10}}{18} + \frac{16}{21}\, x^8 - \frac{256}{35}\, x^6 + \frac{1024}{35}\, x^4\right) J_0(x)Y_0(x) +$$

$$+\left(\frac{2}{9}\, x^9 - \frac{64}{21}\, x^7 + \frac{768}{35}\, x^5 - \frac{2048}{35}\, x^3\right)[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\left(\frac{x^{10}}{18} - \frac{64}{63}\, x^8 + \frac{384}{35}\, x^6 - \frac{2048}{35}\, x^4 + \frac{4096}{35}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^{11}\, J_0(x)Y_0(x)\, dx = \left(\frac{x^{12}}{22} + \frac{100}{99}\, x^{10} - \frac{4000}{231}\, x^8 + \frac{12800}{77}\, x^6 - \frac{51200}{77}\, x^4\right) J_0(x)Y_0(x) +$$

$$+\left(\frac{5}{22}\, x^{11} - \frac{500}{99}\, x^9 + \frac{16000}{231}\, x^7 - \frac{38400}{77}\, x^5 + \frac{102400}{77}\, x^3\right)[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\left(\frac{x^{12}}{22} - \frac{125}{99}\, x^{10} + \frac{16000}{693}\, x^8 - \frac{19200}{77}\, x^6 + \frac{102400}{77}\, x^4 - \frac{204800}{77}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^{13}\, J_0(x)Y_0(x)\, dx =$$

$$= \left(\frac{x^{14}}{26} + \frac{180}{143}\, x^{12} - \frac{4800}{143}\, x^{10} + \frac{576000}{1001}\, x^8 - \frac{5529600}{1001}\, x^6 + \frac{22118400}{1001}\, x^4\right) J_0(x)Y_0(x) +$$

$$+\left(\frac{3}{13}\, x^{13} - \frac{1080}{143}\, x^{11} + \frac{24000}{143}\, x^9 - \frac{2304000}{1001}\, x^7 + \frac{16588800}{1001}\, x^5 - \frac{44236800}{1001}\, x^3\right)[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\left(\frac{x^{14}}{26} - \frac{216}{143}\, x^{12} + \frac{6000}{143}\, x^{10} - \frac{768000}{1001}\, x^8 + \frac{8294400}{1001}\, x^6 - \frac{44236800}{1001}\, x^4 + \frac{88473600}{1001}\, x^2\right)\, J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1}\, J_0(x)Y_0(x)\, dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{(2n^2 + x^2)J_0(x)Y_0(x) + nx[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + x^2 J_1(x)Y_1(x)\right\} -$$

$$-\frac{2n^3}{2n+1}\int x^{2n-1}\, J_0(x)Y_0(x)\, dx$$

b) $\int x^{-2n}\, J_0(x)Y_0(x)\, dx$:

$$\int \frac{J_0(x)Y_0(x)}{x^2}\, dx = -\frac{2x^2+1}{x}\, J_0(x)Y_0(x) + J_0(x)Y_1(x) + J_1(x)Y_0(x) - 2x\, J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_0(x)}{x^4}\, dx = \frac{1}{27\, x^3}\, \left\{(16\, x^4 + 6\, x^2 - 9)\, J_0(x)Y_0(x) + (-8\, x^3 + 3\, x)\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \right.$$
$$\left. + (16\, x^4 - 2\, x^2)\, J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^6}\, dx = \frac{1}{3375\, x^5}\, \left\{(-256\, x^6 - 96\, x^4 + 90\, x^2 - 675)\, J_0(x)Y_0(x) + \right.$$
$$+ (128\, x^5 - 48\, x^3 + 135\, x)\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$\left. + (-256\, x^6 + 32\, x^4 - 54\, x^2)\, J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^8}\, dx = \frac{1}{385875\, x^7}\, \left\{(2048\, x^8 + 768\, x^6 - 720\, x^4 + 3150\, x^2 - 55125)\, J_0(x)Y_0(x) + \right.$$
$$+ (-1024\, x^7 + 384\, x^5 - 1080\, x^3 + 7875\, x)\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$\left. + (2048\, x^8 - 256\, x^6 + 432\, x^4 - 2250\, x^2)\, J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{10}}\, dx = \frac{1}{281302875\, x^7}\, \cdot$$
$$\cdot \left\{(-65536\, x^{10} - 24576\, x^8 + 23040\, x^6 - 100800\, x^4 + 992250\, x^2 - 31255875)\, J_0(x)Y_0(x) + \right.$$
$$(32768\, x^9 - 12288\, x^7 + 34560\, x^5 - 252000\, x^3 + 3472875\, x)\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$\left. + \; (-65536\, x^{10} + 8192\, x^8 - 13824\, x^6 + 72000\, x^4 - 771750\, x^2)\, J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{12}}\, dx = \frac{1}{74882825325\, x^{11}}\, \cdot$$
$$\cdot \left\{(524288\, x^{12} + 196608\, x^{10} - 184320\, x^8 + 806400\, x^6 - 7938000\, x^4 + 137525850\, x^2 - 6807529575)\, J_0(x)Y_0(x) + \right.$$
$$+ (-262144\, x^{11} + 98304\, x^9 - 276480\, x^7 + 2016000\, x^5 - 27783000\, x^3 + 618866325\, x)\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$\left. + (524288\, x^{12} - 65536\, x^{10} + 110592\, x^8 - 576000\, x^6 + 6174000\, x^4 - 112521150\, x^2)\, J_1(x)Y_1(x)\right\}$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_0(x)}{x^{2n+2}}\, dx = \frac{1}{(2n+1)^3} \cdot$$
$$\cdot \left\{ \frac{-(4n^2 + 4n + 1 + 2x^2) J_0(x)Y_0(x) + (2n+1)x\, [J_0(x)Y_1(x) + J_1(x)Y_0(x)] - 2x^2 J_1(x)Y_1(x)}{x^{2n+1}} - \right.$$
$$\left. - \, 8n \int \frac{J_0(x)Y_0(x)}{x^{2n}}\, dx \right\}$$

c) $\int x^{2n}\, J_0(x)Y_1(x)\,dx$:

$$\int x^2\, J_0(x)Y_1(x)\,dx = \frac{x^3}{4} J_0(x)Y_1(x) - \frac{x^3}{4} J_1(x)Y_0(x) + \frac{x^2}{2} J_1(x)Y_1(x)$$

$$\int x^4\, J_0(x)Y_1(x)\,dx =$$

$$= -\frac{x^4}{6} J_0(x)Y_0(x) + \frac{3\,x^5+8\,x^3}{24} J_0(x)Y_1(x) - \frac{3\,x^5-8\,x^3}{24} J_1(x)Y_0(x) + \frac{x^4-2\,x^2}{3} J_1(x)Y_1(x)$$

$$\int x^6\, J_0(x)Y_1(x)\,dx = -\frac{x^6-4\,x^4}{5} J_0(x)Y_0(x) + \frac{5\,x^7+36\,x^5-96\,x^3}{60} J_0(x)Y_1(x) -$$

$$-\frac{5\,x^7-36\,x^5+96\,x^3}{60} J_1(x)Y_0(x) + \frac{3\,x^6-16\,x^4+32\,x^2}{10} J_1(x)Y_1(x)$$

$$\int x^8\, J_0(x)Y_1(x)\,dx = -\frac{15\,x^8-144\,x^6+576\,x^4}{70} J_0(x)Y_0(x) +$$

$$+\frac{35\,x^9+480\,x^7-3456\,x^5+9216\,x^3}{560} J_0(x)Y_1(x) -$$

$$-\frac{35\,x^9-480\,x^7+3456\,x^5-9216\,x^3}{560} J_1(x)Y_0(x) +$$

$$+\frac{10\,x^8-108\,x^6+576\,x^4-1152\,x^2}{35} J_1(x)Y_1(x)$$

$$\int x^{10}\, J_0(x)Y_1(x)\,dx = -\frac{14\,x^{10}-240\,x^8+2304\,x^6-9216\,x^4}{63} J_0(x)Y_0(x) +$$

$$+\frac{63\,x^{11}+1400\,x^9-19200\,x^7+138240\,x^5-368640\,x^3}{1260} J_0(x)Y_1(x) -$$

$$-\frac{63\,x^{11}-1400\,x^9+19200\,x^7-138240\,x^5+368640\,x^3}{1260} J_1(x)Y_0(x) +$$

$$+\frac{35\,x^{10}-640\,x^8+6912\,x^6-36864\,x^4+73728\,x^2}{126} J_1(x)Y_1(x)$$

$$\int x^{12}\, J_0(x)Y_1(x)\,dx = -\frac{105\,x^{12}-2800\,x^{10}+48000\,x^8-460800\,x^6+1843200\,x^4}{462} J_0(x)Y_0(x) +$$

$$+\frac{77\,x^{13}+2520\,x^{11}-56000\,x^9+768000\,x^7-5529600\,x^5+14745600\,x^3}{1848} J_0(x)Y_1(x) -$$

$$-\frac{77\,x^{13}-2520\,x^{11}+56000\,x^9-768000\,x^7+5529600\,x^5-14745600\,x^3}{1848} J_1(x)Y_0(x) +$$

$$+\frac{63\,x^{12}-1750\,x^{10}+32000\,x^8-345600\,x^6+1843200\,x^4-3686400\,x^2}{231} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2}\, J_0(x)Y_1(x)\,dx = x^{2n}\,\left[-\frac{nx^2}{2(2n+1)} J_0(x)Y_0(x) + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)}\right) xJ_0(x)Y_1(x) -\right.$$

$$\left. -\frac{x^3}{4(n+1)} J_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)} J_1(x)Y_1(x)\right] - \frac{2n^2(n+1)}{2n+1} \int x^{2n}\, J_0(x)Y_1(x)\,dx$$

d) $\int x^{-2n-1}\, J_0(x)Y_1(x)\,dx$:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x} = xJ_0(x)Y_0(x) - J_0(x)Y_1(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^3} =$$

$$= -\frac{8\,x^2+3}{9x} J_0(x)Y_0(x) + \frac{4\,x^2-3}{9x^2} J_0(x)Y_1(x) + \frac{4}{9} J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x} J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3} J_0(x)Y_0(x) - \frac{64\,x^4-24\,x^2+135}{675\,x^4} J_0(x)Y_1(x) -$$

$$-\frac{64\,x^2-24}{675\,x^2} J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3} J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5} J_0(x)Y_0(x) +$$

$$+\frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6} J_0(x)Y_1(x) + \frac{512\,x^4-192\,x^2+540}{55125\,x^4} J_1(x)Y_0(x) -$$

$$-\frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5} J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7} J_0(x)Y_0(x) -$$

$$-\frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8} J_0(x)Y_1(x) -$$

$$-\frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6} J_1(x)Y_0(x) +$$

$$+\frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7} J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{11}} =$$

$$-\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9} J_0(x)Y_0(x) +$$

$$+\frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}} J_0(x)Y_1(x) +$$

$$+\frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8} J_1(x)Y_0(x) -$$

$$-\frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{2n+1}} = \frac{1}{x^{2n+1}} \left\{ -\frac{x^2}{4n^2-1} J_0(x)Y_0(x) - \left[\frac{x}{2n+1} + \frac{4nx^3}{(4n^2-1)^2} \right] J_0(x)Y_1(x) + \right.$$

$$\left. + \frac{4nx^3}{(4n^2-1)^2} J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2} J_1(x)Y_1(x) \right\} - \frac{8n}{(2n+1)^2(2n-1)} \int \frac{J_0(x)Y_1(x)\,dx}{x^{2n-1}}$$

e) $\int x^{2n}\, J_1(x)Y_0(x)\, dx$:

$$\int x^2\, J_1(x)Y_0(x)\, dx = -\frac{x^3}{4} J_0(x)Y_1(x) + \frac{x^3}{4} J_1(x)Y_0(x) + \frac{x^2}{2} J_1(x)Y_1(x)$$

$$\int x^4\, J_1(x)Y_0(x)\, dx =$$

$$= -\frac{x^4}{6} J_0(x)Y_0(x) - \frac{3\,x^5 - 8\,x^3}{24} J_0(x)Y_1(x) + \frac{3\,x^5 + 8\,x^3}{24} J_1(x)Y_0(x) + \frac{x^4 - 2\,x^2}{3} J_1(x)Y_1(x)$$

$$\int x^6\, J_1(x)Y_0(x)\, dx = -\frac{x^6 - 4\,x^4}{5} J_0(x)Y_0(x) - \frac{5\,x^7 - 36\,x^5 + 96\,x^3}{60} J_0(x)Y_1(x) +$$

$$+ \frac{5\,x^7 + 36\,x^5 - 96\,x^3}{60} J_1(x)Y_0(x) + \frac{3\,x^6 - 16\,x^4 + 32\,x^2}{10} J_1(x)Y_1(x)$$

$$\int x^8\, J_1(x)Y_0(x)\, dx = -\frac{15\,x^8 - 144\,x^6 + 576\,x^4}{70} J_0(x)Y_0(x) -$$

$$-\frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{560} J_0(x)Y_1(x) +$$

$$+\frac{35\,x^9 + 480\,x^7 - 3456\,x^5 + 9216\,x^3}{560} J_1(x)Y_0(x) +$$

$$+\frac{10\,x^8 - 108\,x^6 + 576\,x^4 - 1152\,x^2}{35} J_1(x)Y_1(x)$$

$$\int x^{10}\, J_1(x)Y_0(x)\, dx = -\frac{14\,x^{10} - 240\,x^8 + 2304\,x^6 - 9216\,x^4}{63} J_0(x)Y_0(x) -$$

$$-\frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{1260} J_0(x)Y_1(x) +$$

$$+\frac{63\,x^{11} + 1400\,x^9 - 19200\,x^7 + 138240\,x^5 - 368640\,x^3}{1260} J_1(x)Y_0(x) +$$

$$+\frac{35\,x^{10} - 640\,x^8 + 6912\,x^6 - 36864\,x^4 + 73728\,x^2}{126} J_1(x)Y_1(x)$$

$$\int x^{12}\, J_1(x)Y_0(x)\, dx = -\frac{105\,x^{12} - 2800\,x^{10} + 48000\,x^8 - 460800\,x^6 + 1843200\,x^4}{462} J_0(x)Y_0(x) -$$

$$-\frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{1848} J_0(x)Y_1(x) +$$

$$+\frac{77\,x^{13} + 2520\,x^{11} - 56000\,x^9 + 768000\,x^7 - 5529600\,x^5 + 14745600\,x^3}{1848} J_1(x)Y_0(x) +$$

$$+\frac{63\,x^{12} - 1750\,x^{10} + 32000\,x^8 - 345600\,x^6 + 1843200\,x^4 - 3686400\,x^2}{231} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2}\, J_1(x)Y_0(x)\, dx = x^{2n} \left[-\frac{nx^2}{2(2n+1)} J_0(x)Y_0(x) - \frac{x^3}{4(n+1)} J_0(x)Y_1(x) - \right.$$

$$\left. + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)} \right) x J_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)} J_1(x)Y_1(x) \right] - \frac{2n^2(n+1)}{2n+1} \int x^{2n}\, J_1(x)Y_0(x)\, dx$$

f) $\int x^{-2n-1}\, J_1(x)Y_0(x)\, dx$:

$$\int \frac{J_1(x)Y_0(x)\, dx}{x} = xJ_0(x)Y_0(x) - J_1(x)Y_0(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^3} =$$
$$= -\frac{8\,x^2+3}{9x} J_0(x)Y_0(x) + \frac{4}{9} J_0(x)Y_1(x) + \frac{4\,x^2-3}{9x^2} J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x} J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3} J_0(x)Y_0(x) - \frac{64\,x^2-24}{675\,x^2} J_0(x)Y_1(x) -$$
$$- \frac{64\,x^4-24\,x^2+135}{675\,x^4} J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3} J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5} J_0(x)Y_0(x) +$$
$$+ \frac{512\,x^4-192\,x^2+540}{55125\,x^4} J_0(x)Y_1(x) + \frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6} J_1(x)Y_0(x) -$$
$$- \frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5} J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7} J_0(x)Y_0(x) -$$
$$- \frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6} J_0(x)Y_1(x) -$$
$$- \frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8} J_1(x)Y_0(x) +$$
$$+ \frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7} J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^{11}} =$$
$$= -\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9} J_0(x)Y_0(x) +$$
$$+ \frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8} J_0(x)Y_1(x) +$$
$$+ \frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}} J_1(x)Y_0(x) -$$
$$- \frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_0(x)\, dx}{x^{2n+1}} = \frac{1}{x^{2n+1}} \left\{ -\frac{x^2}{4n^2-1} J_0(x)Y_0(x) + \frac{4nx^3}{(4n^2-1)^2} J_0(x)Y_1(x) - \right.$$
$$\left. - \left[\frac{x}{2n+1} + \frac{4nx^3}{(4n^2-1)^2} \right] J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2} J_1(x)Y_1(x) \right\} - \frac{8n}{(2n-1)(2n+1)^2} \int \frac{J_1(x)Y_0(x)\, dx}{x^{2n-1}}$$

g) $\int x^{2n+1}\, J_1(x)Y_1(x)\,dx$:

$$\int x\, J_1(x)Y_1(x)\,dx = \frac{x^2}{2}J_0(x)Y_0(x) - xJ_0(x)Y_1(x) + \frac{x^2}{2}J_1(x)Y_1(x)$$

$$\int x^3\, J_1(x)Y_1(x)\,dx =$$

$$= \frac{x^4}{6}J_0(x)Y_0(x) - \frac{x^3}{3}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^4+4\,x^2}{6}J_1(x)Y_1(x)$$

$$\int x^5\, J_1(x)Y_1(x)\,dx = \frac{x^6-4\,x^4}{10}J_0(x)Y_0(x) -$$

$$-\frac{3\,x^5-8\,x^3}{10}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^6+8\,x^4-16\,x^2}{10}J_1(x)Y_1(x)$$

$$\int x^7\, J_1(x)Y_1(x)\,dx = \frac{5\,x^8-48\,x^6+192\,x^4}{70}J_0(x)Y_0(x) -$$

$$-\frac{10\,x^7-72\,x^5+192\,x^3}{35}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{5\,x^8+72\,x^6-384\,x^4+768\,x^2}{70}J_1(x)Y_1(x)$$

$$\int x^9\, J_1(x)Y_1(x)\,dx = \frac{7\,x^{10}-120\,x^8+1152\,x^6-4608\,x^4}{126}J_0(x)Y_0(x) -$$

$$-\frac{35\,x^9-480\,x^7+3456\,x^5-9216\,x^3}{126}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\frac{7\,x^{10}+160\,x^8-1728\,x^6+9216\,x^4-18432\,x^2}{126}J_1(x)Y_1(x)$$

$$\int x^{11}\, J_1(x)Y_1(x)\,dx = \frac{21\,x^{12}-560\,x^{10}+9600\,x^8-92160\,x^6+368640\,x^4}{462}J_0(x)Y_0(x) -$$

$$-\frac{63\,x^{11}-1400\,x^9+19200\,x^7-138240\,x^5+368640\,x^3}{231}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\frac{21\,x^{12}+700\,x^{10}-12800\,x^8+138240\,x^6-737280\,x^4+1474560\,x^2}{462}J_1(x)Y_1(x)$$

$$\int x^{13}\, J_1(x)Y_1(x)\,dx =$$

$$= \frac{11\,x^{14}-420\,x^{12}+11200\,x^{10}-192000\,x^8+1843200\,x^6-7372800\,x^4}{286}J_0(x)Y_0(x) -$$

$$-\frac{77\,x^{13}-2520\,x^{11}+56000\,x^9-768000\,x^7+5529600\,x^5-14745600\,x^3}{286}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+\frac{11\,x^{14}+504\,x^{12}-14000\,x^{10}+256000\,x^8-2764800\,x^6+14745600\,x^4-29491200\,x^2}{286}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1}\, J_1(x)Y_1(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\,\left\{x^2J_0(x)Y_0(x) - (n+1)x\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + [2n(n+1)+x^2]J_1(x)Y_1(x)\right\} -$$

$$-\frac{2n(n^2-1)}{2n+1}\int x^{2n-1}\, J_1(x)Y_1(x)\,dx$$

h) $\int x^{-2n}\, J_1(x)Y_1(x)\, dx$:

Exception:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x} = -\frac{J_0(x)Y_0(x)+J_1(x)Y_1(x)}{2}$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^2} = \frac{2x}{3}J_0(x)Y_0(x) - \frac{1}{3}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] + \frac{2x^2-1}{3x}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^4} =$$

$$= -\frac{16\,x^2+6}{45x}J_0(x)Y_0(x) + \frac{8\,x^2-3}{45x^2}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] - \frac{16\,x^4-2\,x^2+9}{45x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^6} = \frac{256\,x^4+96\,x^2-90}{4725\,x^3}J_0(x)Y_0(x) -$$

$$-\frac{128\,x^4-48\,x^2+135}{4725\,x^4}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] + \frac{256\,x^6-32\,x^4+54\,x^2-675}{4725\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^8} = -\frac{2048\,x^6+768\,x^4-720\,x^2+3150}{496125\,x^5}J_0(x)Y_0(x) +$$

$$+\frac{1024\,x^6-384\,x^4+1080\,x^2-7875}{496125\,x^6}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] -$$

$$-\frac{2048\,x^8-256\,x^6+432\,x^4-2250\,x^2+55125}{496125\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{10}} = \frac{65536\,x^8+24576\,x^6-23040\,x^4+100800\,x^2-992250}{343814625\,x^7}J_0(x)Y_0(x) -$$

$$-\frac{32768\,x^8-12288\,x^6+34560\,x^4-252000\,x^2+3472875}{343814625\,x^8}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] +$$

$$+\frac{65536\,x^{10}-8192\,x^8+13824\,x^6-72000\,x^4+771750\,x^2-31255875}{343814625\,x^9}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{12}} =$$

$$= -\frac{524288\,x^{10}+196608\,x^8-184320\,x^6+806400\,x^4-7938000\,x^2+137525850}{88497884475\,x^9}J_0(x)Y_0(x) +$$

$$+\frac{262144\,x^{10}-98304\,x^8+276480\,x^6-2016000\,x^4+27783000\,x^2-618866325}{88497884475\,x^{10}}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] -$$

$$-\frac{524288\,x^{12}-65536\,x^{10}+110592\,x^8-576000\,x^6+6174000\,x^4-112521150\,x^2+6807529575}{88497884475\,x^{11}}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n+2}} =$$

$$= -\frac{2x^2J_0(x)Y_0(x)+(2n-1)x[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+(4n^2-1+2x^2)J_1(x)Y_1(x)}{(2n+3)(4n^2-1)\,x^{2n+1}} -$$

$$-\frac{8n}{(2n+3)(4n^2-1)}\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n}}$$

2.2. Bessel Functions with different Arguments αx and βx :

See also [10], 4. - 6. .

2.2.1. Integrals of the type $\int x^{2n+1} Z_\nu(\alpha x) Z_\nu(\beta x)\,dx$, $\alpha^2 \neq \beta^2$

Let

$$\int x^{2n+1}\, J_\nu(\alpha x)\, J_\nu(\beta x)\, dx =$$

$$= A_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + B_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + C_\nu(x)\, J_1(\alpha x)\, J_0(\beta x) + D_\nu(x)\, J_1(\alpha x)\, J_1(\beta x)\ ,$$

then in this formula $J_\mu(\alpha x)$ or $J_\mu(\beta x)$ may be substituted by $Y_\mu(\alpha x)$, $H_\mu^{(p)}(\alpha x)$, $p = 1, 2$, or $Y_\mu(\beta x)$, $H_\mu^{(p)}(\beta x)$, $p = 1, 2$, respectively. The functions $A_\mu(x)$, $B_\mu(x)$, $C_\mu(x)$ and $D_\mu(x)$ are always the same in all cases. Therefore the integrals are given with $J_\nu(\alpha x)\, J_\nu(\beta x)$ only.
The same way in

$$\int x^{2n+1}\, J_\nu(\alpha x)\, I_\nu(\beta x)\, dx =$$

$$= P_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + Q_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + R_\nu(x)\, J_1(\alpha x)\, J_0(\beta x) + S_\nu(x)\, J_1(\alpha x)\, J_1(\beta x)\ ,$$

$J_\mu(\alpha x)$ may be substituted by $Y_\mu(\alpha x)$ or $H_\mu^{(p)}(\alpha x)$ without changing the coefficients $P_\nu(x)$ (and so on). The same holds for the integrals $\int x^{2n+1}\, J_\nu(\alpha x)\, K_\nu(\beta x)\, dx$. In both cases the integrals are given with $J_\nu(\alpha x)$ only.

a) $\nu = 0$:

$$\int x \cdot J_0(\alpha x) J_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) J_0(\beta x) - \beta x J_0(\alpha x) J_1(\beta x)}{\alpha^2 - \beta^2}$$

$$\int x \cdot I_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) I_0(\beta x) - \beta x I_0(\alpha x) I_1(\beta x)}{\alpha^2 - \beta^2}$$

$$\int x \cdot K_0(\alpha x) K_0(\beta x)\, dx = -\frac{\alpha x K_1(\alpha x) K_0(\beta x) - \beta x K_0(\alpha x) K_1(\beta x)}{\alpha^2 - \beta^2}$$

$$\int x \cdot J_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) I_0(\beta x) + \beta x J_0(\alpha x) I_1(\beta x)}{\alpha^2 + \beta^2}$$

$$\int x \cdot J_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) K_0(\beta x) - \beta x J_0(\alpha x) K_1(\beta x)}{\alpha^2 + \beta^2}$$

$$\int x \cdot I_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) K_0(\beta x) + \beta x I_0(\alpha x) k_1(\beta x)}{\alpha^2 - \beta^2}$$

$$\int x^3 \cdot J_0(\alpha x) J_0(\beta x)\, dx =$$

$$= \frac{2x^2}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2) J_0(\alpha x) J_0(\beta x) + 2\alpha\beta J_1(\alpha x) J_1(\beta x)\right] +$$

$$+ \left[4x \cdot \frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^3} - \frac{x^3}{\alpha^2-\beta^2}\right] \cdot \left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)\right]$$

$$\int x^3 \cdot I_0(\alpha x) I_0(\beta x)\, dx =$$

$$= \frac{-2x^2}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2) I_0(\alpha x) I_0(\beta x) - 2\alpha\beta I_1(\alpha x) I_1(\beta x)\right] -$$

$$- \left[4x \cdot \frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^3} + \frac{x^3}{\alpha^2-\beta^2}\right] \cdot \left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)\right]$$

$$\int x^3 \cdot K_0(\alpha x) K_0(\beta x)\, dx =$$

$$= \frac{-2x^2}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2)K_0(\alpha x)K_0(\beta x) - 2\alpha\beta K_1(\alpha x)K_1(\beta x)\right] +$$

$$+\left[4x\cdot\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^3}+\frac{x^3}{\alpha^2-\beta^2}\right]\cdot\left[\beta K_0(\alpha x)K_1(\beta x) - \alpha K_1(\alpha x)K_0(\beta x)\right]$$

$$\int x^3\cdot J_0(\alpha x)I_0(\beta x)\,dx \;=$$

$$= \frac{2x^2}{(\alpha^2+\beta^2)^2}\left[(\alpha^2-\beta^2)J_0(\alpha x)I_0(\beta x) - 2\alpha\beta J_1(\alpha x)I_1(\beta x)\right] -$$

$$-\left[4x\cdot\frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^3}-\frac{x^3}{\alpha^2+\beta^2}\right]\cdot\left[\beta J_0(\alpha x)I_1(\beta x) + \alpha J_1(\alpha x)I_0(\beta x)\right]$$

$$\int x^3\cdot J_0(\alpha x)K_0(\beta x)\,dx \;=$$

$$= \frac{2x^2}{(\alpha^2+\beta^2)^2}\left[(\alpha^2-\beta^2)J_0(\alpha x)K_0(\beta x) + 2\alpha\beta J_1(\alpha x)K_1(\beta x)\right] +$$

$$+\left[4x\cdot\frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^3}-\frac{x^3}{\alpha^2+\beta^2}\right]\cdot\left[\beta J_0(\alpha x)K_1(\beta x) - \alpha J_1(\alpha x)K_0(\beta x)\right]$$

$$\int x^3\cdot I_0(\alpha x)K_0(\beta x)\,dx \;=$$

$$= -\frac{2x^2}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2)I_0(\alpha x)K_0(\beta x) + 2\alpha\beta I_1(\alpha x)K_1(\beta x)\right] +$$

$$+\left[4x\cdot\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^3}+\frac{x^3}{\alpha^2-\beta^2}\right]\cdot\left[\beta I_0(\alpha x)K_1(\beta x) + \alpha I_1(\alpha x)K_0(\beta x)\right]$$

Let

$$\int x^m\, F_0(\alpha x)G_0(\beta x)\,dx =$$

$$= P_m^{[FG]}(x)F_0(\alpha x)G_0(\beta x) + Q_m^{[FG]}(x)F_0(\alpha x)G_1(\beta x) + R_m^{[FG]}(x)F_1(\alpha x)G_0(\beta x) + S_m^{[FG]}(x)F_1(\alpha x)G_1(\beta x)\,.$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)}H^{(1)}]} = P_m^{[H^{(2)}H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)}H^{(2)}]}\,,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad\text{and}\quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]}\,.$$

The same holds analogous for the polynomials $Q_m(x)$, $R_m(x)$ and $S_m(x)$.

$$P_5^{[JJ]}(x) = 4\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^4 - 32\,\frac{\alpha^4+\beta^4+4\,\alpha^2\beta^2}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$Q_5^{[JJ]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^5 + 16\,\frac{\beta\,\left(2\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 - 64\,\frac{\beta\,\left(\alpha^4+\beta^4+4\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$R_5^{[JJ]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^5 - 16\,\frac{\alpha\,\left(\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 + 64\,\frac{\alpha\,\left(\alpha^4+\beta^4+4\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$S_5^{[JJ]}(x) = 8\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^4 - 96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$P_5^{[II]}(x) = P_5^{[KK]}(x) = -4\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^4 - 32\,\frac{\alpha^4+\beta^4+4\,\alpha^2\beta^2}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$Q_5^{[II]}(x) = -Q_5^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^5 - 16\,\frac{\beta\,\left(2\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 - 64\,\frac{\beta\,\left(\alpha^4+\beta^4+4\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}$$

$$R_5^{[II]}(x) = -R_5^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^5 + 16\,\frac{\alpha\,\left(\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 + 64\,\frac{\alpha\,\left(\alpha^4+\beta^4+4\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$S_5^{[II]}(x) = S_5^{[KK]}(x) = 8\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^4 + 96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$P_5^{[JI]}(x) = 4\,\frac{\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(\alpha^4-4\,\beta^2\alpha^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$Q_5^{[JI]}(x) = \frac{\beta\,x^5}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(2\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\beta\,\left(\alpha^4-4\,\beta^2\alpha^2+\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$R_5^{[JI]}(x) = \frac{\alpha\,x^5}{\alpha^2+\beta^2} - 16\,\frac{\alpha\,\left(\alpha^2-2\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\alpha^4-4\,\beta^2\alpha^2+\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$S_5^{[JI]}(x) = -8\,\frac{\alpha\,\beta\,x^4}{\left(\alpha^2+\beta^2\right)^2} + 96\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$P_5^{[JK]}(x) = 4\,\frac{\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(\alpha^4-4\,\beta^2\alpha^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$Q_5^{[JK]}(x) = -\frac{\beta\,x^5}{\alpha^2+\beta^2} + 16\,\frac{\beta\,\left(2\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} - 64\,\frac{\beta\,\left(\beta^4-4\,\beta^2\alpha^2+\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$R_5^{[JK]}(x) = \frac{\alpha\,x^5}{\alpha^2+\beta^2} - 16\,\frac{\alpha\,\left(\alpha^2-2\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\beta^4-4\,\beta^2\alpha^2+\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$S_5^{[JK]}(x) = 8\,\frac{\alpha\,\beta\,x^4}{\left(\alpha^2+\beta^2\right)^2} - 96\,\frac{\alpha\,\beta\,\left(-\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$P_5^{[IK]}(x) = -4\,\frac{\left(\alpha^2+\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^2} - 32\,\frac{\left(\beta^4+\alpha^4+4\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^4}$$

$$Q_5^{[IK]}(x) = \frac{\beta\,x^5}{\alpha^2-\beta^2} + 16\,\frac{\beta\,\left(2\,\alpha^2+\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^3} + 64\,\frac{\beta\,\left(\beta^4+\alpha^4+4\,\beta^2\alpha^2\right)x}{\left(\alpha^2-\beta^2\right)^5}$$

$$R_5^{[IK]}(x) = \frac{\alpha\,x^5}{\alpha^2-\beta^2} + 16\,\frac{\alpha\,\left(\alpha^2+2\,\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\beta^4+\alpha^4+4\,\beta^2\alpha^2\right)x}{\left(\alpha^2-\beta^2\right)^5}$$

$$S_5^{[IK]}(x) = -8\,\frac{\alpha\,\beta\,x^4}{\left(\alpha^2-\beta^2\right)^2} - 96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^4}$$

$$P_7^{[JJ]}(x) = 6\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 48\,\frac{3\,\alpha^4+14\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 1152\,\frac{\alpha^6+\beta^6+9\,\alpha^4\beta^2+9\,\alpha^2\beta^4}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$Q_7^{[JJ]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^7 + 12\,\frac{\beta\,\left(7\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 -$$

$$-192\,\frac{\beta\,\left(8\,\alpha^4+19\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 + 2304\,\frac{\beta\,\left(\alpha^6+\beta^6+9\,\alpha^4\beta^2+9\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$R_7^{[JJ]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^7 - 12\,\frac{\alpha\,\left(3\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 +$$

$$+192\,\frac{\alpha\,\left(3\,\alpha^4+19\,\alpha^2\beta^2+8\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 - 2304\,\frac{\alpha\,\left(\alpha^6+\beta^6+9\,\alpha^4\beta^2+9\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$S_7^{[JJ]}(x) = 12\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 384\,\frac{\alpha\,\beta\,\left(38\,\alpha^2\beta^2+11\,\beta^4+11\,\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$P_7^{[II]}(x) = P_7^{[KK]}(x) = -6\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 48\,\frac{3\,\alpha^4+14\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^4 - 1152\,\frac{\beta^6+9\,\alpha^4\beta^2+\alpha^6+9\,\alpha^2\beta^4}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$Q_7^{[II]}(x) = -Q_7^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^7 - 12\,\frac{\beta\,\left(7\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 -$$

$$-192\,\frac{\beta\,\left(8\,\alpha^4+19\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 - 2304\,\frac{\beta\,\left(\beta^6+9\,\alpha^4\beta^2+\alpha^6+9\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$R_7^{[II]}(x) = -R_7^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^7 + 12\,\frac{\alpha\,\left(3\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 +$$

$$+192\,\frac{\alpha\,\left(3\,\alpha^4+19\,\alpha^2\beta^2+8\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 + 2304\,\frac{\alpha\,\left(\beta^6+9\,\alpha^4\beta^2+\alpha^6+9\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$S_7^{[II]}(x) = S_7^{[KK]}(x) = 12\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^6 + 480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 384\,\frac{\alpha\,\beta\,\left(11\,\alpha^4+38\,\alpha^2\beta^2+11\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$P_7^{[JI]}(x) = 6\,\frac{\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^2} - 48\,\frac{\left(3\,\alpha^4-14\,\beta^2\alpha^2+3\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^4} + 1152\,\frac{\left(-\beta^6+\alpha^6-9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$Q_7^{[JI]}(x) = \frac{\beta\,x^7}{\alpha^2+\beta^2} - 12\,\frac{\beta\,\left(7\,\alpha^2-3\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\left(8\,\alpha^4-19\,\beta^2\alpha^2+3\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-2304\,\frac{\beta\,\left(-\beta^6+\alpha^6-9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$R_7^{[JI]}(x) = \frac{\alpha\,x^7}{\alpha^2+\beta^2} - 12\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\alpha\,\left(3\,\alpha^4-19\,\beta^2\alpha^2+8\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-2304\,\frac{\alpha\,\left(-\beta^6+\alpha^6-9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$S_7^{[JI]}(x) = -12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2+\beta^2\right)^2} + 480\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\alpha\,\beta\,\left(11\,\alpha^4-38\,\beta^2\alpha^2+11\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$P_7^{[JK]}(x) = 6\,\frac{\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^2} - 48\,\frac{\left(3\,\alpha^4-14\,\alpha^2\beta^2+3\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^4} + 1152\,\frac{\left(-\beta^6-9\,\beta^2\alpha^4+\alpha^6+9\,\beta^4\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$Q_7^{[JK]}(x) = -\frac{\beta\,x^7}{\alpha^2+\beta^2} + 12\,\frac{\beta\,\left(7\,\alpha^2-3\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\beta\,\left(8\,\alpha^4-19\,\alpha^2\beta^2+3\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+2304\,\frac{\beta\,\left(-\beta^6-9\,\beta^2\alpha^4+\alpha^6+9\,\beta^4\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$R_7^{[JK]}(x) = \frac{\alpha\,x^7}{\alpha^2+\beta^2} - 12\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\alpha\,\left(3\,\alpha^4-19\,\alpha^2\beta^2+8\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-2304\,\frac{\alpha\,\left(-\beta^6-9\,\beta^2\alpha^4+\alpha^6+9\,\beta^4\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$S_7^{[JK]}(x)=12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2+\beta^2\right)^2}-480\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^4}+384\,\frac{\alpha\,\beta\,\left(11\,\alpha^4-38\,\alpha^2\beta^2+11\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$P_7^{[IK]}(x)=-6\,\frac{\left(\alpha^2+\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^2}-48\,\frac{\left(3\,\alpha^4+14\,\beta^2\alpha^2+3\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^4}-1152\,\frac{\left(\alpha^6+\beta^6+9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x^2}{\left(\alpha^2-\beta^2\right)^6}$$

$$Q_7^{[IK]}(x)=\frac{\beta\,x^7}{\alpha^2-\beta^2}+12\,\frac{\beta\,\left(7\,\alpha^2+3\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\beta\,\left(8\,\alpha^4+19\,\beta^2\alpha^2+3\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+2304\,\frac{\beta\,\left(\alpha^6+\beta^6+9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x}{\left(\alpha^2-\beta^2\right)^7}$$

$$R_7^{[IK]}(x)=\frac{\alpha\,x^7}{\alpha^2-\beta^2}+12\,\frac{\alpha\,\left(3\,\alpha^2+7\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\alpha\,\left(3\,\alpha^4+19\,\beta^2\alpha^2+8\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+2304\,\frac{\alpha\,\left(\alpha^6+\beta^6+9\,\beta^2\alpha^4+9\,\alpha^2\beta^4\right)x}{\left(\alpha^2-\beta^2\right)^7}$$

$$S_7^{[IK]}(x)=-12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2-\beta^2\right)^2}-480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^4}-384\,\frac{\alpha\,\beta\,\left(11\,\beta^4+11\,\alpha^4+38\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^6}$$

$$P_9^{[JJ]}(x)=8\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^8-384\,\frac{\alpha^4+5\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^6+3072\,\frac{3\,\alpha^6+32\,\alpha^4\beta^2+32\,\alpha^2\beta^4+3\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^4-$$

$$-73728\,\frac{\beta^8+\alpha^8+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+36\,\alpha^4\beta^4}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$Q_9^{[JJ]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^9+32\,\frac{\beta\,\left(5\,\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7-768\,\frac{\beta\,\left(10\,\alpha^4+22\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5+$$

$$+6144\,\frac{\beta\,\left(19\,\alpha^6+108\,\alpha^4\beta^2+77\,\alpha^2\beta^4+6\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3-147456\,\frac{\beta\,\left(\beta^8+\alpha^8+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+36\,\alpha^4\beta^4\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$R_9^{[JJ]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^9-32\,\frac{\alpha\,\left(2\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7+768\,\frac{\alpha\,\left(3\,\alpha^4+22\,\alpha^2\beta^2+10\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5-$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6+77\,\alpha^4\beta^2+108\,\alpha^2\beta^4+19\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3+147456\,\frac{\alpha\,\left(\beta^8+\alpha^8+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+36\,\alpha^4\beta^4\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$S_9^{[JJ]}(x)=16\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^8-1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^6+3072\,\frac{\beta\,\alpha\,\left(13\,\alpha^4+44\,\alpha^2\beta^2+13\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^4-$$

$$-61440\,\frac{\beta\,\alpha\,\left(5\,\beta^6+5\,\alpha^6+37\,\alpha^2\beta^4+37\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$P_9^{[II]}(x)=P_9^{[KK]}(x)=-8\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^8-384\,\frac{\alpha^4+5\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^6-3072\,\frac{3\,\alpha^6+32\,\alpha^4\beta^2+32\,\alpha^2\beta^4+3\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^4-$$

$$-73728\,\frac{\alpha^8+36\,\alpha^4\beta^4+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$Q_9^{[II]}(x)=-Q_9^{[KK]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^9-32\,\frac{\beta\,\left(5\,\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7-768\,\frac{\beta\,\left(10\,\alpha^4+22\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5-$$

$$-6144\,\frac{\beta\,\left(19\,\alpha^6+108\,\alpha^4\beta^2+77\,\alpha^2\beta^4+6\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3-147456\,\frac{\beta\,\left(\alpha^8+36\,\alpha^4\beta^4+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$R_9^{[II]}(x)=-R_9^{[KK]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^9+32\,\frac{\alpha\,\left(2\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7+768\,\frac{\alpha\,\left(3\,\alpha^4+22\,\alpha^2\beta^2+10\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5+$$

$$+6144\,\frac{\alpha\,\left(6\,\alpha^6+77\,\alpha^4\beta^2+108\,\alpha^2\beta^4+19\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3+147456\,\frac{\alpha\,\left(\alpha^8+36\,\alpha^4\beta^4+16\,\alpha^2\beta^6+16\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$S_9^{[II]}(x)=S_9^{[KK]}(x)=16\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^8+1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^6+3072\,\frac{\beta\,\alpha\,\left(13\,\alpha^4+44\,\alpha^2\beta^2+13\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^4+$$

$$+61440\,\frac{\beta\,\alpha\,\left(5\,\beta^6+5\,\alpha^6+37\,\alpha^4\beta^2+37\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$P_9^{[JI]}(x)=8\,\frac{\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^2}-384\,\frac{\left(\alpha^4-5\,\alpha^2\beta^2+\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^4}+3072\,\frac{\left(3\,\alpha^6-32\,\alpha^4\beta^2+32\,\alpha^2\beta^4-3\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-73728\,\frac{\left(36\,\alpha^4\beta^4-16\,\alpha^2\beta^6-16\,\alpha^6\beta^2+\alpha^8+\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$Q_9^{[JI]}(x)=\frac{\beta\,x^9}{\alpha^2+\beta^2}-32\,\frac{\beta\,\left(5\,\alpha^2-2\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}+768\,\frac{\beta\,\left(10\,\alpha^4-22\,\alpha^2\beta^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\beta\,\left(19\,\alpha^6-108\,\alpha^4\beta^2+77\,\alpha^2\beta^4-6\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+147456\,\frac{\beta\,\left(36\,\alpha^4\beta^4-16\,\alpha^2\beta^6-16\,\alpha^6\beta^2+\alpha^8+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$R_9^{[JI]}(x)=\frac{\alpha\,x^9}{\alpha^2+\beta^2}-32\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}+768\,\frac{\alpha\,\left(3\,\alpha^4-22\,\alpha^2\beta^2+10\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6-77\,\alpha^4\beta^2+108\,\alpha^2\beta^4-19\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+147456\,\frac{\alpha\,\left(36\,\alpha^4\beta^4-16\,\alpha^2\beta^6-16\,\alpha^6\beta^2+\alpha^8+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$S_9^{[JI]}(x)=-16\,\frac{\alpha\,\beta\,x^8}{\left(\alpha^2+\beta^2\right)^2}+1344\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^4}-3072\,\frac{\alpha\,\beta\,\left(13\,\alpha^4-44\,\alpha^2\beta^2+13\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+61440\,\frac{\alpha\,\beta\,\left(37\,\alpha^2\beta^4-5\,\beta^6-37\,\alpha^4\beta^2+5\,\alpha^6\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$P_9^{[JK]}(x)=8\,\frac{\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^2}-384\,\frac{\left(\alpha^4-5\,\beta^2\alpha^2+\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^4}+3072\,\frac{\left(3\,\alpha^6-32\,\alpha^4\beta^2+32\,\alpha^2\beta^4-3\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-73728\,\frac{\left(\alpha^8-16\,\alpha^2\beta^6-16\,\beta^2\alpha^6+\beta^8+36\,\alpha^4\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$Q_9^{[JK]}(x)=-\frac{\beta\,x^9}{\alpha^2+\beta^2}+32\,\frac{\beta\,\left(5\,\alpha^2-2\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}-768\,\frac{\beta\,\left(10\,\alpha^4-22\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\,\left(19\,\alpha^6-108\,\alpha^4\beta^2+77\,\alpha^2\beta^4-6\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}-147456\,\frac{\beta\,\left(\alpha^8-16\,\alpha^2\beta^6-16\,\beta^2\alpha^6+\beta^8+36\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$R_9^{[JK]}(x)=\frac{\alpha\,x^9}{\alpha^2+\beta^2}-32\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}+768\,\frac{\alpha\,\left(3\,\alpha^4-22\,\beta^2\alpha^2+10\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6-77\,\alpha^4\beta^2+108\,\alpha^2\beta^4-19\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+147456\,\frac{\alpha\,\left(\alpha^8-16\,\alpha^2\beta^6-16\,\beta^2\alpha^6+\beta^8+36\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$S_9^{[JK]}(x)=16\,\frac{\beta\,\alpha\,x^8}{\left(\alpha^2+\beta^2\right)^2}-1344\,\frac{\beta\,\alpha\,\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^4}+3072\,\frac{\beta\,\alpha\,\left(13\,\alpha^4-44\,\beta^2\alpha^2+13\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-61440\,\frac{\beta\,\alpha\,\left(37\,\alpha^2\beta^4-5\,\beta^6-37\,\alpha^4\beta^2+5\,\alpha^6\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$P_9^{[IK]}(x)=-8\,\frac{\left(\alpha^2+\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^2}-384\,\frac{\left(\alpha^4+5\,\beta^2\alpha^2+\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-3072\,\frac{\left(3\,\alpha^6+32\,\beta^2\alpha^4+32\,\alpha^2\beta^4+3\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-73728\,\frac{\left(36\,\alpha^4\beta^4+16\,\alpha^6\beta^2+\alpha^8+\beta^8+16\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$Q_9^{[IK]}(x)=\frac{\beta\,x^9}{\alpha^2-\beta^2}+32\,\frac{\beta\,\left(5\,\alpha^2+2\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+768\,\frac{\beta\,\left(10\,\alpha^4+22\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\,\left(19\,\alpha^6+108\,\beta^2\alpha^4+77\,\alpha^2\beta^4+6\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+147456\,\frac{\beta\,\left(36\,\alpha^4\beta^4+16\,\alpha^6\beta^2+\alpha^8+\beta^8+16\,\alpha^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$R_9^{[IK]}(x)=\frac{\alpha\,x^9}{\alpha^2-\beta^2}+32\,\frac{\alpha\,\left(2\,\alpha^2+5\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+768\,\frac{\alpha\,\left(3\,\alpha^4+22\,\beta^2\alpha^2+10\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\alpha\,\left(6\,\alpha^6+77\,\beta^2\alpha^4+108\,\alpha^2\beta^4+19\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+147456\,\frac{\alpha\,\left(36\,\alpha^4\beta^4+16\,\alpha^6\beta^2+\alpha^8+\beta^8+16\,\alpha^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$S_9^{[IK]}(x)=-16\,\frac{\beta\,\alpha\,x^8}{\left(\alpha^2-\beta^2\right)^2}-1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-3072\,\frac{\beta\,\alpha\,\left(13\,\alpha^4+44\,\beta^2\alpha^2+13\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-61440\,\frac{\beta\,\alpha\,\left(37\,\alpha^2\beta^4+5\,\alpha^6+37\,\beta^2\alpha^4+5\,\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$P_{11}^{[JJ]}(x)=10\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-160\,\frac{5\,\alpha^4+26\,\alpha^2\beta^2+5\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^8+7680\,\frac{5\,\alpha^6+58\,\alpha^4\beta^2+58\,\alpha^2\beta^4+5\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-61440\,\frac{15\,\alpha^8+283\,\alpha^6\beta^2+664\,\alpha^4\beta^4+283\,\alpha^2\beta^6+15\,\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+7372800\,\frac{\beta^{10}+100\,\beta^4\alpha^6+25\,\beta^2\alpha^8+25\,\beta^8\alpha^2+\alpha^{10}+100\,\beta^6\alpha^4}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$Q_{11}^{[JJ]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^{11}+20\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9-640\,\frac{\beta\,\left(37\,\alpha^4+79\,\alpha^2\beta^2+10\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7+$$

$$+15360\,\frac{\beta\,\left(62\,\alpha^6+332\,\alpha^4\beta^2+221\,\alpha^2\beta^4+15\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5-$$

$$-122880\,\frac{\beta\,\left(107\,\alpha^8+1119\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+643\,\alpha^2\beta^6+30\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3+$$

$$+14745600\,\frac{\beta\,\left(\beta^{10}+100\,\beta^4\alpha^6+25\,\beta^2\alpha^8+25\,\beta^8\alpha^2+\alpha^{10}+100\,\beta^6\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$R_{11}^{[JJ]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^{11}-20\,\frac{\alpha\,\left(5\,\alpha^2+13\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9+640\,\frac{\alpha\,\left(10\,\alpha^4+79\,\alpha^2\beta^2+37\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7-$$

$$-15360\,\frac{\alpha\,\left(15\,\alpha^6+221\,\alpha^4\beta^2+332\,\alpha^2\beta^4+62\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8+643\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+1119\,\alpha^2\beta^6+107\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3-$$

$$-14745600\,\frac{\alpha\,\left(\beta^{10}+100\,\beta^4\alpha^6+25\,\beta^2\alpha^8+25\,\beta^8\alpha^2+\alpha^{10}+100\,\beta^6\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$S_{11}^{[JJ]}(x)=20\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^8+3840\,\frac{\alpha\,\beta\,\left(47\,\alpha^4+158\,\alpha^2\beta^2+47\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-430080\,\frac{\alpha\,\beta\,\left(11\,\alpha^6+79\,\alpha^4\beta^2+79\,\alpha^2\beta^4+11\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+245760\,\frac{\alpha\,\beta\,\left(137\,\alpha^8+1762\,\alpha^6\beta^2+3762\,\alpha^4\beta^4+1762\,\alpha^2\beta^6+137\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x$$

$$P_{11}^{[II]}(x)=P_{11}^{[KK]}(x)=-10\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-160\,\frac{5\,\alpha^4+26\,\alpha^2\beta^2+5\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^8-7680\,\frac{5\,\alpha^6+58\,\alpha^4\beta^2+58\,\alpha^2\beta^4+5\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-61440\,\frac{15\,\alpha^8+283\,\alpha^6\beta^2+664\,\alpha^4\beta^4+283\,\alpha^2\beta^6+15\,\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^4-$$

$$-7372800\,\frac{\alpha^{10}+\beta^{10}+25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+25\,\alpha^2\beta^8+100\,\alpha^4\beta^6}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$Q_{11}^{[II]}(x)=-Q_{11}^{[KK]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^{11}-20\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9-640\,\frac{\beta\,\left(37\,\alpha^4+79\,\alpha^2\beta^2+10\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7-$$

$$-15360\,\frac{\beta\,\left(62\,\alpha^6+332\,\alpha^4\beta^2+221\,\alpha^2\beta^4+15\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5-$$

$$-122880\,\frac{\beta\,\left(107\,\alpha^8+1119\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+643\,\alpha^2\beta^6+30\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3-$$

$$-14745600\,\frac{\beta\,\left(\alpha^{10}+\beta^{10}+25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+25\,\alpha^2\beta^8+100\,\alpha^4\beta^6\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$R_{11}^{[II]}(x)=-R_{11}^{[KK]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^{11}+20\,\frac{\alpha\,\left(5\,\alpha^2+13\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9+640\,\frac{\alpha\,\left(10\,\alpha^4+79\,\alpha^2\beta^2+37\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7+$$

$$+15360\,\frac{\alpha\,\left(15\,\alpha^6+221\,\alpha^4\beta^2+332\,\alpha^2\beta^4+62\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8+643\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+1119\,\alpha^2\beta^6+107\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3+$$

$$+14745600\,\frac{\alpha\,\left(\alpha^{10}+\beta^{10}+25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+25\,\alpha^2\beta^8+100\,\alpha^4\beta^6\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$S_{11}(x)=S_{11}^{[KK]}(x)=20\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}+2880\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^8+3840\,\frac{\alpha\,\beta\,\left(47\,\alpha^4+158\,\alpha^2\beta^2+47\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^6+$$

$$+430080\,\frac{\alpha\,\beta\,\left(11\,\alpha^6+79\,\alpha^4\beta^2+79\,\alpha^2\beta^4+11\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+245760\,\frac{\alpha\,\beta\,\left(137\,\alpha^8+3762\,\alpha^4\beta^4+1762\,\alpha^2\beta^6+1762\,\alpha^6\beta^2+137\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$P_{11}^{[JI]}(x)=10\,\frac{\left(\alpha^2-\beta^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-160\,\frac{\left(5\,\alpha^4-26\,\beta^2\alpha^2+5\,\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+7680\,\frac{\left(5\,\alpha^6-58\,\beta^2\alpha^4+58\,\beta^4\alpha^2-5\,\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-61440\,\frac{\left(15\,\alpha^8-283\,\beta^2\alpha^6+664\,\alpha^4\beta^4-283\,\alpha^2\beta^6+15\,\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+7372800\,\frac{\left(-25\,\beta^2\alpha^8+100\,\beta^4\alpha^6-100\,\beta^6\alpha^4-\beta^{10}+\alpha^{10}+25\,\beta^8\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$Q_{11}^{[JI]}(x)=\frac{\beta\,x^{11}}{\alpha^2+\beta^2}-20\,\frac{\beta\,\left(13\,\alpha^2-5\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+640\,\frac{\beta\,\left(37\,\alpha^4-79\,\beta^2\alpha^2+10\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-15360\,\frac{\beta\,\left(62\,\alpha^6-332\,\beta^2\alpha^4+221\,\beta^4\alpha^2-15\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+122880\,\frac{\beta\,\left(107\,\alpha^8-1119\,\beta^2\alpha^6+1881\,\alpha^4\beta^4-643\,\alpha^2\beta^6+30\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-14745600\,\frac{\beta\,\left(-25\,\beta^2\alpha^8+100\,\beta^4\alpha^6-100\,\beta^6\alpha^4-\beta^{10}+\alpha^{10}+25\,\beta^8\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$R_{11}^{[JI]}(x)=\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}-20\,\frac{\alpha\,\left(5\,\alpha^2-13\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+640\,\frac{\alpha\,\left(10\,\alpha^4-79\,\beta^2\alpha^2+37\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-15360\,\frac{\alpha\,\left(15\,\alpha^6-221\,\beta^2\alpha^4+332\,\beta^4\alpha^2-62\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8-643\,\beta^2\alpha^6+1881\,\alpha^4\beta^4-1119\,\alpha^2\beta^6+107\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-14745600\,\frac{\alpha\,\left(-25\,\beta^2\alpha^8+100\,\beta^4\alpha^6-100\,\beta^6\alpha^4-\beta^{10}+\alpha^{10}+25\,\beta^8\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$S_{11}^{[JI]}(x)=-20\,\frac{\beta\,\alpha\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}+2880\,\frac{\beta\,\alpha\,\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}-3840\,\frac{\beta\,\alpha\,\left(47\,\alpha^4-158\,\beta^2\alpha^2+47\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+430080\,\frac{\beta\,\alpha\,\left(11\,\alpha^6-79\,\beta^2\alpha^4+79\,\beta^4\alpha^2-11\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}-$$

$$-245760\,\frac{\beta\,\alpha\,\left(-1762\,\beta^2\alpha^6+137\,\beta^8+137\,\alpha^8-1762\,\alpha^2\beta^6+3762\,\alpha^4\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$P_{11}^{[JK]}(x)=10\,\frac{\left(\alpha^2-\beta^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-160\,\frac{\left(5\,\alpha^4-26\,\alpha^2\beta^2+5\,\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+7680\,\frac{\left(5\,\alpha^6-58\,\beta^2\alpha^4+58\,\beta^4\alpha^2-5\,\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-61440\,\frac{\left(15\,\alpha^8-283\,\alpha^6\beta^2+664\,\alpha^4\beta^4-283\,\alpha^2\beta^6+15\,\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+7372800\,\frac{\left(-\beta^{10}-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8+100\,\alpha^6\beta^4+\alpha^{10}-25\,\alpha^8\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$Q_{11}^{[JK]}(x)=-\frac{\beta\,x^{11}}{\alpha^2+\beta^2}+20\,\frac{\beta\,\left(13\,\alpha^2-5\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}-640\,\frac{\beta\,\left(37\,\alpha^4-79\,\alpha^2\beta^2+10\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+15360\,\frac{\beta\,\left(62\,\alpha^6-332\,\beta^2\alpha^4+221\,\beta^4\alpha^2-15\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-122880\,\frac{\beta\,\left(107\,\alpha^8-1119\,\alpha^6\beta^2+1881\,\alpha^4\beta^4-643\,\alpha^2\beta^6+30\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}+$$

$$+14745600\,\frac{\beta\,\left(-\beta^{10}-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8+100\,\alpha^6\beta^4+\alpha^{10}-25\,\alpha^8\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$R_{11}^{[JK]}(x)=\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}-20\,\frac{\alpha\,\left(5\,\alpha^2-13\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+640\,\frac{\alpha\,\left(10\,\alpha^4-79\,\alpha^2\beta^2+37\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-15360\,\frac{\alpha\,\left(15\,\alpha^6-221\,\beta^2\alpha^4+332\,\beta^4\alpha^2-62\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8-643\,\alpha^6\beta^2+1881\,\alpha^4\beta^4-1119\,\alpha^2\beta^6+107\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-14745600\,\frac{\alpha\,\left(-\beta^{10}-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8+100\,\alpha^6\beta^4+\alpha^{10}-25\,\alpha^8\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$S_{11}^{[JK]}(x)=20\,\frac{\alpha\,\beta\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+3840\,\frac{\alpha\,\beta\,\left(47\,\alpha^4-158\,\alpha^2\beta^2+47\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-430080\,\frac{\alpha\,\beta\,\left(11\,\alpha^6-79\,\beta^2\alpha^4+79\,\beta^4\alpha^2-11\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+245760\,\frac{\alpha\,\beta\,\left(137\,\alpha^8+137\,\beta^8+3762\,\alpha^4\beta^4-1762\,\alpha^2\beta^6-1762\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$P_{11}^{[IK]}(x)=-10\,\frac{\left(\alpha^2+\beta^2\right)x^{10}}{\left(\alpha^2-\beta^2\right)^2}-160\,\frac{\left(5\,\alpha^4+26\,\beta^2\alpha^2+5\,\beta^4\right)x^8}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-7680\,\frac{\left(5\,\alpha^6+58\,\beta^2\alpha^4+58\,\beta^4\alpha^2+5\,\beta^6\right)x^6}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-61440\,\frac{\left(15\,\alpha^8+283\,\alpha^6\beta^2+664\,\alpha^4\beta^4+283\,\alpha^2\beta^6+15\,\beta^8\right)x^4}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-7372800\,\frac{\left(25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+\beta^{10}+25\,\alpha^2\beta^8+\alpha^{10}+100\,\alpha^4\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$Q_{11}^{[IK]}(x)=\frac{\beta\,x^{11}}{\alpha^2-\beta^2}+20\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^3}+640\,\frac{\beta\,\left(37\,\alpha^4+79\,\beta^2\alpha^2+10\,\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+15360\,\frac{\beta\,\left(62\,\alpha^6+332\,\beta^2\alpha^4+221\,\beta^4\alpha^2+15\,\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+122880\,\frac{\beta\,\left(107\,\alpha^8+1119\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+643\,\alpha^2\beta^6+30\,\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^9}+$$

$$+14745600\,\frac{\beta\,\left(25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+\beta^{10}+25\,\alpha^2\beta^8+\alpha^{10}+100\,\alpha^4\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^{11}}$$

$$R_{11}^{[IK]}(x)=\frac{\alpha\,x^{11}}{\alpha^2-\beta^2}+20\,\frac{\alpha\,\left(5\,\alpha^2+13\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^3}+640\,\frac{\alpha\,\left(10\,\alpha^4+79\,\beta^2\alpha^2+37\,\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+15360\,\frac{\alpha\,\left(15\,\alpha^6+221\,\beta^2\alpha^4+332\,\beta^4\alpha^2+62\,\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8+643\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+1119\,\alpha^2\beta^6+107\,\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^9}+$$

$$+14745600\,\frac{\alpha\,\left(25\,\alpha^8\beta^2+100\,\alpha^6\beta^4+\beta^{10}+25\,\alpha^2\beta^8+\alpha^{10}+100\,\alpha^4\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^{11}}$$

$$S_{11}^{[IK]}(x) = -20\,\frac{\alpha\,\beta\,x^{10}}{(\alpha^2-\beta^2)^2} - 2880\,\frac{\alpha\,\beta\,(\alpha^2+\beta^2)\,x^8}{(\alpha^2-\beta^2)^4} - 3840\,\frac{\alpha\,\beta\,(47\,\alpha^4+158\,\beta^2\alpha^2+47\,\beta^4)\,x^6}{(\alpha^2-\beta^2)^6} -$$

$$-430080\,\frac{\alpha\,\beta\,(11\,\alpha^6+79\,\beta^2\alpha^4+79\,\beta^4\alpha^2+11\,\beta^6)\,x^4}{(\alpha^2-\beta^2)^8} -$$

$$-245760\,\frac{\alpha\,\beta\,(3762\,\alpha^4\beta^4+137\,\alpha^8+137\,\beta^8+1762\,\alpha^6\beta^2+1762\,\alpha^2\beta^6)\,x^2}{(\alpha^2-\beta^2)^{10}}$$

Recurrence formulas: see page 262.

b) $\nu = 1$:

$$\int x\cdot J_1(\alpha x)J_1(\beta x)\,dx = \frac{x}{\alpha^2-\beta^2}\,[\beta J_1(\alpha x)J_0(\beta x) - \alpha J_0(\alpha x)J_1(\beta x)]$$

$$\int x\cdot I_1(\alpha x)I_1(\beta x)\,dx = \frac{x}{\alpha^2-\beta^2}\,[\alpha I_0(\alpha x)I_1(\beta x) - \beta I_1(\alpha x)I_0(\beta x)]$$

$$\int x\cdot K_1(\alpha x)K_1(\beta x)\,dx = \frac{x}{\alpha^2-\beta^2}\,[\beta K_1(\alpha x)K_0(\beta x) - \alpha K_0(\alpha x)K_1(\beta x)]$$

$$\int x\cdot J_1(\alpha x)I_1(\beta x)\,dx = \frac{x}{\alpha^2+\beta^2}\,[\beta J_1(\alpha x)I_0(\beta x) - \alpha J_0(\alpha x)I_1(\beta x)]$$

$$\int x\cdot J_1(\alpha x)K_1(\beta x)\,dx = -\frac{x}{\alpha^2+\beta^2}\,[\beta J_1(\alpha x)K_0(\beta x) + \alpha J_0(\alpha x)K_1(\beta x)]$$

$$\int x\cdot I_1(\alpha x)K_1(\beta x)\,dx = \frac{x}{\alpha^2-\beta^2}\,[\beta I_1(\alpha x)K_0(\beta x) + \alpha I_0(\alpha x)K_1(\beta x)]$$

$$\int x^3\cdot J_1(\alpha x)J_1(\beta x)\,dx =$$

$$= \frac{2x^2}{(\alpha^2-\beta^2)^2}\left[2\alpha\beta J_0(\alpha x)J_0(\beta x) + (\alpha^2+\beta^2)J_1(\alpha x)J_1(\beta x)\right] +$$

$$+\frac{8\alpha\beta x}{(\alpha^2-\beta^2)^3}\cdot[\beta J_0(\alpha x)J_1(\beta x) - \alpha J_1(\alpha x)J_0(\beta x)] -$$

$$-\frac{x^3}{\alpha^2-\beta^2}\cdot[\alpha J_0(\alpha x)J_1(\beta x) - \beta J_1(\alpha x)J_0(\beta x)]$$

$$\int x^3\cdot I_1(\alpha x)I_1(\beta x)\,dx =$$

$$= \frac{2x^2}{(\alpha^2-\beta^2)^2}\left[2\alpha\beta I_0(\alpha x)I_0(\beta x) - (\alpha^2+\beta^2)I_1(\alpha x)I_1(\beta x)\right] +$$

$$+\frac{8\alpha\beta x}{(\alpha^2-\beta^2)^3}\cdot[\beta I_0(\alpha x)I_1(\beta x) - \alpha I_1(\alpha x)I_0(\beta x)] +$$

$$+\frac{x^3}{\alpha^2-\beta^2}\cdot[\alpha I_0(\alpha x)I_1(\beta x) - \beta I_1(\alpha x)I_0(\beta x)]$$

$$\int x^3\cdot K_1(\alpha x)K_1(\beta x)\,dx =$$

$$= \frac{2x^2}{(\alpha^2-\beta^2)^2}\left[2\alpha\beta K_0(\alpha x)K_0(\beta x) - (\alpha^2+\beta^2)K_1(\alpha x)K_1(\beta x)\right] -$$

$$-\frac{8\alpha\beta x}{(\alpha^2-\beta^2)^3}\cdot[\beta K_0(\alpha x)K_1(\beta x)-\alpha K_1(\alpha x)K_0(\beta x)]-$$

$$-\frac{x^3}{\alpha^2-\beta^2}\cdot[\alpha K_0(\alpha x)K_1(\beta x)-\beta K_1(\alpha x)K_0(\beta x)]$$

$$\int x^3\cdot J_1(\alpha x)I_1(\beta x)\,dx\ =$$

$$=\ \frac{2x^2}{(\alpha^2+\beta^2)^2}\left[2\alpha\beta J_0(\alpha x)I_0(\beta x)+(\alpha^2-\beta^2)J_1(\alpha x)I_1(\beta x)\right]-$$

$$-\frac{8\alpha\beta x}{(\alpha^2+\beta^2)^3}\cdot[\beta J_0(\alpha x)I_1(\beta x)+\alpha J_1(\alpha x)I_0(\beta x)]-$$

$$-\frac{x^3}{\alpha^2+\beta^2}\cdot[\alpha J_0(\alpha x)I_1(\beta x)-\beta J_1(\alpha x)I_0(\beta x)]$$

$$\int x^3\cdot J_1(\alpha x)K_1(\beta x)\,dx\ =$$

$$=\ \frac{2x^2}{(\alpha^2-\beta^2)^2}\left[-2\alpha\beta J_0(\alpha x)K_0(\beta x)+(\alpha^2-\beta^2)J_1(\alpha x)K_1(\beta x)\right]-$$

$$-\frac{8\alpha\beta x}{(\alpha^2-\beta^2)^3}\cdot[\beta J_0(\alpha x)K_1(\beta x)-\alpha J_1(\alpha x)K_0(\beta x)]-$$

$$-\frac{x^3}{\alpha^2-\beta^2}\cdot[\alpha J_0(\alpha x)K_1(\beta x)+\beta J_1(\alpha x)K_0(\beta x)]$$

$$\int x^3\cdot I_1(\alpha x)K_1(\beta x)\,dx\ =$$

$$=\ -\frac{2x^2}{(\alpha^2-\beta^2)^2}\left[2\alpha\beta I_0(\alpha x)K_0(\beta x)+(\alpha^2+\beta^2)I_1(\alpha x)K_1(\beta x)\right]+$$

$$+\frac{8\alpha\beta x}{(\alpha^2-\beta^2)^3}\cdot[\beta I_0(\alpha x)K_1(\beta x)+\alpha I_1(\alpha x)K_0(\beta x)]+$$

$$+\frac{x^3}{\alpha^2-\beta^2}\cdot[\alpha I_0(\alpha x)K_1(\beta x)+\beta I_1(\alpha x)K_0(\beta x)]$$

Let

$$\int x^m\,F_1(\alpha x)G_1(\beta x)\,dx=$$

$$=T_m^{[FG]}(x)F_0(\alpha x)G_0(\beta x)+U_m^{[FG]}(x)F_0(\alpha x)G_1(\beta x)+V_m^{[FG]}(x)F_1(\alpha x)G_0(\beta x)+W_m^{[FG]}(x)F_1(\alpha x)G_1(\beta x)\,.$$

One has

$$T_m^{[JJ]}=T_m^{[YY]}=T_m^{[H^{(1)}H^{(1)}]}=T_m^{[H^{(2)}H^{(2)}]}=T_m^{[JY]}=T_m^{[JH^{(1)}]}=T_m^{[JH^{(2)}]}=T_m^{[YH^{(1)}]}=T_m^{[YH^{(2)}]}=T_m^{[H^{(1)}H^{(2)}]}\,,$$

$$T_m^{[JI]}=T_m^{[YI]}=T_m^{[H^{(1)}I]}=T_m^{[H^{(2)}I]}\quad\text{and}\quad T_m^{[JK]}=T_m^{[YK]}=T_m^{[H^{(1)}K]}=T_m^{[H^{(2)}K]}\,.$$

The same holds analogous for the polynomials $Q_m(x),\ R_m(x)$ and $S_m(x)$.

$$T_5^{[JJ]}(x)=8\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^4-96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$U_5^{[JJ]}(x)=-\frac{\alpha}{\alpha^2-\beta^2}\,x^5+8\,\frac{\alpha\,\left(\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3-192\,\frac{\alpha\,\beta^2\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$V_5^{[JJ]}(x)=\frac{\beta}{\alpha^2-\beta^2}\,x^5-8\,\frac{\beta\,\left(5\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3+192\,\frac{\beta\,\alpha^2\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$W_5^{[JJ]}(x) = 4\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^4 - 16\,\frac{\beta^4+\alpha^4+10\,\alpha^2\beta^2}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$T_5^{[II]}(x) = T_5^{[KK]}(x) = 8\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^4 + 96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$U_5^{[II]}(x) = -U_5^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^5 + 8\,\frac{\alpha\,\left(\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 + 192\,\frac{\alpha\,\beta^2\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$V_5^{[II]}(x) = -V_5^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^5 - 8\,\frac{\beta\,\left(5\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^3 - 192\,\frac{\beta\,\alpha^2\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x$$

$$W_5^{[II]}(x) = W_5^{[KK]}(x) = -4\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^4 - 16\,\frac{\beta^4+\alpha^4+10\,\alpha^2\beta^2}{\left(\alpha^2-\beta^2\right)^4}\,x^2$$

$$T_5^{[JI]}(x) = 8\,\frac{\beta\,\alpha\,x^4}{\left(\alpha^2+\beta^2\right)^2} - 96\,\frac{\beta\,\alpha\,\left(\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$U_5^{[JI]}(x) = -\frac{\alpha\,x^5}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(\alpha^2-5\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\alpha\,\beta^2\left(\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$V_5^{[JI]}(x) = \frac{\beta\,x^5}{\alpha^2+\beta^2} - 8\,\frac{\beta\,\left(5\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\alpha^2\left(\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$W_5^{[JI]}(x) = 4\,\frac{\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\left(\alpha^4-10\,\beta^2\alpha^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$T_5^{[JK]}(x) = -8\,\frac{\alpha\,\beta\,x^4}{\left(\alpha^2+\beta^2\right)^2} + 96\,\frac{\alpha\,\beta\,\left(-\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$U_5^{[JK]}(x) = -\frac{\alpha\,x^5}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(\alpha^2-5\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\alpha\,\beta^2\left(-\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$V_5^{[JK]}(x) = -\frac{\beta\,x^5}{\alpha^2+\beta^2} + 8\,\frac{\beta\,\left(5\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\beta\,\alpha^2\left(-\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^5}$$

$$W_5^{[JK]}(x) = 4\,\frac{\left(-\beta^2+\alpha^2\right)x^4}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\left(\alpha^4-10\,\beta^2\alpha^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$T_5^{[IK]}(x) = -8\,\frac{\alpha\,\beta\,x^4}{\left(\alpha^2-\beta^2\right)^2} - 96\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^4}$$

$$U_5^{[IK]}(x) = \frac{\alpha\,x^5}{\alpha^2-\beta^2} + 8\,\frac{\alpha\,\left(\alpha^2+5\,\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\alpha\,\beta^2\left(\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^5}$$

$$V_5^{[IK]}(x) = \frac{\beta\,x^5}{\alpha^2-\beta^2} + 8\,\frac{\beta\,\left(5\,\alpha^2+\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\beta\,\alpha^2\left(\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^5}$$

$$W_5^{[IK]}(x) = -4\,\frac{\left(\alpha^2+\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^2} - 16\,\frac{\left(\beta^4+\alpha^4+10\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^4}$$

$$T_7^{[JJ]}(x) = 12\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 4608\,\frac{\alpha\,\beta\,\left(\alpha^4+\beta^4+3\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$U_7^{[JJ]}(x) = -\frac{\alpha}{\alpha^2-\beta^2}\,x^7 + 24\,\frac{\alpha\,\left(\alpha^2+4\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 -$$

$$-192\,\frac{\alpha\,\left(\alpha^4+18\,\alpha^2\beta^2+11\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 + 9216\,\frac{\alpha\,\beta^2\,\left(\alpha^4+\beta^4+3\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$V_7^{[JJ]}(x) = \frac{\beta}{\alpha^2-\beta^2}\,x^7 - 24\,\frac{\beta\,\left(4\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 +$$

$$+192\,\frac{\beta\,\left(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 - 9216\,\frac{\beta\,\alpha^2\,\left(\alpha^4+\beta^4+3\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$W_7^{[JJ]}(x) = 6\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 96\,\frac{\alpha^4+8\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 384\,\frac{\beta^6+29\,\alpha^4\beta^2+29\,\alpha^2\beta^4+\alpha^6}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$T_7^{[II]}(x) = T_7^{[KK]}(x) = 12\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^6 + 480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^4 + 4608\,\frac{\alpha\,\beta\,\left(\beta^4+3\,\alpha^2\beta^2+\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$U_7^{[II]}(x) = -U_7^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^7 + 24\,\frac{\alpha\,\left(\alpha^2+4\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 +$$

$$+192\,\frac{\alpha\,\left(\alpha^4+18\,\alpha^2\beta^2+11\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 + 9216\,\frac{\alpha\,\beta^2\,\left(\beta^4+3\,\alpha^2\beta^2+\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$V_7^{[II]}(x) = -V_7^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^7 - 24\,\frac{\beta\,\left(4\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^5 -$$

$$-192\,\frac{\beta\,\left(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^3 - 9216\,\frac{\beta\,\alpha^2\,\left(\beta^4+3\,\alpha^2\beta^2+\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x$$

$$W_7^{[II]}(x) = W_7^{[KK]}(x) = -6\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^6 - 96\,\frac{\alpha^4+8\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^4 - 384\,\frac{\alpha^6+29\,\alpha^4\beta^2+29\,\alpha^2\beta^4+\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^2$$

$$T_7^{[JI]}(x) = 12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2+\beta^2\right)^2} - 480\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^4} + 4608\,\frac{\alpha\,\beta\,\left(\beta^4+\alpha^4-3\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$U_7^{[JI]}(x) = -\frac{\alpha\,x^7}{\alpha^2+\beta^2} + 24\,\frac{\alpha\,\left(\alpha^2-4\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(\alpha^4-18\,\beta^2\alpha^2+11\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-9216\,\frac{\alpha\,\beta^2\,\left(\beta^4+\alpha^4-3\,\beta^2\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$V_7^{[JI]}(x) = \frac{\beta\,x^7}{\alpha^2+\beta^2} - 24\,\frac{\beta\,\left(4\,\alpha^2-\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\left(11\,\alpha^4-18\,\beta^2\alpha^2+\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-9216\,\frac{\beta\,\alpha^2\,\left(\beta^4+\alpha^4-3\,\beta^2\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$W_7^{[JI]}(x) = 6\,\frac{\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^2} - 96\,\frac{\left(\alpha^4-8\,\beta^2\alpha^2+\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\left(-\beta^6+\alpha^6-29\,\beta^2\alpha^4+29\,\alpha^2\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$T_7^{[JK]}(x) = -12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2+\beta^2\right)^2} + 480\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^4} - 4608\,\frac{\alpha\,\beta\,\left(\alpha^4-3\,\alpha^2\beta^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$U_7^{[JK]}(x) = -\frac{\alpha\,x^7}{\alpha^2+\beta^2} + 24\,\frac{\alpha\,\left(\alpha^2-4\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(\alpha^4-18\,\alpha^2\beta^2+11\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-9216\,\frac{\alpha\,\beta^2\,\left(\alpha^4-3\,\alpha^2\beta^2+\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$V_7^{[JK]}(x) = -\frac{\beta\, x^7}{\alpha^2+\beta^2} + 24\,\frac{\beta\,\left(4\,\alpha^2-\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\beta\,\left(11\,\alpha^4-18\,\alpha^2\beta^2+\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+9216\,\frac{\beta\,\alpha^2\left(\alpha^4-3\,\alpha^2\beta^2+\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^7}$$

$$W_7^{[JK]}(x) = 6\,\frac{\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^2} - 96\,\frac{\left(\alpha^4-8\,\alpha^2\beta^2+\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\left(\alpha^6-\beta^6-29\,\beta^2\alpha^4+29\,\beta^4\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$T_7^{[IK]}(x) = -12\,\frac{\alpha\,\beta\,x^6}{\left(\alpha^2-\beta^2\right)^2} - 480\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^4} - 4608\,\frac{\alpha\,\beta\,\left(\beta^4+3\,\beta^2\alpha^2+\alpha^4\right)x^2}{\left(\alpha^2-\beta^2\right)^6}$$

$$U_7^{[IK]}(x) = \frac{\alpha\,x^7}{\alpha^2-\beta^2} + 24\,\frac{\alpha\,\left(\alpha^2+4\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\alpha\,\left(\alpha^4+18\,\beta^2\alpha^2+11\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^5} +$$

$$+9216\,\frac{\alpha\,\beta^2\left(\beta^4+3\,\beta^2\alpha^2+\alpha^4\right)x}{\left(\alpha^2-\beta^2\right)^7}$$

$$V_7^{[IK]}(x) = \frac{\beta\,x^7}{\alpha^2-\beta^2} + 24\,\frac{\beta\,\left(4\,\alpha^2+\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\beta\,\left(11\,\alpha^4+18\,\beta^2\alpha^2+\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^5} +$$

$$+9216\,\frac{\beta\,\alpha^2\left(\beta^4+3\,\beta^2\alpha^2+\alpha^4\right)x}{\left(\alpha^2-\beta^2\right)^7}$$

$$W_7^{[IK]}(x) = -6\,\frac{\left(\alpha^2+\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^2} - 96\,\frac{\left(\alpha^4+8\,\beta^2\alpha^2+\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^4} - 384\,\frac{\left(\beta^6+29\,\beta^2\alpha^4+\alpha^6+29\,\alpha^2\beta^4\right)x^2}{\left(\alpha^2-\beta^2\right)^6}$$

$$T_9^{[JJ]}(x) = 16\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^8 - 1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^6 + 1536\,\frac{\beta\,\alpha\,\left(27\,\alpha^4+86\,\alpha^2\beta^2+27\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^4 -$$

$$-368640\,\frac{\beta\,\alpha\,\left(\beta^6+\alpha^6+6\,\alpha^2\beta^4+6\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$U_9^{[JJ]}(x) = -\frac{\alpha}{\alpha^2-\beta^2}\,x^9 + 16\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7 - 384\,\frac{\alpha\,\left(3\,\alpha^4+43\,\alpha^2\beta^2+24\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5 +$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6+121\,\alpha^4\beta^2+239\,\alpha^2\beta^4+57\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3 - 737280\,\frac{\alpha\,\beta^2\left(\beta^6+\alpha^6+6\,\alpha^2\beta^4+6\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$V_9^{[JJ]}(x) = \frac{\beta}{\alpha^2-\beta^2}\,x^9 - 16\,\frac{\beta\,\left(11\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7 + 384\,\frac{\beta\,\left(24\,\alpha^4+43\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5 -$$

$$-3072\,\frac{\beta\,\left(57\,\alpha^6+239\,\alpha^4\beta^2+121\,\alpha^2\beta^4+3\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3 + 737280\,\frac{\beta\,\alpha^2\left(\beta^6+\alpha^6+6\,\alpha^2\beta^4+6\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$W_9^{[JJ]}(x) = 8\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^8 - 96\,\frac{3\,\alpha^4+22\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^6 + 1536\,\frac{3\,\alpha^6+67\,\alpha^4\beta^2+67\,\alpha^2\beta^4+3\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^4 -$$

$$-6144\,\frac{3\,\beta^8+3\,\alpha^8+178\,\alpha^2\beta^6+178\,\alpha^6\beta^2+478\,\alpha^4\beta^4}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$T_9^{[II]}(x) = T_9^{[KK]}(x) = 16\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^8 + 1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^6 + 1536\,\frac{\beta\,\alpha\,\left(27\,\alpha^4+86\,\alpha^2\beta^2+27\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^4 +$$

$$+368640\,\frac{\beta\,\alpha\,\left(\beta^6+\alpha^6+6\,\alpha^4\beta^2+6\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$U_9^{[II]}(x) = -U_9^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^9 + 16\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7 + 384\,\frac{\alpha\,\left(3\,\alpha^4+43\,\alpha^2\beta^2+24\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5 +$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6+121\,\alpha^4\beta^2+239\,\alpha^2\beta^4+57\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3 + 737280\,\frac{\alpha\,\beta^2\,\left(\beta^6+\alpha^6+6\,\alpha^4\beta^2+6\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$V_9^{[II]}(x) = -V_9^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^9 - 16\,\frac{\beta\,\left(11\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7 - 384\,\frac{\beta\,\left(24\,\alpha^4+43\,\alpha^2\beta^2+3\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5 -$$

$$-3072\,\frac{\beta\,\left(57\,\alpha^6+239\,\alpha^4\beta^2+121\,\alpha^2\beta^4+3\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3 - 737280\,\frac{\beta\,\alpha^2\,\left(\beta^6+\alpha^6+6\,\alpha^4\beta^2+6\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$W_9^{[II]}(x) = W_9^{[KK]}(x) = -8\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^8 - 96\,\frac{3\,\alpha^4+22\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^6 - 1536\,\frac{3\,\alpha^6+67\,\alpha^4\beta^2+67\,\alpha^2\beta^4+3\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^4 -$$

$$-6144\,\frac{3\,\alpha^8+478\,\alpha^4\beta^4+178\,\alpha^2\beta^6+178\,\alpha^6\beta^2+3\,\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^2$$

$$T_9^{[JI]}(x) = 16\,\frac{\alpha\,\beta\,x^8}{\left(\alpha^2+\beta^2\right)^2} - 1344\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^4} + 1536\,\frac{\alpha\,\beta\,\left(27\,\alpha^4-86\,\alpha^2\beta^2+27\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-368640\,\frac{\alpha\,\beta\,\left(6\,\alpha^2\beta^4-\beta^6+\alpha^6-6\,\alpha^4\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$U_9^{[JI]}(x) = -\frac{\alpha\,x^9}{\alpha^2+\beta^2} + 16\,\frac{\alpha\,\left(3\,\alpha^2-11\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3} - 384\,\frac{\alpha\,\left(3\,\alpha^4-43\,\alpha^2\beta^2+24\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6-121\,\alpha^4\beta^2+239\,\alpha^2\beta^4-57\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7} + 737280\,\frac{\alpha\,\beta^2\,\left(6\,\alpha^2\beta^4-\beta^6+\alpha^6-6\,\alpha^4\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$V_9^{[JI]}(x) = \frac{\beta\,x^9}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3} + 384\,\frac{\beta\,\left(24\,\alpha^4-43\,\alpha^2\beta^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-3072\,\frac{\beta\,\left(57\,\alpha^6-239\,\alpha^4\beta^2+121\,\alpha^2\beta^4-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7} + 737280\,\frac{\beta\,\alpha^2\,\left(6\,\alpha^2\beta^4-\beta^6+\alpha^6-6\,\alpha^4\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$W_9^{[JI]}(x) = 8\,\frac{\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^2} - 96\,\frac{\left(3\,\alpha^4-22\,\alpha^2\beta^2+3\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^4} +$$

$$+1536\,\frac{\left(3\,\alpha^6-67\,\alpha^4\beta^2+67\,\alpha^2\beta^4-3\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-6144\,\frac{\left(-178\,\beta^2\alpha^6+3\,\beta^8-178\,\alpha^2\beta^6+478\,\alpha^4\beta^4+3\,\alpha^8\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$T_9^{[JK]}(x) = -16\,\frac{\beta\,\alpha\,x^8}{\left(\alpha^2+\beta^2\right)^2} + 1344\,\frac{\beta\,\alpha\,\left(\alpha^2-\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^4} - 1536\,\frac{\beta\,\alpha\,\left(27\,\alpha^4-86\,\beta^2\alpha^2+27\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^6} +$$

$$+368640\,\frac{\beta\,\alpha\,\left(6\,\beta^4\alpha^2-\beta^6-6\,\beta^2\alpha^4+\alpha^6\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$U_9^{[JK]}(x) = -\frac{\alpha\,x^9}{\alpha^2+\beta^2} + 16\,\frac{\alpha\,\left(3\,\alpha^2-11\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3} - 384\,\frac{\alpha\,\left(3\,\alpha^4-43\,\beta^2\alpha^2+24\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6-121\,\beta^2\alpha^4+239\,\beta^4\alpha^2-57\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7} + 737280\,\frac{\alpha\,\beta^2\,\left(-6\,\beta^2\alpha^4+6\,\beta^4\alpha^2-\beta^6+\alpha^6\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$V_9^{[JK]}(x) = -\frac{\beta\,x^9}{\alpha^2+\beta^2} + 16\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3} - 384\,\frac{\beta\,\left(24\,\alpha^4-43\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+3072\,\frac{\beta\,\left(57\,\alpha^6-239\,\beta^2\alpha^4+121\,\beta^4\alpha^2-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}-737280\,\frac{\beta\,\alpha^2\left(-6\,\beta^2\alpha^4+6\,\beta^4\alpha^2-\beta^6+\alpha^6\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$W_9^{[JK]}(x)=8\,\frac{\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^2}-96\,\frac{\left(3\,\alpha^4-22\,\beta^2\alpha^2+3\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+1536\,\frac{\left(3\,\alpha^6-67\,\beta^2\alpha^4+67\,\beta^4\alpha^2-3\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-6144\,\frac{\left(3\,\alpha^8+3\,\beta^8+478\,\alpha^4\beta^4-178\,\alpha^2\beta^6-178\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$T_9^{[JK]}(x)=-16\,\frac{\beta\,\alpha\,x^8}{\left(\alpha^2-\beta^2\right)^2}-1344\,\frac{\beta\,\alpha\,\left(\alpha^2+\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-1536\,\frac{\beta\,\alpha\,\left(27\,\alpha^4+86\,\beta^2\alpha^2+27\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-368640\,\frac{\beta\,\alpha\,\left(6\,\alpha^2\beta^4+\alpha^6+\beta^6+6\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$U_9^{[JK]}(x)=\frac{\alpha\,x^9}{\alpha^2-\beta^2}+16\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\alpha\,\left(3\,\alpha^4+43\,\beta^2\alpha^2+24\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6+121\,\beta^2\alpha^4+239\,\alpha^2\beta^4+57\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+737280\,\frac{\alpha\,\beta^2\left(6\,\alpha^2\beta^4+\alpha^6+\beta^6+6\,\beta^2\alpha^4\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$V_9^{[JK]}(x)=\frac{\beta\,x^9}{\alpha^2-\beta^2}+16\,\frac{\beta\,\left(11\,\alpha^2+3\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\beta\,\left(24\,\alpha^4+43\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$-3072\,\frac{\beta\,\left(57\,\alpha^6+239\,\beta^2\alpha^4+121\,\alpha^2\beta^4+3\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+737280\,\frac{\beta\,\alpha^2\left(6\,\alpha^2\beta^4+\alpha^6+\beta^6+6\,\beta^2\alpha^4\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$W_9^{[JK]}(x)=-8\,\frac{\left(\alpha^2+\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^2}-96\,\frac{\left(3\,\alpha^4+22\,\beta^2\alpha^2+3\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1536\,\frac{\left(3\,\alpha^6+67\,\beta^2\alpha^4+67\,\alpha^2\beta^4+3\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-6144\,\frac{\left(478\,\alpha^4\beta^4+178\,\alpha^6\beta^2+3\,\alpha^8+3\,\beta^8+178\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$T_9^{[IK]}(x)=-16\,\frac{\alpha\,\beta\,x^8}{\left(\alpha^2-\beta^2\right)^2}-1344\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-1536\,\frac{\alpha\,\beta\,\left(27\,\alpha^4+86\,\beta^2\alpha^2+27\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-368640\,\frac{\alpha\,\beta\,\left(6\,\alpha^2\beta^4+\beta^6+6\,\alpha^4\beta^2+\alpha^6\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$U_9^{[IK]}(x)=\frac{\alpha\,x^9}{\alpha^2-\beta^2}+16\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\alpha\,\left(3\,\alpha^4+43\,\beta^2\alpha^2+24\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+3072\,\frac{\alpha\,\left(3\,\alpha^6+121\,\alpha^4\beta^2+239\,\alpha^2\beta^4+57\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+737280\,\frac{\alpha\,\beta^2\left(6\,\alpha^2\beta^4+\beta^6+6\,\alpha^4\beta^2+\alpha^6\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$V_9^{[IK]}(x)=\frac{\beta\,x^9}{\alpha^2-\beta^2}+16\,\frac{\beta\,\left(11\,\alpha^2+3\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\beta\,\left(24\,\alpha^4+43\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+3072\,\frac{\beta\,\left(57\,\alpha^6+239\,\alpha^4\beta^2+121\,\alpha^2\beta^4+3\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^7}+737280\,\frac{\beta\,\alpha^2\left(6\,\alpha^2\beta^4+\beta^6+6\,\alpha^4\beta^2+\alpha^6\right)x}{\left(\alpha^2-\beta^2\right)^9}$$

$$W_9^{[IK]}(x)=-8\,\frac{\left(\alpha^2+\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^2}-96\,\frac{\left(3\,\alpha^4+22\,\beta^2\alpha^2+3\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1536\,\frac{\left(3\,\alpha^6+67\,\alpha^4\beta^2+67\,\alpha^2\beta^4+3\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-6144\,\frac{\left(478\,\alpha^4\beta^4+178\,\alpha^2\beta^6+3\,\alpha^8+3\,\beta^8+178\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^8}$$

$$T_{11}^{[JJ]}(x)=20\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^8+46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4+13\,\alpha^2\beta^2+4\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6+13\,\alpha^4\beta^2+13\,\alpha^2\beta^4+2\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+44236800\,\frac{\alpha\,\beta\,\left(\alpha^8+10\,\alpha^6\beta^2+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$U_{11}^{[JJ]}(x)=-\frac{\alpha}{\alpha^2-\beta^2}\,x^{11}+40\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9-3840\,\frac{\alpha\,\left(\alpha^4+13\,\alpha^2\beta^2+7\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7+$$

$$+92160\,\frac{\alpha\,\left(\alpha^6+32\,\alpha^4\beta^2+59\,\alpha^2\beta^4+13\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5-$$

$$-737280\,\frac{\alpha\,\left(\alpha^8+73\,\alpha^6\beta^2+300\,\alpha^4\beta^4+227\,\alpha^2\beta^6+29\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3+$$

$$+88473600\,\frac{\alpha\,\beta^2\,\left(\alpha^8+10\,\alpha^6\beta^2+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$V_{11}^{[JJ]}(x)=\frac{\beta}{\alpha^2-\beta^2}\,x^{11}-40\,\frac{\beta\,\left(7\,\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9+3840\,\frac{\beta\,\left(7\,\alpha^4+13\,\alpha^2\beta^2+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7-$$

$$-92160\,\frac{\beta\,\left(13\,\alpha^6+59\,\alpha^4\beta^2+32\,\alpha^2\beta^4+\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5+$$

$$+737280\,\frac{\beta\,\left(29\,\alpha^8+227\,\alpha^6\beta^2+300\,\alpha^4\beta^4+73\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3-$$

$$-88473600\,\frac{\beta\,\alpha^2\,\left(\alpha^8+10\,\alpha^6\beta^2+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$W_{11}^{[JJ]}(x)=10\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-640\,\frac{\alpha^4+7\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^8+23040\,\frac{\alpha^6+20\,\alpha^4\beta^2+20\,\alpha^2\beta^4+\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-368640\,\frac{\alpha^8+45\,\alpha^6\beta^2+118\,\alpha^4\beta^4+45\,\alpha^2\beta^6+\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+1474560\,\frac{\beta^{10}+527\,\beta^4\alpha^6+102\,\beta^2\alpha^8+102\,\beta^8\alpha^2+\alpha^{10}+527\,\beta^6\alpha^4}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$T_{11}^{[II]}(x)=T_{11}^{[KK]}(x)=20\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}+2880\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^8+46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4+13\,\alpha^2\beta^2+4\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^6+$$

$$+2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6+13\,\alpha^4\beta^2+13\,\alpha^2\beta^4+2\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^4+$$

$$+44236800\,\frac{\alpha\,\beta\,\left(\alpha^8+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+10\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$U_{11}^{[II]}(x)=-U_{11}^{[KK]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^{11}+40\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9+3840\,\frac{\alpha\,\left(\alpha^4+13\,\alpha^2\beta^2+7\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7+$$

$$+92160\,\frac{\alpha\,\left(\alpha^6+32\,\alpha^4\beta^2+59\,\alpha^2\beta^4+13\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5+$$

$$+737280\,\frac{\alpha\,\left(\alpha^8+73\,\alpha^6\beta^2+300\,\alpha^4\beta^4+227\,\alpha^2\beta^6+29\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3+$$

$$+88473600\,\frac{\alpha\,\beta^2\,\left(\alpha^8+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+10\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$V_{11}^{[II]}(x)=-V_{11}^{[KK]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^{11}-40\,\frac{\beta\,\left(7\,\alpha^2+2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^9-3840\,\frac{\beta\,\left(7\,\alpha^4+13\,\alpha^2\beta^2+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^7-$$

$$-92160\,\frac{\beta\,\left(13\,\alpha^6+59\,\alpha^4\beta^2+32\,\alpha^2\beta^4+\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^5-$$

$$-737280\,\frac{\beta\,\left(29\,\alpha^8+227\,\alpha^6\beta^2+300\,\alpha^4\beta^4+73\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^3-$$

$$-88473600\,\frac{\beta\,\alpha^2\,\left(\alpha^8+20\,\alpha^4\beta^4+10\,\alpha^2\beta^6+10\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{11}}\,x$$

$$W_{11}^{[II]}(x)=W_{11}^{[KK]}(x)=-10\,\frac{\alpha^2+\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^{10}-640\,\frac{\alpha^4+7\,\alpha^2\beta^2+\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^8-23040\,\frac{\alpha^6+20\,\alpha^4\beta^2+20\,\alpha^2\beta^4+\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^6-$$

$$-368640\,\frac{\alpha^8+45\,\alpha^6\beta^2+118\,\alpha^4\beta^4+45\,\alpha^2\beta^6+\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^4-$$

$$-1474560\,\frac{\alpha^{10}+\beta^{10}+102\,\alpha^8\beta^2+527\,\alpha^6\beta^4+102\,\alpha^2\beta^8+527\,\alpha^4\beta^6}{\left(\alpha^2-\beta^2\right)^{10}}\,x^2$$

$$T_{11}^{[JI]}(x)=20\,\frac{\alpha\,\beta\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4-13\,\beta^2\alpha^2+4\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6-13\,\beta^2\alpha^4+13\,\beta^4\alpha^2-2\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+44236800\,\frac{\alpha\,\beta\,\left(-10\,\beta^2\alpha^6+\beta^8+\alpha^8-10\,\alpha^2\beta^6+20\,\alpha^4\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$U_{11}^{[JI]}(x)=-\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}+40\,\frac{\alpha\,\left(2\,\alpha^2-7\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}-3840\,\frac{\alpha\,\left(\alpha^4-13\,\beta^2\alpha^2+7\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+92160\,\frac{\alpha\,\left(\alpha^6-32\,\beta^2\alpha^4+59\,\beta^4\alpha^2-13\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-737280\,\frac{\alpha\,\left(\alpha^8-73\,\beta^2\alpha^6+300\,\alpha^4\beta^4-227\,\alpha^2\beta^6+29\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-88473600\,\frac{\alpha\,\beta^2\,\left(-10\,\beta^2\alpha^6+\beta^8+\alpha^8-10\,\alpha^2\beta^6+20\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$V_{11}^{[JI]}(x)=\frac{\beta\,x^{11}}{\alpha^2+\beta^2}-40\,\frac{\beta\,\left(7\,\alpha^2-2\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+3840\,\frac{\beta\,\left(7\,\alpha^4-13\,\beta^2\alpha^2+\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-92160\,\frac{\beta\,\left(13\,\alpha^6-59\,\beta^2\alpha^4+32\,\beta^4\alpha^2-\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+737280\,\frac{\beta\,\left(29\,\alpha^8-227\,\beta^2\alpha^6+300\,\alpha^4\beta^4-73\,\alpha^2\beta^6+\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-88473600\,\frac{\beta\,\alpha^2\left(-10\,\beta^2\alpha^6+\beta^8+\alpha^8-10\,\alpha^2\beta^6+20\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$W_{11}^{[JI]}(x)=10\,\frac{\left(\alpha^2-\beta^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-640\,\frac{\left(\alpha^4-7\,\beta^2\alpha^2+\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+23040\,\frac{\left(\alpha^6-20\,\beta^2\alpha^4+20\,\beta^4\alpha^2-\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-368640\,\frac{\left(\alpha^8-45\,\beta^2\alpha^6+118\,\alpha^4\beta^4-45\,\alpha^2\beta^6+\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+1474560\,\frac{\left(-102\,\beta^2\alpha^8+527\,\beta^4\alpha^6-527\,\beta^6\alpha^4-\beta^{10}+\alpha^{10}+102\,\beta^8\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$T_{11}^{[JK]}(x)=-20\,\frac{\alpha\,\beta\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}+2880\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}-46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4-13\,\alpha^2\beta^2+4\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6-13\,\beta^2\alpha^4+13\,\beta^4\alpha^2-2\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}-$$

$$-44236800\,\frac{\alpha\,\beta\,\left(\alpha^8+\beta^8+20\,\alpha^4\beta^4-10\,\alpha^2\beta^6-10\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$U_{11}^{[JK]}(x)=-\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}+40\,\frac{\alpha\,\left(2\,\alpha^2-7\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}-3840\,\frac{\alpha\,\left(\alpha^4-13\,\beta^2\alpha^2+7\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+92160\,\frac{\alpha\,\left(\alpha^6-32\,\beta^2\alpha^4+59\,\beta^4\alpha^2-13\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-737280\,\frac{\alpha\,\left(\alpha^8-73\,\alpha^6\beta^2+300\,\alpha^4\beta^4-227\,\alpha^2\beta^6+29\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-88473600\,\frac{\alpha\,\beta^2\left(-10\,\alpha^6\beta^2+20\,\alpha^4\beta^4+\beta^8+\alpha^8-10\,\alpha^2\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$V_{11}^{[JK]}(x)=-\frac{\beta\,x^{11}}{\alpha^2+\beta^2}+40\,\frac{\beta\,\left(7\,\alpha^2-2\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}-3840\,\frac{\beta\,\left(7\,\alpha^4-13\,\beta^2\alpha^2+\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+92160\,\frac{\beta\,\left(13\,\alpha^6-59\,\beta^2\alpha^4+32\,\beta^4\alpha^2-\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-737280\,\frac{\beta\,\left(29\,\alpha^8-227\,\alpha^6\beta^2+300\,\alpha^4\beta^4-73\,\alpha^2\beta^6+\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}+$$

$$+88473600\,\frac{\beta\,\alpha^2\left(\alpha^8+20\,\alpha^4\beta^4-10\,\alpha^2\beta^6-10\,\alpha^6\beta^2+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$W_{11}^{[JK]}(x)=10\,\frac{\left(\alpha^2-\beta^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-640\,\frac{\left(\alpha^4-7\,\beta^2\alpha^2+\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+23040\,\frac{\left(\alpha^6-20\,\beta^2\alpha^4+20\,\beta^4\alpha^2-\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-368640\,\frac{\left(\alpha^8-45\,\alpha^6\beta^2+118\,\alpha^4\beta^4-45\,\alpha^2\beta^6+\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+1474560\,\frac{\left(-\beta^{10}-527\,\alpha^4\beta^6+102\,\alpha^2\beta^8+527\,\alpha^6\beta^4+\alpha^{10}-102\,\alpha^8\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$T_{11}^{[IK]}(x)=-20\,\frac{\alpha\,\beta\,x^{10}}{\left(\alpha^2-\beta^2\right)^2}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^4}-46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4+13\,\beta^2\alpha^2+4\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6+13\,\beta^2\alpha^4+13\,\beta^4\alpha^2+2\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-44236800\,\frac{\alpha\,\beta\,\left(20\,\alpha^4\beta^4+\alpha^8+\beta^8+10\,\alpha^6\beta^2+10\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$U_{11}^{[IK]}(x)=\frac{\alpha\,x^{11}}{\alpha^2-\beta^2}+40\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^3}+3840\,\frac{\alpha\,\left(\alpha^4+13\,\beta^2\alpha^2+7\,\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+92160\,\frac{\alpha\,\left(\alpha^6+32\,\beta^2\alpha^4+59\,\beta^4\alpha^2+13\,\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+737280\,\frac{\alpha\,\left(\alpha^8+73\,\alpha^6\beta^2+300\,\alpha^4\beta^4+227\,\alpha^2\beta^6+29\,\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^9}+$$

$$+88473600\,\frac{\alpha\,\beta^2\left(20\,\alpha^4\beta^4+\alpha^8+\beta^8+10\,\alpha^6\beta^2+10\,\alpha^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^{11}}$$

$$V_{11}^{[IK]}(x)=\frac{\beta\,x^{11}}{\alpha^2-\beta^2}+40\,\frac{\beta\,\left(7\,\alpha^2+2\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^3}+3840\,\frac{\beta\,\left(7\,\alpha^4+13\,\beta^2\alpha^2+\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+92160\,\frac{\beta\,\left(13\,\alpha^6+59\,\beta^2\alpha^4+32\,\beta^4\alpha^2+\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+737280\,\frac{\beta\,\left(29\,\alpha^8+227\,\alpha^6\beta^2+300\,\alpha^4\beta^4+73\,\alpha^2\beta^6+\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^9}+$$

$$+88473600\,\frac{\beta\,\alpha^2\left(20\,\alpha^4\beta^4+\alpha^8+\beta^8+10\,\alpha^6\beta^2+10\,\alpha^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^{11}}$$

$$W_{11}^{[IK]}(x)=-10\,\frac{\left(\alpha^2+\beta^2\right)x^{10}}{\left(\alpha^2-\beta^2\right)^2}-640\,\frac{\left(\alpha^4+7\,\beta^2\alpha^2+\beta^4\right)x^8}{\left(\alpha^2-\beta^2\right)^4}-23040\,\frac{\left(\alpha^6+20\,\beta^2\alpha^4+20\,\beta^4\alpha^2+\beta^6\right)x^6}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-368640\,\frac{\left(\alpha^8+45\,\alpha^6\beta^2+118\,\alpha^4\beta^4+45\,\alpha^2\beta^6+\beta^8\right)x^4}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-1474560\,\frac{\left(102\,\alpha^8\beta^2+527\,\alpha^6\beta^4+\beta^{10}+102\,\alpha^2\beta^8+\alpha^{10}+527\,\alpha^4\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^{10}}$$

Recurrence formulas: see page 262.

2.2.2. Integrals of the type $\int x^{2n} Z_0(\alpha x) Z_1(\beta x)\, dx\,,\ \alpha^2 \neq \beta^2$

See the remark in 2.2.1, page 228.

$$\int x^2 \cdot J_0(\alpha x) J_1(\beta x)\, dx \; =$$

$$= \; \frac{2\beta x}{(\alpha^2-\beta^2)^2}\,[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)] + \frac{x^2}{\alpha^2-\beta^2}\,[\beta J_0(\alpha x) J_0(\beta x) + \alpha J_1(\alpha x) J_1(\beta x)]$$

$$\int x^2 \cdot I_0(\alpha x) I_1(\beta x)\, dx \; =$$

$$= \; -\frac{2\beta x}{(\alpha^2-\beta^2)^2}\,[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)] - \frac{x^2}{\alpha^2-\beta^2}\,[\beta I_0(\alpha x) I_0(\beta x) - \alpha I_1(\alpha x) I_1(\beta x)]$$

$$\int x^2 \cdot K_0(\alpha x) K_1(\beta x)\, dx \; =$$

$$= \; -\frac{2\beta x}{(\alpha^2-\beta^2)^2}\,[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x)] + \frac{x^2}{\alpha^2-\beta^2}\,[\beta K_0(\alpha x) K_0(\beta x) - \alpha K_1(\alpha x) K_1(\beta x)]$$

$$\int x^2 \cdot J_0(\alpha x) I_1(\beta x)\, dx \; =$$

$$= \; -\frac{2\beta x}{(\alpha^2+\beta^2)^2}\,[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x)] + \frac{x^2}{\alpha^2+\beta^2}\,[\beta J_0(\alpha x) I_0(\beta x) + \alpha J_1(\alpha x) I_1(\beta x)]$$

$$\int x^2 \cdot J_1(\alpha x) I_0(\beta x)\, dx \; = -\frac{\alpha x^2}{\alpha^2+\beta^2}\, J_0(\alpha x)\, I_0(\beta x) + \frac{2\,\alpha\beta\, x}{(\alpha^2+\beta^2)^2}\, J_0(\alpha x)\, I_1(\beta x) +$$

$$+\frac{2\,\alpha^2 x}{(\alpha^2+\beta^2)^2}\, J_1(\alpha x)\, I_0(\beta x) + \frac{\beta x^2}{\alpha^2+\beta^2}\, J_1(\alpha x)\, I_1(\beta x)$$

$$\int x^2 \cdot J_0(\alpha x) K_1(\beta x)\, dx \; =$$

$$= \; -\frac{2\beta x}{(\alpha^2+\beta^2)^2}\,[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x)] - \frac{x^2}{\alpha^2+\beta^2}\,[\beta J_0(\alpha x) K_0(\beta x) - \alpha J_1(\alpha x) K_1(\beta x)]$$

$$\int x^2 \cdot J_1(\alpha x) K_0(\beta x)\, dx \; =$$

$$= \; \frac{2\alpha x}{(\alpha^2+\beta^2)^2}\,[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x)] - \frac{x^2}{\alpha^2+\beta^2}\,[\beta J_0(\alpha x) K_0(\beta x) + \alpha J_1(\alpha x) K_1(\beta x)]$$

$$\int x^2 \cdot I_0(\alpha x) K_1(\beta x)\, dx \; =$$

$$= \; -\frac{2\beta x}{(\alpha^2-\beta^2)^2}\,[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x)] + \frac{x^2}{\alpha^2-\beta^2}\,[\beta I_0(\alpha x) K_0(\beta x) + \alpha I_1(\alpha x) K_1(\beta x)]$$

$$\int x^2 \cdot I_1(\alpha x) K_0(\beta x)\, dx \; =$$

$$= \; -\frac{2\alpha x}{(\alpha^2-\beta^2)^2}\,[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x)] + \frac{x^2}{\alpha^2-\beta^2}\,[\beta I_0(\alpha x) K_0(\beta x) + \alpha I_1(\alpha x) K_1(\beta x)]$$

Let

$$\int x^m\, U_0(\alpha x) W_1(\beta x)\, dx =$$

$$= P_m^{[UW]}(x) U_0(\alpha x) W_0(\beta x) + Q_m^{[UW]}(x) U_0(\alpha x) W_1(\beta x) + R_m^{[UW]}(x) U_1(\alpha x) W_0(\beta x) + S_m^{[UW]}(x) U_1(\alpha x) W_1(\beta x)\ .$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)}H^{(1)}]} = P_m^{[H^{(2)}H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)}H^{(2)}]} ,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad \text{and} \quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]} .$$

The same holds analogous for the polynomials $Q_m(x)$, $R_m(x)$ and $S_m(x)$.

$$P_4^{[JJ]}(x) = \frac{\beta x^4}{\alpha^2-\beta^2} - \frac{8\beta\left(2\,\alpha^2+\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$Q_4^{[JJ]}(x) = \frac{2\,(\alpha^2+2\,\beta^2)x^3}{\left(\alpha^2-\beta^2\right)^2} - \frac{16\beta^2\left(2\,\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4}$$

$$R_4^{[JJ]}(x) = \frac{16\,\beta\,\alpha\,\left(2\,\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4} - \frac{6\,\beta\,\alpha x^3}{\left(\alpha^2-\beta^2\right)^2}$$

$$S_4^{[JJ]}(x) = \frac{\alpha x^4}{\alpha^2-\beta^2} - \frac{4\,\alpha\,\left(\alpha^2+5\,\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$P_4^{[II]}(x) = -P_4^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^4 - 8\,\frac{\beta\,\left(2\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^2$$

$$Q_4^{[II]}(x) = Q_4^{[KK]}(x) = -2\,\frac{\alpha^2+2\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^3 - 16\,\frac{\beta^2\left(2\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x$$

$$R_4^{[II]}(x) = R_4^{[KK]}(x) = 6\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^3 + 16\,\frac{\alpha\,\beta\,\left(2\,\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x$$

$$S_4^{[II]}(x) = -S_4^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^4 + 4\,\frac{\alpha\,\left(\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^2$$

$$P_4^{[JI]}(x) = \frac{\beta\,x^4}{\alpha^2+\beta^2} - 8\,\frac{\beta\,\left(2\,\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$Q_4^{[JI]}(x) = 2\,\frac{\left(\alpha^2-2\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\,\frac{\beta^2\,\left(2\,\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$R_4^{[JI]}(x) = -6\,\frac{\beta\,\alpha\,x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\,\frac{\beta\,\alpha\,\left(2\,\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$S_4^{[JI]}(x) = \frac{\alpha\,x^4}{\alpha^2+\beta^2} - 4\,\frac{\alpha\,\left(\alpha^2-5\,\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$P_4^{[IJ]}(x) = -\frac{\alpha\,x^4}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(-2\,\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$Q_4^{[IJ]}(x) = 6\,\frac{\alpha\,\beta\,x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha\,\beta\,\left(-2\,\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$R_4^{[IJ]}(x) = 2\,\frac{\left(2\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha^2\,\left(-2\,\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$S_4^{[IJ]}(x) = \frac{\beta\,x^4}{\alpha^2+\beta^2} - 4\,\frac{\beta\,\left(-\beta^2+5\,\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$P_4^{[JK]}(x) = -\frac{\beta\,x^4}{\alpha^2+\beta^2} + 8\,\frac{\beta\,\left(2\,\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$Q_4^{[JK]}(x) = 2\,\frac{\left(\alpha^2-2\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\,\frac{\beta^2\left(2\,\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$R_4^{[JK]}(x) = 6\,\frac{\alpha\,\beta\,x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha\,\beta\,\left(2\,\alpha^2-\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$S_4^{[JK]}(x) = \frac{\alpha\,x^4}{\alpha^2+\beta^2} - 4\,\frac{\alpha\,\left(-5\,\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$P_4^{[KJ]}(x) = -\frac{\alpha\,x^4}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(-2\,\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$Q_4^{[KJ]}(x) = -6\,\frac{\alpha\,\beta\,x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\,\frac{\alpha\,\beta\,\left(-2\,\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$R_4^{[KJ]}(x) = 2\,\frac{\left(2\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha^2\left(-2\,\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^4}$$

$$S_4^{[KJ]}(x) = -\frac{\beta\,x^4}{\alpha^2+\beta^2} + 4\,\frac{\beta\,\left(5\,\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}$$

$$P_4^{[IK]}(x) = \frac{\beta\,x^4}{\alpha^2-\beta^2} + 8\,\frac{\beta\,\left(2\,\alpha^2+\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$Q_4^{[IK]}(x) = -2\,\frac{\left(\alpha^2+2\,\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^2} - 16\,\frac{\beta^2\left(2\,\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4}$$

$$R_4^{[IK]}(x) = -6\,\frac{\beta\,\alpha\,x^3}{\left(\alpha^2-\beta^2\right)^2} - 16\,\frac{\beta\,\alpha\,\left(2\,\alpha^2+\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4}$$

$$S_4^{[IK]}(x) = \frac{\alpha\,x^4}{\alpha^2-\beta^2} + 4\,\frac{\alpha\,\left(5\,\beta^2+\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$P_4^{[KI]}(x) = \frac{\alpha\,x^4}{\alpha^2-\beta^2} + 8\,\frac{\alpha\,\left(\alpha^2+2\,\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$Q_4^{[KI]}(x) = -6\,\frac{\beta\,\alpha\,x^3}{\left(\alpha^2-\beta^2\right)^2} - 16\,\frac{\beta\,\alpha\,\left(\alpha^2+2\,\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4}$$

$$R_4^{[KI]}(x) = -2\,\frac{\left(2\,\alpha^2+\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^2} - 16\,\frac{\alpha^2\left(\alpha^2+2\,\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^4}$$

$$S_4^{[KI]}(x) = \frac{\beta\,x^4}{\alpha^2-\beta^2} + 4\,\frac{\beta\,\left(5\,\alpha^2+\beta^2\right)x^2}{\left(\alpha^2-\beta^2\right)^3}$$

$$P_6^{[JJ]}(x) = \frac{\beta}{\alpha^2-\beta^2}\,x^6 - 8\,\frac{\beta\,\left(7\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^4 + 192\,\frac{\beta\,\left(3\,\alpha^4+\beta^4+6\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^2$$

$$Q_6^{[JJ]}(x) = 2\,\frac{2\,\alpha^2+3\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^5 - 32\,\frac{\alpha^4+11\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^3 + 384\,\frac{\beta^2\left(3\,\alpha^4+\beta^4+6\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^6}\,x$$

$$R_6^{[JJ]}(x) = -10\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^5 + 32\,\frac{\beta\,\alpha\,\left(8\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^3 - 384\,\frac{\beta\,\alpha\,\left(3\,\alpha^4+\beta^4+6\,\alpha^2\beta^2\right)}{\left(\alpha^2-\beta^2\right)^6}\,x$$

$$S_6^{[JJ]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^6 - 16\,\frac{\alpha\,\left(\alpha^2+4\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^4 + 64\,\frac{\alpha\,\left(19\,\alpha^2\beta^2+10\,\beta^4+\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^2$$

$$P_6^{[II]}(x) = -P_6^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^6 - 8\,\frac{\beta\,\left(7\,\alpha^2+3\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^4 - 192\,\frac{\beta\,\left(6\,\alpha^2\beta^2+3\,\alpha^4+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^2$$

$$Q_6^{[II]}(x) = Q_6^{[KK]}(x) = -2\,\frac{2\,\alpha^2+3\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^5 - 32\,\frac{\alpha^4+11\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^3 - 384\,\frac{\beta^2\,\left(6\,\alpha^2\beta^2+3\,\alpha^4+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x$$

$$R_6^{[II]}(x) = R_6^{[KK]}(x) = 10\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^5 + 32\,\frac{\alpha\,\beta\,\left(8\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^3 + 384\,\frac{\alpha\,\beta\,\left(6\,\alpha^2\beta^2+3\,\alpha^4+\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x$$

$$S_6^{[II]}(x) = -S_6^{[KK]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^6 + 16\,\frac{\alpha\,\left(\alpha^2+4\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^4 + 64\,\frac{\alpha\,\left(10\,\beta^4+19\,\alpha^2\beta^2+\alpha^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^2$$

$$P_6^{[JI]}(x) = \frac{\beta\,x^6}{\alpha^2+\beta^2} - 8\,\frac{\beta\,\left(7\,\alpha^2-3\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\left(\beta^4+3\,\alpha^4-6\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6^{[JI]}(x) = 2\,\frac{\left(2\,\alpha^2-3\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(\alpha^4-11\,\beta^2\alpha^2+3\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\beta^2\,\left(\beta^4+3\,\alpha^4-6\,\beta^2\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6^{[JI]}(x) = -10\,\frac{\beta\,\alpha\,x^5}{\left(\alpha^2+\beta^2\right)^2} + 32\,\frac{\beta\,\alpha\,\left(8\,\alpha^2-7\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\beta\,\alpha\,\left(\beta^4+3\,\alpha^4-6\,\beta^2\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6^{[JI]}(x) = \frac{\alpha\,x^6}{\alpha^2+\beta^2} - 16\,\frac{\alpha\,\left(\alpha^2-4\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\alpha^4-19\,\beta^2\alpha^2+10\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_6^{[IJ]}(x) = -\frac{\alpha\,x^6}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(\alpha^4-6\,\beta^2\alpha^2+3\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6^{[IJ]}(x) = 10\,\frac{\beta\,\alpha\,x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\beta\,\alpha\,\left(7\,\alpha^2-8\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\beta\,\alpha\,\left(\alpha^4-6\,\beta^2\alpha^2+3\,\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6^{[IJ]}(x) = 2\,\frac{\left(3\,\alpha^2-2\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(3\,\alpha^4-11\,\beta^2\alpha^2+\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\alpha^2\,\left(\alpha^4-6\,\beta^2\alpha^2+3\,\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6^{[IJ]}(x) = \frac{\beta\,x^6}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(4\,\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\beta\,\left(10\,\alpha^4-19\,\beta^2\alpha^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_6^{[JK]}(x) = -\frac{\beta\,x^6}{\alpha^2+\beta^2} + 8\,\frac{\beta\,\left(7\,\alpha^2-3\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\beta\,\left(-6\,\alpha^2\beta^2+\beta^4+3\,\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6^{[JK]}(x) = 2\,\frac{\left(2\,\alpha^2-3\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(\alpha^4-11\,\alpha^2\beta^2+3\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\beta^2\,\left(-6\,\alpha^2\beta^2+\beta^4+3\,\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6^{[JK]}(x) = 10\,\frac{\alpha\,\beta\,x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\alpha\,\beta\,\left(8\,\alpha^2-7\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\alpha\,\beta\,\left(-6\,\alpha^2\beta^2+\beta^4+3\,\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6^{[JK]}(x) = \frac{\alpha\,x^6}{\alpha^2+\beta^2} - 16\,\frac{\alpha\,\left(\alpha^2-4\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\alpha^4-19\,\alpha^2\beta^2+10\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_6^{[KJ]}(x) = -\frac{\alpha\,x^6}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(\alpha^4+3\,\beta^4-6\,\alpha^2\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6^{[KJ]}(x) = -10\,\frac{\alpha\,\beta\,x^5}{\left(\alpha^2+\beta^2\right)^2} + 32\,\frac{\alpha\,\beta\,\left(7\,\alpha^2-8\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\alpha\,\beta\,\left(\alpha^4+3\,\beta^4-6\,\alpha^2\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6^{[KJ]}(x) = 2\,\frac{\left(3\,\alpha^2 - 2\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(3\,\alpha^4 - 11\,\alpha^2\beta^2 + \beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\alpha^2\left(\alpha^4 + 3\,\beta^4 - 6\,\alpha^2\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6^{[KJ]}(x) = -\frac{\beta\,x^6}{\alpha^2+\beta^2} + 16\,\frac{\beta\,\left(4\,\alpha^2 - \beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} - 64\,\frac{\beta\,\left(10\,\alpha^4 + \beta^4 - 19\,\alpha^2\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_6^{[IK]}(x) = \frac{\beta\,x^6}{\alpha^2-\beta^2} + 8\,\frac{\beta\,\left(7\,\alpha^2 + 3\,\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\beta\,\left(\beta^4 + 3\,\alpha^4 + 6\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^5}$$

$$Q_6^{[IK]}(x) = -2\,\frac{\left(2\,\alpha^2 + 3\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^2} - 32\,\frac{\left(\alpha^4 + 11\,\beta^2\alpha^2 + 3\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^4} - 384\,\frac{\beta^2\left(\beta^4 + 3\,\alpha^4 + 6\,\beta^2\alpha^2\right)x}{\left(\alpha^2-\beta^2\right)^6}$$

$$R_6^{[IK]}(x) = -10\,\frac{\beta\,\alpha\,x^5}{\left(\alpha^2-\beta^2\right)^2} - 32\,\frac{\beta\,\alpha\,\left(8\,\alpha^2 + 7\,\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^4} - 384\,\frac{\beta\,\alpha\,\left(\beta^4 + 3\,\alpha^4 + 6\,\beta^2\alpha^2\right)x}{\left(\alpha^2-\beta^2\right)^6}$$

$$S_6^{[IK]}(x) = \frac{\alpha\,x^6}{\alpha^2-\beta^2} + 16\,\frac{\alpha\,\left(\alpha^2 + 4\,\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\alpha^4 + 10\,\beta^4 + 19\,\beta^2\alpha^2\right)x^2}{\left(\alpha^2-\beta^2\right)^5}$$

$$P_6^{[KI]}(x) = \frac{\alpha\,x^6}{\alpha^2-\beta^2} + 8\,\frac{\alpha\,\left(3\,\alpha^2 + 7\,\beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^3} + 192\,\frac{\alpha\,\left(\alpha^4 + 6\,\beta^2\alpha^2 + 3\,\beta^4\right)x^2}{\left(\alpha^2-\beta^2\right)^5}$$

$$Q_6^{[KI]}(x) = -10\,\frac{\beta\,\alpha\,x^5}{\left(\alpha^2-\beta^2\right)^2} - 32\,\frac{\beta\,\alpha\,\left(7\,\alpha^2 + 8\,\beta^2\right)x^3}{\left(\alpha^2-\beta^2\right)^4} - 384\,\frac{\beta\,\alpha\,\left(\alpha^4 + 6\,\beta^2\alpha^2 + 3\,\beta^4\right)x}{\left(\alpha^2-\beta^2\right)^6}$$

$$R_6^{[KI]}(x) = -2\,\frac{\left(3\,\alpha^2 + 2\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^2} - 32\,\frac{\left(3\,\alpha^4 + 11\,\beta^2\alpha^2 + \beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^4} - 384\,\frac{\alpha^2\left(\alpha^4 + 6\,\beta^2\alpha^2 + 3\,\beta^4\right)x}{\left(\alpha^2-\beta^2\right)^6}$$

$$S_6^{[KI]}(x) = \frac{\beta\,x^6}{\alpha^2-\beta^2} + 16\,\frac{\beta\,\left(4\,\alpha^2 + \beta^2\right)x^4}{\left(\alpha^2-\beta^2\right)^3} + 64\,\frac{\beta\,\left(10\,\alpha^4 + 19\,\beta^2\alpha^2 + \beta^4\right)x^2}{\left(\alpha^2-\beta^2\right)^5}$$

$$P_8^{[JJ]}(x) = \frac{\beta}{\alpha^2-\beta^2}\,x^8 - 24\,\frac{\beta\,\left(5\,\alpha^2 + 2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^6 + 192\,\frac{\beta\,\left(21\,\alpha^4 + 43\,\alpha^2\beta^2 + 6\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^4 -$$

$$-9216\,\frac{\beta\,\left(\beta^6 + 4\,\alpha^6 + 12\,\alpha^2\beta^4 + 18\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^2$$

$$Q_8^{[JJ]}(x) = 2\,\frac{3\,\alpha^2 + 4\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^7 - 48\,\frac{3\,\alpha^4 + 26\,\alpha^2\beta^2 + 6\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^5 + 384\,\frac{3\,\alpha^6 + 86\,\alpha^4\beta^2 + 109\,\alpha^2\beta^4 + 12\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^3 -$$

$$-18432\,\frac{\beta^2\left(\beta^6 + 4\,\alpha^6 + 12\,\alpha^2\beta^4 + 18\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^8}\,x$$

$$R_8^{[JJ]}(x) = -14\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^7 + 48\,\frac{\beta\,\alpha\,\left(18\,\alpha^2 + 17\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^5 - 1920\,\frac{\beta\,\alpha\,\left(9\,\alpha^4 + 26\,\alpha^2\beta^2 + 7\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^3 +$$

$$+18432\,\frac{\beta\,\alpha\,\left(\beta^6 + 4\,\alpha^6 + 12\,\alpha^2\beta^4 + 18\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^8}\,x$$

$$S_8^{[JJ]}(x) = \frac{\alpha}{\alpha^2-\beta^2}\,x^8 - 12\,\frac{\alpha\,\left(3\,\alpha^2 + 11\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^6 + 192\,\frac{\alpha\,\left(3\,\alpha^4 + 44\,\alpha^2\beta^2 + 23\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^4 -$$

$$-768\,\frac{\alpha\,\left(47\,\beta^6 + 3\,\alpha^6 + 239\,\alpha^2\beta^4 + 131\,\alpha^4\beta^2\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^2$$

$$P_8^{[II]}(x) = -P_8^{[KK]}(x) = -\frac{\beta}{\alpha^2-\beta^2}\,x^8 - 24\,\frac{\beta\,\left(5\,\alpha^2 + 2\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^6 - 192\,\frac{\beta\,\left(21\,\alpha^4 + 43\,\alpha^2\beta^2 + 6\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^4 -$$

$$-9216\,\frac{\beta\,\left(\beta^6+4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^2$$

$$Q_8^{[II]}(x)=Q_8^{[KK]}(x)=-2\,\frac{3\,\alpha^2+4\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^7-48\,\frac{3\,\alpha^4+26\,\alpha^2\beta^2+6\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^5-$$

$$-384\,\frac{3\,\alpha^6+86\,\alpha^4\beta^2+109\,\alpha^2\beta^4+12\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^3--18432\,\frac{\beta^2\left(\beta^6+4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^8}\,x$$

$$R_8^{[II]}(x)=R_8^{[KK]}(x)=14\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^7+48\,\frac{\beta\,\alpha\,\left(18\,\alpha^2+17\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^5+1920\,\frac{\beta\,\alpha\,\left(9\,\alpha^4+26\,\alpha^2\beta^2+7\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^3+$$

$$+18432\,\frac{\beta\,\alpha\,\left(\beta^6+4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^8}\,x$$

$$S_8^{[II]}(x)=-S_8^{[KK]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^8+12\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^6+192\,\frac{\alpha\,\left(3\,\alpha^4+44\,\alpha^2\beta^2+23\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^4+$$

$$+768\,\frac{\alpha\,\left(47\,\beta^6+3\,\alpha^6+131\,\alpha^4\beta^2+239\,\alpha^2\beta^4\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^2$$

$$P_8^{[JI]}(x)=\frac{\beta\,x^8}{\alpha^2+\beta^2}-24\,\frac{\beta\,\left(5\,\alpha^2-2\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}+192\,\frac{\beta\,\left(21\,\alpha^4-43\,\beta^2\alpha^2+6\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-9216\,\frac{\beta\,\left(4\,\alpha^6-\beta^6+12\,\alpha^2\beta^4-18\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8^{[JI]}(x)=2\,\frac{\left(3\,\alpha^2-4\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\left(3\,\alpha^4-26\,\beta^2\alpha^2+6\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+384\,\frac{\left(3\,\alpha^6-86\,\beta^2\alpha^4+109\,\alpha^2\beta^4-12\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^6}+18432\,\frac{\beta^2\left(4\,\alpha^6-\beta^6+12\,\alpha^2\beta^4-18\,\beta^2\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8^{[JI]}(x)=-14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2+\beta^2\right)^2}+48\,\frac{\alpha\,\beta\,\left(18\,\alpha^2-17\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^4}-1920\,\frac{\alpha\,\beta\,\left(9\,\alpha^4-26\,\beta^2\alpha^2+7\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+18432\,\frac{\alpha\,\beta\,\left(-18\,\beta^2\alpha^4+12\,\alpha^2\beta^4+4\,\alpha^6-\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8^{[JI]}(x)=\frac{\alpha\,x^8}{\alpha^2+\beta^2}-12\,\frac{\alpha\,\left(3\,\alpha^2-11\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}+192\,\frac{\alpha\,\left(3\,\alpha^4-44\,\beta^2\alpha^2+23\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-768\,\frac{\alpha\,\left(3\,\alpha^6-47\,\beta^6+239\,\alpha^2\beta^4-131\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_8^{[IJ]}(x)=-\frac{\alpha\,x^8}{\alpha^2+\beta^2}+24\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}-192\,\frac{\alpha\,\left(6\,\alpha^4-43\,\beta^2\alpha^2+21\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+9216\,\frac{\alpha\,\left(-12\,\beta^2\alpha^4+18\,\alpha^2\beta^4+\alpha^6-4\,\beta^6\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8^{[IJ]}(x)=14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\alpha\,\beta\,\left(17\,\alpha^2-18\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+1920\,\frac{\alpha\,\beta\,\left(7\,\alpha^4-26\,\beta^2\alpha^2+9\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-18432\,\frac{\alpha\,\beta\,\left(-12\,\beta^2\alpha^4+18\,\alpha^2\beta^4+\alpha^6-4\,\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8^{[IJ]}(x)=2\,\frac{\left(4\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\left(6\,\alpha^4-26\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+384\,\frac{\left(12\,\alpha^6-109\,\beta^2\alpha^4+86\,\alpha^2\beta^4-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^6}-18432\,\frac{\alpha^2\left(-12\,\beta^2\alpha^4+18\,\alpha^2\beta^4+\alpha^6-4\,\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8^{[IJ]}(x)=\frac{\beta\,x^8}{\alpha^2+\beta^2}-12\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}+192\,\frac{\beta\,\left(23\,\alpha^4-44\,\beta^2\alpha^2+3\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-768\,\frac{\beta\,\left(-239\,\beta^2\alpha^4+131\,\alpha^2\beta^4+47\,\alpha^6-3\,\beta^6\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_8^{[JK]}(x)=-\frac{\beta\,x^8}{\alpha^2+\beta^2}+24\,\frac{\beta\,\left(5\,\alpha^2-2\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}-192\,\frac{\beta\,\left(21\,\alpha^4-43\,\alpha^2\beta^2+6\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+9216\,\frac{\beta\,\left(4\,\alpha^6+12\,\beta^4\alpha^2-\beta^6-18\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8^{[JK]}(x)=2\,\frac{\left(3\,\alpha^2-4\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\left(3\,\alpha^4-26\,\alpha^2\beta^2+6\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+384\,\frac{\left(3\,\alpha^6-86\,\beta^2\alpha^4+109\,\beta^4\alpha^2-12\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^6}+18432\,\frac{\beta^2\left(4\,\alpha^6+12\,\beta^4\alpha^2-\beta^6-18\,\beta^2\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8^{[JK]}(x)=14\,\frac{\beta\,\alpha\,x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\beta\,\alpha\,\left(18\,\alpha^2-17\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+1920\,\frac{\beta\,\alpha\,\left(9\,\alpha^4-26\,\alpha^2\beta^2+7\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-18432\,\frac{\beta\,\alpha\,\left(4\,\alpha^6+12\,\beta^4\alpha^2-\beta^6-18\,\beta^2\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8^{[JK]}(x)=\frac{\alpha\,x^8}{\alpha^2+\beta^2}-12\,\frac{\alpha\,\left(3\,\alpha^2-11\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}+192\,\frac{\alpha\,\left(3\,\alpha^4-44\,\alpha^2\beta^2+23\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-768\,\frac{\alpha\,\left(3\,\alpha^6+239\,\beta^4\alpha^2-47\,\beta^6-131\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_8^{[KJ]}(x)=-\frac{\alpha\,x^8}{\alpha^2+\beta^2}+24\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}-192\,\frac{\alpha\,\left(6\,\alpha^4-43\,\alpha^2\beta^2+21\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+9216\,\frac{\alpha\,\left(\alpha^6+18\,\beta^4\alpha^2-4\,\beta^6-12\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8^{[KJ]}(x)=-14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2+\beta^2\right)^2}+48\,\frac{\alpha\,\beta\,\left(17\,\alpha^2-18\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^4}-1920\,\frac{\alpha\,\beta\,\left(7\,\alpha^4-26\,\alpha^2\beta^2+9\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+18432\,\frac{\alpha\,\beta\,\left(\alpha^6+18\,\beta^4\alpha^2-4\,\beta^6-12\,\beta^2\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8^{[KJ]}(x)=2\,\frac{\left(4\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^2}-48\,\frac{\left(6\,\alpha^4-26\,\alpha^2\beta^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+384\,\frac{\left(12\,\alpha^6-109\,\beta^2\alpha^4+86\,\beta^4\alpha^2-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^6}-18432\,\frac{\alpha^2\left(\alpha^6+18\,\beta^4\alpha^2-4\,\beta^6-12\,\beta^2\alpha^4\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8^{[KJ]}(x)=-\frac{\beta\,x^8}{\alpha^2+\beta^2}+12\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3}-192\,\frac{\beta\,\left(23\,\alpha^4-44\,\alpha^2\beta^2+3\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+768\,\frac{\beta\,\left(47\,\alpha^6+131\,\beta^4\alpha^2-3\,\beta^6-239\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_8^{[IK]}(x)=\frac{\beta\,x^8}{\alpha^2-\beta^2}+24\,\frac{\beta\,\left(5\,\alpha^2+2\,\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\beta\,\left(21\,\alpha^4+43\,\beta^2\alpha^2+6\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+9216\,\frac{\beta\,\left(18\,\beta^2\alpha^4+12\,\alpha^2\beta^4+4\,\alpha^6+\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^7}$$

$$Q_8^{[IK]}(x)=-2\,\frac{\left(3\,\alpha^2+4\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^2}-48\,\frac{\left(3\,\alpha^4+26\,\beta^2\alpha^2+6\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-384\,\frac{\left(3\,\alpha^6+86\,\beta^2\alpha^4+109\,\alpha^2\beta^4+12\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^6}-18432\,\frac{\beta^2\left(18\,\beta^2\alpha^4+12\,\alpha^2\beta^4+4\,\alpha^6+\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^8}$$

$$R_8^{[IK]}(x)=-14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2-\beta^2\right)^2}-48\,\frac{\alpha\,\beta\,\left(18\,\alpha^2+17\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1920\,\frac{\alpha\,\beta\,\left(9\,\alpha^4+26\,\beta^2\alpha^2+7\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^6}-18432\,\frac{\alpha\,\beta\,\left(18\,\beta^2\alpha^4+12\,\alpha^2\beta^4+4\,\alpha^6+\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^8}$$

$$S_8^{[IK]}(x)=\frac{\alpha\,x^8}{\alpha^2-\beta^2}+12\,\frac{\alpha\,\left(3\,\alpha^2+11\,\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\alpha\,\left(3\,\alpha^4+44\,\beta^2\alpha^2+23\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+768\,\frac{\alpha\,\left(47\,\beta^6+239\,\alpha^2\beta^4+3\,\alpha^6+131\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2-\beta^2\right)^7}$$

$$P_8^{[KI]}(x)=\frac{\alpha\,x^8}{\alpha^2-\beta^2}+24\,\frac{\alpha\,\left(2\,\alpha^2+5\,\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\alpha\,\left(6\,\alpha^4+43\,\beta^2\alpha^2+21\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+9216\,\frac{\alpha\,\left(\alpha^6+18\,\alpha^2\beta^4+4\,\beta^6+12\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2-\beta^2\right)^7}$$

$$Q_8^{[KI]}(x)=-14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2-\beta^2\right)^2}-48\,\frac{\alpha\,\beta\,\left(17\,\alpha^2+18\,\beta^2\right)x^5}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1920\,\frac{\alpha\,\beta\,\left(7\,\alpha^4+26\,\beta^2\alpha^2+9\,\beta^4\right)x^3}{\left(\alpha^2-\beta^2\right)^6}-18432\,\frac{\alpha\,\beta\,\left(\alpha^6+18\,\alpha^2\beta^4+4\,\beta^6+12\,\beta^2\alpha^4\right)x}{\left(\alpha^2-\beta^2\right)^8}$$

$$R_8^{[KI]}(x)=-2\,\frac{\left(4\,\alpha^2+3\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^2}-48\,\frac{\left(6\,\alpha^4+26\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-384\,\frac{\left(12\,\alpha^6+109\,\beta^2\alpha^4+86\,\alpha^2\beta^4+3\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^6}-18432\,\frac{\alpha^2\left(4\,\beta^6+18\,\alpha^2\beta^4+12\,\beta^2\alpha^4+\alpha^6\right)x}{\left(\alpha^2-\beta^2\right)^8}$$

$$S_8^{[KI]}(x)=\frac{\beta\,x^8}{\alpha^2-\beta^2}+12\,\frac{\beta\,\left(11\,\alpha^2+3\,\beta^2\right)x^6}{\left(\alpha^2-\beta^2\right)^3}+192\,\frac{\beta\,\left(23\,\alpha^4+44\,\beta^2\alpha^2+3\,\beta^4\right)x^4}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+768\,\frac{\beta\,\left(47\,\alpha^6+131\,\alpha^2\beta^4+3\,\beta^6+239\,\beta^2\alpha^4\right)x^2}{\left(\alpha^2-\beta^2\right)^7}$$

$$P_{10}^{[JJ]}=\frac{\beta}{\alpha^2-\beta^2}\,x^{10}-16\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^8+768\,\frac{\beta\,\left(19\,\alpha^4+39\,\alpha^2\beta^2+5\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^6-$$

$$-6144\,\frac{\beta\,\left(69\,\alpha^6+332\,\alpha^4\beta^2+214\,\alpha^2\beta^4+15\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^4+$$

$$+737280\,\frac{\beta\,\left(5\,\alpha^8+40\,\alpha^6\beta^2+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^2$$

$$Q_{10}^{[JJ]}=2\,\frac{4\,\alpha^2+5\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^9-64\,\frac{6\,\alpha^4+47\,\alpha^2\beta^2+10\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^7+1536\,\frac{6\,\alpha^6+136\,\alpha^4\beta^2+158\,\alpha^2\beta^4+15\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^5-$$

$$-12288\,\frac{6\,\alpha^8+337\,\alpha^6\beta^2+1018\,\alpha^4\beta^4+499\,\alpha^2\beta^6+30\,\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^3+$$

$$+1474560\,\frac{\beta^2\left(5\,\alpha^8+40\,\alpha^6\beta^2+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x$$

$$R_{10}^{[JJ]}=-18\,\frac{\beta\,\alpha}{\left(\alpha^2-\beta^2\right)^2}\,x^9+64\,\frac{\beta\,\alpha\,\left(32\,\alpha^2+31\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^7-10752\,\frac{\beta\,\alpha\,\left(9\,\alpha^4+28\,\alpha^2\beta^2+8\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^5+$$

$$+12288\,\frac{\beta\,\alpha\,\left(144\,\alpha^6+863\,\alpha^4\beta^2+782\,\alpha^2\beta^4+101\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^3-$$

$$-1474560\,\frac{\beta\,\alpha\,\left(5\,\alpha^8+40\,\alpha^6\beta^2+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x$$

$$S_{10}^{[JJ]}=\frac{\alpha}{\alpha^2-\beta^2}\,x^{10}-32\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^8+384\,\frac{\alpha\,\left(6\,\alpha^4+79\,\alpha^2\beta^2+41\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^6-$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6+199\,\alpha^4\beta^2+354\,\alpha^2\beta^4+71\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^4+$$

$$+24576\,\frac{\alpha\,\left(6\,\alpha^8+481\,\alpha^6\beta^2+1881\,\alpha^4\beta^4+1281\,\alpha^2\beta^6+131\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^2$$

$$P_{10}^{[II]}(x)=-P_{10}^{[KK]}(x)=-\frac{\beta}{\alpha^2-\beta^2}\,x^{10}-16\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^8-768\,\frac{\beta\,\left(19\,\alpha^4+39\,\alpha^2\beta^2+5\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^6-$$

$$-6144\,\frac{\beta\,\left(69\,\alpha^6+332\,\alpha^4\beta^2+214\,\alpha^2\beta^4+15\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^4-$$

$$-737280\,\frac{\beta\,\left(5\,\alpha^8+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+40\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x^2$$

$$Q_{10}^{[II]}(x)=Q_{10}^{[KK]}(x)=-2\,\frac{4\,\alpha^2+5\,\beta^2}{\left(\alpha^2-\beta^2\right)^2}\,x^9-64\,\frac{6\,\alpha^4+47\,\alpha^2\beta^2+10\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\,x^7-$$

$$-1536\,\frac{6\,\alpha^6+136\,\alpha^4\beta^2+158\,\alpha^2\beta^4+15\,\beta^6}{\left(\alpha^2-\beta^2\right)^6}\,x^5-$$

$$-12288\,\frac{6\,\alpha^8+337\,\alpha^6\beta^2+1018\,\alpha^4\beta^4+499\,\alpha^2\beta^6+30\,\beta^8}{\left(\alpha^2-\beta^2\right)^8}\,x^3-$$

$$-1474560\,\frac{\beta^2\left(5\,\alpha^8+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+40\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x$$

$$R_{10}^{[II]}(x)=R_{10}^{[KK]}(x)=18\,\frac{\alpha\,\beta}{\left(\alpha^2-\beta^2\right)^2}\,x^9+64\,\frac{\alpha\,\beta\,\left(32\,\alpha^2+31\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\,x^7+$$

$$+10752\,\frac{\alpha\,\beta\,\left(9\,\alpha^4+28\,\alpha^2\beta^2+8\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^6}\,x^5+12288\,\frac{\alpha\,\beta\,\left(144\,\alpha^6+863\,\alpha^4\beta^2+782\,\alpha^2\beta^4+101\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^8}\,x^3+$$

$$+1474560\,\frac{\alpha\,\beta\,\left(5\,\alpha^8+60\,\alpha^4\beta^4+20\,\alpha^2\beta^6+40\,\alpha^6\beta^2+\beta^8\right)}{\left(\alpha^2-\beta^2\right)^{10}}\,x$$

$$S_{10}^{[II]}(x)=-S_{10}^{[KK]}(x)=\frac{\alpha}{\alpha^2-\beta^2}\,x^9+32\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)}{\left(\alpha^2-\beta^2\right)^3}\,x^7+384\,\frac{\alpha\,\left(6\,\alpha^4+79\,\alpha^2\beta^2+41\,\beta^4\right)}{\left(\alpha^2-\beta^2\right)^5}\,x^5+$$

$$+6144\,\frac{\alpha\,\left(6\,\alpha^6+199\,\alpha^4\beta^2+354\,\alpha^2\beta^4+71\,\beta^6\right)}{\left(\alpha^2-\beta^2\right)^7}\,x^3+$$

$$+24576\,\frac{\alpha\,\left(6\,\alpha^8+1881\,\alpha^4\beta^4+1281\,\alpha^2\beta^6+481\,\alpha^6\beta^2+131\,\beta^8\right)}{\left(\alpha^2-\beta^2\right)^9}\,x$$

$$
\begin{aligned}
P_{10}^{[JI]}(x) = {} & \frac{\beta\, x^{10}}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(13\,\alpha^2 - 5\,\beta^2\right) x^8}{\left(\alpha^2+\beta^2\right)^3} + 768\,\frac{\beta\,\left(19\,\alpha^4 - 39\,\alpha^2\beta^2 + 5\,\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^5} - \\
& -6144\,\frac{\beta\,\left(69\,\alpha^6 - 332\,\alpha^4\beta^2 + 214\,\alpha^2\beta^4 - 15\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^7} + \\
& +737280\,\frac{\beta\,\left(-40\,\beta^2\alpha^6 + \beta^8 + 5\,\alpha^8 - 20\,\alpha^2\beta^6 + 60\,\alpha^4\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^9}
\end{aligned}
$$

$$
\begin{aligned}
Q_{10}^{[JI]}(x) = {} & 2\,\frac{\left(4\,\alpha^2 - 5\,\beta^2\right) x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\left(6\,\alpha^4 - 47\,\alpha^2\beta^2 + 10\,\beta^4\right) x^7}{\left(\alpha^2+\beta^2\right)^4} + \\
& +1536\,\frac{\left(6\,\alpha^6 - 136\,\alpha^4\beta^2 + 158\,\alpha^2\beta^4 - 15\,\beta^6\right) x^5}{\left(\alpha^2+\beta^2\right)^6} - \\
& -12288\,\frac{\left(6\,\alpha^8 - 337\,\beta^2\alpha^6 + 1018\,\alpha^4\beta^4 - 499\,\alpha^2\beta^6 + 30\,\beta^8\right) x^3}{\left(\alpha^2+\beta^2\right)^8} - \\
& -1474560\,\frac{\beta^2\left(-40\,\beta^2\alpha^6 + \beta^8 + 5\,\alpha^8 - 20\,\alpha^2\beta^6 + 60\,\alpha^4\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^{10}}
\end{aligned}
$$

$$
\begin{aligned}
R_{10}^{[JI]}(x) = {} & -18\,\frac{\alpha\,\beta\, x^9}{\left(\alpha^2+\beta^2\right)^2} + 64\,\frac{\alpha\,\beta\,\left(32\,\alpha^2 - 31\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^4} - 10752\,\frac{\alpha\,\beta\,\left(9\,\alpha^4 - 28\,\beta^2\alpha^2 + 8\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^6} + \\
& +12288\,\frac{\alpha\,\beta\,\left(144\,\alpha^6 - 863\,\beta^2\alpha^4 + 782\,\beta^4\alpha^2 - 101\,\beta^6\right) x^3}{\left(\alpha^2+\beta^2\right)^8} - \\
& -1474560\,\frac{\alpha\,\beta\,\left(-40\,\beta^2\alpha^6 + \beta^8 + 5\,\alpha^8 - 20\,\alpha^2\beta^6 + 60\,\alpha^4\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^{10}}
\end{aligned}
$$

$$
\begin{aligned}
S_{10}^{[JI]}(x) = {} & \frac{\alpha\, x^{10}}{\alpha^2+\beta^2} - 32\,\frac{\alpha\,\left(2\,\alpha^2 - 7\,\beta^2\right) x^8}{\left(\alpha^2+\beta^2\right)^3} + 384\,\frac{\alpha\,\left(6\,\alpha^4 - 79\,\beta^2\alpha^2 + 41\,\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^5} - \\
& -6144\,\frac{\alpha\,\left(6\,\alpha^6 - 199\,\beta^2\alpha^4 + 354\,\beta^4\alpha^2 - 71\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^7} + \\
& +24576\,\frac{\alpha\,\left(-481\,\beta^2\alpha^6 + 131\,\beta^8 + 6\,\alpha^8 - 1281\,\alpha^2\beta^6 + 1881\,\alpha^4\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^9}
\end{aligned}
$$

$$
\begin{aligned}
P_{10}^{[IJ]}(x) = {} & -\frac{\alpha\, x^{10}}{\alpha^2+\beta^2} + 16\,\frac{\alpha\,\left(5\,\alpha^2 - 13\,\beta^2\right) x^8}{\left(\alpha^2+\beta^2\right)^3} - 768\,\frac{\alpha\,\left(5\,\alpha^4 - 39\,\beta^2\alpha^2 + 19\,\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^5} + \\
& +6144\,\frac{\alpha\,\left(15\,\alpha^6 - 214\,\beta^2\alpha^4 + 332\,\beta^4\alpha^2 - 69\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^7} - \\
& -737280\,\frac{\alpha\,\left(-20\,\beta^2\alpha^6 + 5\,\beta^8 + \alpha^8 - 40\,\alpha^2\beta^6 + 60\,\alpha^4\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^9}
\end{aligned}
$$

$$
\begin{aligned}
Q_{10}^{[IJ]}(x) = {} & 18\,\frac{\alpha\,\beta\, x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\alpha\,\beta\,\left(31\,\alpha^2 - 32\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^4} + 10752\,\frac{\alpha\,\beta\,\left(8\,\alpha^4 - 28\,\beta^2\alpha^2 + 9\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^6} - \\
& -12288\,\frac{\alpha\,\beta\,\left(101\,\alpha^6 - 782\,\beta^2\alpha^4 + 863\,\beta^4\alpha^2 - 144\,\beta^6\right) x^3}{\left(\alpha^2+\beta^2\right)^8} + \\
& +1474560\,\frac{\alpha\,\beta\,\left(-20\,\beta^2\alpha^6 + 5\,\beta^8 + \alpha^8 - 40\,\alpha^2\beta^6 + 60\,\alpha^4\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^{10}}
\end{aligned}
$$

$$
R_{10}^{[IJ]}(x) = 2\,\frac{\left(5\,\alpha^2 - 4\,\beta^2\right) x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\left(10\,\alpha^4 - 47\,\beta^2\alpha^2 + 6\,\beta^4\right) x^7}{\left(\alpha^2+\beta^2\right)^4} +
$$

$$+1536\,\frac{\left(15\,\alpha^6-158\,\beta^2\alpha^4+136\,\beta^4\alpha^2-6\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-12288\,\frac{\left(30\,\alpha^8-499\,\beta^2\alpha^6+1018\,\alpha^4\beta^4-337\,\alpha^2\beta^6+6\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+1474560\,\frac{\alpha^2\left(-20\,\beta^2\alpha^6+5\,\beta^8+\alpha^8-40\,\alpha^2\beta^6+60\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$S_{10}^{[IJ]}(x)=\frac{\beta\,x^{10}}{\alpha^2+\beta^2}-32\,\frac{\beta\left(7\,\alpha^2-2\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3}+384\,\frac{\beta\left(41\,\alpha^4-79\,\beta^2\alpha^2+6\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\beta\left(71\,\alpha^6-354\,\beta^2\alpha^4+199\,\beta^4\alpha^2-6\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+24576\,\frac{\beta\left(-1281\,\beta^2\alpha^6+6\,\beta^8+131\,\alpha^8-481\,\alpha^2\beta^6+1881\,\alpha^4\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$P_{10}^{[JK]}(x)=-\frac{\beta\,x^{10}}{\alpha^2+\beta^2}+16\,\frac{\beta\left(13\,\alpha^2-5\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3}-768\,\frac{\beta\left(19\,\alpha^4-39\,\alpha^2\beta^2+5\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\left(69\,\alpha^6-332\,\beta^2\alpha^4+214\,\beta^4\alpha^2-15\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-737280\,\frac{\beta\left(\beta^8+60\,\alpha^4\beta^4+5\,\alpha^8-20\,\alpha^2\beta^6-40\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$Q_{10}^{[JK]}(x)=2\,\frac{\left(4\,\alpha^2-5\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^2}-64\,\frac{\left(6\,\alpha^4-47\,\beta^2\alpha^2+10\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+1536\,\frac{\left(6\,\alpha^6-136\,\beta^2\alpha^4+158\,\beta^4\alpha^2-15\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-12288\,\frac{\left(6\,\alpha^8-337\,\alpha^6\beta^2+1018\,\alpha^4\beta^4-499\,\alpha^2\beta^6+30\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^8}-$$

$$-1474560\,\frac{\beta^2\left(-20\,\alpha^2\beta^6+60\,\alpha^4\beta^4+\beta^8+5\,\alpha^8-40\,\alpha^6\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$R_{10}^{[JK]}(x)=18\,\frac{\alpha\,\beta\,x^9}{\left(\alpha^2+\beta^2\right)^2}-64\,\frac{\alpha\,\beta\left(32\,\alpha^2-31\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^4}+10752\,\frac{\alpha\,\beta\left(9\,\alpha^4-28\,\beta^2\alpha^2+8\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-12288\,\frac{\alpha\,\beta\left(144\,\alpha^6-863\,\beta^2\alpha^4+782\,\beta^4\alpha^2-101\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+1474560\,\frac{\alpha\,\beta\left(-20\,\alpha^2\beta^6+60\,\alpha^4\beta^4+\beta^8+5\,\alpha^8-40\,\alpha^6\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$S_{10}^{[JK]}(x)=\frac{\alpha\,x^{10}}{\alpha^2+\beta^2}-32\,\frac{\alpha\left(2\,\alpha^2-7\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3}+384\,\frac{\alpha\left(6\,\alpha^4-79\,\beta^2\alpha^2+41\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\alpha\left(6\,\alpha^6-199\,\beta^2\alpha^4+354\,\beta^4\alpha^2-71\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+24576\,\frac{\alpha\left(131\,\beta^8+1881\,\alpha^4\beta^4+6\,\alpha^8-1281\,\alpha^2\beta^6-481\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$P_{10}^{[KJ]}(x)=-\frac{\alpha\,x^{10}}{\alpha^2+\beta^2}+16\,\frac{\alpha\left(5\,\alpha^2-13\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3}-768\,\frac{\alpha\left(5\,\alpha^4-39\,\beta^2\alpha^2+19\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+6144\,\frac{\alpha\,\left(15\,\alpha^6-214\,\beta^2\alpha^4+332\,\beta^4\alpha^2-69\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-737280\,\frac{\alpha\,\left(5\,\beta^8+60\,\alpha^4\beta^4+\alpha^8-40\,\alpha^2\beta^6-20\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$Q_{10}^{[KJ]}(x)=-18\,\frac{\alpha\,\beta\,x^9}{\left(\alpha^2+\beta^2\right)^2}+64\,\frac{\alpha\,\beta\,\left(31\,\alpha^2-32\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^4}-10752\,\frac{\alpha\,\beta\,\left(8\,\alpha^4-28\,\beta^2\alpha^2+9\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+12288\,\frac{\alpha\,\beta\,\left(101\,\alpha^6-782\,\beta^2\alpha^4+863\,\beta^4\alpha^2-144\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^8}-$$

$$-1474560\,\frac{\alpha\,\beta\,\left(5\,\beta^8+60\,\alpha^4\beta^4+\alpha^8-40\,\alpha^2\beta^6-20\,\alpha^6\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$R_{10}^{[KJ]}(x)=2\,\frac{\left(5\,\alpha^2-4\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^2}-64\,\frac{\left(10\,\alpha^4-47\,\beta^2\alpha^2+6\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^4}+$$

$$+1536\,\frac{\left(15\,\alpha^6-158\,\beta^2\alpha^4+136\,\beta^4\alpha^2-6\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-12288\,\frac{\left(30\,\alpha^8-499\,\alpha^6\beta^2+1018\,\alpha^4\beta^4-337\,\alpha^2\beta^6+6\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+1474560\,\frac{\alpha^2\,\left(\alpha^8+5\,\beta^8-40\,\alpha^2\beta^6-20\,\alpha^6\beta^2+60\,\alpha^4\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$S_{10}^{[KJ]}(x)=-\frac{\beta\,x^{10}}{\alpha^2+\beta^2}+32\,\frac{\beta\,\left(7\,\alpha^2-2\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3}-384\,\frac{\beta\,\left(41\,\alpha^4-79\,\beta^2\alpha^2+6\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\,\left(71\,\alpha^6-354\,\beta^2\alpha^4+199\,\beta^4\alpha^2-6\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-24576\,\frac{\beta\,\left(6\,\beta^8+1881\,\alpha^4\beta^4+131\,\alpha^8-481\,\alpha^2\beta^6-1281\,\alpha^6\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$P_{10}^{[IK]}(x)=\frac{\beta\,x^{10}}{\alpha^2-\beta^2}+16\,\frac{\beta\,\left(13\,\alpha^2+5\,\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^3}+768\,\frac{\beta\,\left(19\,\alpha^4+39\,\beta^2\alpha^2+5\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\,\left(69\,\alpha^6+332\,\beta^2\alpha^4+214\,\alpha^2\beta^4+15\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+737280\,\frac{\beta\,\left(60\,\alpha^4\beta^4+40\,\alpha^6\beta^2+5\,\alpha^8+\beta^8+20\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^9}$$

$$Q_{10}^{[IK]}(x)=-2\,\frac{\left(4\,\alpha^2+5\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^2}-64\,\frac{\left(6\,\alpha^4+47\,\beta^2\alpha^2+10\,\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1536\,\frac{\left(6\,\alpha^6+136\,\beta^2\alpha^4+158\,\beta^4\alpha^2+15\,\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-12288\,\frac{\left(6\,\alpha^8+337\,\alpha^6\beta^2+1018\,\alpha^4\beta^4+499\,\alpha^2\beta^6+30\,\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-1474560\,\frac{\beta^2\,\left(20\,\alpha^2\beta^6+5\,\alpha^8+60\,\alpha^4\beta^4+\beta^8+40\,\alpha^6\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$R_{10}^{[IK]}(x)=-18\,\frac{\alpha\,\beta\,x^9}{\left(\alpha^2-\beta^2\right)^2}-64\,\frac{\alpha\,\beta\,\left(32\,\alpha^2+31\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^4}-10752\,\frac{\alpha\,\beta\,\left(9\,\alpha^4+28\,\beta^2\alpha^2+8\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-12288\,\frac{\alpha\,\beta\,\left(144\,\alpha^6+863\,\beta^2\alpha^4+782\,\beta^4\alpha^2+101\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-1474560\,\frac{\alpha\,\beta\,\left(20\,\alpha^2\beta^6+5\,\alpha^8+60\,\alpha^4\beta^4+\beta^8+40\,\alpha^6\beta^2\right)x}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$S_{10}^{[IK]}(x)=\frac{\alpha\,x^{10}}{\alpha^2-\beta^2}+32\,\frac{\alpha\,\left(2\,\alpha^2+7\,\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\alpha\,\left(6\,\alpha^4+79\,\beta^2\alpha^2+41\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\alpha\,\left(6\,\alpha^6+199\,\beta^2\alpha^4+354\,\beta^4\alpha^2+71\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+24576\,\frac{\alpha\,\left(1881\,\alpha^4\beta^4+481\,\alpha^6\beta^2+6\,\alpha^8+131\,\beta^8+1281\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^9}$$

$$P_{10}^{[KI]}(x)=\frac{\alpha\,x^{10}}{\alpha^2-\beta^2}+16\,\frac{\alpha\,\left(5\,\alpha^2+13\,\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^3}+768\,\frac{\alpha\,\left(5\,\alpha^4+39\,\beta^2\alpha^2+19\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\alpha\,\left(15\,\alpha^6+214\,\beta^2\alpha^4+332\,\beta^4\alpha^2+69\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+737280\,\frac{\alpha\,\left(60\,\alpha^4\beta^4+20\,\alpha^6\beta^2+\alpha^8+5\,\beta^8+40\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^9}$$

$$Q_{10}^{[KI]}(x)=-18\,\frac{\alpha\,\beta\,x^9}{\left(\alpha^2-\beta^2\right)^2}-64\,\frac{\alpha\,\beta\,\left(31\,\alpha^2+32\,\beta^2\right)x^7}{\left(\alpha^2-\beta^2\right)^4}-10752\,\frac{\alpha\,\beta\,\left(8\,\alpha^4+28\,\beta^2\alpha^2+9\,\beta^4\right)x^5}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-12288\,\frac{\alpha\,\beta\,\left(101\,\alpha^6+782\,\beta^2\alpha^4+863\,\beta^4\alpha^2+144\,\beta^6\right)x^3}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-1474560\,\frac{\alpha\,\beta\,\left(60\,\alpha^4\beta^4+20\,\alpha^6\beta^2+\alpha^8+5\,\beta^8+40\,\alpha^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$R_{10}^{[KI]}(x)=-2\,\frac{\left(5\,\alpha^2+4\,\beta^2\right)x^9}{\left(\alpha^2-\beta^2\right)^2}-64\,\frac{\left(10\,\alpha^4+47\,\beta^2\alpha^2+6\,\beta^4\right)x^7}{\left(\alpha^2-\beta^2\right)^4}-$$

$$-1536\,\frac{\left(15\,\alpha^6+158\,\beta^2\alpha^4+136\,\beta^4\alpha^2+6\,\beta^6\right)x^5}{\left(\alpha^2-\beta^2\right)^6}-$$

$$-12288\,\frac{\left(30\,\alpha^8+499\,\alpha^6\beta^2+1018\,\alpha^4\beta^4+337\,\alpha^2\beta^6+6\,\beta^8\right)x^3}{\left(\alpha^2-\beta^2\right)^8}-$$

$$-1474560\,\frac{\alpha^2\,\left(5\,\beta^8+40\,\alpha^2\beta^6+60\,\alpha^4\beta^4+20\,\alpha^6\beta^2+\alpha^8\right)x}{\left(\alpha^2-\beta^2\right)^{10}}$$

$$S_{10}^{[KI]}(x)=\frac{\beta\,x^{10}}{\alpha^2-\beta^2}+32\,\frac{\beta\,\left(7\,\alpha^2+2\,\beta^2\right)x^8}{\left(\alpha^2-\beta^2\right)^3}+384\,\frac{\beta\,\left(41\,\alpha^4+79\,\beta^2\alpha^2+6\,\beta^4\right)x^6}{\left(\alpha^2-\beta^2\right)^5}+$$

$$+6144\,\frac{\beta\,\left(71\,\alpha^6+354\,\beta^2\alpha^4+199\,\beta^4\alpha^2+6\,\beta^6\right)x^4}{\left(\alpha^2-\beta^2\right)^7}+$$

$$+24576\,\frac{\beta\,\left(1881\,\alpha^4\beta^4+1281\,\alpha^6\beta^2+131\,\alpha^8+6\,\beta^8+481\,\alpha^2\beta^6\right)x^2}{\left(\alpha^2-\beta^2\right)^9}$$

Recurrence formulas:

Let

$$Z^{(m)}_{\mu\nu,UW} = \int x^m U_\mu(\alpha x) W_\nu(\beta x)\, dx \ ,$$

then the following systems hold:

$$Z^{(2n+3)}_{00,JJ} = x^{2n+2}\cdot Z^{(1)}_{00,JJ} - \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+2)}_{10,JJ} - \beta Z^{(2n+2)}_{01,JJ}\Big]$$

$$Z^{(2n+3)}_{11,JJ} = x^{2n+2}\cdot Z^{(1)}_{11,JJ} - \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+2)}_{10,JJ} - \alpha Z^{(2n+2)}_{01,JJ}\Big]$$

$$Z^{(2n+2)}_{01,JJ} = x^{2n}\cdot Z^{(2)}_{01,JJ} - \frac{4n\beta}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,JJ} - \alpha Z^{(2n)}_{10,JJ}\Big] - \frac{2n}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+1)}_{00,JJ} + \alpha Z^{(2n+1)}_{11,JJ}\Big]$$

$$Z^{(2n+2)}_{10,JJ} = x^{2n}\cdot Z^{(2)}_{10,JJ} + \frac{4n\alpha}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,JJ} - \alpha Z^{(2n)}_{10,JJ}\Big] + \frac{2n}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+1)}_{00,JJ} + \beta Z^{(2n+1)}_{11,JJ}\Big] \ ,$$

$$Z^{(2n+3)}_{00,II} = x^{2n+2}\cdot Z^{(1)}_{00,II} - \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+2)}_{10,II} - \beta Z^{(2n+2)}_{01,II}\Big]$$

$$Z^{(2n+3)}_{11,II} = x^{2n+2}\cdot Z^{(1)}_{11,II} + \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+2)}_{10,II} - \alpha Z^{(2n+2)}_{01,II}\Big]$$

$$Z^{(2n+2)}_{01,II} = x^{2n}\cdot Z^{(2)}_{01,II} + \frac{4n\beta}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,II} - \alpha Z^{(2n)}_{10,II}\Big] + \frac{2n}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+1)}_{00,II} - \alpha Z^{(2n+1)}_{11,II}\Big]$$

$$Z^{(2n+2)}_{10,II} = x^{2n}\cdot Z^{(2)}_{10,II} - \frac{4n\alpha}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,II} - \alpha Z^{(2n)}_{10,II}\Big] - \frac{2n}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+1)}_{00,II} - \beta Z^{(2n+1)}_{11,II}\Big] \ ,$$

$$Z^{(2n+3)}_{00,KK} = -\,x^{2n+2}\cdot Z^{(1)}_{00,KK} + \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+2)}_{10,KK} - \beta Z^{(2n+2)}_{01,KK}\Big]$$

$$Z^{(2n+3)}_{11,KK} = x^{2n+2}\cdot Z^{(1)}_{11,KK} - \frac{2(n+1)}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+2)}_{10,KK} - \alpha Z^{(2n+2)}_{01,KK}\Big]$$

$$Z^{(2n+2)}_{01,KK} = x^{2n}\cdot Z^{(2)}_{01,KK} + \frac{4n\beta}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,KK} - \alpha Z^{(2n)}_{10,KK}\Big] - \frac{2n}{\alpha^2-\beta^2}\,\Big[\beta Z^{(2n+1)}_{00,KK} - \alpha Z^{(2n+1)}_{11,KK}\Big]$$

$$Z^{(2n+2)}_{10,KK} = x^{2n}\cdot Z^{(2)}_{10,KK} - \frac{4n\alpha}{(\alpha^2-\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,KK} - \alpha Z^{(2n)}_{10,KK}\Big] + \frac{2n}{\alpha^2-\beta^2}\,\Big[\alpha Z^{(2n+1)}_{00,KK} - \beta Z^{(2n+1)}_{11,KK}\Big] \ ,$$

$$Z^{(2n+3)}_{00,JI} = x^{2n+2}\cdot Z^{(1)}_{00,JI} - \frac{2(n+1)}{\alpha^2+\beta^2}\,\Big[\alpha Z^{(2n+2)}_{10,JI} + \beta Z^{(2n+2)}_{01,JI}\Big]$$

$$Z^{(2n+3)}_{11,JI} = x^{2n+2}\cdot Z^{(1)}_{11,JI} - \frac{2(n+1)}{\alpha^2+\beta^2}\,\Big[\beta Z^{(2n+2)}_{10,JI} - \alpha Z^{(2n+2)}_{01,JI}\Big]$$

$$Z^{(2n+2)}_{01,JI} = x^{2n}\cdot Z^{(2)}_{01,JI} + \frac{4n\beta}{(\alpha^2+\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,JI} + \alpha Z^{(2n)}_{10,JI}\Big] - \frac{2n}{\alpha^2+\beta^2}\,\Big[\beta Z^{(2n+1)}_{00,JI} + \alpha Z^{(2n+1)}_{11,JI}\Big]$$

$$Z^{(2n+2)}_{10,JI} = x^{2n}\cdot Z^{(2)}_{10,JI} - \frac{4n\alpha}{(\alpha^2+\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,JI} + \alpha Z^{(2n)}_{10,JI}\Big] + \frac{2n}{\alpha^2+\beta^2}\,\Big[\alpha Z^{(2n+1)}_{00,JI} - \beta Z^{(2n+1)}_{11,JI}\Big] \ ,$$

$$Z^{(2n+3)}_{00,JK} = x^{2n+2}\cdot Z^{(1)}_{00,JK} - \frac{2(n+1)}{\alpha^2+\beta^2}\,\Big[\alpha Z^{(2n+2)}_{10,JK} - \beta Z^{(2n+2)}_{01,JK}\Big]$$

$$Z^{(2n+3)}_{11,JK} = x^{2n+2}\cdot Z^{(1)}_{11,JK} + \frac{2(n+1)}{\alpha^2+\beta^2}\,\Big[\beta Z^{(2n+2)}_{10,JK} + \alpha Z^{(2n+2)}_{01,JK}\Big]$$

$$Z^{(2n+2)}_{01,JK} = x^{2n}\cdot Z^{(2)}_{01,JK} + \frac{4n\beta}{(\alpha^2+\beta^2)^2}\,\Big[\beta Z^{(2n)}_{01,JK} - \alpha Z^{(2n)}_{10,JK}\Big] + \frac{2n}{\alpha^2+\beta^2}\,\Big[\beta Z^{(2n+1)}_{00,JK} - \alpha Z^{(2n+1)}_{11,JK}\Big]$$

$$Z_{10,JK}^{(2n+2)} = x^{2n} \cdot Z_{10,JK}^{(2)} + \frac{4n\alpha}{(\alpha^2+\beta^2)^2} \left[\beta Z_{01,JK}^{(2n)} - \alpha Z_{10,JK}^{(2n)}\right] + \frac{2n}{\alpha^2+\beta^2} \left[\alpha Z_{00,JK}^{(2n+1)} + \beta Z_{11,JK}^{(2n+1)}\right] \ ,$$

$$Z_{00,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{00,IK}^{(1)} - \frac{2(n+1)}{\alpha^2-\beta^2} \left[\alpha Z_{10,IK}^{(2n+2)} + \beta Z_{01,IK}^{(2n+2)}\right]$$

$$Z_{11,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,IK}^{(1)} - \frac{2(n+1)}{\alpha^2-\beta^2} \left[\beta Z_{10,IK}^{(2n+2)} + \alpha Z_{01,IK}^{(2n+2)}\right]$$

$$Z_{01,IK}^{(2n+2)} = x^{2n} \cdot Z_{01,IK}^{(2)} + \frac{4n\beta}{(\alpha^2-\beta^2)^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)}\right] - \frac{2n}{\alpha^2-\beta^2} \left[\beta Z_{00,IK}^{(2n+1)} + \alpha Z_{11,IK}^{(2n+1)}\right]$$

$$Z_{10,IK}^{(2n+2)} = x^{2n} \cdot Z_{10,IK}^{(2)} + \frac{4n\alpha}{(\alpha^2-\beta^2)^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)}\right] - \frac{2n}{\alpha^2-\beta^2} \left[\alpha Z_{00,IK}^{(2n+1)} + \beta Z_{11,IK}^{(2n+1)}\right] \ ,$$

2.2.3. Integrals of the type $\int x^{2n} \cdot Z_\nu(\alpha x) \cdot Z_\nu(\beta x)\, dx$

a) *Basic Integrals:*

In the case $\alpha = \beta$ it was necessary to define the new functions $\Theta(x)$ and $\Omega(x)$ (see page 195). The more there is no solution expected in the described class of functions if $\alpha \neq \beta$.
Let $0 < \beta < \alpha$ and $\gamma = \beta/\alpha < 1$. The integrals may be reduced to the single parameter γ by

$$\int x^{2n} \cdot Z_\nu(\alpha x) \cdot Z_\nu(\beta x)\, dx = \alpha^{-2n-1} \int t^{2n} \cdot Z_\nu(t) \cdot Z_\nu(\gamma t)\, dt\,, \quad t = \alpha x\,.$$

The functions $\Theta(x)$ and $\Omega(x)$ from page 195 are generalized to

$$\Theta_0(x;\gamma) = \int_0^x J_0(s) \cdot J_0(\gamma s)\, ds \qquad \text{and} \qquad \Omega_0(x;\gamma) = \int_0^x I_0(s) \cdot I_0(\gamma s)\, ds\,.$$

Power series:

$$\Theta_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot (1-\gamma^2)^k \cdot P_k\left(\frac{1+\gamma^2}{1-\gamma^2}\right) x^{2k+1}$$

and

$$\Omega_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(1-\gamma^2)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot P_k\left(\frac{1+\gamma^2}{1-\gamma^2}\right) x^{2k+1}\,,$$

where

$$P_n(x) = \frac{(2n)!}{2^n \cdot (n!)^2}\, x^n + \ldots$$

denotes the Legendre polynoms. Their values may be found by the recurrence relation

$$P_{n+1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{2n+1}{n+1} \cdot \frac{1+\gamma^2}{1-\gamma^2} \cdot P_n\left(\frac{1+\gamma^2}{1-\gamma^2}\right) - \frac{n}{n+1} P_{n-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)$$

with

$$P_0\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = 1 \qquad \text{and} \qquad P_1\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{1+\gamma^2}{1-\gamma^2}\,.$$

Some first terms of the power series:

$$\Theta_0(x;\gamma) = x - \frac{\gamma^2+1}{12}\, x^3 + \frac{\gamma^4+4\,\gamma^2+1}{320}\, x^5 - \frac{\gamma^6+9\,\gamma^4+9\,\gamma^2+1}{16128}\, x^7 + \frac{\gamma^8+16\,\gamma^6+36\,\gamma^4+16\,\gamma^2+1}{1327104}\, x^9 -$$
$$- \frac{\gamma^{10}+25\,\gamma^8+100\,\gamma^6+100\,\gamma^4+25\,\gamma^2+1}{162201600}\, x^{11} + \frac{\gamma^{12}+36\,\gamma^{10}+225\,\gamma^8+400\,\gamma^6+225\,\gamma^4+36\,\gamma^2+1}{27603763200}\, x^{13} -$$
$$- \frac{\gamma^{14}+49\,\gamma^{12}+441\,\gamma^{10}+1225\,\gamma^8+1225\,\gamma^6+441\,\gamma^4+49\,\gamma^2+1}{6242697216000}\, x^{15} +$$
$$+ \frac{\gamma^{16}+64\,\gamma^{14}+784\,\gamma^{12}+3136\,\gamma^{10}+4900\,\gamma^8+3136\,\gamma^6+784\,\gamma^4+64\,\gamma^2+1}{1811214552268800}\, x^{17} -$$
$$- \frac{\gamma^{18}+81\,\gamma^{16}+1296\,\gamma^{14}+7056\,\gamma^{12}+15876\,\gamma^{10}+15876\,\gamma^8+7056\,\gamma^6+1296\,\gamma^4+81\,\gamma^2+1}{655872751986278400}\, x^{19} + \ldots$$

If $x > \gamma x >> 1$ one has

$$\Omega_0(x;\gamma) \quad \approx \quad \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}\,.$$

Let $\Theta_0(x;\gamma)$ be computed with n decimal signs, then in the case $x > \gamma x >> 1$ the loss of significant digits can be expected. Only about

$$n - \lg \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}$$

significant digits are left.
The upper integral with primary parameters:

$$\int_0^x J_0(\alpha s) \cdot J_0(\beta s)\, ds = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot (\alpha^2-\beta^2)^k \cdot P_k\left(\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\right) x^{2k+1}$$

Asymptotic series for $x > \gamma x >> 1$:

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma) + \frac{1}{\pi\sqrt{\gamma}\,x}\left[\frac{\sin(1-\gamma)x}{1-\gamma} - \frac{\cos(\gamma+1)x}{\gamma+1}\right] +$$

$$+\frac{1}{8\pi\gamma^{3/2}\,x^2}\left[\frac{\gamma^2-10\,\gamma+1}{(1-\gamma)^2}\,\cos(1-\gamma)x - \frac{\gamma^2+10\,\gamma+1}{(\gamma+1)^2}\,\sin(\gamma+1)x\right] +$$

$$+\frac{1}{128\pi\gamma^{5/2}\,x^3}\left[\frac{9\,\gamma^4+52\,\gamma^3+342\,\gamma^2+52\,\gamma+9}{(\gamma+1)^3}\,\cos(\gamma+1)x - \frac{9\,\gamma^4-52\,\gamma^3+342\,\gamma^2-52\,\gamma+9}{(1-\gamma)^3}\,\sin(1-\gamma)x\right] +$$

$$+\frac{3}{1024\pi\gamma^{7/2}\,x^4}\left[\frac{25\,\gamma^6+150\,\gamma^5+503\,\gamma^4+2804\,\gamma^3+503\,\gamma^2+150\,\gamma+25}{(\gamma+1)^4}\,\sin(\gamma+1)x - \right.$$

$$\left. - \frac{25\,\gamma^6-150\,\gamma^5+503\,\gamma^4-2804\,\gamma^3+503\,\gamma^2-150\,\gamma+25}{(1-\gamma)^4}\,\cos(1-\gamma)x\right] + \ldots\,,$$

where $\mathbf{K}$ denotes the complete elliptic integral of the first kind, see [1] or [5].
Particulary follows

$$\lim_{x\to\infty}\Theta_0(x;\gamma) = \frac{2}{\pi}\,\mathbf{K}(\gamma)\,.$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	1.0000	1.0000	1.0001	1.0002	1.0004	1.0006	1.0009	1.0012	1.0016	1.0020
0.1	1.0025	1.0030	1.0036	1.0043	1.0050	1.0057	1.0065	1.0073	1.0083	1.0092
0.2	1.0102	1.0113	1.0124	1.0136	1.0149	1.0162	1.0176	1.0190	1.0205	1.0221
0.3	1.0237	1.0254	1.0272	1.0290	1.0309	1.0329	1.0350	1.0371	1.0394	1.0417
0.4	1.0441	1.0465	1.0491	1.0518	1.0545	1.0574	1.0603	1.0634	1.0665	1.0698
0.5	1.0732	1.0767	1.0803	1.0841	1.0880	1.0920	1.0962	1.1006	1.1051	1.1097
0.6	1.1146	1.1196	1.1248	1.1302	1.1359	1.1417	1.1479	1.1542	1.1609	1.1678
0.7	1.1750	1.1826	1.1905	1.1988	1.2074	1.2166	1.2262	1.2363	1.2470	1.2583
0.8	1.2702	1.2830	1.2965	1.3110	1.3265	1.3432	1.3613	1.3809	1.4023	1.4258
0.9	1.4518	1.4810	1.5139	1.5517	1.5959	1.6489	1.7145	1.8004	1.9232	2.1369

The following picture shows $\Theta_0(x;1/2)$ (solid line), the asymptotic approximation with x^{-1} (long dashes) and with x^{-2} (short dashes):

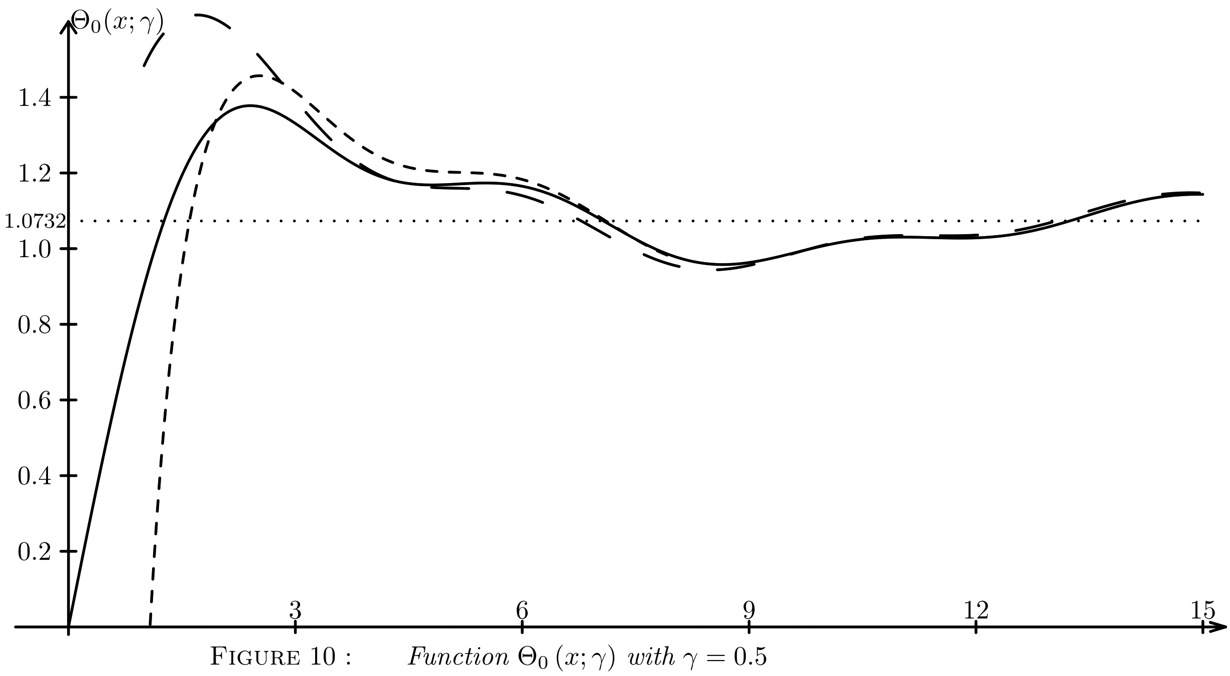


FIGURE 10 : *Function* $\Theta_0\,(x;\gamma)$ *with* $\gamma = 0.5$

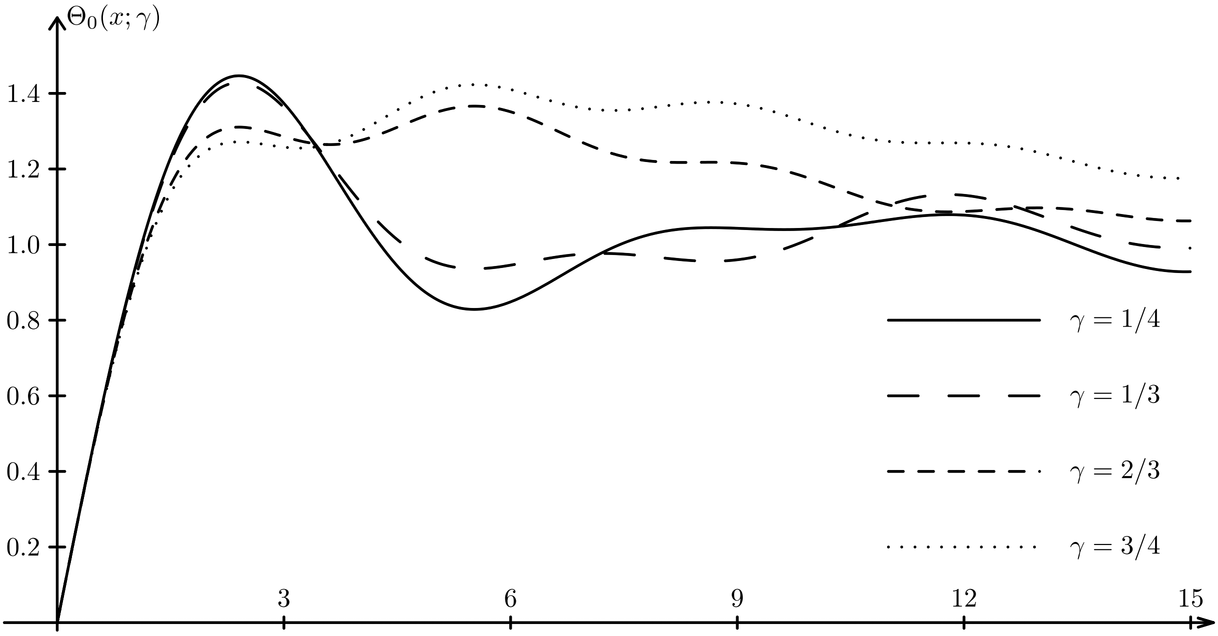


FIGURE 11 : *Some Functions* $\Theta_0\,(x;\gamma)$

Let

$$\Theta_1(x;\gamma) = \int_0^x J_1(s)\cdot J_1(\gamma s)\,ds \qquad \text{and} \qquad \Omega_1(x;\gamma) = \int_0^x I_1(s)\cdot I_1(\gamma s)\,ds\ .$$

Power series:

$$\Theta_1(x;\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2\cdot 4^{k+1}\cdot(k+1)\cdot(2k+3)}\cdot(1-\gamma^2)^{k+1}\cdot P_{k+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)\,x^{2k+3}$$

and

$$\Omega_1(x;\gamma) = \sum_{k=0}^{\infty} \frac{(1-\gamma^2)^{k+1}}{(k!)^2\cdot 4^{k+1}\cdot(k+1)\cdot(2k+3)}\cdot P_{k+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)\,x^{2k+3}\ ,$$

where $P_n^{-1}\,(x)$ denotes the associated Legendre functions of the first kind. Their values may be found by the recurrence relation, starting with $n = 0$:

$$P_{n+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{2n+1}{n+2}\cdot\frac{1+\gamma^2}{1-\gamma^2}\cdot P_n^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) - \frac{n-1}{n+2}\,P_{n-1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)$$

with

$$P_0^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \gamma \qquad \text{and} \qquad P_1^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{\gamma}{1-\gamma^2}\ .$$

Some first terms of the power series:

$$\Theta_1(x;\gamma) = \frac{\gamma}{12}\,x^3 - \frac{\gamma^3+\gamma}{160}\,x^5 + \frac{\gamma^5+3\,\gamma^3+\gamma}{5376}\,x^7 - \frac{\gamma^7+6\,\gamma^5+6\,\gamma^3+\gamma}{331776}\,x^9 + \frac{\gamma^9+10\,\gamma^7+20\,\gamma^5+10\,\gamma^3+\gamma}{32440320}\,x^{11} -$$
$$-\frac{\gamma^{11}+15\,\gamma^9+50\,\gamma^7+50\,\gamma^5+15\,\gamma^3+\gamma}{4600627200}\,x^{13} + \frac{\gamma^{13}+21\,\gamma^{11}+105\,\gamma^9+175\,\gamma^7+105\,\gamma^5+21\,\gamma^3+\gamma}{891813888000}\,x^{15} -$$
$$-\frac{\gamma^{15}+28\,\gamma^{13}+196\,\gamma^{11}+490\,\gamma^9+490\,\gamma^7+196\,\gamma^5+28\,\gamma^3+\gamma}{226401819033600}\,x^{17} +$$
$$+\frac{\gamma^{17}+36\,\gamma^{15}+336\,\gamma^{13}+1176\,\gamma^{11}+1764\,\gamma^9+1176\,\gamma^7+336\,\gamma^5+36\,\gamma^3+\gamma}{72874750220697600}\,x^{19} - \ldots$$

In the case $x > \gamma x >> 1$ one has once again

$$\Omega_1(x;\gamma) \quad\approx\quad \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}\ .$$

Asymptotic series for $x > \gamma x >> 1$:

$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\left[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\right] + \frac{1}{\pi\sqrt{\gamma}}\left[\frac{1}{x}\left(\frac{\cos(1+\gamma)x}{1+\gamma} - \frac{\sin(1-\gamma)x}{1-\gamma}\right) - \right.$$

$$-\frac{1}{8g\,x^2}\left(\frac{3\gamma^2-2\gamma+3}{(1+\gamma)^2}\sin(1+\gamma)x + \frac{3\gamma^2+2\gamma+3}{(1-\gamma)^2}\cos(1-\gamma)x\right) +$$

$$+\frac{1}{128g^3\,x^3}\left(\frac{15\gamma^4+108\gamma^3-70\gamma^2+108\gamma+15}{(1+\gamma)^3}\cos(1+\gamma)x - \frac{15\gamma^4-108\gamma^3-70\gamma^2-108\gamma+15}{(1-\gamma)^3}\sin(1-\gamma)x\right) -$$

$$-\frac{3}{1024\gamma^3\,x^4}\left(\frac{35\gamma^6+210\gamma^5+909\gamma^4-580\gamma^3+909\gamma^2+210\gamma+35}{(1+\gamma)^4}\sin(1+\gamma)x + \right.$$

$$\left.\left. + \frac{35\gamma^6-210\gamma^5+909\gamma^4+580\gamma^3+909\gamma^2-210\gamma+35}{(1-\gamma)^4}\cos(1-\gamma)x\right) + \ldots\right]$$

with the complete elliptic integrals of the first and second kind.
Particulary follows

$$\lim_{x\to\infty}\Theta_1(x;\gamma) = \frac{2}{\pi\gamma}\left[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\right] .$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0050	0.0100	0.0150	0.0200	0.0250	0.0300	0.0351	0.0401	0.0451
0.1	0.0502	0.0553	0.0603	0.0654	0.0705	0.0756	0.0808	0.0859	0.0911	0.0963
0.2	0.1015	0.1068	0.1121	0.1174	0.1227	0.1280	0.1334	0.1389	0.1443	0.1498
0.3	0.1554	0.1609	0.1666	0.1722	0.1780	0.1837	0.1895	0.1954	0.2013	0.2073
0.4	0.2134	0.2195	0.2257	0.2319	0.2382	0.2446	0.2511	0.2577	0.2643	0.2711
0.5	0.2779	0.2849	0.2919	0.2991	0.3064	0.3138	0.3214	0.3290	0.3369	0.3448
0.6	0.3530	0.3613	0.3698	0.3785	0.3873	0.3964	0.4058	0.4153	0.4252	0.4353
0.7	0.4457	0.4564	0.4674	0.4789	0.4907	0.5029	0.5157	0.5289	0.5427	0.5571
0.8	0.5721	0.5879	0.6046	0.6221	0.6407	0.6605	0.6816	0.7042	0.7287	0.7552
0.9	0.7844	0.8165	0.8525	0.8934	0.9407	0.9967	1.0654	1.1543	1.2803	1.4971

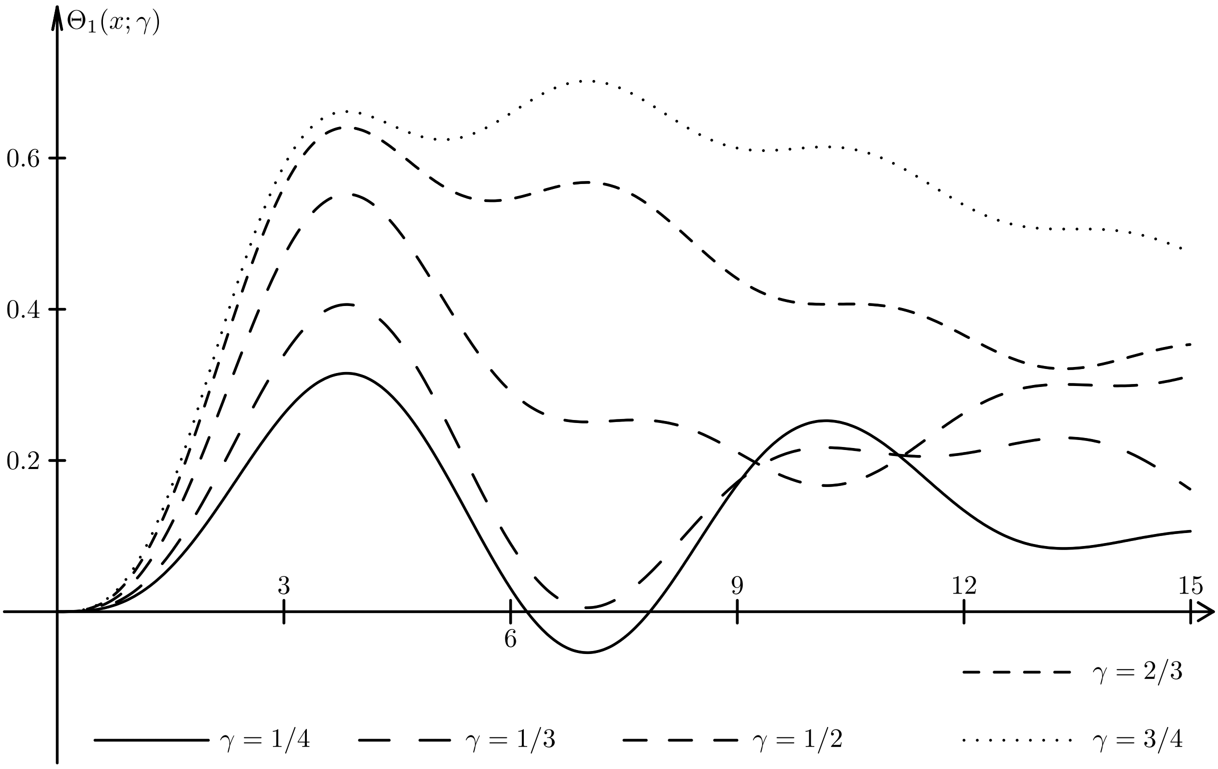


FIGURE 12 : *Some Functions* $\Theta_1(x;\gamma)$

The value of x may be too large to use the power series for $\Theta_0(x;\gamma)$ and γx may be too small to apply the asymptotic formula. In this case

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma) +$$
$$+\frac{A_0(x;\gamma)\;\cos x\,J_0(\gamma x) + A_1(x;\gamma)\;\cos x\,J_1(\gamma x) + B_0(x;\gamma)\;\sin x\,J_0(\gamma x) + B_1(x;\gamma)\;\sin x\,J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable. Let

$$A_\mu(x;\gamma) = \sum_{k=0}^\infty \frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k} \qquad \text{and} \qquad B_\mu(x;\gamma) = \sum_{k=0}^\infty \frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\ ,$$

then holds

$$a_0^{(0)}(x;\gamma) = -1\ , \quad a_1^{(0)}(x;\gamma) = -\frac{11\gamma^2+5}{8}\ , \quad a_2^{(0)}(x;\gamma) = -\frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128}\ ,$$
$$a_3^{(0)}(x;\gamma) = \frac{3\,\left(\gamma^2+15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024}\ ,$$
$$a_4^{(0)}(x;\gamma) = \frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768}\ ,$$
$$a_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144}\ ,$$
$$a_6^{(0)}(x;\gamma) =$$
$$-\frac{45\,[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649]}{4194304}$$

$$a_0^{(1)}(x;\gamma) = -\gamma\ , \quad a_1^{(1)}(x;\gamma) = -\frac{\gamma\left(\gamma^2-17\right)}{8}\ , \quad a_2^{(1)}(x;\gamma) = \frac{\gamma\,\left(9\,\gamma^4 + 206\,\gamma^2 + 809\right)}{128}\ ,$$
$$a_3^{(1)}(x;\gamma) = \frac{\gamma\,\left(75\,\gamma^6 + 143\,\gamma^4 - 24063\,\gamma^2 - 25307\right)}{1024}\ ,$$
$$a_4^{(1)}(x;\gamma) = -\frac{3\gamma\,\left(1225\,\gamma^8 - 1892\,\gamma^6 + 201078\,\gamma^4 + 2678812\,\gamma^2 + 1315081\right)}{32768}\ ,$$
$$a_5^{(1)}(x;\gamma) = -\frac{3\gamma\,\left(19845\,\gamma^{10} - 67625\,\gamma^8 + 467314\,\gamma^6 - 58112658\,\gamma^4 - 216355367\,\gamma^2 - 61495829\right)}{262144}\ ,$$
$$a_6^{(1)}(x;\gamma) =$$
$$\frac{3\gamma\,\left(800415\,\gamma^{12} - 3869530\,\gamma^{10} + 12921201\,\gamma^8 + 680167252\,\gamma^6 + 20763422609\,\gamma^4 + 36255061542\,\gamma^2 + 6716005951\right)}{4194304}$$

$$b_0^{(0)}(x;\gamma) = 1\ , \quad b_1^{(0)}(x;\gamma) = -\frac{11\gamma^2+5}{8} = a_1^{(0)}\ , \quad b_2^{(0)}(x;\gamma) = \frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128} = -a_2^{(0)}\ ,$$
$$b_3^{(0)}(x;\gamma) = \frac{3\,\left(\gamma^2+15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024} = a_3^{(0)}\ ,$$
$$b_4^{(0)}(x;\gamma) = -\frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768} = -a_4^{(0)}\ ,$$
$$b_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144} = a_5^{(0)}\ ,$$
$$b_6^{(0)}(x;\gamma) = -a_6^{(0)} =$$
$$\frac{45\,[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649]}{4194304}$$

$$b_0^{(1)}(x;\gamma) = -\gamma\ , \quad b_1^{(1)}(x;\gamma) = \frac{\gamma\ \left(\gamma^2-17\right)}{8} = a_1^{(1)}\ , \quad b_2^{(1)}(x;\gamma) = \frac{\gamma\ \left(9\,\gamma^4+206\,\gamma^2+809\right)}{128} = a_2^{(1)}\ ,$$

$$b_3^{(1)}(x;\gamma) = -\frac{\gamma\ \left(75\,\gamma^6+143\,\gamma^4-24063\,\gamma^2-25307\right)}{1024} = -a_3^{(0)}\ ,$$

$$b_4^{(1)}(x;\gamma) = -\frac{3\gamma\ \left(1225\,\gamma^8-1892\,\gamma^6+201078\,\gamma^4+2678812\,\gamma^2+1315081\right)}{32768} = a_4^{(1)}\ ,$$

$$b_5^{(1)}(x;\gamma) = \frac{3\gamma\ \left(19845\,\gamma^{10}-67625\,\gamma^8+467314\,\gamma^6-58112658\,\gamma^4-216355367\,\gamma^2-61495829\right)}{262144} = -a_5^{(1)}\ ,$$

$$b_6^{(1)}(x;\gamma) = a_6^{(1)} =$$

$$\frac{3\gamma\ \left(800415\,\gamma^{12}-3869530\,\gamma^{10}+12921201\,\gamma^8+680167252\,\gamma^6+20763422609\,\gamma^4+36255061542\,\gamma^2+6716005951\right)}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0(x;\gamma) \approx A_0(x;0) = -1 - \frac{0.625}{x} + \frac{1.0078}{x^2} + \frac{2.5928}{x^3} - \frac{9.1869}{x^4} - \frac{41.568}{x^5} + \frac{229.20}{x^6} + \ldots$$

$$B_0(x;\gamma) \approx B_0(x;0) = 1 - \frac{0.625}{x} - \frac{1.0078}{x^2} + \frac{2.5928}{x^3} + \frac{9.1869}{x^4} - \frac{41.568}{x^5} - \frac{229.20}{x^6} + \ldots$$

$$A_1(x;\gamma) \approx \gamma\,\frac{\partial A_1}{\partial\gamma}\,(x;0) = \gamma\ \left[-1 + \frac{2.125}{x} + \frac{6.3203}{x^2} - \frac{24.714}{x^3} - \frac{120.40}{x^4} + \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$B_1(x;\gamma) \approx \gamma\,\frac{\partial B_1}{\partial\gamma}\,(x;0) = \gamma\ \left[-1 - \frac{2.125}{x} + \frac{6.3203}{x^2} + \frac{24.714}{x^3} - \frac{120.40}{x^4} - \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$\left|\frac{a_3^{(0)}(x;0)}{a_2^{(0)}(x;0)}\right| = \left|\frac{b_3^{(0)}(x;0)}{b_2^{(0)}(x;0)}\right| = 2.57\ , \quad \left|\frac{a_4^{(0)}(x;0)}{a_3^{(0)}(x;0)}\right| = 3.54\ , \quad \left|\frac{a_5^{(0)}(x;0)}{a_4^{(0)}(x;0)}\right| = 4.52\ , \quad \left|\frac{a_6^{(0)}(x;0)}{a_5^{(0)}(x;0)}\right| = 5.51$$

The summand $a_k^{(0)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0)}(x;\gamma)$.
Let

$$\Delta_n(x;\gamma) = -\Theta_0(x;\gamma) +$$

$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n)}(x;\gamma)\ \cos x\, J_0(\gamma x) + A_1^{(n)}(x;\gamma)\ \cos x\, J_1(\gamma x) + B_0^{(n)}(x;\gamma)\ \sin x\, J_0(\gamma x) + B_1^{(n)}(x;\gamma)\ \sin x\, J_1(\gamma x)\right]$$

with

$$A_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n} \frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k} \qquad \text{and} \qquad B_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n} \frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\ .$$

For the case $\gamma = 0.1$ some of these differences are shown:

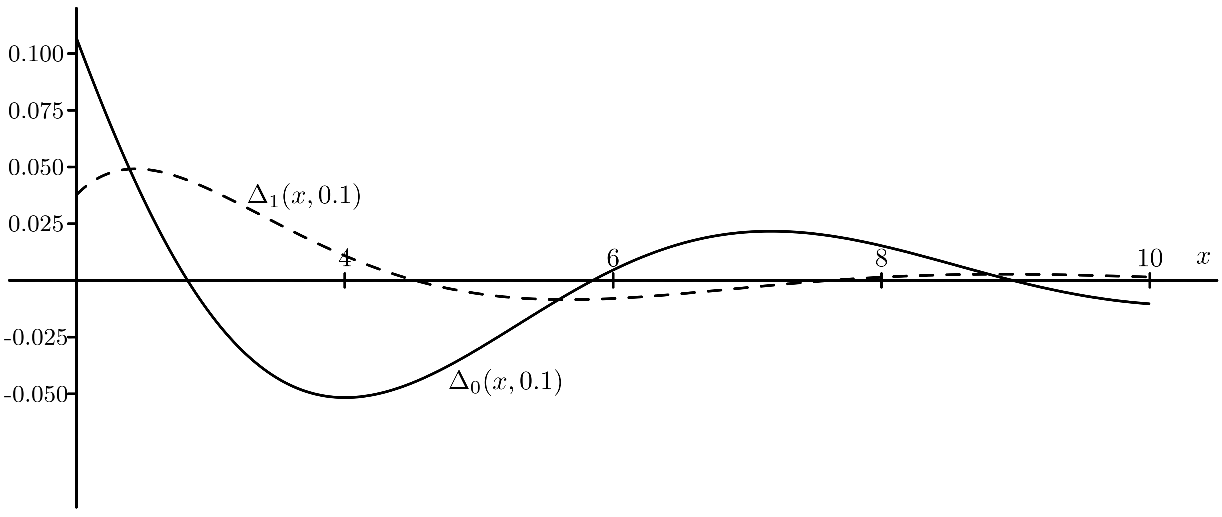


FIGURE 13 : *Differences $\Delta_0(x;\gamma)$ and $\Delta_1(x;\gamma)$ with $\gamma = 0.1$*

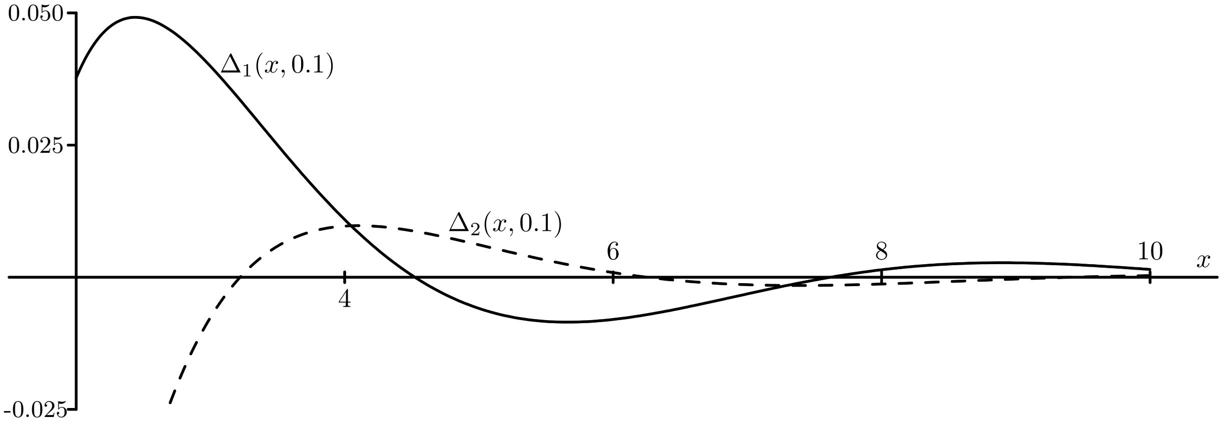


FIGURE 14 : *Differences $\Delta_1(x;\gamma)$ and $\Delta_2(x;\gamma)$ with $\gamma = 0.1$*

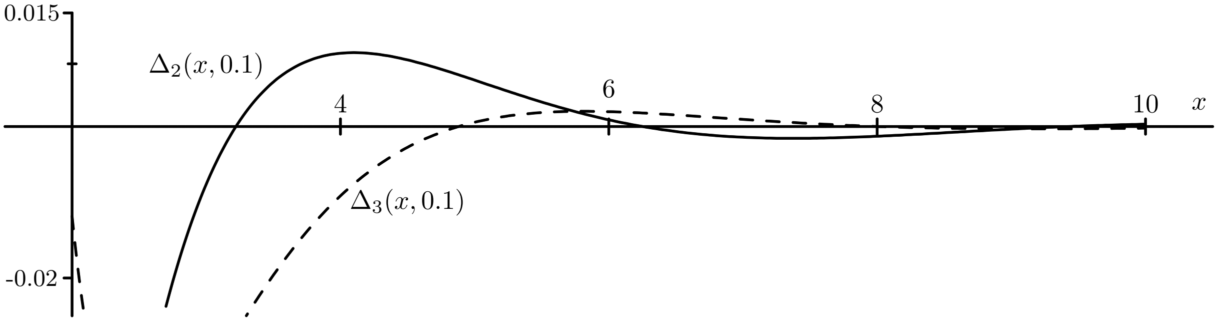


FIGURE 15 : *Differences $\Delta_2(x;\gamma)$ and $\Delta_3(x;\gamma)$ with $\gamma = 0.1$*

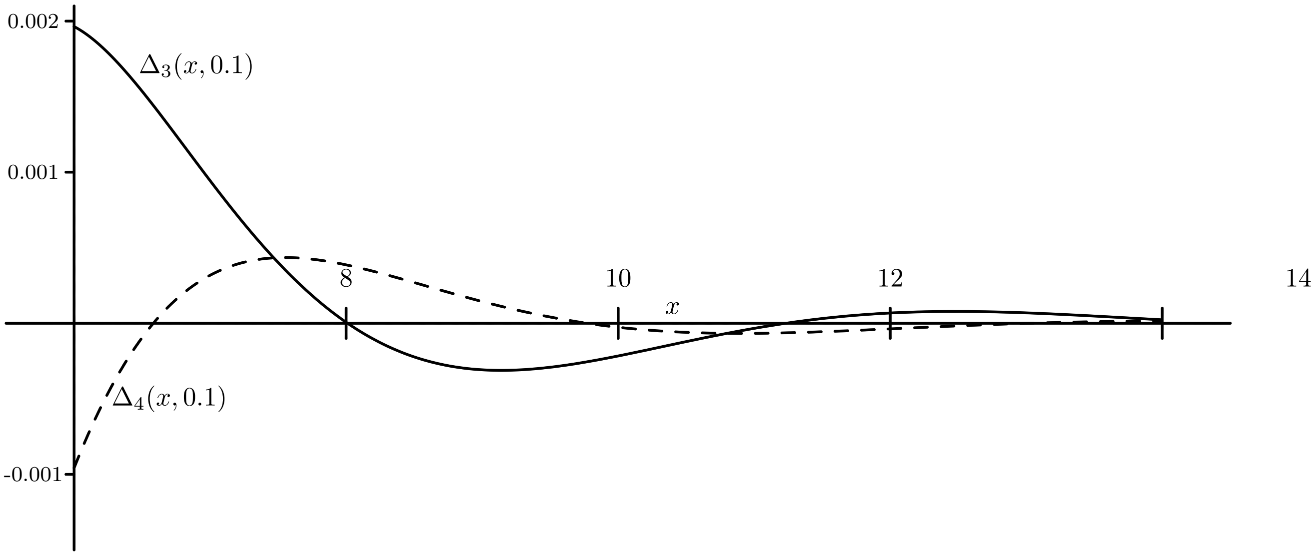


FIGURE 16 : *Differences $\Delta_3(x;\gamma)$ and $\Delta_4(x;\gamma)$ with $\gamma = 0.1$*

The same way the asymptotic expansion

$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\,[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)] +$$

$$+\frac{A_0^*(x;\gamma)\ \cos x\, J_0(\gamma x) + A_1^*(x;\gamma)\ \cos x\, J_1(\gamma x) + B_0^*(x;\gamma)\ \sin x\, J_0(\gamma x) + B_1^*(x;\gamma)\ \sin x\, J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable in the case $x >> 1$ and $\gamma x \approx 1$. Let

$$A_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k}\ ,$$

then holds

$$a_0^{(0,*)}(x;\gamma) = -\gamma\ , \quad a_1^{(0,*)}(x;\gamma) = -\frac{\gamma\ \left(3\,\gamma^2+13\right)}{8}\ , \quad a_2^{(0,*)}(x;\gamma) = -\frac{\gamma\ \left(15\,\gamma^4-382\,\gamma^2-657\right)}{128}\ ,$$

$$a_3^{(0,*)}(x;\gamma) = \frac{3\,\gamma\ \left(35\,\gamma^6+327\,\gamma^4+8457\,\gamma^2+7565\right)}{1024}$$

$$a_4^{(0,*)}(x;\gamma) = \frac{3\,\gamma\ \left(1575\,\gamma^8-860\,\gamma^6-455382\,\gamma^4-2435292\,\gamma^2-1304345\right)}{32768}$$

$$a_5^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\ \left(24255\,\gamma^{10}-74795\,\gamma^8+1750326\,\gamma^6+76252650\,\gamma^4+190548859\,\gamma^2+67043025\right)}{262144}$$

$$a_6^{(0,*)}(x;\gamma) = \frac{45\,\gamma\ \left(63063\,\gamma^{12}-295722\,\gamma^{10}+1295545\,\gamma^8-137114124\,\gamma^6-1469273511\,\gamma^4-2157119402\,\gamma^2-532523145\right)}{4194304}$$

$$a_0^{(1,*)}(x;\gamma) = -1\ , \quad a_1^{(1,*)}(x;\gamma) = \frac{7\,\gamma^2+9}{8}\ , \quad a_2^{(1,*)}(x;\gamma) = \frac{57\,\gamma^4+622\,\gamma^2+345}{128}\ ,$$

$$a_3^{(1,*)}(x;\gamma) = \frac{195\,\gamma^6-8921\,\gamma^4-30871\,\gamma^2-9555}{1024}$$

$$a_4^{(1,*)}(x;\gamma) = -\frac{7035\,\gamma^8+100692\,\gamma^6+4097826\,\gamma^4+7006164\,\gamma^2+1371195}{32768}$$

$$a_5^{(1,*)}(x;\gamma) = -\frac{97335\,\gamma^{10}-38595\,\gamma^8-54339354\,\gamma^6-442588230\,\gamma^4-449504301\,\gamma^2-60259815}{262144}$$

$$a_6^{(1,*)}(x;\gamma) = \frac{3565485\,\gamma^{12}-12841710\,\gamma^{10}+423532419\,\gamma^8+25838749116\,\gamma^6+96291507171\,\gamma^4+64464832914\,\gamma^2+6264182925}{4194304}$$

$$b_0^{(0,*)}(x;\gamma) = \gamma\ , \quad b_1^{(0,*)}(x;\gamma) = -\frac{\gamma\left(3\,\gamma^2+13\right)}{8} = a_1^{(0,*)}\ , \quad b_2^{(0,*)}(x;\gamma) = \frac{\gamma\ \left(15\,\gamma^4-382\,\gamma^2-657\right)}{128} = -a_2^{(0,*)}\ ,$$

$$b_3^{(0,*)}(x;\gamma) = \frac{3\,\gamma\ \left(35\,\gamma^6+327\,\gamma^4+8457\,\gamma^2+7565\right)}{1024} = a_3^{(0,*)}$$

$$b_4^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\ \left(1575\,\gamma^8-860\,\gamma^6-455382\,\gamma^4-2435292\,\gamma^2-1304345\right)}{32768} = -a_4^{(0,*)}$$

$$b_5^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\ \left(24255\,\gamma^{10}-74795\,\gamma^8+1750326\,\gamma^6+76252650\,\gamma^4+190548859\,\gamma^2+67043025\right)}{262144} = a_5^{(0,*)}$$

$$b_6^{(0,*)}(x;\gamma) = -a_6^{(0,*)} = \frac{45\,\gamma\ \left(63063\,\gamma^{12}-295722\,\gamma^{10}+1295545\,\gamma^8-137114124\,\gamma^6-1469273511\,\gamma^4-2157119402\,\gamma^2-532523145\right)}{4194304}$$

$$b_0^{(1,*)}(x;\gamma) = -1\ , \quad b_1^{(1,*)}(x;\gamma) = -\frac{7\,\gamma^2+9}{8} = -a_1^{(1,*)}\ , \quad b_2^{(1,*)}(x;\gamma) = \frac{57\,\gamma^4+622\,\gamma^2+345}{128} = a_2^{(1,*)}\ ,$$

$$b_3^{(1,*)}(x;\gamma) = \frac{195\,\gamma^6-8921\,\gamma^4-30871\,\gamma^2-9555}{1024} = -a_3^{(1,*)}$$

$$b_4^{(1,*)}(x;\gamma) = -\frac{7035\,\gamma^8+100692\,\gamma^6+4097826\,\gamma^4+7006164\,\gamma^2+1371195}{32768} = a_4^{(1,*)}$$

$$b_5^{(1,*)}(x;\gamma) = \frac{97335\,\gamma^{10}-38595\,\gamma^8-54339354\,\gamma^6-442588230\,\gamma^4-449504301\,\gamma^2-60259815}{262144} = -a_5^{(1,*)}$$

$$b_6^{(1,*)}(x;\gamma) = a_6^{(1,*)} =$$

$$\frac{3565485\,\gamma^{12} - 12841710\,\gamma^{10} + 423532419\,\gamma^{8} + 25838749116\,\gamma^{6} + 96291507171\,\gamma^{4} + 64464832914\,\gamma^{2} + 6264182925}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0^*(x;\gamma) \approx \gamma\,\frac{\partial A_0^*}{\partial\gamma}\,(x;0) = \gamma\,\left[-1 - \frac{1.625}{x} + \frac{5.1328}{x^2} + \frac{22.163}{x^3} - \frac{119.42}{x^4} - \frac{767.25}{x^5} + \frac{5713.4}{x^6} + \ldots\right]$$

$$B_0^*(x;\gamma) \approx \gamma\,\frac{\partial B_0^*}{\partial\gamma}\,(x;0) = \gamma\,\left[1 - \frac{1.625}{x} - \frac{5.1328}{x^2} + \frac{22.163}{x^3} + \frac{119.42}{x^4} - \frac{767.25}{x^5} - \frac{5713.4}{x^6} + \ldots\right]$$

$$A_1^*(x;\gamma) \approx A_1^*(x;0) = -1 + \frac{1.125}{x} + \frac{2.6953}{x^2} - \frac{9.3311}{x^3} - \frac{41.846}{x^4} + \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$B_1^*(x;\gamma) \approx B_1^*(x;0) = -1 - \frac{1.125}{x} + \frac{2.6953}{x^2} + \frac{9.3311}{x^3} - \frac{41.846}{x^4} - \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$\left|\frac{a_3^{(1,*)}(x;0)}{a_2^{(1,*)}(x;0)}\right| = \left|\frac{b_3^{(1,*)}(x;0)}{b_2^{(1,*)}(x;0)}\right| = 3.46\ , \quad \left|\frac{a_4^{(1,*)}(x;0)}{a_3^{(1,*)}(x;0)}\right| = 4.48\ , \quad \left|\frac{a_5^{(1,*)}(x;0)}{a_4^{(1,*)}(x;0)}\right| = 5.49\ , \quad \left|\frac{a_6^{(1,*)}(x;0)}{a_5^{(1,*)}(x;0)}\right| = 6.50$$

The summand $a_k^{(0,*)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0,*)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0,*)}(x;\gamma)$.
Let

$$\Delta_n^*(x;\gamma) = -\Theta_1(x;\gamma) +$$

$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n,*)}(x;\gamma)\ \cos x\,J_0(\gamma x) + A_1^{(n,*)}(x;\gamma)\ \cos x\,J_1(\gamma x) + B_0^{(n,*)}(x;\gamma)\ \sin x\,J_0(\gamma x) + B_1^{(n,*)}(x;\gamma)\ \sin x\,J_1(\gamma x)\right]$$

with

$$A_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n}\frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k} \qquad \text{and} \qquad B_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n}\frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\ .$$

For the case $\gamma = 0.1$ some of these differences are shown:

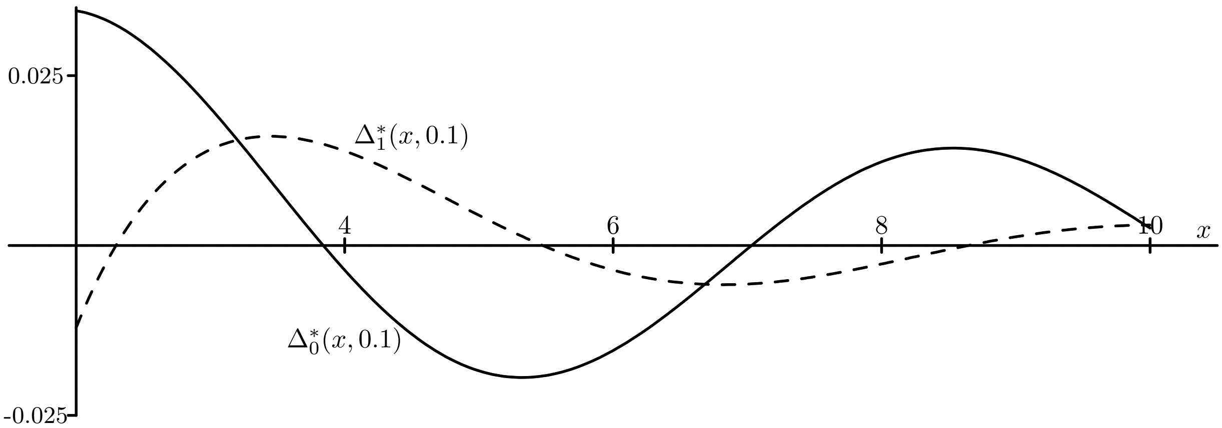


FIGURE 17 : *Differences $\Delta_0^*(x;\gamma)$ and $\Delta_1^*(x;\gamma)$ with $\gamma = 0.1$*

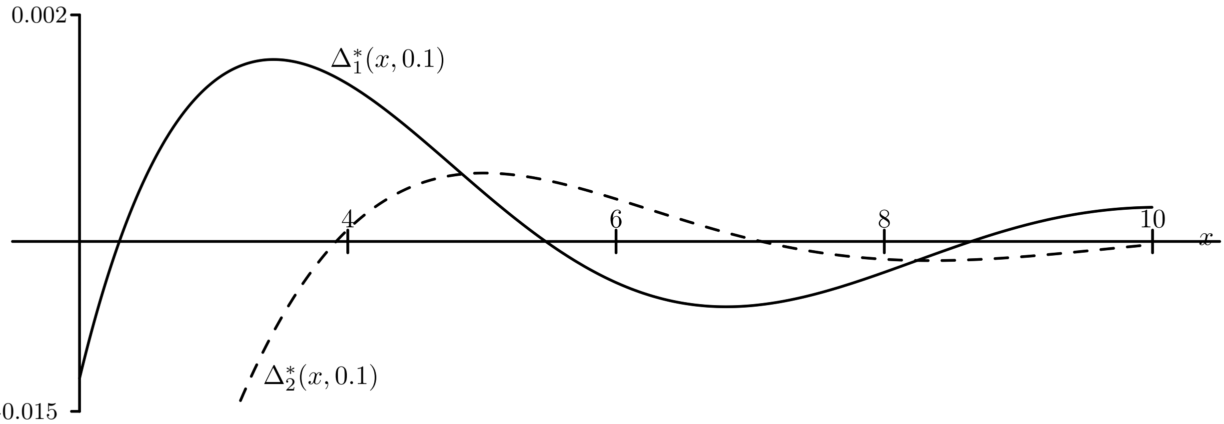


FIGURE 18 : *Differences* $\Delta_1^*(x;\gamma)$ *and* $\Delta_2^*(x;\gamma)$ *with* $\gamma = 0.1$

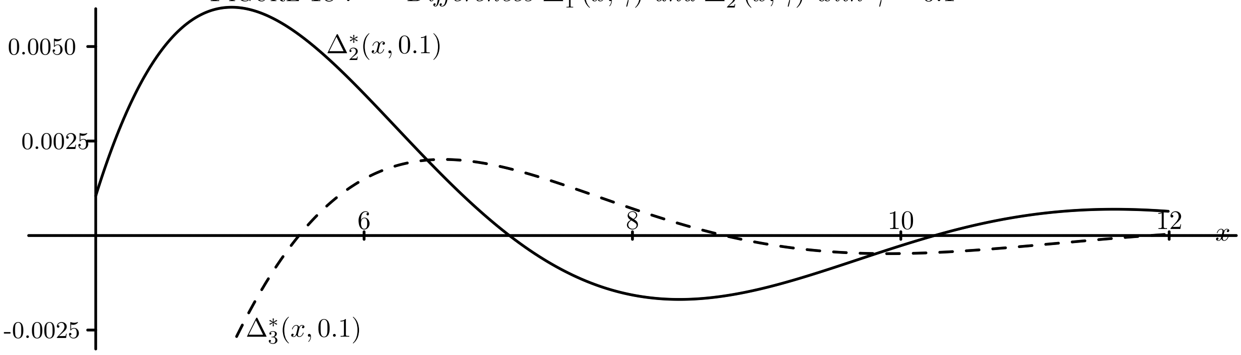


FIGURE 19 : *Differences* $\Delta_2^*(x;\gamma)$ *and* $\Delta_3^*(x;\gamma)$ *with* $\gamma = 0.1$

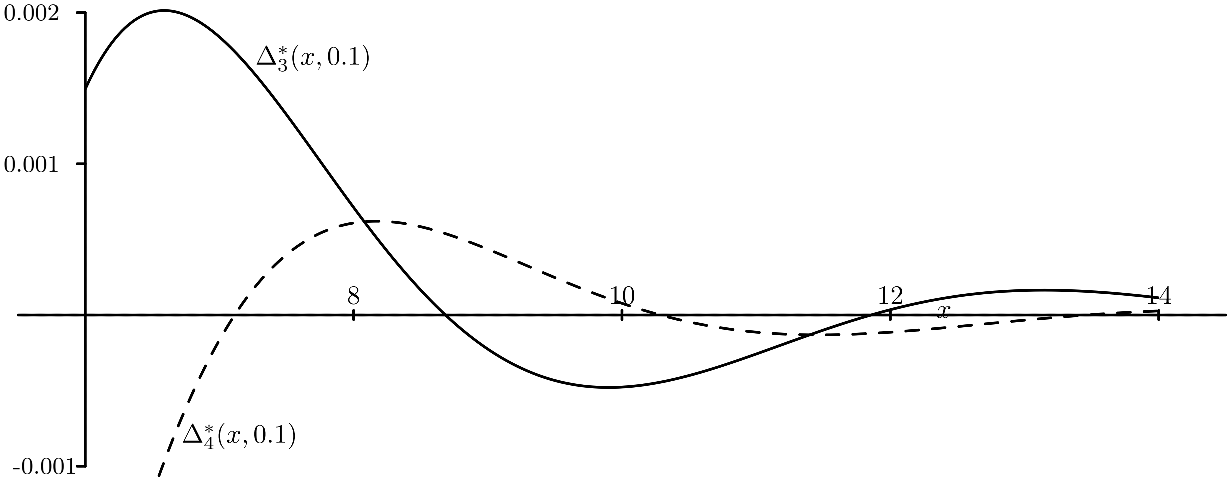


FIGURE 20 : *Differences* $\Delta_3^*(x;\gamma)$ *and* $\Delta_4^*(x;\gamma)$ *with* $\gamma = 0.1$

b) Integrals:

Holds (with $0 < \beta < \alpha$ and $\beta/\alpha = \gamma < 1$)

$$\int J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_0 \left(ax; \frac{b}{a} \right) \ , \qquad \int J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_1 \left(ax; \frac{b}{a} \right)$$

$$\int I_0(\alpha x)\, I_0(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_0 \left(ax; \frac{b}{a} \right) \ , \qquad \int I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_1 \left(ax; \frac{b}{a} \right)$$

(Θ_ν and Ω_ν as definded on pages 264 and 266.)

$$\int x^2 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{x\left(\beta^2+\alpha^2\right)}{\left(\alpha^2-\beta^2\right)^2} J_0(\alpha x)\, J_0(\beta x) - \frac{\beta\, x^2}{\alpha^2-\beta^2} J_0(\alpha x)\, J_1(\beta x) + \frac{\alpha\, x^2}{\alpha^2-\beta^2} J_1(\alpha x)\, J_0(\beta x) +$$
$$+2\, \frac{x\beta\, \alpha}{\left(\alpha^2-\beta^2\right)^2} J_1(\alpha x)\, J_1(\beta x) + \frac{\beta^2+\alpha^2}{\left(\alpha^2-\beta^2\right)^2} \int J_0(\alpha x)\, J_0(\beta x)\, dx - 2\, \frac{\beta\, \alpha}{\left(\alpha^2-\beta^2\right)^2} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = 2\, \frac{x\beta\, \alpha}{\left(\alpha^2-\beta^2\right)^2} J_0(\alpha x)\, J_0(\beta x) - \frac{\alpha\, x^2}{\alpha^2-\beta^2} J_0(\alpha x)\, J_1(\beta x) + \frac{\beta\, x^2}{\alpha^2-\beta^2} J_1(\alpha x)\, J_0(\beta x) +$$
$$+\frac{x\left(\beta^2+\alpha^2\right)}{\left(\alpha^2-\beta^2\right)^2} J_1(\alpha x)\, J_1(\beta x) + \frac{2\beta\, \alpha}{\left(\alpha^2-\beta^2\right)^2} \int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{\beta^2+\alpha^2}{\left(\alpha^2-\beta^2\right)^2} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{(\alpha^2+\beta^2)x}{(\alpha^2-\beta^2)^2}\, I_0(\alpha x)\, I_0(\beta x) - \frac{\beta\, x^2}{\alpha^2-\beta^2}\, I_0(\alpha x)\, I_1(\beta x) + \frac{\alpha\, x^2}{\alpha^2-\beta^2}\, I_1(\alpha x)\, I_0(\beta x) +$$
$$+\frac{2\, \beta\, \alpha x}{(\alpha^2-\beta^2)^2}\, I_1(\alpha x)\, I_1(\beta x) - \frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2} \int I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{2\, \beta\, \alpha}{(\alpha^2-\beta^2)^2} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^2 \cdot I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{2\, x\alpha\, \beta}{(\alpha^2-\beta^2)^2}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\alpha\, x^2}{\alpha^2-\beta^2}\, I_0(\alpha x)\, I_1(\beta x) - \frac{\beta\, x^2}{\alpha^2-\beta^2}\, I_1(\alpha x)\, I_0(\beta x) -$$
$$-\frac{(\alpha^2+\beta^2)x}{(\alpha^2-\beta^2)^2}\, I_1(\alpha x)\, I_1(\beta x) + \frac{2\, \alpha\, \beta}{(\alpha^2-\beta^2)^2} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^4 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = 3\, \frac{\left(x^2\alpha^6 - 3\, \alpha^4 - \alpha^4 x^2 \beta^2 - \alpha^2 x^2 \beta^4 - 10\, \alpha^2\beta^2 - 3\, \beta^4 + x^2\beta^6\right) x}{\left(\alpha^2-\beta^2\right)^4} J_0(\alpha x)\, J_0(\beta x) -$$
$$-\frac{\left(x^2\alpha^4 - 2\, \alpha^2 x^2 \beta^2 - 15\, \alpha^2 - 9\, \beta^2 + x^2\beta^4\right) \beta\, x^2}{\left(\alpha^2-\beta^2\right)^3} J_0(\alpha x)\, J_1(\beta x) +$$
$$+\frac{\left(x^2\alpha^4 - 2\, \alpha^2 x^2 \beta^2 - 9\, \alpha^2 - 15\, \beta^2 + x^2\beta^4\right) \alpha\, x^2}{\left(\alpha^2-\beta^2\right)^3} J_1(\alpha x)\, J_0(\beta x) +$$
$$+6\, \frac{\left(x^2\alpha^4 - 2\, \alpha^2 x^2 \beta^2 - 4\, \alpha^2 - 4\, \beta^2 + x^2\beta^4\right) \beta\, \alpha\, x}{\left(\alpha^2-\beta^2\right)^4} J_1(\alpha x)\, J_1(\beta x) -$$
$$-\frac{9\, \alpha^4 + 30\, \alpha^2\beta^2 + 9\, \beta^4}{\left(\alpha^2-\beta^2\right)^4} \int J_0(\alpha x)\, J_0(\beta x)\, dx + 24\, \frac{\beta\, \alpha\, \left(\beta^2+\alpha^2\right)}{\left(\alpha^2-\beta^2\right)^4} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^4 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = 6\, \frac{\left(x^2\alpha^4 - 4\, \alpha^2 - 2\, x^2\alpha^2\beta^2 + x^2\beta^4 - 4\, \beta^2\right) \beta\, \alpha\, x}{\left(\alpha^2-\beta^2\right)^4} J_0(\alpha x)\, J_0(\beta x) -$$

$$-\frac{\left(x^2\alpha^4-2\,\alpha^2x^2\beta^2-3\,\alpha^2-21\,\beta^2+x^2\beta^4\right)\alpha\,x^2}{\left(\alpha^2-\beta^2\right)^3}J_0(\alpha x)\,J_1(\beta x)+$$

$$+\frac{\left(x^2\alpha^4-21\,\alpha^2-2\,\alpha^2x^2\beta^2-3\,\beta^2+x^2\beta^4\right)x^2\beta}{\left(\alpha^2-\beta^2\right)^3}J_1(\alpha x)\,J_0(\beta x)+$$

$$+3\,\frac{\left(x^2\alpha^6-\alpha^4-x^2\alpha^4\beta^2-14\,\alpha^2\beta^2-x^2\alpha^2\beta^4-\beta^4+x^2\beta^6\right)x}{\left(\alpha^2-\beta^2\right)^4}J_1(\alpha x)\,J_1(\beta x)-$$

$$-24\,\frac{\alpha\,\beta\,\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^4}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{3\,\alpha^4+42\,\alpha^2\beta^2+3\,\beta^4}{\left(\alpha^2-\beta^2\right)^4}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^4\cdot I_0(\alpha x)\,I_0(\beta x)\,dx=-\frac{3x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+\left(\beta^2+3\,\alpha^2\right)\left(3\,\beta^2+\alpha^2\right)]}{(\alpha^2-\beta^2)^4}\,I_0(\alpha x)\,I_0(\beta x)-$$

$$-\frac{\beta x^2[\left(\alpha^2-\beta^2\right)^2x^2+15\,\alpha^2+9\,\beta^2]}{(\alpha^2-\beta^2)^3}\,I_0(\alpha x)\,I_1(\beta x)+\frac{\alpha\,x^2[\left(\alpha^2-\beta^2\right)^2x^2+9\,\alpha^2+15\,\beta^2]}{(\alpha^2-\beta^2)^3}\,I_1(\alpha x)\,I_0(\beta x)+$$

$$+\frac{6\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^2x^2+4\,\alpha^2+4\,\beta^2]}{(\alpha^2-\beta^2)^4}\,I_1(\alpha x)\,I_1(\beta x)-$$

$$-\frac{3\,\left(\beta^2+3\,\alpha^2\right)\left(3\,\beta^2+\alpha^2\right)}{(\alpha^2-\beta^2)^4}\int I_0(\alpha x)\,I_0(\beta x)\,dx-\frac{24\,\beta\,\alpha\,\left(\alpha^2+\beta^2\right)}{(\alpha^2-\beta^2)^4}\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^4\cdot I_1(\alpha x)\,I_1(\beta x)\,dx=\frac{6\,\left(\alpha^2-\beta^2\right)^2\alpha\,\beta\,x^3+24\,\left(\alpha^2+\beta^2\right)\beta\,\alpha x}{(\alpha^2-\beta^2)^4}\,I_0(\alpha x)\,I_0(\beta x)+$$

$$+\frac{\left(\alpha^2-\beta^2\right)^2\alpha\,x^4+3\,\left(\alpha^2+7\,\beta^2\right)\alpha\,x^2}{(\alpha^2-\beta^2)^3}\,I_0(\alpha x)\,I_1(\beta x)-\frac{\left(\alpha^2-\beta^2\right)^2\beta\,x^4+3\,\left(7\,\alpha^2+\beta^2\right)\beta\,x^2}{(\alpha^2-\beta^2)^3}\,I_1(\alpha x)\,I_0(\beta x)-$$

$$-\frac{3\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+3\,\alpha^4+42\,\beta^2\alpha^2+3\,\beta^4}{(\alpha^2-\beta^2)^4}\,I_1(\alpha x)\,I_1(\beta x)+$$

$$+\frac{24\,\left(\alpha^2+\beta^2\right)\beta\,\alpha}{(\alpha^2-\beta^2)^4}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{3\,\alpha^4+42\,\beta^2\alpha^2+3\,\beta^4}{(\alpha^2-\beta^2)^4}\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

Let

$$\int x^n\cdot J_\nu(\alpha x)\,J_\nu(\beta x)\,dx=\frac{P_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\,J_0(\alpha x)\,J_0(\beta x)+\frac{Q_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^{n-1}}\,J_0(\alpha x)\,J_1(\beta x)+$$

$$+\frac{R_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^{n-1}}\,J_1(\alpha x)\,J_0(\beta x)+\frac{S_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\,J_1(\alpha x)\,J_1(\beta x)+$$

$$+\frac{U_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{V_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

and

$$\int x^n\cdot I_\nu(\alpha x)\,I_\nu(\beta x)\,dx=\frac{\bar{P}_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\,I_0(\alpha x)\,I_0(\beta x)+\frac{\bar{Q}_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^{n-1}}\,I_0(\alpha x)\,I_1(\beta x)+$$

$$+\frac{\bar{R}_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^{n-1}}\,I_1(\alpha x)\,I_0(\beta x)+\frac{\bar{S}_n^{(\nu)}(x)}{(\alpha^2-\beta^2)^n}\,I_1(\alpha x)\,I_1(\beta x)+$$

$$+\frac{\bar{U}_n^{(\nu)}}{(\alpha^2-\beta^2)^n}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{\bar{V}_n^{(\nu)}}{(\alpha^2-\beta^2)^n}\int I_1(\alpha x)\,I_1(\beta x)\,dx\;,$$

then holds

$$P_6^{(0)}=5\,x[\left(\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^4x^4-3\,\left(5\,\alpha^4+22\,\beta^2\alpha^2+5\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+3\left(\beta^2+\alpha^2\right)\left(15\,\beta^4+98\,\beta^2\alpha^2+15\,\alpha^4\right)]$$

$$Q_6^{(0)}=-\beta\,x^2[\left(\alpha^2-\beta^2\right)^4x^4-5\left(11\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+465\,\alpha^4+1230\,\beta^2\alpha^2+225\,\beta^4]$$

$$R_6^{(0)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^4x^4-5\left(5\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+225\,\alpha^4+1230\,\beta^2\alpha^2+465\,\beta^4]$$

$$S_6^{(0)}=10\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^4x^4-24\left(\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^2x^2+246\,\beta^2\alpha^2+69\,\alpha^4+69\,\beta^4]$$

$$U_6^{(0)}=15\left(\beta^2+\alpha^2\right)\left(15\,\beta^4+98\,\beta^2\alpha^2+15\,\alpha^4\right)$$

$$V_6^{(0)}=-30\,\beta\,\alpha\left(23\,\beta^4+23\,\alpha^4+82\,\beta^2\alpha^2\right)$$

$$P_6^{(1)}=10\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^4x^4-24\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+81\,\alpha^4+222\,\beta^2\alpha^2+81\,\beta^4]$$

$$Q_6^{(1)}=-\alpha\,x^2[\left(\alpha^2-\beta^2\right)^4x^4-5\left(3\,\alpha^2+13\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+45\,\alpha^4+765\,\beta^4+1110\,\beta^2\alpha^2]$$

$$R_6^{(1)}=x^2\beta[\left(\alpha^2-\beta^2\right)^4x^4-5\left(13\,\alpha^2+3\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+45\,\beta^4+765\,\alpha^4+1110\,\beta^2\alpha^2]$$

$$S_6^{(1)}=5\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4-$$

$$-3\left(3\,\alpha^4+26\,\beta^2\alpha^2+3\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+3\left(\alpha^2+\beta^2\right)\left(3\,\beta^4+122\,\beta^2\alpha^2+3\,\alpha^4\right)]$$

$$U_6^{(1)}=30\left(27\,\alpha^4+74\,\beta^2\alpha^2+27\,\beta^4\right)\beta\,\alpha$$

$$V_6^{(1)}=-15\left(\alpha^2+\beta^2\right)\left(3\,\beta^4+122\,\beta^2\alpha^2+3\,\alpha^4\right)$$

$$\bar{P}_6^{(0)}=-5\,x[\left(\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^4x^4+3\left(5\,\alpha^4+22\,\beta^2\alpha^2+5\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+3\left(\beta^2+\alpha^2\right)\left(15\,\beta^4+98\,\beta^2\alpha^2+15\,\alpha^4\right)]$$

$$\bar{Q}_6^{(0)}=-\beta\,x^2[\left(\alpha^2-\beta^2\right)^4x^4+5\left(11\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+465\,\alpha^4+1230\,\beta^2\alpha^2+225\,\beta^4]$$

$$\bar{R}_6^{(0)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^4x^4+5\left(5\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+225\,\alpha^4+1230\,\beta^2\alpha^2+465\,\beta^4]$$

$$\bar{S}_6^{(0)}=10\,\beta\,\alpha\,x[\left(\alpha^2-\beta^2\right)^4x^4+24\left(\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^2x^2+69\,\alpha^4+69\,\beta^4+246\,\beta^2\alpha^2]$$

$$\bar{U}_6^{(0)}=-15\left(\beta^2+\alpha^2\right)\left(15\,\beta^4+98\,\beta^2\alpha^2+15\,\alpha^4\right)$$

$$\bar{V}_6^{(0)}=-30\,\beta\,\alpha\left(23\,\beta^4+23\,\alpha^4+82\,\beta^2\alpha^2\right)$$

$$\bar{P}_6^{(1)}=10\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^4x^4+24\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+222\,\beta^2\alpha^2+81\,\alpha^4+81\,\beta^4]$$

$$\bar{Q}_6^{(1)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^4x^4+5\left(3\,\alpha^2+13\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+765\,\beta^4+45\,\alpha^4+1110\,\beta^2\alpha^2]$$

$$\bar{R}_6^{(1)}=-x^2\beta[\left(\alpha^2-\beta^2\right)^4x^4+5\left(13\,\alpha^2+3\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2x^2+45\,\beta^4+765\,\alpha^4+1110\,\beta^2\alpha^2]$$

$$\bar{S}_6^{(1)}=-5\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+3\left(3\,\alpha^4+26\,\beta^2\alpha^2+3\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+3\left(\alpha^2+\beta^2\right)\left(3\,\beta^4+122\,\beta^2\alpha^2+3\,\alpha^4\right)]$$

$$\bar{U}_6^{(1)}=30\left(27\,\alpha^4+74\,\beta^2\alpha^2+27\,\beta^4\right)\beta\,\alpha$$

$$\bar{V}_6^{(1)}=15\left(\alpha^2+\beta^2\right)\left(3\,\beta^4+122\,\beta^2\alpha^2+3\,\alpha^4\right)$$

$$P_8^{(0)}=7\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6-5\left(7\,\alpha^4+34\,\beta^2\alpha^2+7\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+15\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+314\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2-$$

$$-1575\,\beta^8-1575\,\alpha^8-21540\,\alpha^2\beta^6-45930\,\alpha^4\beta^4-21540\,\alpha^6\beta^2]$$

$$Q_8^{(0)}=-\beta\,x^2[\left(\alpha^2-\beta^2\right)^6x^6-7\left(17\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+35\,\left(107\,\alpha^4+242\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-160755\,\alpha^4\beta^2-25935\,\alpha^6-124845\,\alpha^2\beta^4-11025\,\beta^6]$$
$$R_8^{(0)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^6x^6-7\,\left(7\,\alpha^2+17\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+35\,\left(35\,\alpha^4+242\,\beta^2\alpha^2+107\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-160755\,\alpha^2\beta^4-25935\,\beta^6-11025\,\alpha^6-124845\,\alpha^4\beta^2]$$
$$S_8^{(0)}=14\,\beta\,\alpha\,x[\left(\alpha^2-\beta^2\right)^6x^6-60\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+15\,\left(71\,\alpha^4+242\,\beta^2\alpha^2+71\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-240\,\left(\alpha^2+\beta^2\right)\left(11\,\beta^4+74\,\beta^2\alpha^2+11\,\alpha^4\right)]$$
$$U_8^{(0)}=-11025\,\alpha^8-150780\,\alpha^6\beta^2-321510\,\alpha^4\beta^4-150780\,\alpha^2\beta^6-11025\,\beta^8$$
$$V_8^{(0)}=3360\,\alpha\,\beta\,\left(\alpha^2+\beta^2\right)\left(11\,\beta^4+74\,\beta^2\alpha^2+11\,\alpha^4\right)$$

$$P_8^{(1)}=14\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^6x^6-60\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+45\,\left(25\,\alpha^4+78\,\beta^2\alpha^2+25\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2-720\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+22\,\beta^2\alpha^2+5\,\alpha^4\right)]$$
$$Q_8^{(1)}=-\alpha\,x^2[\left(\alpha^2-\beta^2\right)^6x^6-7\,\left(5\,\alpha^2+19\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\,\left(3\,\beta^2+5\,\alpha^2\right)\left(15\,\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^2x^2-186165\,\beta^4\alpha^2-1575\,\alpha^6-48825\,\beta^6-85995\,\beta^2\alpha^4]$$
$$R_8^{(1)}=\beta\,x^2[\left(\alpha^2-\beta^2\right)^6x^6-7\,\left(5\,\beta^2+19\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\,\left(5\,\beta^2+3\,\alpha^2\right)\left(\beta^2+15\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^2x^2-85995\,\beta^4\alpha^2-48825\,\alpha^6-1575\,\beta^6-186165\,\beta^2\alpha^4]$$
$$S_8^{(1)}=7\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6-5\,\left(5\,\alpha^4+38\,\beta^2\alpha^2+5\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+45\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+118\,\beta^2\alpha^2+5\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-19260\,\beta^2\alpha^6-225\,\beta^8-19260\,\beta^6\alpha^2-53190\,\beta^4\alpha^4-225\,\alpha^8]$$
$$U_8^{(1)}=-10080\,\beta\,\alpha\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+22\,\beta^2\alpha^2+5\,\alpha^4\right)$$
$$V_8^{(1)}=1575\,\alpha^8+134820\,\beta^2\alpha^6+372330\,\beta^4\alpha^4+134820\,\beta^6\alpha^2+1575\,\beta^8$$

$$\bar{P}_8^{(0)}=-7\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+5\,\left(7\,\alpha^4+34\,\beta^2\alpha^2+7\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+15\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+314\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+21540\,\beta^2\alpha^6+1575\,\beta^8+21540\,\beta^6\alpha^2+45930\,\beta^4\alpha^4+1575\,\alpha^8]$$
$$\bar{Q}_8^{(0)}=-\beta\,x^2[\left(\alpha^2-\beta^2\right)^6x^6+7\,\left(17\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+35\,\left(107\,\alpha^4+242\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+160755\,\beta^2\alpha^4+11025\,\beta^6+124845\,\beta^4\alpha^2+25935\,\alpha^6]$$
$$\bar{R}_8^{(0)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^6x^6+7\,\left(7\,\alpha^2+17\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+35\,\left(35\,\alpha^4+242\,\beta^2\alpha^2+107\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+124845\,\beta^2\alpha^4+11025\,\alpha^6+160755\,\beta^4\alpha^2+25935\,\beta^6]$$
$$\bar{S}_8^{(0)}=14\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^6x^6+60\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+15\,\left(71\,\alpha^4+242\,\beta^2\alpha^2+71\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+240\,\left(\alpha^2+\beta^2\right)\left(11\,\beta^4+74\,\beta^2\alpha^2+11\,\alpha^4\right)]$$
$$\bar{U}_8^{(0)}=-11025\,\alpha^8-150780\,\beta^2\alpha^6-321510\,\beta^4\alpha^4-150780\,\beta^6\alpha^2-11025\,\beta^8$$

$$\bar{V}_8^{(0)} = -3360\,\beta\,\alpha\,\left(\alpha^2+\beta^2\right)\left(11\,\beta^4+74\,\beta^2\alpha^2+11\,\alpha^4\right)$$

$$\bar{P}_8^{(1)} = 14\,\beta\,\alpha\,x[\left(\alpha^2-\beta^2\right)^6 x^6+60\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4+$$
$$+45\,\left(25\,\alpha^4+78\,\beta^2\alpha^2+25\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2+720\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+22\,\beta^2\alpha^2+5\,\alpha^4\right)]$$
$$\bar{Q}_8^{(1)} = \alpha\,x^2[\left(\alpha^2-\beta^2\right)^6 x^6+7\,\left(5\,\alpha^2+19\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4+$$
$$+105\,\left(3\,\beta^2+5\,\alpha^2\right)\left(15\,\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^2 x^2+85995\,\beta^2\alpha^4+1575\,\alpha^6+186165\,\beta^4\alpha^2+48825\,\beta^6]$$
$$\bar{R}_8^{(1)} = -\beta\,x^2[\left(\alpha^2-\beta^2\right)^6 x^6+7\,\left(19\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4+$$
$$+105\,\left(5\,\beta^2+3\,\alpha^2\right)\left(\beta^2+15\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^2 x^2+85995\,\beta^4\alpha^2+1575\,\beta^6+186165\,\beta^2\alpha^4+48825\,\alpha^6]$$
$$\bar{S}_8^{(1)} = -7\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6+5\,\left(5\,\alpha^4+38\,\beta^2\alpha^2+5\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4+$$
$$+45\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+118\,\beta^2\alpha^2+5\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2 x^2+$$
$$+19260\,\beta^2\alpha^6+225\,\beta^8+19260\,\beta^6\alpha^2+53190\,\beta^4\alpha^4+225\,\alpha^8]$$
$$\bar{U}_8^{(1)} = 10080\,\left(\alpha^2+\beta^2\right)\left(5\,\beta^4+22\,\beta^2\alpha^2+5\,\alpha^4\right)\alpha\,\beta$$
$$\bar{V}_8^{(1)} = 1575\,\alpha^8+134820\,\beta^2\alpha^6+372330\,\beta^4\alpha^4+134820\,\beta^6\alpha^2+1575\,\beta^8$$

$$P_{10}^{(0)} = 9\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^8 x^8-7\,\left(9\,\alpha^4+46\,\beta^2\alpha^2+9\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6 x^6+$$
$$+105\,\left(\alpha^2+\beta^2\right)\left(21\,\beta^4+214\,\beta^2\alpha^2+21\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^4 x^4-$$
$$-315\,\left(\beta^2+15\,\alpha^2\right)\left(15\,\beta^2+\alpha^2\right)\left(7\,\beta^4+18\,\beta^2\alpha^2+7\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2 x^2+$$
$$+315\,\left(\alpha^2+\beta^2\right)\left(315\,\beta^8+6548\,\alpha^2\beta^6+19042\,\alpha^4\beta^4+6548\,\alpha^6\beta^2+315\,\alpha^8\right)]$$
$$Q_{10}^{(0)} = -\beta\,x^2[\left(\alpha^2-\beta^2\right)^8 x^8-9\,\left(9\,\beta^2+23\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6 x^6+$$
$$+63\,\left(223\,\alpha^4+482\,\alpha^2\beta^2+63\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4-$$
$$-945\,\left(391\,\alpha^6+2139\,\beta^2\alpha^4+1461\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2 x^2+$$
$$+25957260\,\beta^2\alpha^6+2299185\,\alpha^8+46590390\,\alpha^4\beta^4+17157420\,\alpha^2\beta^6+893025\,\beta^8]$$
$$R_{10}^{(0)} = \alpha\,x^2[\left(\alpha^2-\beta^2\right)^8 x^8-9\,\left(9\,\alpha^2+23\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6+$$
$$+63\,\left(63\,\alpha^4+482\,\alpha^2\beta^2+223\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4-$$
$$-945\,\left(105\,\alpha^6+1461\,\beta^2\alpha^4+2139\,\beta^4\alpha^2+391\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2 x^2+$$
$$+17157420\,\beta^2\alpha^6+893025\,\alpha^8+46590390\,\alpha^4\beta^4+25957260\,\alpha^2\beta^6+2299185\,\beta^8]$$
$$S_{10}^{(0)} = 18\,\beta\,\alpha\,x[\left(\alpha^2-\beta^2\right)^8 x^8-112\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6+$$
$$+35\,\left(143\,\alpha^4+482\,\alpha^2\beta^2+143\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4-$$
$$-2520\,\left(\alpha^2+\beta^2\right)\left(31\,\beta^4+194\,\alpha^2\beta^2+31\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2 x^2+$$
$$+2395260\,\beta^2\alpha^6+177345\,\alpha^8+5176710\,\alpha^4\beta^4+2395260\,\alpha^2\beta^6+177345\,\beta^8]$$
$$U_{10}^{(0)} = 2835\,\left(\alpha^2+\beta^2\right)\left(315\,\beta^8+6548\,\alpha^2\beta^6+19042\,\alpha^4\beta^4+6548\,\beta^2\alpha^6+315\,\alpha^8\right)$$
$$V_{10}^{(0)} = -5670\,\beta\,\alpha\,\left(563\,\beta^8+563\,\alpha^8+7604\,\beta^2\alpha^6+16434\,\alpha^4\beta^4+7604\,\alpha^2\beta^6\right)$$

$$P_{10}^{(1)} = 18\,x\alpha\,\beta[\left(\alpha^2-\beta^2\right)^8 x^8-112\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6+$$
$$+105\,\left(49\,\alpha^4+158\,\beta^2\alpha^2+49\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4-$$

$$-2520\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+186\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+2485980\,\beta^2\alpha^6+275625\,\beta^8+2485980\,\beta^6\alpha^2+4798710\,\beta^4\alpha^4+275625\,\alpha^8]$$

$$Q_{10}^{(1)}=-\alpha\,x^2[\left(\alpha^2-\beta^2\right)^8x^8-9\,\left(7\,\alpha^2+25\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+63\,\left(35\,\alpha^4+474\,\beta^2\alpha^2+259\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4-$$

$$-945\,\left(35\,\alpha^6+1223\,\beta^2\alpha^4+2313\,\beta^4\alpha^2+525\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2(\beta+\alpha)^2x^2+$$

$$+9718380\,\beta^2\alpha^6+4862025\,\beta^8+35029260\,\beta^6\alpha^2+43188390\,\beta^4\alpha^4+99225\,\alpha^8]$$

$$R_{10}^{(1)}=\beta\,x^2[\left(\alpha^2-\beta^2\right)^8x^8-9\,\left(7\,\beta^2+25\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+63\,\left(259\,\alpha^4+474\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4-$$

$$-945\,\left(525\,\alpha^6+2313\,\beta^2\alpha^4+1223\,\beta^4\alpha^2+35\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+35029260\,\beta^2\alpha^6+99225\,\beta^8+9718380\,\beta^6\alpha^2+43188390\,\beta^4\alpha^4+4862025\,\alpha^8]$$

$$S_{10}^{(1)}=9\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8-7\,\left(7\,\beta^2+\alpha^2\right)\left(\beta^2+7\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+35\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+698\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^4x^4-$$

$$-315\,\left(35\,\alpha^8+1748\,\beta^2\alpha^6+4626\,\beta^4\alpha^4+1748\,\beta^6\alpha^2+35\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+315\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^8+5108\,\beta^6\alpha^2+22482\,\beta^4\alpha^4+5108\,\beta^2\alpha^6+35\,\alpha^8\right)]$$

$$U_{10}^{(1)}=5670\,\left(875\,\alpha^8+7892\,\beta^2\alpha^6+15234\,\beta^4\alpha^4+7892\,\beta^6\alpha^2+875\,\beta^8\right)\beta\,\alpha$$

$$V_{10}^{(1)}=-2835\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^8+5108\,\beta^6\alpha^2+22482\,\beta^4\alpha^4+5108\,\beta^2\alpha^6+35\,\alpha^8\right)$$

$$\bar{P}_{10}^{(0)}=-9\,x[\left(\beta^2+\alpha^2\right)\left(\alpha^2-\beta^2\right)^8x^8+7\,\left(9\,\alpha^4+46\,\beta^2\alpha^2+9\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+105\,\left(\beta^2+\alpha^2\right)\left(21\,\beta^4+214\,\beta^2\alpha^2+21\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+315\,\left(\beta^2+15\,\alpha^2\right)\left(15\,\beta^2+\alpha^2\right)\left(7\,\beta^4+18\,\beta^2\alpha^2+7\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+315\,\left(\beta^2+\alpha^2\right)\left(315\,\beta^8+6548\,\beta^6\alpha^2+19042\,\beta^4\alpha^4+6548\,\beta^2\alpha^6+315\,\alpha^8\right)]$$

$$\bar{Q}_{10}^{(0)}=-\beta\,x^2[\left(\alpha^2-\beta^2\right)^8x^8+9\,\left(23\,\alpha^2+9\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+63\,\left(223\,\alpha^4+482\,\beta^2\alpha^2+63\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+945\,\left(391\,\alpha^6+2139\,\alpha^4\beta^2+1461\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+46590390\,\beta^4\alpha^4+25957260\,\beta^2\alpha^6+17157420\,\beta^6\alpha^2+2299185\,\alpha^8+893025\,\beta^8]$$

$$\bar{R}_{10}^{(0)}=\alpha\,x^2[\left(\alpha^2-\beta^2\right)^8x^8+9\,\left(9\,\alpha^2+23\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+63\,\left(63\,\alpha^4+482\,\beta^2\alpha^2+223\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+945\,\left(105\,\alpha^6+1461\,\beta^2\alpha^4+2139\,\beta^4\alpha^2+391\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+2299185\,\beta^8+893025\,\alpha^8+46590390\,\beta^4\alpha^4+25957260\,\beta^6\alpha^2+17157420\,\beta^2\alpha^6]$$

$$\bar{S}_{10}^{(0)}=18\,\beta\,\alpha\,x[\left(\alpha^2-\beta^2\right)^8x^8+112\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+35\,\left(143\,\alpha^4+482\,\beta^2\alpha^2+143\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+2520\,\left(\alpha^2+\beta^2\right)\left(31\,\beta^4+194\,\beta^2\alpha^2+31\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+177345\,\beta^8+177345\,\alpha^8+5176710\,\beta^4\alpha^4+2395260\,\beta^6\alpha^2+2395260\,\beta^2\alpha^6]$$

$$\bar{U}_{10}^{(0)}=-2835\,\left(\alpha^2+\beta^2\right)\left(315\,\beta^8+6548\,\beta^6\alpha^2+19042\,\beta^4\alpha^4+6548\,\beta^2\alpha^6+315\,\alpha^8\right)$$

$$\bar{V}_{10}^{(0)}=-5670\,\beta\,\alpha\,\left(16434\,\beta^4\alpha^4+7604\,\beta^2\alpha^6+563\,\alpha^8+7604\,\beta^6\alpha^2+563\,\beta^8\right)$$

$$\bar{P}_{10}^{(1)} = 18\,\alpha\,\beta\,x[\left(\alpha^2-\beta^2\right)^8 x^8 + 112\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6 +$$
$$+105\,\left(49\,\alpha^4+158\,\beta^2\alpha^2+49\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+2520\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+186\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^2 (\beta+\alpha)^2 x^2 +$$
$$+2485980\,\beta^2\alpha^6+275625\,\beta^8+2485980\,\beta^6\alpha^2+4798710\,\beta^4\alpha^4+275625\,\alpha^8]$$
$$\bar{Q}_{10}^{(1)} = \alpha\,x^2[\left(\alpha^2-\beta^2\right)^8 x^8 + 9\,\left(7\,\alpha^2+25\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6 (\beta+\alpha)^6 x^6 +$$
$$+63\,\left(35\,\alpha^4+474\,\beta^2\alpha^2+259\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+945\,\left(35\,\alpha^6+1223\,\beta^2\alpha^4+2313\,\beta^4\alpha^2+525\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2 x^2 +$$
$$+9718380\,\beta^2\alpha^6+4862025\,\beta^8+35029260\,\beta^6\alpha^2+43188390\,\beta^4\alpha^4+99225\,\alpha^8]$$
$$\bar{R}_{10}^{(1)} = -\beta\,x^2[\left(\alpha^2-\beta^2\right)^8 x^8 + 9\,\left(7\,\beta^2+25\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6 (\beta+\alpha)^6 x^6 +$$
$$+63\,\left(259\,\alpha^4+474\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+945\,\left(525\,\alpha^6+2313\,\beta^2\alpha^4+1223\,\beta^4\alpha^2+35\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2 x^2 +$$
$$+35029260\,\beta^2\alpha^6+99225\,\beta^8+9718380\,\beta^6\alpha^2+43188390\,\beta^4\alpha^4+4862025\,\alpha^8]$$
$$\bar{S}_{10}^{(1)} = -9\,x[\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^8 x^8 + 7\,\left(7\,\beta^2+\alpha^2\right)\left(\beta^2+7\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6 x^6 +$$
$$+35\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^4+698\,\beta^2\alpha^2+35\,\alpha^4\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+315\,\left(35\,\alpha^8+1748\,\beta^2\alpha^6+4626\,\beta^4\alpha^4+1748\,\beta^6\alpha^2+35\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2 x^2 +$$
$$+315\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^8+5108\,\beta^6\alpha^2+22482\,\beta^4\alpha^4+5108\,\beta^2\alpha^6+35\,\alpha^8\right)]$$
$$\bar{U}_{10}^{(1)} = 5670\,\left(875\,\alpha^8+7892\,\beta^2\alpha^6+15234\,\beta^4\alpha^4+7892\,\beta^6\alpha^2+875\,\beta^8\right)\beta\,\alpha$$
$$\bar{V}_{10}^{(1)} = 2835\,\left(\alpha^2+\beta^2\right)\left(35\,\beta^8+5108\,\beta^6\alpha^2+22482\,\beta^4\alpha^4+5108\,\beta^2\alpha^6+35\,\alpha^8\right)$$

2.2.4. Integrals of the type $\int x^{2n+1}\cdot Z_0(\alpha x)\cdot Z_1(\beta x)\,dx$

Holds (with $0<\beta<\alpha$ and $\beta/\alpha=\gamma<1$)

$$\int J_0(\alpha x)\,J_0(\beta x)\,dx=\frac{1}{\alpha}\,\Theta_0\left(ax;\frac{b}{a}\right)\;,\qquad \int J_1(\alpha x)\,J_1(\beta x)\,dx=\frac{1}{\alpha}\,\Theta_1\left(ax;\frac{b}{a}\right)\;,$$

$$\int I_0(\alpha x)\,I_0(\beta x)\,dx=\frac{1}{\alpha}\,\Omega_0\left(ax;\frac{b}{a}\right)\;,\qquad \int I_1(\alpha x)\,I_1(\beta x)\,dx=\frac{1}{\alpha}\,\Omega_1\left(ax;\frac{b}{a}\right)\;.$$

(Θ_ν and Ω_ν as definded on pages 264 and 266. In these integrals both Bessel functions are of the same order, so one can suppose $\beta<\alpha$. This relation is not presumed for the product $Z_0(\alpha x)\cdot Z_1(\beta x)$, that means for the following integrals.)

$$\int\frac{J_0(\alpha x)\,J_1(\beta x)\,dx}{x}=-J_0(\alpha x)\,J_1(\beta x)+\beta\int J_0(\alpha x)\,J_0(\beta x)\,dx-\alpha\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int\frac{I_0(\alpha x)\,I_1(\beta x)\,dx}{x}=-I_0(\alpha x)\,I_1(\beta x)+\beta\int I_0(\alpha x)\,I_1(\beta x)\,dx+\alpha\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x\,J_0(\alpha x)\,J_1(\beta x)\,dx=$$

$$=\frac{x}{\alpha^2-\beta^2}\left[\beta\,J_0(\alpha x)\,J_0(\beta x)+\alpha\,J_1(\alpha x)\,J_1(\beta x)\right]-\frac{\beta}{\alpha^2-\beta^2}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{\alpha}{\alpha^2-\beta^2}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x\,I_0(\alpha x)\,I_1(\beta x)\,dx=$$

$$=\frac{x}{\alpha^2-\beta^2}\left[\alpha\,I_1(\alpha x)\,I_1(\beta x)-\beta\,I_0(\alpha x)\,I_0(\beta x)\right]+\frac{\beta}{\alpha^2-\beta^2}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{\alpha}{\alpha^2-\beta^2}\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

Let

$$\int x^n\cdot J_0(\alpha x)\,J_1(\beta x)\,dx=\frac{P_n(x)}{(\alpha^2-\beta^2)^n}\,J_0(\alpha x)\,J_0(\beta x)+\frac{Q_n(x)}{(\alpha^2-\beta^2)^{n-1}}\,J_0(\alpha x)\,J_1(\beta x)+$$

$$+\frac{R_n(x)}{(\alpha^2-\beta^2)^{n-1}}\,J_1(\alpha x)\,J_0(\beta x)+\frac{S_n(x)}{(\alpha^2-\beta^2)^n}\,J_1(\alpha x)\,J_1(\beta x)+$$

$$+\frac{U_n}{(\alpha^2-\beta^2)^n}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{V_n}{(\alpha^2-\beta^2)^n}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

and

$$\int x^n\cdot I_0(\alpha x)\,I_1(\beta x)\,dx=\frac{\bar P_n(x)}{(\alpha^2-\beta^2)^n}\,I_0(\alpha x)\,I_0(\beta x)+\frac{\bar Q_n(x)}{(\alpha^2-\beta^2)^{n-1}}\,I_0(\alpha x)\,I_1(\beta x)+$$

$$+\frac{\bar R_n(x)}{(\alpha^2-\beta^2)^{n-1}}\,I_1(\alpha x)\,I_0(\beta x)+\frac{\bar S_n(x)}{(\alpha^2-\beta^2)^n}\,I_1(\alpha x)\,I_1(\beta x)+$$

$$+\frac{\bar U_n}{(\alpha^2-\beta^2)^n}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{\bar V_n}{(\alpha^2-\beta^2)^n}\int I_1(\alpha x)\,I_1(\beta x)\,dx\;,$$

then holds

$$P_3^{(0)}=\beta\,x[\left(\alpha^2-\beta^2\right)^2x^2-3\,\beta^2-5\,\alpha^2]\;,\quad Q_3^{(0)}=x^2[\alpha^2+3\,\beta^2]$$

$$R_3^{(0)}=-4\,\alpha\,\beta\,x^2\;,\quad S_3^{(0)}=\alpha\,x[\left(\alpha^2-\beta^2\right)^2x^2-\alpha^2-7\,\beta^2]$$

$$U_3^{(0)}=-\beta\left(5\,\alpha^2+3\,\beta^2\right)\;,\quad V_3^{(0)}=\alpha\,\left(7\,\beta^2+\alpha^2\right)$$

$$\bar P_3^{(0)}=-\beta\,x[\left(\alpha^2-\beta^2\right)^2x^2+5\,\alpha^2+3\,\beta^2]\;,\quad \bar Q_3^{(0)}=-[\alpha^2+3\,\beta^2]x^2$$

$$\bar R_3^{(0)}=4\,\alpha\,\beta\,x^2\;,\quad \bar S_3^{(0)}=\alpha\,x\left(\alpha^2-\beta^2\right)^2x^2+7\,\beta^2+\alpha^2$$

$$\bar U_3^{(0)}=-\beta\left(5\,\alpha^2+3\,\beta^2\right)\;,\quad \bar V_3^{(0)}=-\alpha\,\left(7\,\beta^2+\alpha^2\right)$$

$$P_5^{(0)} = \beta\, x[\left(\alpha^2-\beta^2\right)^4 x^4 - 3\,\left(11\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 3\,\left(3\,\beta^2+13\,\alpha^2\right)\left(3\,\alpha^2+5\,\beta^2\right)]$$

$$Q_5^{(0)} = x^2[\left(3\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 - 3\,\left(15\,\beta^2+\alpha^2\right)\left(\beta^2+3\,\alpha^2\right)]$$

$$R_5^{(0)} = -4\,\beta\,\alpha\, x^2[2\,\left(\alpha^2-\beta^2\right)^2 x^2 - 27\,\alpha^2 - 21\,\beta^2]$$

$$S_5^{(0)} = \alpha\, x[\left(\alpha^2-\beta^2\right)^4 x^4 - 3\,\left(3\,\alpha^2+13\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 9\,\alpha^4 + 246\,\alpha^2\beta^2 + 129\,\beta^4]$$

$$U_5^{(0)} = 3\beta\,\left(3\,\beta^2+13\,\alpha^2\right)\left(3\,\alpha^2+5\,\beta^2\right)\;,\quad V_5^{(0)} = -3\,\alpha\,\left(43\,\beta^4+82\,\alpha^2\beta^2+3\,\alpha^4\right)$$

$$\bar{P}_5^{(0)} = -\beta\, x[\left(\alpha^2-\beta^2\right)^4 x^4 + 3\,\left(11\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 3\,\left(3\,\beta^2+13\,\alpha^2\right)\left(3\,\alpha^2+5\,\beta^2\right)]$$

$$\bar{Q}_5^{(0)} = -x^2[\left(3\,\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 3\,\left(15\,\beta^2+\alpha^2\right)\left(\beta^2+3\,\alpha^2\right)]$$

$$\bar{R}_5^{(0)} = 4\,\beta\,\alpha\, x^2[2\,\left(\alpha^2-\beta^2\right)^2 x^2 + 27\,\alpha^2 + 21\,\beta^2]$$

$$\bar{S}_5^{(0)} = \alpha\, x[\left(\alpha^2-\beta^2\right)^4 x^4 + 3\,\left(3\,\alpha^2+13\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 9\,\alpha^4 + 129\,\beta^4 + 246\,\beta^2\alpha^2]$$

$$\bar{U}_5^{(0)} = -3\,\left(3\,\beta^2+13\,\alpha^2\right)\left(3\,\alpha^2+5\,\beta^2\right)\beta$$

$$\bar{V}_5^{(0)} = -3\,\alpha\,\left(43\,\beta^4+82\,\beta^2\alpha^2+3\,\alpha^4\right)$$

$$P_7^{(0)} = \beta\, x[\left(\alpha^2-\beta^2\right)^6 x^6 - 5\,\left(17\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+15\,\left(115\,\alpha^4+234\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 - 1575\,\beta^6 - 5625\,\alpha^6 - 15915\,\beta^4\alpha^2 - 22965\,\beta^2\alpha^4]$$

$$Q_7^{(0)} = x^2[\left(5\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 - 5\,\left(15\,\alpha^4+142\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 +$$
$$+1575\,\beta^6 + 8805\,\beta^2\alpha^4 + 225\,\alpha^6 + 12435\,\beta^4\alpha^2]$$

$$R_7^{(0)} = -4\,\beta\,\alpha\, x^2[3\,\left(\alpha^2-\beta^2\right)^4 x^4 - 5\,\left(25\,\alpha^2+23\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 1350\,\alpha^4 + 3540\,\beta^2\alpha^2 + 870\,\beta^4]$$

$$S_7^{(0)} = \alpha\, x[\left(\alpha^2-\beta^2\right)^6 x^6 - 5\,\left(19\,\beta^2+5\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+15\,\left(15\,\alpha^4+242\,\beta^2\alpha^2+127\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 - 14205\,\beta^2\alpha^4 - 5055\,\beta^6 - 26595\,\beta^4\alpha^2 - 225\,\alpha^6]$$

$$U_7^{(0)} = -15\,\left(375\,\alpha^6+1531\,\beta^2\alpha^4+1061\,\beta^4\alpha^2+105\,\beta^6\right)\beta$$

$$V_7^{(0)} = 15\,\alpha\,\left(947\,\beta^2\alpha^4+15\,\alpha^6+337\,\beta^6+1773\,\beta^4\alpha^2\right)$$

$$\bar{P}_7^{(0)} = -\beta\, x[\left(\alpha^2-\beta^2\right)^6 x^6 + 5\,\left(17\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+15\,\left(115\,\alpha^4+234\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 1575\,\beta^6 + 5625\,\alpha^6 + 15915\,\beta^4\alpha^2 + 22965\,\beta^2\alpha^4]$$

$$\bar{Q}_7^{(0)} = -x^2[\left(5\,\alpha^2+7\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 + 5\,\left(15\,\alpha^4+142\,\beta^2\alpha^2+35\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 +$$
$$+225\,\alpha^6 + 8805\,\beta^2\alpha^4 + 12435\,\beta^4\alpha^2 + 1575\,\beta^6]$$

$$\bar{R}_7^{(0)} = 4\,\beta\,\alpha\, x^2[3\,\left(\alpha^2-\beta^2\right)^4 x^4 + 5\,\left(25\,\alpha^2+23\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 1350\,\alpha^4 + 3540\,\beta^2\alpha^2 + 870\,\beta^4]$$

$$\bar{S}_7^{(0)} = \alpha\, x[\left(\alpha^2-\beta^2\right)^6 x^6 + 5\,\left(19\,\beta^2+5\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 +$$
$$+15\,\left(15\,\alpha^4+242\,\beta^2\alpha^2+127\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2 x^2 + 14205\,\beta^2\alpha^4 + 225\,\alpha^6 + 26595\,\beta^4\alpha^2 + 5055\,\beta^6]$$

$$\bar{U}_7^{(0)} = -15\,\left(375\,\alpha^6+1531\,\beta^2\alpha^4+1061\,\beta^4\alpha^2+105\,\beta^6\right)\beta$$

$$\bar{V}_7^{(0)} = -15\,\alpha\,\left(947\,\beta^2\alpha^4+15\,\alpha^6+337\,\beta^6+1773\,\beta^4\alpha^2\right)$$

$$P_9^{(0)} = \beta\, x[\left(\alpha^2-\beta^2\right)^8 x^8 - 7\,\left(23\,\alpha^2+9\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6 x^6 + 105\,\left(\beta^2+7\,\alpha^2\right)\left(21\,\beta^2+11\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^4 x^4 -$$

$$-315\ \left(455\,\alpha^6+2139\,\beta^2\alpha^4+1397\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+3262140\,\beta^2\alpha^6+452025\,\alpha^8+4798710\,\beta^4\alpha^4+1709820\,\beta^6\alpha^2+99225\,\beta^8]$$
$$Q_9^{(0)}=x^2[\left(9\,\beta^2+7\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^6x^6-7\ \left(35\,\alpha^4+286\,\beta^2\alpha^2+63\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\ \left(35\,\alpha^6+867\,\beta^2\alpha^4+1041\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-2749950\,\beta^4\alpha^4-99225\,\beta^8-835380\,\beta^2\alpha^6-11025\,\alpha^8-1465380\,\beta^6\alpha^2]$$
$$R_9^{(0)}=-4\,\beta\,\alpha\,x^2[4\ \left(\alpha^2-\beta^2\right)^6x^6-7\ \left(49\,\alpha^2+47\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\ \left(105\,\alpha^4+318\,\beta^2\alpha^2+89\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2-630\ \left(\beta^2+7\,\alpha^2\right)\left(97\,\beta^4+134\,\beta^2\alpha^2+25\,\alpha^4\right)]$$
$$S_9^{(0)}=\alpha\,x[\left(\alpha^2-\beta^2\right)^8x^8-7\ \left(7\,\alpha^2+25\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+35\ \left(35\,\alpha^4+482\,\beta^2\alpha^2+251\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4-$$
$$-315\ \left(35\,\alpha^6+1287\,\beta^2\alpha^4+2313\,\beta^4\alpha^2+461\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+343665\,\beta^8+5176710\,\beta^4\alpha^4+3514140\,\beta^6\alpha^2+1276380\,\beta^2\alpha^6+11025\,\alpha^8]$$
$$U_9^{(0)}=315\,\beta\ \left(15234\,\beta^4\alpha^4+5428\,\beta^6\alpha^2+315\,\beta^8+1435\,\alpha^8+10356\,\beta^2\alpha^6\right)$$
$$V_9^{(0)}=-315\ \left(35\,\alpha^8+4052\,\beta^2\alpha^6+16434\,\beta^4\alpha^4+11156\,\beta^6\alpha^2+1091\,\beta^8\right)\alpha$$

$$\bar{P}_9^{(0)}=-\beta\,x[\left(\alpha^2-\beta^2\right)^8x^8+7\ \left(23\,\alpha^2+9\,\beta^2\right)\left(\alpha^2-\beta^2)\right)^6x^6+105\ \left(\beta^2+7\,\alpha^2\right)\left(21\,\beta^2+11\,\alpha^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+315\ \left(455\,\alpha^6+2139\,\beta^2\alpha^4+1397\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+1709820\,\beta^6\alpha^2+3262140\,\beta^2\alpha^6+99225\,\beta^8+452025\,\alpha^8+4798710\,\beta^4\alpha^4]$$
$$\bar{Q}_9^{(0)}=-x^2[\left(7\,\alpha^2+9\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+7\ \left(35\,\alpha^4+286\,\beta^2\alpha^2+63\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\ \left(35\,\alpha^6+867\,\beta^2\alpha^4+1041\,\beta^4\alpha^2+105\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+99225\,\beta^8+1465380\,\beta^6\alpha^2+2749950\,\beta^4\alpha^4+835380\,\beta^2\alpha^6+11025\,\alpha^8]$$
$$\bar{R}_9^{(0)}=4\,\beta\,\alpha\,x^2[4\ \left(\alpha^2-\beta^2\right)^6x^6+7\ \left(49\,\alpha^2+47\,\beta^2\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+105\ \left(105\,\alpha^4+318\,\beta^2\alpha^2+89\,\beta^4\right)\left(\alpha^2-\beta^2\right)^2x^2+630\ \left(\beta^2+7\,\alpha^2\right)\left(97\,\beta^4+134\,\beta^2\alpha^2+25\,\alpha^4\right)]$$
$$\bar{S}_9^{(0)}=\alpha\,x[\left(\alpha^2-\beta^2\right)^8x^8+7\ \left(7\,\alpha^2+25\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+35\ \left(35\,\alpha^4+482\,\beta^2\alpha^2+251\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+315\ \left(35\,\alpha^6+1287\,\beta^2\alpha^4+2313\,\beta^4\alpha^2+461\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$
$$+3514140\,\beta^6\alpha^2+5176710\,\beta^4\alpha^4+1276380\,\beta^2\alpha^6+343665\,\beta^8+11025\,\alpha^8]$$
$$\bar{U}_9^{(0)}=-315\ \left(1435\,\alpha^8+10356\,\beta^2\alpha^6+15234\,\beta^4\alpha^4+5428\,\beta^6\alpha^2+315\,\beta^8\right)\beta$$
$$\bar{V}_9^{(0)}=-315\,\alpha\ \left(35\,\alpha^8+4052\,\beta^2\alpha^6+16434\,\beta^4\alpha^4+11156\,\beta^6\alpha^2+1091\,\beta^8\right)$$

$$P_{11}^{(0)}=\beta\,x[\left(\alpha^2-\beta^2\right)^{10}x^{10}-9\ \left(29\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8+$$
$$+63\ \left(387\,\alpha^4+794\,\beta^2\alpha^2+99\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6-$$
$$-945\ \left(1113\,\alpha^6+5429\,\beta^2\alpha^4+3467\,\beta^4\alpha^2+231\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4+$$
$$+2835\ \left(6195\,\alpha^8+52196\,\beta^2\alpha^6+78882\,\beta^4\alpha^4+25412\,\beta^6\alpha^2+1155\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2-$$
$$-54474525\,\alpha^{10}-1575415170\,\beta^4\alpha^6-1200752910\,\beta^6\alpha^4-9823275\,\beta^{10}-258673905\,\beta^8\alpha^2-616751415\,\beta^2\alpha^8]$$
$$Q_{11}^{(0)}=x^2[\left(9\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8-9\ \left(63\,\alpha^4+478\,\beta^2\alpha^2+99\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6+$$
$$+63\ \left(315\,\alpha^6+6719\,\beta^2\alpha^4+7633\,\beta^4\alpha^2+693\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4-$$
$$-945\ \left(315\,\alpha^8+15308\,\beta^2\alpha^6+44346\,\beta^4\alpha^4+20796\,\beta^6\alpha^2+1155\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+893025\,\alpha^{10}+674225370\,\beta^4\alpha^6+232489845\,\beta^8\alpha^2+112756455\,\beta^2\alpha^8+827757630\,\beta^6\alpha^4+9823275\,\beta^{10}]$$

$$R_{11}^{(0)}=-4\,\beta\,\alpha\,x^2[5\,\left(\alpha^2-\beta^2\right)^8x^8-9\,\left(81\,\alpha^2+79\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+252\,\left(189\,\alpha^4+598\,\beta^2\alpha^2+173\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4-$$

$$-1890\,\left(735\,\alpha^6+4611\,\beta^2\alpha^4+4317\,\beta^4\alpha^2+577\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+125998740\,\beta^2\alpha^6+6546015\,\beta^8+93248820\,\beta^6\alpha^2+225297450\,\beta^4\alpha^4+13395375\,\alpha^8]$$

$$S_{11}^{(0)}=\alpha\,x[\left(\alpha^2-\beta^2\right)^{10}x^{10}-9\,\left(9\,\alpha^2+31\,\beta^2\right)\left(\alpha^2-\beta^2)\right)^8x^8+$$

$$+63\,\left(63\,\alpha^4+802\,\beta^2\alpha^2+415\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6-$$

$$-315\,\left(315\,\alpha^6+9743\,\beta^2\alpha^4+17201\,\beta^4\alpha^2+3461\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+2835\,\left(315\,\alpha^8+21188\,\beta^2\alpha^6+81234\,\beta^4\alpha^4+55332\,\beta^6\alpha^2+5771\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2-$$

$$-166337955\,\beta^2\alpha^8-1728947430\,\beta^6\alpha^4-893025\,\alpha^{10}-605485125\,\beta^8\alpha^2-1178220330\,\beta^4\alpha^6-36007335\,\beta^{10}]$$

$$U_{11}^{(0)}=-2835\,\left(19215\,\alpha^{10}+217549\,\beta^2\alpha^8+555702\,\beta^4\alpha^6+423546\,\beta^6\alpha^4+91243\,\beta^8\alpha^2+3465\,\beta^{10}\right)\beta$$

$$V_{11}^{(0)}=2835\,\alpha\,\left(315\,\alpha^{10}+58673\,\beta^2\alpha^8+415598\,\beta^4\alpha^6+609858\,\beta^6\alpha^4+213575\,\beta^8\alpha^2+12701\,\beta^{10}\right)$$

$$\bar{P}_{11}^{(0)}=-\beta\,x[\left(\alpha^2-\beta^2\right)^{10}x^{10}+9\,\left(29\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8+$$

$$+63\,\left(387\,\alpha^4+794\,\beta^2\alpha^2+99\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+945\,\left(1113\,\alpha^6+5429\,\beta^2\alpha^4+3467\,\beta^4\alpha^2+231\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+2835\,\left(6195\,\alpha^8+52196\,\beta^2\alpha^6+78882\,\beta^4\alpha^4+25412\,\beta^6\alpha^2+1155\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+258673905\,\beta^8\alpha^2+54474525\,\alpha^{10}+1575415170\,\beta^4\alpha^6+616751415\,\beta^2\alpha^8+1200752910\,\beta^6\alpha^4+9823275\,\beta^{10}]$$

$$\bar{Q}_{11}^{(0)}=-x^2[\left(9\,\alpha^2+11\,\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8+9\,\left(63\,\alpha^4+478\,\beta^2\alpha^2+99\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+63\,\left(315\,\alpha^6+6719\,\beta^2\alpha^4+7633\,\beta^4\alpha^2+693\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+945\,\left(315\,\alpha^8+15308\,\beta^2\alpha^6+44346\,\beta^4\alpha^4+20796\,\beta^6\alpha^2+1155\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+893025\,\alpha^{10}+674225370\,\beta^4\alpha^6+232489845\,\beta^8\alpha^2+9823275\,\beta^{10}+112756455\,\beta^2\alpha^8+827757630\,\beta^6\alpha^4]$$

$$\bar{R}_{11}^{(0)}=4\,\beta\,\alpha\,x^2[\left(\alpha^2-\beta^2\right)^8x^8+9\,\left(81\,\alpha^2+79\,\beta^2\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+252\,\left(189\,\alpha^4+598\,\beta^2\alpha^2+173\,\beta^4\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+1890\,\left(735\,\alpha^6+4611\,\beta^2\alpha^4+4317\,\beta^4\alpha^2+577\,\beta^6\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+93248820\,\beta^6\alpha^2+225297450\,\beta^4\alpha^4+125998740\,\beta^2\alpha^6+6546015\,\beta^8+13395375\,\alpha^8]$$

$$\bar{S}_{11}^{(0)}=\alpha\,x[\left(\alpha^2-\beta^2\right)^{10}x^{10}+9\,\left(9\,\alpha^2+31\,\beta^2\right)\left(\alpha^2-\beta^2\right)^8x^8+63\,\left(63\,\alpha^4+802\,\beta^2\alpha^2+415\,\beta^4\right)\left(\alpha^2-\beta^2\right)^6x^6+$$

$$+315\,\left(315\,\alpha^6+9743\,\beta^2\alpha^4+17201\,\beta^4\alpha^2+3461\,\beta^6\right)\left(\alpha^2-\beta^2\right)^4x^4+$$

$$+2835\,\left(315\,\alpha^8+21188\,\beta^2\alpha^6+81234\,\beta^4\alpha^4+55332\,\beta^6\alpha^2+5771\,\beta^8\right)\left(\alpha^2-\beta^2\right)^2x^2+$$

$$+893025\,\alpha^{10}+1178220330\,\beta^4\alpha^6+166337955\,\beta^2\alpha^8+1728947430\,\beta^6\alpha^4+36007335\,\beta^{10}+605485125\,\beta^8\alpha^2]$$

$$\bar{U}_{11}^{(0)}=-2835\,\left(19215\,\alpha^{10}+217549\,\beta^2\alpha^8+555702\,\beta^4\alpha^6+423546\,\beta^6\alpha^4+91243\,\beta^8\alpha^2+3465\,\beta^{10}\right)\beta$$

$$\bar{V}_{11}^{(0)}=-2835\,\alpha\,\left(315\,\alpha^{10}+58673\,\beta^2\alpha^8+415598\,\beta^4\alpha^6+609858\,\beta^6\alpha^4+213575\,\beta^8\alpha^2+12701\,\beta^{10}\right)$$

2.2.5. Integrals of the type $\int x^{2n+1} \cdot J_0(\alpha x) \cdot I_0(\beta x)\, dx$

$$\int x J_0(\alpha x) \cdot I_0(\beta x)\, dx = \frac{\beta\, x}{\alpha^2+\beta^2}\, J_0(\alpha x) \cdot I_1(\beta x) + \frac{\alpha\, x}{\alpha^2+\beta^2}\, J_1(\alpha x) \cdot I_0(\beta x)$$

$$\int x^3 J_0(\alpha x) \cdot I_0(\beta x)\, dx =$$

$$= 2\,\frac{\left(\alpha^2-\beta^2\right) x^2}{\left(\alpha^2+\beta^2\right)^2}\, J_0(\alpha x) \cdot I_0(\beta x) + \left(\frac{\beta\, x^3}{\alpha^2+\beta^2} - 4\,\frac{\beta\,\left(-\beta^2+\alpha^2\right) x}{\left(\alpha^2+\beta^2\right)^3}\right) J_0(\alpha x) \cdot I_1(\beta x) +$$

$$+ \left(\frac{\alpha\, x^3}{\alpha^2+\beta^2} - 4\,\frac{\alpha\,\left(\alpha^2-\beta^2\right) x}{\left(\alpha^2+\beta^2\right)^3}\right) J_1(\alpha x) \cdot I_0(\beta x) - 4\,\frac{\beta\,\alpha\, x^2}{\left(\alpha^2+\beta^2\right)^2}\, J_1(\alpha x) \cdot I_1(\beta x)$$

With

$$\int x^n \cdot J_0(\alpha x) \cdot I_0(\beta x)\, dx =$$

$$= P_n(x) J_0(\alpha x) \cdot I_0(\beta x) + Q_n(x) J_0(\alpha x) \cdot I_1(\beta x) + R_n(x) J_1(\alpha x) \cdot I_0(\beta x) + S_n(x) J_1(\alpha x) \cdot I_1(\beta x)$$

holds

$$P_5(x) = 4\,\frac{\left(-\beta^2+\alpha^2\right) x^4}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(\alpha^4-4\,\alpha^2\beta^2+\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$Q_5(x) = \frac{\beta\, x^5}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(2\,\alpha^2-\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\beta\,\left(\alpha^4-4\,\alpha^2\beta^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^5}$$

$$R_5(x) = \frac{\alpha\, x^5}{\alpha^2+\beta^2} - 16\,\frac{\alpha\,\left(\alpha^2-2\,\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\alpha\,\left(\alpha^4-4\,\alpha^2\beta^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^5}$$

$$S_5(x) = -8\,\frac{\alpha\,\beta\, x^4}{\left(\alpha^2+\beta^2\right)^2} + 96\,\frac{\alpha\,\beta\,\left(-\beta^2+\alpha^2\right) x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$P_7(x) = 6\,\frac{\left(-\beta^2+\alpha^2\right) x^6}{\left(\alpha^2+\beta^2\right)^2} - 48\,\frac{\left(3\,\alpha^4-14\,\alpha^2\beta^2+3\,\beta^4\right) x^4}{\left(\alpha^2+\beta^2\right)^4} + 1152\,\frac{\left(\alpha^6-9\,\alpha^4\beta^2+9\,\alpha^2\beta^4-\beta^6\right) x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$Q_7(x) = \frac{\beta\, x^7}{\alpha^2+\beta^2} - 12\,\frac{\beta\,\left(7\,\alpha^2-3\,\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\left(8\,\alpha^4-19\,\alpha^2\beta^2+3\,\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-2304\,\frac{\beta\,\left(\alpha^6-9\,\alpha^4\beta^2+9\,\alpha^2\beta^4-\beta^6\right) x}{\left(\alpha^2+\beta^2\right)^7}$$

$$R_7(x) = \frac{\alpha\, x^7}{\alpha^2+\beta^2} - 12\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\alpha\,\left(3\,\alpha^4-19\,\alpha^2\beta^2+8\,\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-2304\,\frac{\alpha\,\left(\alpha^6-9\,\alpha^4\beta^2+9\,\alpha^2\beta^4-\beta^6\right) x}{\left(\alpha^2+\beta^2\right)^7}$$

$$S_7(x) = -12\,\frac{\beta\,\alpha\, x^6}{\left(\alpha^2+\beta^2\right)^2} + 480\,\frac{\beta\,\alpha\,\left(-\beta^2+\alpha^2\right) x^4}{\left(\alpha^2+\beta^2\right)^4} - 384\,\frac{\beta\,\alpha\,\left(11\,\alpha^4-38\,\alpha^2\beta^2+11\,\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$P_9(x) = 8\,\frac{\left(-\beta^2+\alpha^2\right) x^8}{\left(\alpha^2+\beta^2\right)^2} - 384\,\frac{\left(\alpha^4-5\,\alpha^2\beta^2+\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^4} + 3072\,\frac{\left(3\,\alpha^6-32\,\alpha^4\beta^2+32\,\alpha^2\beta^4-3\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-73728\,\frac{\left(\alpha^8-16\,\beta^2\alpha^6+36\,\beta^4\alpha^4-16\,\beta^6\alpha^2+\beta^8\right) x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$Q_9(x) = \frac{\beta\, x^9}{\alpha^2+\beta^2} - 32\,\frac{\beta\,\left(5\,\alpha^2-2\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^3} + 768\,\frac{\beta\,\left(10\,\alpha^4-22\,\alpha^2\beta^2+3\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-6144\,\frac{\beta\,\left(19\,\alpha^6-108\,\alpha^4\beta^2+77\,\alpha^2\beta^4-6\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+147456\,\frac{\beta\,\left(\alpha^8-16\,\beta^2\alpha^6+36\,\beta^4\alpha^4-16\,\beta^6\alpha^2+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$R_9(x)=\frac{\alpha\,x^9}{\alpha^2+\beta^2}-32\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}+768\,\frac{\alpha\,\left(3\,\alpha^4-22\,\alpha^2\beta^2+10\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6-77\,\alpha^4\beta^2+108\,\alpha^2\beta^4-19\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+147456\,\frac{\alpha\,\left(\alpha^8-16\,\beta^2\alpha^6+36\,\beta^4\alpha^4-16\,\beta^6\alpha^2+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$S_9(x)=-16\,\frac{\alpha\,\beta\,x^8}{\left(\alpha^2+\beta^2\right)^2}+1344\,\frac{\alpha\,\beta\,\left(-\beta^2+\alpha^2\right)x^6}{\left(\alpha^2+\beta^2\right)^4}-3072\,\frac{\alpha\,\beta\,\left(13\,\alpha^4-44\,\alpha^2\beta^2+13\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+61440\,\frac{\alpha\,\beta\,\left(5\,\alpha^6-37\,\alpha^4\beta^2+37\,\alpha^2\beta^4-5\,\beta^6\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$P_{11}(x)=10\,\frac{\left(-\beta^2+\alpha^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-160\,\frac{\left(5\,\alpha^4-26\,\alpha^2\beta^2+5\,\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+7680\,\frac{\left(5\,\alpha^6-58\,\alpha^4\beta^2+58\,\alpha^2\beta^4-5\,\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-61440\,\frac{\left(15\,\alpha^8-283\,\beta^2\alpha^6+664\,\beta^4\alpha^4-283\,\beta^6\alpha^2+15\,\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+7372800\,\frac{\left(\alpha^{10}-25\,\alpha^8\beta^2+100\,\alpha^6\beta^4-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8-\beta^{10}\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$Q_{11}(x)=\frac{\beta\,x^{11}}{\alpha^2+\beta^2}-20\,\frac{\beta\,\left(13\,\alpha^2-5\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+640\,\frac{\beta\,\left(37\,\alpha^4-79\,\alpha^2\beta^2+10\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-15360\,\frac{\beta\,\left(62\,\alpha^6-332\,\alpha^4\beta^2+221\,\alpha^2\beta^4-15\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+122880\,\frac{\beta\,\left(107\,\alpha^8-1119\,\beta^2\alpha^6+1881\,\beta^4\alpha^4-643\,\beta^6\alpha^2+30\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-14745600\,\frac{\beta\,\left(\alpha^{10}-25\,\alpha^8\beta^2+100\,\alpha^6\beta^4-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8-\beta^{10}\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$R_{11}(x)=\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}-20\,\frac{\alpha\,\left(5\,\alpha^2-13\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+640\,\frac{\alpha\,\left(10\,\alpha^4-79\,\alpha^2\beta^2+37\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-15360\,\frac{\alpha\,\left(15\,\alpha^6-221\,\alpha^4\beta^2+332\,\alpha^2\beta^4-62\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+122880\,\frac{\alpha\,\left(30\,\alpha^8-643\,\beta^2\alpha^6+1881\,\beta^4\alpha^4-1119\,\beta^6\alpha^2+107\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-14745600\,\frac{\alpha\,\left(\alpha^{10}-25\,\alpha^8\beta^2+100\,\alpha^6\beta^4-100\,\alpha^4\beta^6+25\,\alpha^2\beta^8-\beta^{10}\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$S_{11}(x)=-20\,\frac{\beta\,\alpha\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}+2880\,\frac{\beta\,\alpha\,\left(-\beta^2+\alpha^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}-3840\,\frac{\beta\,\alpha\,\left(47\,\alpha^4-158\,\alpha^2\beta^2+47\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}+$$

$$+430080\,\frac{\beta\,\alpha\,\left(11\,\alpha^6-79\,\alpha^4\beta^2+79\,\alpha^2\beta^4-11\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}-$$

$$-245760\,\frac{\beta\,\alpha\,\left(137\,\alpha^8-1762\,\beta^2\alpha^6+3762\,\beta^4\alpha^4-1762\,\beta^6\alpha^2+137\,\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

2.2.6. Integrals of the type $\int x^{2n} \cdot J_0(\alpha x) \cdot I_1(\beta x)\, dx$

$$\int x^2 J_0(\alpha x) \cdot I_1(\beta x)\, dx =$$

$$= \frac{\beta\, x^2}{\alpha^2+\beta^2}\, J_0(\alpha x) \cdot I_0(\beta x) - 2\, \frac{\beta^2 x}{\left(\alpha^2+\beta^2\right)^2}\, J_0(\alpha x) \cdot I_1(\beta x) - 2\, \frac{\beta \alpha x}{\left(\alpha^2+\beta^2\right)^2}\, J_1(\alpha x) \cdot I_0(\beta x) + \frac{\alpha\, x^2}{\alpha^2+\beta^2}\, J_1(\alpha x) \cdot I_1(\beta x)$$

$$\int x^4 J_0(\alpha x) \cdot I_1(\beta x)\, dx = \left(\frac{\beta\, x^4}{\alpha^2+\beta^2} - 8\, \frac{\beta\, \left(-\beta^2+2\,\alpha^2\right) x^2}{\left(\alpha^2+\beta^2\right)^3} \right) J_0(\alpha x) \cdot I_0(\beta x) +$$

$$+ \left(2\, \frac{\left(\alpha^2-2\,\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\, \frac{\beta^2 \left(-\beta^2+2\,\alpha^2\right) x}{\left(\alpha^2+\beta^2\right)^4} \right) J_0(\alpha x) \cdot I_1(\beta x) +$$

$$+ \left(-6\, \frac{\beta\, \alpha\, x^3}{\left(\alpha^2+\beta^2\right)^2} + 16\, \frac{\beta\, \alpha\, \left(2\,\alpha^2-\beta^2\right) x}{\left(\alpha^2+\beta^2\right)^4} \right) J_1(\alpha x) \cdot I_0(\beta x) + \left(\frac{\alpha\, x^4}{\alpha^2+\beta^2} - 4\, \frac{\alpha\, \left(\alpha^2-5\,\beta^2\right) x^2}{\left(\alpha^2+\beta^2\right)^3} \right) J_1(\alpha x) \cdot I_1(\beta x)$$

With

$$\int x^n \cdot J_0(\alpha x) \cdot I_1(\beta x)\, dx =$$

$$= P_n(x) J_0(\alpha x) \cdot I_0(\beta x) + Q_n(x) J_0(\alpha x) \cdot I_1(\beta x) + R_n(x) J_1(\alpha x) \cdot I_0(\beta x) + S_n(x) J_1(\alpha x) \cdot I_1(\beta x)$$

holds

$$P_6(x) = \frac{\beta\, x^6}{\alpha^2+\beta^2} - 8\, \frac{\beta\, \left(7\,\alpha^2-3\,\beta^2\right) x^4}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\beta\, \left(3\,\alpha^4-6\,\beta^2\alpha^2+\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6(x) = 2\, \frac{\left(2\,\alpha^2-3\,\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\, \frac{\left(\alpha^4-11\,\beta^2\alpha^2+3\,\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\, \frac{\beta^2 \left(3\,\alpha^4-6\,\beta^2\alpha^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6(x) = -10\, \frac{\alpha\, \beta\, x^5}{\left(\alpha^2+\beta^2\right)^2} + 32\, \frac{\alpha\, \beta\, \left(8\,\alpha^2-7\,\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^4} - 384\, \frac{\alpha\, \beta\, \left(3\,\alpha^4-6\,\beta^2\alpha^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6(x) = \frac{\alpha\, x^6}{\alpha^2+\beta^2} - 16\, \frac{\alpha\, \left(\alpha^2-4\,\beta^2\right) x^4}{\left(\alpha^2+\beta^2\right)^3} + 64\, \frac{\alpha\, \left(\alpha^4-19\,\beta^2\alpha^2+10\,\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_8(x) = \frac{\beta\, x^8}{\alpha^2+\beta^2} - 24\, \frac{\beta\, \left(5\,\alpha^2-2\,\beta^2\right) x^6}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\beta\, \left(21\,\alpha^4-43\,\beta^2\alpha^2+6\,\beta^4\right) x^4}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-9216\, \frac{\beta\, \left(4\,\alpha^6-18\,\alpha^4\beta^2+12\,\alpha^2\beta^4-\beta^6\right) x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8(x) = 2\, \frac{\left(3\,\alpha^2-4\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^2} - 48\, \frac{\left(3\,\alpha^4-26\,\beta^2\alpha^2+6\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^4} + 384\, \frac{\left(3\,\alpha^6-86\,\alpha^4\beta^2+109\,\alpha^2\beta^4-12\,\beta^6\right) x^3}{\left(\alpha^2+\beta^2\right)^6} +$$

$$+18432\, \frac{\beta^2 \left(4\,\alpha^6-18\,\alpha^4\beta^2+12\,\alpha^2\beta^4-\beta^6\right) x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8(x) = -14\, \frac{\alpha\, \beta\, x^7}{\left(\alpha^2+\beta^2\right)^2} + 48\, \frac{\alpha\, \beta\, \left(18\,\alpha^2-17\,\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^4} - 1920\, \frac{\alpha\, \beta\, \left(9\,\alpha^4-26\,\beta^2\alpha^2+7\,\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^6} +$$

$$+18432\, \frac{\alpha\, \beta\, \left(4\,\alpha^6-18\,\alpha^4\beta^2+12\,\alpha^2\beta^4-\beta^6\right) x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8(x) = \frac{\alpha\, x^8}{\alpha^2+\beta^2} - 12\, \frac{\alpha\, \left(3\,\alpha^2-11\,\beta^2\right) x^6}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\alpha\, \left(3\,\alpha^4-44\,\beta^2\alpha^2+23\,\beta^4\right) x^4}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-768\, \frac{\alpha\, \left(3\,\alpha^6-131\,\alpha^4\beta^2+239\,\alpha^2\beta^4-47\,\beta^6\right) x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_{10}(x) = \frac{\beta\, x^{10}}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(13\,\alpha^2 - 5\,\beta^2\right) x^8}{\left(\alpha^2+\beta^2\right)^3} + 768\,\frac{\beta\,\left(19\,\alpha^4 - 39\,\beta^2\alpha^2 + 5\,\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-6144\,\frac{\beta\,\left(69\,\alpha^6 - 332\,\alpha^4\beta^2 + 214\,\alpha^2\beta^4 - 15\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^7} +$$

$$+737280\,\frac{\beta\,\left(5\,\alpha^8 - 40\,\alpha^6\beta^2 + 60\,\alpha^4\beta^4 - 20\,\alpha^2\beta^6 + \beta^8\right) x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$Q_{10}(x) = 2\,\frac{\left(4\,\alpha^2 - 5\,\beta^2\right) x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\left(6\,\alpha^4 - 47\,\beta^2\alpha^2 + 10\,\beta^4\right) x^7}{\left(\alpha^2+\beta^2\right)^4} + 1536\,\frac{\left(6\,\alpha^6 - 136\,\alpha^4\beta^2 + 158\,\alpha^2\beta^4 - 15\,\beta^6\right) x^5}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-12288\,\frac{\left(6\,\alpha^8 - 337\,\alpha^6\beta^2 + 1018\,\alpha^4\beta^4 - 499\,\alpha^2\beta^6 + 30\,\beta^8\right) x^3}{\left(\alpha^2+\beta^2\right)^8} -$$

$$-1474560\,\frac{\beta^2\left(5\,\alpha^8 - 40\,\alpha^6\beta^2 + 60\,\alpha^4\beta^4 - 20\,\alpha^2\beta^6 + \beta^8\right) x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$R_{10}(x) = -18\,\frac{\alpha\,\beta\, x^9}{\left(\alpha^2+\beta^2\right)^2} + 64\,\frac{\alpha\,\beta\,\left(32\,\alpha^2 - 31\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^4} - 10752\,\frac{\alpha\,\beta\,\left(9\,\alpha^4 - 28\,\beta^2\alpha^2 + 8\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^6} +$$

$$+12288\,\frac{\alpha\,\beta\,\left(144\,\alpha^6 - 863\,\alpha^4\beta^2 + 782\,\alpha^2\beta^4 - 101\,\beta^6\right) x^3}{\left(\alpha^2+\beta^2\right)^8} -$$

$$-1474560\,\frac{\alpha\,\beta\,\left(5\,\alpha^8 - 40\,\alpha^6\beta^2 + 60\,\alpha^4\beta^4 - 20\,\alpha^2\beta^6 + \beta^8\right) x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$S_{10}(x) = \frac{\alpha\, x^{10}}{\alpha^2+\beta^2} - 32\,\frac{\alpha\,\left(2\,\alpha^2 - 7\,\beta^2\right) x^8}{\left(\alpha^2+\beta^2\right)^3} + 384\,\frac{\alpha\,\left(6\,\alpha^4 - 79\,\beta^2\alpha^2 + 41\,\beta^4\right) x^6}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-6144\,\frac{\alpha\,\left(6\,\alpha^6 - 199\,\alpha^4\beta^2 + 354\,\alpha^2\beta^4 - 71\,\beta^6\right) x^4}{\left(\alpha^2+\beta^2\right)^7} +$$

$$+24576\,\frac{\alpha\,\left(6\,\alpha^8 - 481\,\alpha^6\beta^2 + 1881\,\alpha^4\beta^4 - 1281\,\alpha^2\beta^6 + 131\,\beta^8\right) x^2}{\left(\alpha^2+\beta^2\right)^9}$$

2.2.7. Integrals of the type $\int x^{2n} \cdot J_1(\alpha x) \cdot I_0(\beta x)\,dx$

$$\int x^2 J_1(\alpha x) \cdot I_0(\beta x)\,dx =$$

$$= -\frac{\alpha\,x^2}{\alpha^2+\beta^2}\,J_0(\alpha x)\cdot I_0(\beta x) + 2\,\frac{\alpha\,x\beta}{\left(\alpha^2+\beta^2\right)^2}\,J_0(\alpha x)\cdot I_1(\beta x) + 2\,\frac{\alpha^2 x}{\left(\alpha^2+\beta^2\right)^2}\,J_1(\alpha x)\cdot I_0(\beta x) + \frac{\beta\,x^2}{\alpha^2+\beta^2}\,J_1(\alpha x)\cdot I_1(\beta x)$$

$$\int x^4 J_1(\alpha x)\cdot I_0(\beta x)\,dx = \left(-\frac{\alpha\,x^4}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(-2\,\beta^2+\alpha^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}\right) J_0(\alpha x)\cdot I_0(\beta x) +$$

$$+\left(6\,\frac{\alpha\,\beta\,x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha\,\beta\,\left(-2\,\beta^2+\alpha^2\right)x}{\left(\alpha^2+\beta^2\right)^4}\right) J_0(\alpha x)\cdot I_1(\beta x) +$$

$$+\left(2\,\frac{\left(2\,\alpha^2-\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^2} - 16\,\frac{\alpha^2\left(\alpha^2-2\,\beta^2\right)x}{\left(\alpha^2+\beta^2\right)^4}\right) J_1(\alpha x)\cdot I_0(\beta x) + \left(\frac{\beta\,x^4}{\alpha^2+\beta^2} - 4\,\frac{\beta\,\left(5\,\alpha^2-\beta^2\right)x^2}{\left(\alpha^2+\beta^2\right)^3}\right) J_1(\alpha x)\cdot I_1(\beta x)$$

With

$$\int x^n \cdot J_1(\alpha x)\cdot I_0(\beta x)\,dx =$$

$$= P_n(x) J_0(\alpha x)\cdot I_0(\beta x) + Q_n(x) J_0(\alpha x)\cdot I_1(\beta x) + R_n(x) J_1(\alpha x)\cdot I_0(\beta x) + S_n(x) J_1(\alpha x)\cdot I_1(\beta x)$$

holds

$$P_6(x) = -\frac{\alpha\,x^6}{\alpha^2+\beta^2} + 8\,\frac{\alpha\,\left(3\,\alpha^2-7\,\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(\alpha^4-6\,\alpha^2\beta^2+3\,\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$Q_6(x) = 10\,\frac{\alpha\,\beta\,x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\alpha\,\beta\,\left(7\,\alpha^2-8\,\beta^2\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\alpha\,\beta\,\left(\alpha^4-6\,\alpha^2\beta^2+3\,\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$R_6(x) = 2\,\frac{\left(3\,\alpha^2-2\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^2} - 32\,\frac{\left(3\,\alpha^4-11\,\alpha^2\beta^2+\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\alpha^2\left(\alpha^4-6\,\alpha^2\beta^2+3\,\beta^4\right)x}{\left(\alpha^2+\beta^2\right)^6}$$

$$S_6(x) = \frac{\beta\,x^6}{\alpha^2+\beta^2} - 16\,\frac{\beta\,\left(4\,\alpha^2-\beta^2\right)x^4}{\left(\alpha^2+\beta^2\right)^3} + 64\,\frac{\beta\,\left(10\,\alpha^4-19\,\alpha^2\beta^2+\beta^4\right)x^2}{\left(\alpha^2+\beta^2\right)^5}$$

$$P_8(x) = -\frac{\alpha\,x^8}{\alpha^2+\beta^2} + 24\,\frac{\alpha\,\left(2\,\alpha^2-5\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3} - 192\,\frac{\alpha\,\left(6\,\alpha^4-43\,\alpha^2\beta^2+21\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+9216\,\frac{\alpha\,\left(\alpha^6-12\,\alpha^4\beta^2+18\,\alpha^2\beta^4-4\,\beta^6\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$Q_8(x) = 14\,\frac{\alpha\,\beta\,x^7}{\left(\alpha^2+\beta^2\right)^2} - 48\,\frac{\alpha\,\beta\,\left(17\,\alpha^2-18\,\beta^2\right)x^5}{\left(\alpha^2+\beta^2\right)^4} + 1920\,\frac{\alpha\,\beta\,\left(7\,\alpha^4-26\,\alpha^2\beta^2+9\,\beta^4\right)x^3}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-18432\,\frac{\alpha\,\beta\,\left(\alpha^6-12\,\alpha^4\beta^2+18\,\alpha^2\beta^4-4\,\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$R_8(x) = 2\,\frac{\left(4\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^2} - 48\,\frac{\left(6\,\alpha^4-26\,\alpha^2\beta^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^4} + 384\,\frac{\left(12\,\alpha^6-109\,\alpha^4\beta^2+86\,\alpha^2\beta^4-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-18432\,\frac{\alpha^2\left(\alpha^6-12\,\alpha^4\beta^2+18\,\alpha^2\beta^4-4\,\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^8}$$

$$S_8(x) = \frac{\beta\,x^8}{\alpha^2+\beta^2} - 12\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^6}{\left(\alpha^2+\beta^2\right)^3} + 192\,\frac{\beta\,\left(23\,\alpha^4-44\,\alpha^2\beta^2+3\,\beta^4\right)x^4}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-768\,\frac{\beta\,\left(47\,\alpha^6-239\,\alpha^4\beta^2+131\,\alpha^2\beta^4-3\,\beta^6\right)x^2}{\left(\alpha^2+\beta^2\right)^7}$$

$$P_{10}(x) = -\frac{\alpha\, x^{10}}{\alpha^2+\beta^2} + 16\,\frac{\alpha\,\left(5\,\alpha^2 - 13\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3} - 768\,\frac{\alpha\,\left(5\,\alpha^4 - 39\,\alpha^2\beta^2 + 19\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+\,6144\,\frac{\alpha\,\left(15\,\alpha^6 - 214\,\alpha^4\beta^2 + 332\,\alpha^2\beta^4 - 69\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7} -$$

$$-737280\,\frac{\alpha\,\left(\alpha^8 - 20\,\alpha^6\beta^2 + 60\,\beta^4\alpha^4 - 40\,\beta^6\alpha^2 + 5\,\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

$$Q_{10}(x) = 18\,\frac{\alpha\,\beta\, x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\alpha\,\beta\,\left(31\,\alpha^2 - 32\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^4} + 10752\,\frac{\alpha\,\beta\,\left(8\,\alpha^4 - 28\,\alpha^2\beta^2 + 9\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-12288\,\frac{\alpha\,\beta\,\left(101\,\alpha^6 - 782\,\alpha^4\beta^2 + 863\,\alpha^2\beta^4 - 144\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^8} +$$

$$+\,1474560\,\frac{\alpha\,\beta\,\left(\alpha^8 - 20\,\alpha^6\beta^2 + 60\,\beta^4\alpha^4 - 40\,\beta^6\alpha^2 + 5\,\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$R_{10}(x) = 2\,\frac{\left(5\,\alpha^2 - 4\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^2} - 64\,\frac{\left(10\,\alpha^4 - 47\,\alpha^2\beta^2 + 6\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^4} +$$

$$+\,1536\,\frac{\left(15\,\alpha^6 - 158\,\alpha^4\beta^2 + 136\,\alpha^2\beta^4 - 6\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-12288\,\frac{\left(30\,\alpha^8 - 499\,\alpha^6\beta^2 + 1018\,\beta^4\alpha^4 - 337\,\beta^6\alpha^2 + 6\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^8} +$$

$$+\,1474560\,\frac{\alpha^2\left(\alpha^8 - 20\,\alpha^6\beta^2 + 60\,\beta^4\alpha^4 - 40\,\beta^6\alpha^2 + 5\,\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$S_{10}(x) = \frac{\beta\, x^{10}}{\alpha^2+\beta^2} - 32\,\frac{\beta\,\left(7\,\alpha^2 - 2\,\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^3} + 384\,\frac{\beta\,\left(41\,\alpha^4 - 79\,\alpha^2\beta^2 + 6\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-\,6144\,\frac{\beta\,\left(71\,\alpha^6 - 354\,\alpha^4\beta^2 + 199\,\alpha^2\beta^4 - 6\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^7} +$$

$$+\,24576\,\frac{\beta\,\left(131\,\alpha^8 - 1281\,\alpha^6\beta^2 + 1881\,\beta^4\alpha^4 - 481\,\beta^6\alpha^2 + 6\,\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^9}$$

2.2.8. Integrals of the type $\int x^{2n+1} \cdot J_1(\alpha x) \cdot I_1(\beta x)\, dx$

$$\int x J_1(\alpha x) \cdot I_1(\beta x)\, dx = -\frac{\alpha\, x}{\alpha^2+\beta^2}\, J_0(\alpha x) \cdot I_1(\beta x) + \frac{\beta\, x}{\alpha^2+\beta^2}\, J_1(\alpha x) \cdot I_0(\beta x)$$

$$\int x^3 J_1(\alpha x) \cdot I_1(\beta x)\, dx = 4\, \frac{\beta\, \alpha\, x^2}{\left(\alpha^2+\beta^2\right)^2}\, J_0(\alpha x) \cdot I_0(\beta x) +$$

$$+\left(-\frac{\alpha\, x^3}{\alpha^2+\beta^2} - 8\, \frac{\beta^2 \alpha\, x}{\left(\alpha^2+\beta^2\right)^3}\right) J_0(\alpha x) \cdot I_1(\beta x) + \left(\frac{\beta\, x^3}{\alpha^2+\beta^2} - 8\, \frac{\alpha^2 \beta\, x}{\left(\alpha^2+\beta^2\right)^3}\right) J_1(\alpha x) \cdot I_0(\beta x) +$$

$$+2\, \frac{\left(\alpha^2-\beta^2\right) x^2}{\left(\alpha^2+\beta^2\right)^2}\, J_1(\alpha x) \cdot I_1(\beta x)$$

With

$$\int x^n \cdot J_1(\alpha x) \cdot I_1(\beta x)\, dx =$$

$$= P_n(x) J_0(\alpha x) \cdot I_0(\beta x) + Q_n(x) J_0(\alpha x) \cdot I_1(\beta x) + R_n(x) J_1(\alpha x) \cdot I_0(\beta x) + S_n(x) J_1(\alpha x) \cdot I_1(\beta x)$$

holds

$$P_5(x) = 8\, \frac{\alpha\, \beta\, x^4}{\left(\alpha^2+\beta^2\right)^2} - 96\, \frac{\alpha\, \beta\, \left(\alpha^2-\beta^2\right) x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$Q_5(x) = -\frac{\alpha\, x^5}{\alpha^2+\beta^2} + 8\, \frac{\alpha\, \left(\alpha^2-5\,\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\beta^2 \alpha\, \left(\alpha^2-\beta^2\right) x}{\left(\alpha^2+\beta^2\right)^5}$$

$$R_5(x) = \frac{\beta\, x^5}{\alpha^2+\beta^2} - 8\, \frac{\beta\, \left(5\,\alpha^2-\beta^2\right) x^3}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\alpha^2 \beta\, \left(\alpha^2-\beta^2\right) x}{\left(\alpha^2+\beta^2\right)^5}$$

$$S_5(x) = 4\, \frac{\left(\alpha^2-\beta^2\right) x^4}{\left(\alpha^2+\beta^2\right)^2} - 16\, \frac{\left(\alpha^4-10\,\beta^2\alpha^2+\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^4}$$

$$P_7(x) = 12\, \frac{\alpha\, \beta\, x^6}{\left(\alpha^2+\beta^2\right)^2} - 480\, \frac{\alpha\, \beta\, \left(\alpha^2-\beta^2\right) x^4}{\left(\alpha^2+\beta^2\right)^4} + 4608\, \frac{\alpha\, \beta\, \left(\alpha^4-3\,\beta^2\alpha^2+\beta^4\right) x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$Q_7(x) = -\frac{\alpha\, x^7}{\alpha^2+\beta^2} + 24\, \frac{\alpha\, \left(\alpha^2-4\,\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^3} - 192\, \frac{\alpha\, \left(\alpha^4-18\,\beta^2\alpha^2+11\,\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-\,9216\, \frac{\beta^2 \alpha\, \left(\alpha^4-3\,\beta^2\alpha^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^7}$$

$$R_7(x) = \frac{\beta\, x^7}{\alpha^2+\beta^2} - 24\, \frac{\beta\, \left(4\,\alpha^2-\beta^2\right) x^5}{\left(\alpha^2+\beta^2\right)^3} + 192\, \frac{\beta\, \left(11\,\alpha^4-18\,\beta^2\alpha^2+\beta^4\right) x^3}{\left(\alpha^2+\beta^2\right)^5} -$$

$$-\,9216\, \frac{\alpha^2 \beta\, \left(\alpha^4-3\,\beta^2\alpha^2+\beta^4\right) x}{\left(\alpha^2+\beta^2\right)^7}$$

$$S_7(x) = 6\, \frac{\left(\alpha^2-\beta^2\right) x^6}{\left(\alpha^2+\beta^2\right)^2} - 96\, \frac{\left(\alpha^4-8\,\beta^2\alpha^2+\beta^4\right) x^4}{\left(\alpha^2+\beta^2\right)^4} + 384\, \frac{\left(\alpha^6-29\,\alpha^4\beta^2+29\,\beta^4\alpha^2-\beta^6\right) x^2}{\left(\alpha^2+\beta^2\right)^6}$$

$$P_9(x) = 16\, \frac{\alpha\, \beta\, x^8}{\left(\alpha^2+\beta^2\right)^2} - 1344\, \frac{\alpha\, \beta\, \left(\alpha^2-\beta^2\right) x^6}{\left(\alpha^2+\beta^2\right)^4} + 1536\, \frac{\alpha\, \beta\, \left(27\,\alpha^4-86\,\beta^2\alpha^2+27\,\beta^4\right) x^4}{\left(\alpha^2+\beta^2\right)^6} -$$

$$-\,368640\, \frac{\alpha\, \beta\, \left(\alpha^6-6\,\alpha^4\beta^2+6\,\beta^4\alpha^2-\beta^6\right) x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$Q_9(x) = -\frac{\alpha\, x^9}{\alpha^2+\beta^2} + 16\, \frac{\alpha\, \left(3\,\alpha^2-11\,\beta^2\right) x^7}{\left(\alpha^2+\beta^2\right)^3} - 384\, \frac{\alpha\, \left(3\,\alpha^4-43\,\beta^2\alpha^2+24\,\beta^4\right) x^5}{\left(\alpha^2+\beta^2\right)^5} +$$

$$+\,3072\,\frac{\alpha\,\left(3\,\alpha^6-121\,\alpha^4\beta^2+239\,\beta^4\alpha^2-57\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+737280\,\frac{\alpha\,\beta^2\,\left(\alpha^6-6\,\alpha^4\beta^2+6\,\beta^4\alpha^2-\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$R_9(x)=\frac{\beta\,x^9}{\alpha^2+\beta^2}-16\,\frac{\beta\,\left(11\,\alpha^2-3\,\beta^2\right)x^7}{\left(\alpha^2+\beta^2\right)^3}+384\,\frac{\beta\,\left(24\,\alpha^4-43\,\beta^2\alpha^2+3\,\beta^4\right)x^5}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-\,3072\,\frac{\beta\,\left(57\,\alpha^6-239\,\alpha^4\beta^2+121\,\beta^4\alpha^2-3\,\beta^6\right)x^3}{\left(\alpha^2+\beta^2\right)^7}+737280\,\frac{\alpha^2\beta\,\left(\alpha^6-6\,\alpha^4\beta^2+6\,\beta^4\alpha^2-\beta^6\right)x}{\left(\alpha^2+\beta^2\right)^9}$$

$$S_9(x)=8\,\frac{\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^2}-96\,\frac{\left(3\,\alpha^4-22\,\beta^2\alpha^2+3\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^4}+1536\,\frac{\left(3\,\alpha^6-67\,\alpha^4\beta^2+67\,\beta^4\alpha^2-3\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-\,6144\,\frac{\left(3\,\alpha^8-178\,\alpha^6\beta^2+478\,\alpha^4\beta^4-178\,\alpha^2\beta^6+3\,\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^8}$$

$$P_{11}(x)=20\,\frac{\alpha\,\beta\,x^{10}}{\left(\alpha^2+\beta^2\right)^2}-2880\,\frac{\alpha\,\beta\,\left(\alpha^2-\beta^2\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+46080\,\frac{\alpha\,\beta\,\left(4\,\alpha^4-13\,\beta^2\alpha^2+4\,\beta^4\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-\,2580480\,\frac{\alpha\,\beta\,\left(2\,\alpha^6-13\,\alpha^4\beta^2+13\,\beta^4\alpha^2-2\,\beta^6\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+\,44236800\,\frac{\alpha\,\beta\,\left(\alpha^8-10\,\alpha^6\beta^2+20\,\alpha^4\beta^4-10\,\alpha^2\beta^6+\beta^8\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

$$Q_{11}(x)=-\frac{\alpha\,x^{11}}{\alpha^2+\beta^2}+40\,\frac{\alpha\,\left(2\,\alpha^2-7\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}-3840\,\frac{\alpha\,\left(\alpha^4-13\,\beta^2\alpha^2+7\,\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}+$$

$$+\,92160\,\frac{\alpha\,\left(\alpha^6-32\,\alpha^4\beta^2+59\,\beta^4\alpha^2-13\,\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}-$$

$$-\,737280\,\frac{\alpha\,\left(\alpha^8-73\,\alpha^6\beta^2+300\,\alpha^4\beta^4-227\,\alpha^2\beta^6+29\,\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-\,88473600\,\frac{\beta^2\alpha\,\left(\alpha^8-10\,\alpha^6\beta^2+20\,\alpha^4\beta^4-10\,\alpha^2\beta^6+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$R_{11}(x)=\frac{\beta\,x^{11}}{\alpha^2+\beta^2}-40\,\frac{\beta\,\left(7\,\alpha^2-2\,\beta^2\right)x^9}{\left(\alpha^2+\beta^2\right)^3}+3840\,\frac{\beta\,\left(7\,\alpha^4-13\,\beta^2\alpha^2+\beta^4\right)x^7}{\left(\alpha^2+\beta^2\right)^5}-$$

$$-\,92160\,\frac{\beta\,\left(13\,\alpha^6-59\,\alpha^4\beta^2+32\,\beta^4\alpha^2-\beta^6\right)x^5}{\left(\alpha^2+\beta^2\right)^7}+$$

$$+\,737280\,\frac{\beta\,\left(29\,\alpha^8-227\,\alpha^6\beta^2+300\,\alpha^4\beta^4-73\,\alpha^2\beta^6+\beta^8\right)x^3}{\left(\alpha^2+\beta^2\right)^9}-$$

$$-\,88473600\,\frac{\alpha^2\beta\,\left(\alpha^8-10\,\alpha^6\beta^2+20\,\alpha^4\beta^4-10\,\alpha^2\beta^6+\beta^8\right)x}{\left(\alpha^2+\beta^2\right)^{11}}$$

$$S_{11}(x)=10\,\frac{\left(\alpha^2-\beta^2\right)x^{10}}{\left(\alpha^2+\beta^2\right)^2}-640\,\frac{\left(\alpha^4-7\,\beta^2\alpha^2+\beta^4\right)x^8}{\left(\alpha^2+\beta^2\right)^4}+23040\,\frac{\left(\alpha^6-20\,\alpha^4\beta^2+20\,\beta^4\alpha^2-\beta^6\right)x^6}{\left(\alpha^2+\beta^2\right)^6}-$$

$$-\,368640\,\frac{\left(\alpha^8-45\,\alpha^6\beta^2+118\,\alpha^4\beta^4-45\,\alpha^2\beta^6+\beta^8\right)x^4}{\left(\alpha^2+\beta^2\right)^8}+$$

$$+\,1474560\,\frac{\left(\alpha^{10}-102\,\alpha^8\beta^2+527\,\beta^4\alpha^6-527\,\beta^6\alpha^4+102\,\beta^8\alpha^2-\beta^{10}\right)x^2}{\left(\alpha^2+\beta^2\right)^{10}}$$

2.2.9. Integrals of the type $\int x^{2n+1} \cdot J_\nu(\alpha x)\, Y_\nu(\beta x)\, dx$

Compare with 2.2.1. .

$$\int x\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{x}{\alpha^2 - \beta^2}\left[\alpha\, J_1(\alpha x)\, Y_0(\beta x) - \beta J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{x}{\alpha^2 - \beta^2}\left[\beta\, J_1(\alpha x)\, Y_0(\beta x) - \alpha J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x^3\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{2\,(\alpha^2+\beta^2)\,x^2}{(\alpha^2-\beta^2)^2}\, J_0(\alpha x)\, Y_0(\beta x) + \frac{4\,\alpha\,\beta x^2}{(\alpha^2-\beta^2)^2}\, J_1(\alpha x)\, Y_1(\beta x) +$$

$$+ \frac{(\alpha^2-\beta^2)^2 x^3 - 4\left(\alpha^2+\beta^2\right) x}{(\alpha^2-\beta^2)^3}\left[\alpha\, J_1(\alpha x)\, Y_0(\beta x) - \beta\, J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x^3\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{4\,\alpha\,\beta\, x^2}{(\alpha^2-\beta^2)^2}\, J_0(\alpha x)\, Y_0(\beta x) + \frac{2\,(\alpha^2+\beta^2)\,x^2}{(\alpha^2-\beta^2)^2}\, J_1(\alpha x)\, Y_1(\beta x) -$$

$$- \frac{\alpha[(\alpha^2-\beta^2)^2\, x^3 - 8\beta^2\, x]}{(\alpha^2-\beta^2)^3}\, J_0(\alpha x)\, Y_1(\beta x) + \frac{\beta[(\alpha^2-\beta^2)^2\, x^3 - 8\alpha^2\, x]}{(\alpha^2-\beta^2)^3}\, J_1(\alpha x)\, Y_0(\beta x)$$

$$\int x^5\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{4\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^4 - 32(\,\alpha^4 + 4\,\alpha^2\beta^2 + \beta^4)x^2}{(\alpha^2-\beta^2)^4}\, J_0(\alpha x)\, Y_0(\beta x) -$$

$$- \frac{\beta\left[\left(\alpha^2-\beta^2\right)^4 x^5 - 16\,\left(2\,\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^3 + 64\,\left(\alpha^4 + 4\,\alpha^2\beta^2 + \beta^4\right) x\right]}{(\alpha^2-\beta^2)^5}\, J_0(\alpha x)\, Y_1(\beta x) +$$

$$+ \frac{\alpha\left[\left(\alpha^2-\beta^2\right)^4 x^5 - 16\,\left(\alpha^2+2\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2 x^3 + 64\,\left(\alpha^4 + 4\,\alpha^2\beta^2 + \beta^4\right) x\right]}{(\alpha^2-\beta^2)^5}\, J_1(\alpha x)\, Y_0(\beta x) +$$

$$+ \frac{8\,\alpha\,\beta\,\left(\alpha^2-\beta^2\right)^2 x^4 - 96\,\alpha\,\beta\,\left(\alpha^2+\beta^2\right) x^2}{(\alpha^2-\beta^2)^4}\, J_1(\alpha x)\, Y_1(\beta x)$$

$$\int x^5\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{8\,\alpha\,\beta\,\left(\alpha^2-\beta^2\right)^2\, x^4 - 96\,\alpha\,\beta\,\left(\alpha^2+\beta^2\right)\, x^2}{(\alpha^2-\beta^2)^4}\, J_0(\alpha x)\, Y_0(\beta x) -$$

$$- \frac{\alpha[(\alpha^2-\beta^2)^4\, x^5 - 8\,\left(\alpha^2+5\,\beta^2\right)\left(\alpha^2-\beta^2\right)^2\, x^3 + 192\,\beta^2\,\left(\alpha^2+\beta^2\right)\, x]}{(\alpha^2-\beta^2)^5}\, J_0(\alpha x)\, Y_1(\beta x)] +$$

$$+ \frac{\beta[(\alpha^2-\beta^2)^4\, x^5 - 8\,\left(5\,\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2\, x^3 + 192\,\alpha^2\,\left(\alpha^2+\beta^2\right)\, x]}{(\alpha^2-\beta^2)^5}\, J_1(\alpha x)\, Y_0(\beta x) +$$

$$+ \frac{4\,\left(\alpha^2+\beta^2\right)\left(\alpha^2-\beta^2\right)^2\, x^4 - 16\,(\alpha^4 + 10\,\alpha^2\beta^2 +\, \beta^4)\, x^2}{(\alpha^2-\beta^2)^4}\, J_1(\alpha x)\, Y_1(\beta x)$$

2.3.1. Integrals of the type $\int x^{-1}\, Z_\nu(x+\alpha)Z_1(x)\,dx$ and $\int [x(x+\alpha)]^{-1}\, Z_1(x+\alpha)Z_1(x)\,dx$

$$\int \frac{J_1(x)J_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\Big(J_0(x)J_1(x+\alpha) - J_1(x)J_0(x+\alpha)\Big)$$

$$\int \frac{I_1(x)I_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\Big(I_0(x)I_1(x+\alpha) - I_1(x)I_0(x+\alpha)\Big)$$

$$\int \frac{K_1(x)I_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\Big(K_0(x)K_1(x+\alpha) - K_1(x)K_0(x+\alpha)\Big)$$

$$\int \frac{J_1(x)J_1(x+\alpha)}{x}\,dx =$$

$$= -\frac{x+\alpha}{\alpha}J_0(x)J_0(x+\alpha) - \frac{x}{\alpha^2}J_1(x)J_0(x+\alpha) + \frac{x+\alpha}{\alpha^2}J_0(x)J_1(x+\alpha) - \frac{x+\alpha}{\alpha}J_1(x)J_1(x+\alpha)$$

$$\int \frac{I_1(x)I_1(x+\alpha)}{x}\,dx =$$

$$= \frac{x+\alpha}{\alpha}I_0(x)I_0(x+\alpha) + \frac{x}{\alpha^2}I_1(x)I_0(x+\alpha) - \frac{x+\alpha}{\alpha^2}I_0(x)I_1(x+\alpha) - \frac{x+\alpha}{\alpha}I_1(x)I_1(x+\alpha)$$

$$\int \frac{K_1(x)K_1(x+\alpha)}{x}\,dx =$$

$$= \frac{x+\alpha}{\alpha}K_0(x)K_0(x+\alpha) - \frac{x}{\alpha^2}K_1(x)K_0(x+\alpha) + \frac{x+\alpha}{\alpha^2}K_0(x)K_1(x+\alpha) - \frac{x+\alpha}{\alpha}K_1(x)K_1(x+\alpha)$$

From this

$$\int \frac{J_1(x)\,J_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\,\{2(x+\alpha)J_1(x)\,J_0(x+\alpha) - 2x\,J_0(x)\,J_1(x+\alpha) - \alpha(2x+\alpha)\,[\,J_0(x)\,J_0(x+\alpha) + J_1(x)\,J_1(x+\alpha)]\}$$

$$\int \frac{I_1(x)\,I_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\,\{-2(x+\alpha)I_1(x)\,I_0(x+\alpha) + 2x\,I_0(x)\,I_1(x+\alpha) + \alpha(2x+\alpha)\,[\,I_0(x)\,I_0(x+\alpha) - I_1(x)\,I_1(x+\alpha)]\}$$

$$\int \frac{K_1(x)\,K_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\,\{2(x+\alpha)K_1(x)\,K_0(x+\alpha) - 2x\,K_0(x)\,K_1(x+\alpha) + \alpha(2x+\alpha)\,[\,K_0(x)\,K_0(x+\alpha) - K_1(x)\,K_1(x+\alpha)]\}$$

2.4.1. Integrals of the type $\int x^{2n+1}\ \ln x\, Z_\nu^2(x)\,dx$ and $\int x^{2n}\ \ln x\, Z_0(x)\,Z_1(x)\,dx$

In the following integrals $J_\nu(x)$ may be substituted by $Y_\nu(x)$, $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$.

$$\int x\ \ln x\, J_0^2(x)\,dx = \frac{x^2(\ln x-1)}{2}\left[J_0^2(x)+J_1^2(x)\right]+\frac{x}{2}\,J_0(x)\,J_1(x)$$

$$\int x\ \ln x\, I_0^2(x)\,dx = \frac{x^2(\ln x-1)}{2}\left[I_0^2(x)-I_1^2(x)\right]+\frac{x}{2}\,I_0(x)\,I_1(x)$$

$$\int x\ \ln x\, K_0^2(x)\,dx = \frac{x^2(\ln x-1)}{2}\left[K_0^2(x)-K_1^2(x)\right]-\frac{x}{2}\,K_0(x)\,K_1(x)$$

$$\int x\ \ln x\, J_1^2(x)\,dx = \frac{x^2(\ln x-1)-1}{2}\,J_0^2(x)+\frac{x(1-2\ln x)}{2}\,J_0(x)\,J_1(x)+\frac{x^2(\ln x-1)}{2}\,J_1^2(x)$$

$$\int x\ \ln x\, I_1^2(x)\,dx = \frac{x^2(1-\ln x)-1}{2}\,I_0^2(x)+\frac{x(2\ln x-1)}{2}\,I_0(x)\,I_1(x)+\frac{x^2(\ln x-1)}{2}\,I_1^2(x)$$

$$\int x\ \ln x\, K_1^2(x)\,dx = \frac{x^2(\ln x-1)-1}{2}\,K_0^2(x)-\frac{x(2\ln x-1)}{2}\,K_0(x)\,K_1(x)+\frac{x^2(\ln x-1)}{2}\,K_1^2(x)$$

$$\int x^2\ \ln x\, J_0(x)\,J_1(x)\,dx = -\frac{x^2}{4}\,J_0^2(x)+\frac{x}{2}\,J_0(x)\,J_1(x)+\frac{x^2(2\ln x-1)}{4}\,J_1^2(x)$$

$$\int x^2\ \ln x\, I_0(x)\,I_1(x)\,dx = \frac{x^2}{4}\,I_0^2(x)-\frac{x}{2}\,I_0(x)\,I_1(x)+\frac{x^2(2\ln x-1)}{4}\,I_1^2(x)$$

$$\int x^2\ \ln x\, K_0(x)\,K_1(x)\,dx = -\frac{x^2}{4}\,K_0^2(x)-\frac{x}{2}\,K_0(x)\,K_1(x)+\frac{x^2(1-2\ln x)}{4}\,K_1^2(x)$$

$$\int x^3\ \ln x\, J_0^2(x)\,dx =$$
$$= \frac{3\,x^2-x^4+3\,x^4\ln x}{18}\,J_0^2(x)+\frac{x^3-6\,x+6\,x^3\ln x}{18}\,J_0(x)\,J_1(x)-\frac{x^2+x^4+(6\,x^2-3\,x^4)\ln x}{18}\,J_1^2(x)$$

$$\int x^3\ \ln x\, I_0^2(x)\,dx =$$
$$= \frac{-3\,x^2-x^4+3\,x^4\ln x}{18}\,I_0^2(x)+\frac{x^3+6\,x+6\,x^3\ln x}{18}\,I_0(x)\,I_1(x)-\frac{x^2-x^4+(6\,x^2+3\,x^4)\ln x}{18}\,I_1^2(x)$$

$$\int x^3\ \ln x\, K_0^2(x)\,dx =$$
$$= \frac{-3\,x^2-x^4+3\,x^4\ln x}{18}\,K_0^2(x)-\frac{x^3+6\,x+6\,x^3\ln x}{18}\,K_0(x)\,K_1(x)-\frac{x^2-x^4+(6\,x^2+3\,x^4)\ln x}{18}\,K_1^2(x)$$

$$\int x^3\ \ln x\, J_1^2(x)\,dx =$$
$$= -\frac{6\,x^2+x^4-3\,x^4\ln x}{18}\,J_0^2(x)+\frac{12\,x+x^3-12\,x^3\ln x}{18}\,J_0(x)\,J_1(x)-\frac{x^2+x^4-(12x^2+3\,x^4)\ln x}{18}\,J_1^2(x)$$

$$\int x^3\ \ln x\, I_1^2(x)\,dx =$$
$$= \frac{-6\,x^2+x^4-3\,x^4\ln x}{18}\,I_0^2(x)+\frac{12\,x-x^3+12\,x^3\ln x}{18}\,I_0(x)\,I_1(x)+\frac{x^2-x^4-(12x^2-3\,x^4)\ln x}{18}\,I_1^2(x)$$

$$\int x^3\ \ln x\, K_1^2(x)\,dx =$$

$$= \frac{-6\,x^2+x^4-3\,x^4\ln x}{18}\,K_0^2(x) - \frac{12\,x-x^3+12\,x^3\ln x}{18}\,K_0(x)\,K_1(x) + \frac{x^2-x^4-(12x^2-3\,x^4)\ln x}{18}\,K_1^2(x)$$

$$\int x^4\,\ln x\,J_0(x)\,J_1(x)\,dx = \frac{12\,x^2-x^4-6\,x^4\ln x}{36}\,J_0^2(x) +$$

$$+\frac{5\,x^3-12\,x+12\,x^3\ln x}{18}\,J_0(x)\,J_1(x) - \frac{10\,x^2+x^4+(24x^2-12\,x^4)\ln x}{36}\,J_1^2(x)$$

$$\int x^4\,\ln x\,I_0(x)\,I_1(x)\,dx = \frac{12\,x^2+x^4+6\,x^4\ln x}{36}\,I_0^2(x) -$$

$$-\frac{5\,x^3+12\,x+12\,x^3\ln x}{18}\,I_0(x)\,I_1(x) + \frac{10\,x^2-x^4+(24x^2+12\,x^4)\ln x}{36}\,I_1^2(x)$$

$$\int x^4\,\ln x\,K_0(x)\,K_1(x)\,dx = -\frac{12\,x^2+x^4+6\,x^4\ln x}{36}\,K_0^2(x) -$$

$$-\frac{5\,x^3+12\,x+12\,x^3\ln x}{18}\,K_0(x)\,K_1(x) - \frac{10\,x^2-x^4+(24x^2+12\,x^4)\ln x}{36}\,K_1^2(x)$$

Let

$$\int x^{2n+1}\,\ln x\,J_0^2(x)\,dx =$$

$$= \frac{A_n(x)+B_n(x)\,\ln x}{N_n^{(20,20)}}\,J_0^2(x) + \frac{C_n(x)+D_n(x)\,\ln x}{N_n^{(20,11)}}\,J_0(x)\,J_1(x) + \frac{E_n(x)+F_n(x)\,\ln x}{N_n^{(20,02)}}\,J_1^2(x)\,,$$

$$\int x^{2n}\,\ln x\,J_0(x)\,J_1(x)\,dx =$$

$$= \frac{G_n(x)+H_n(x)\,\ln x}{N_n^{(11,20)}}\,J_0^2(x) + \frac{I_n(x)+K_n(x)\,\ln x}{N_n^{(11,11)}}\,J_0(x)\,J_1(x) + \frac{L_n(x)+M_n(x)\,\ln x}{N_n^{(11,02)}}\,J_1^2(x)\,,$$

$$\int x^{2n+1}\,\ln x\,J_1^2(x)\,dx =$$

$$= \frac{P_n(x)+Q_n(x)\,\ln x}{N_n^{(02,20)}}\,J_0^2(x) + \frac{R_n(x)+S_n(x)\,\ln x}{N_n^{(02,11)}}\,J_0(x)\,J_1(x) + \frac{T_n(x)+U_n(x)\,\ln x}{N_n^{(02,02)}}\,J_1^2(x)\,,$$

and let the integrals with $I_\nu(x)$ be described with the polynomials $A_n^*(x),\ldots$ and such with $K_\nu(x)$ written with $A_n^{**}(x)\,\ldots$ (the denominators $N_n^{(\mu,\nu,\lambda,\kappa)}$ are the same), then holds

$$N_2^{(20,20)} = 450\,,\quad A_2(x) = -9\,x^6+56\,x^4-240\,x^2\,,\quad B_2(x) = 45\,x^6+120\,x^4\,,$$
$$N_2^{(20,11)} = 450\,,\quad C_2(x) = 9\,x^5-344\,x^3+480\,x\,,\quad D_2(x) = 180\,x^5-480\,x^3\,,$$
$$N_2^{(20,02)} = 450\,,\quad E_2(x) = -9\,x^6-52\,x^4+344\,x^2\,,\quad F_2(x) = 45\,x^6-240\,x^4+480\,x^2\,,$$

$$A_2^*(x) = -9\,x^6-56\,x^4-240\,x^2\,,\quad B_2^*(x) = 45\,x^6-120\,x^4\,,$$
$$C_2^*(x) = 9\,x^5+344\,x^3+480\,x\,,\quad D_2^*(x) = 180\,x^5+480\,x^3\,,$$
$$E_2^*(x) = 9\,x^6-52\,x^4-344\,x^2\,,\quad F_2^*(x) = -45\,x^6-240\,x^4-480\,x^2\,,$$

$$A_2^{**}(x) = -9\,x^6-56\,x^4-240\,x^2\,,\quad B_2^{**}(x) = 45\,x^6-120\,x^4\,,$$
$$C_2^{**}(x) = -9\,x^5-344\,x^3-480\,x\,,\quad D_2^{**}(x) = -180\,x^5-480\,x^3\,,$$
$$E_2^{**}(x) = 9\,x^6-52\,x^4-344\,x^2\,,\quad F_2^{**}(x) = -45\,x^6-240\,x^4-480\,x^2\,,$$

$\int x^4\,\ln x\,Z_0(x)\,Z_1(x)\,dx$ see before.

$$N_2^{(02,02)} = 150\,,\quad P_2(x) = -3x^6-23\,x^4+120\,x^2\,,\quad Q_2(x) = 15\,x^6-60\,x^4\,,$$

$$N_2^{(02,11)} = 150\,,\quad R_2(x) = 3\,x^5+152\,x^3-240\,x\,,\quad S_2(x) = -90\,x^5+240\,x^3\,,$$

$$N_2^{(02,02)} = 150\,, \quad T_2(x) = -3\,x^6 + 16\,x^4 - 152\,x^2\,, \quad U_2(x) = 15\,x^6 + 120\,x^4 - 240\,x^2\,,$$

$$P_2^*(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^*(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^*(x) = -3\,x^5 + 152\,x^3 + 240\,x\,, \quad S_2^*(x) = 90\,x^5 + 240\,x^3\,,$$
$$T_2^*(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^*(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$P_2^{**}(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^{**}(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^{**}(x) = 3\,x^5 - 152\,x^3 - 240\,x\,, \quad S_2^{**}(x) = -90\,x^5 - 240\,x^3\,,$$
$$T_2^{**}(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^{**}(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$N_3^{(20,20)} = 2450\,, \quad A_3(x) = -25\,x^8 + 303\,x^6 - 4152\,x^4 + 10080\,x^2\,, \quad B_3(x) = 175\,x^8 + 1260\,x^6 - 5040\,x^4\,,$$
$$N_3^{(20,11)} = 2450\,, \quad C_3(x) = 25\,x^7 - 3078\,x^5 + 21648\,x^3 - 20160\,x\,, \quad D_3(x) = 1050\,x^7 - 7560\,x^5 + 20160\,x^3\,,$$
$$N_3^{(20,02)} = 2450\,, \quad E_3(x) = -25\,x^8 - 297\,x^6 + 5784\,x^4 - 21648\,x^2\,, \quad F_3(x) = 175\,x^8 - 1890\,x^6 + 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^*(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^*(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^*(x) = 25\,x^7 + 3078\,x^5 + 21648\,x^3 + 20160\,x\,, \quad D_3^*(x) = 1050\,x^7 + 7560\,x^5 + 20160\,x^3\,,$$
$$E_3^*(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^*(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^{**}(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^{**}(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^{**}(x) = -25\,x^7 - 3078\,x^5 - 21648\,x^3 - 20160\,x\,, \quad D_3^{**}(x) = -1050\,x^7 - 7560\,x^5 - 20160\,x^3\,,$$
$$E_3^{**}(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^{**}(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$N_3^{(11,20)} = 300\,, \quad G_3(x) = -3\,x^6 + 152\,x^4 - 480\,x^2\,, \quad H_3(x) = -60\,x^6 + 240\,x^4\,,$$
$$N_3^{(11,11)} = 150\,, \quad I_3(x) = 39\,x^5 - 424\,x^3 + 480\,x\,, \quad K_3(x) = 180\,x^5 - 480\,x^3\,,$$
$$N_3^{(11,02)} = 300\,, \quad L_3(x) = -3\,x^6 - 184\,x^4 + 848\,x^2\,, \quad M_3(x) = 90\,x^6 - 480\,x^4 + 960\,x^2\,,$$

$$G_3^*(x) = 3\,x^6 + 152\,x^4 + 480\,x^2\,, \quad H_3^*(x) = 60\,x^6 + 240\,x^4\,,$$
$$I_3^*(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^*(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^*(x) = -3\,x^6 + 184\,x^4 + 848\,x^2\,, \quad M_3^*(x) = 90\,x^6 + 480\,x^4 + 960\,x^2\,,$$

$$G_3^{**}(x) = -3\,x^6 - 152\,x^4 - 480\,x^2\,, \quad H_3^{**}(x) = -60\,x^6 - 240\,x^4\,,$$
$$I_3^{**}(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^{**}(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^{**}(x) = 3\,x^6 - 184\,x^4 - 848\,x^2\,, \quad M_3^{**}(x) = -90\,x^6 - 480\,x^4 - 960\,x^2\,,$$

$$N_3^{(02,20)} = 2450\,, \quad P_3(x) = -25\,x^8 - 334\,x^6 + 5256\,x^4 - 13440\,x^2\,, \quad Q_3(x) = 175\,x^8 - 1680\,x^6 + 6720\,x^4\,,$$
$$N_3^{(02,11)} = 2450\,, \quad R_3(x) = 25\,x^7 + 3684\,x^5 - 27744\,x^3 + 26880\,x\,, \quad S_3(x) = -1400\,x^7 + 10080\,x^5 - 26880\,x^3\,,$$
$$N_3^{(02,02)} = 2450\,, \quad T_3(x) = -25\,x^8 + 291\,x^6 - 7152\,x^4 + 27744\,x^2\,, \quad U_3(x) = 175\,x^8 + 2520\,x^6 - 13440\,x^4 + 26880\,x^2\,,$$

$$P_3^*(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^*(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$
$$R_3^*(x) = -25\,x^7 + 3684\,x^5 + 27744\,x^3 + 26880\,x\,, \quad S_3^*(x) = 1400\,x^7 + 10080\,x^5 + 26880\,x^3\,,$$

$$T_3^*(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^*(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

$$P_3^{**}(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^{**}(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$
$$R_3^{**}(x) = 25\,x^7 - 3684\,x^5 - 27744\,x^3 - 26880\,x\,, \quad S_3^{**}(x) = -1400\,x^7 - 10080\,x^5 - 26880\,x^3\,,$$
$$T_3^{**}(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^{**}(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

$$N_4^{(20,20)} = 198450\,, \quad A_4(x) = -1225\,x^{10} + 24600\,x^8 - 732096\,x^6 + 6315264\,x^4 - 11612160\,x^2\,,$$
$$B_4(x) = 11025\,x^{10} + 151200\,x^8 - 1451520\,x^6 + 5806080\,x^4\,,$$
$$N_4^{(20,11)} = 198450\,, \quad C_4(x) = 1225\,x^9 - 348000\,x^7 + 5844096\,x^5 - 31067136\,x^3 + 23224320\,x\,,$$
$$D_4(x) = 88200\,x^9 - 1209600\,x^7 + 8709120\,x^5 - 23224320\,x^3\,,$$
$$N_4^{(20,02)} = 198450\,, \quad E_4(x) = -1225\,x^{10} - 24400\,x^8 + 916704\,x^6 - 9727488\,x^4 + 31067136\,x^2\,,$$
$$F_4(x) = 11025\,x^{10} - 201600\,x^8 + 2177280\,x^6 - 11612160\,x^4 + 23224320\,x^2\,,$$

$$A_4^*(x) = -1225\,x^{10} - 24600\,x^8 - 732096\,x^6 - 6315264\,x^4 - 11612160\,x^2\,,$$
$$B_4^*(x) = 11025\,x^{10} - 151200\,x^8 - 1451520\,x^6 - 5806080\,x^4\,,$$
$$C_4^*(x) = 1225\,x^9 + 348000\,x^7 + 5844096\,x^5 + 31067136\,x^3 + 23224320\,x\,,$$
$$D_4^*(x) = 88200\,x^9 + 1209600\,x^7 + 8709120\,x^5 + 23224320\,x^3\,,$$
$$E_4^*(x) = 1225\,x^{10} - 24400\,x^8 - 916704\,x^6 - 9727488\,x^4 - 31067136\,x^2\,,$$
$$F_4^*(x) = -11025\,x^{10} - 201600\,x^8 - 2177280\,x^6 - 11612160\,x^4 - 23224320\,x^2\,,$$

$$A_4^{**}(x) = -1225\,x^{10} - 24600\,x^8 - 732096\,x^6 - 6315264\,x^4 - 11612160\,x^2\,,$$
$$B_4^{**}(x) = 11025\,x^{10} - 151200\,x^8 - 1451520\,x^6 - 5806080\,x^4\,,$$
$$C_4^{**}(x) = -1225\,x^9 - 348000\,x^7 - 5844096\,x^5 - 31067136\,x^3 - 23224320\,x\,,$$
$$D_4^{**}(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$
$$E_4^{**}(x) = 1225\,x^{10} - 24400\,x^8 - 916704\,x^6 - 9727488\,x^4 - 31067136\,x^2\,,$$
$$F_4^{**}(x) = -11025\,x^{10} - 201600\,x^8 - 2177280\,x^6 - 11612160\,x^4 - 23224320\,x^2\,,$$

$$N_4^{(11,20)} = 4900\,, \quad G_4(x) = -25\,x^8 + 3684\,x^6 - 38256\,x^4 + 80640\,x^2\,, \quad H_4(x) = -1050\,x^8 + 10080\,x^6 - 40320\,x^4\,,$$
$$N_4^{(11,11)} = 2450\,, \quad I_4(x) = 625\,x^7 - 16092\,x^5 + 96672\,x^3 - 80640\,x\,, \quad K_4(x) = 4200\,x^7 - 30240\,x^5 + 80640\,x^3\,,$$
$$N_4^{(11,02)} = 4900\,, \quad L_4(x) = -25\,x^8 - 4266\,x^6 + 56352\,x^4 - 193344\,x^2\,,$$
$$M_4(x) = 1400\,x^8 - 15120\,x^6 + 80640\,x^4 - 161280\,x^2\,,$$

$$G_4^*(x) = 25\,x^8 + 3684\,x^6 + 38256\,x^4 + 80640\,x^2\,, \quad H_4^*(x) = 1050\,x^8 + 10080\,x^6 + 40320\,x^4\,,$$
$$I_4^*(x) = -625\,x^7 - 16092\,x^5 - 96672\,x^3 - 80640\,x\,, \quad K_4^*(x) = -4200\,x^7 - 30240\,x^5 - 80640\,x^3\,,$$
$$L_4^*(x) = -25\,x^8 + 4266\,x^6 + 56352\,x^4 + 193344\,x^2\,, \quad M_4^*(x) = 1400\,x^8 + 15120\,x^6 + 80640\,x^4 + 161280\,x^2\,,$$

$$G_4^{**}(x) = -25\,x^8 - 3684\,x^6 - 38256\,x^4 - 80640\,x^2\,, \quad H_4^{**}(x) = -1050\,x^8 - 10080\,x^6 - 40320\,x^4\,,$$
$$I_4^{**}(x) = -625\,x^7 - 16092\,x^5 - 96672\,x^3 - 80640\,x\,, \quad K_4^{**}(x) = -4200\,x^7 - 30240\,x^5 - 80640\,x^3\,,$$

$$L_4^{**}(x) = 25\,x^8 - 4266\,x^6 - 56352\,x^4 - 193344\,x^2\,, \quad M_4^{**}(x) = -1400\,x^8 - 15120\,x^6 - 80640\,x^4 - 161280\,x^2\,,$$

$$N_4^{(02,20)} = 39690\,, \quad P_4(x) = -245\,x^{10} - 5205\,x^8 + 173952\,x^6 - 1542528\,x^4 + 2903040\,x^2\,,$$
$$Q_4(x) = 2205\,x^{10} - 37800\,x^8 + 362880\,x^6 - 1451520\,x^4\,,$$
$$N_4^{(02,11)} = 39690\,, \quad R_4(x) = 245\,x^9 + 79440\,x^7 - 1406592\,x^5 + 7621632\,x^3 - 5806080\,x\,,$$
$$S_4(x) = -22050\,x^9 + 302400\,x^7 - 2177280\,x^5 + 5806080\,x^3\,,$$
$$N_4^{(02,02)} = 39690\,, \quad T_4(x) = -245\,x^{10} + 4840\,x^8 - 215568\,x^6 + 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4(x) = 2205\,x^{10} + 50400\,x^8 - 544320\,x^6 + 2903040\,x^4 - 5806080\,x^2\,,$$

$$P_4^*(x) = 245\,x^{10} - 5205\,x^8 - 173952\,x^6 - 1542528\,x^4 - 2903040\,x^2\,,$$
$$Q_4^*(x) = -2205\,x^{10} - 37800\,x^8 - 362880\,x^6 - 1451520\,x^4\,,$$
$$R_4^*(x) = -245\,x^9 + 79440\,x^7 + 1406592\,x^5 + 7621632\,x^3 + 5806080\,x\,,$$
$$S_4^*(x) = 22050\,x^9 + 302400\,x^7 + 2177280\,x^5 + 5806080\,x^3\,,$$
$$T_4^*(x) = -245\,x^{10} - 4840\,x^8 - 215568\,x^6 - 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4^*(x) = 2205\,x^{10} - 50400\,x^8 - 544320\,x^6 - 2903040\,x^4 - 5806080\,x^2\,,$$

$$P_4^{**}(x) = 245\,x^{10} - 5205\,x^8 - 173952\,x^6 - 1542528\,x^4 - 2903040\,x^2\,,$$
$$Q_4^{**}(x) = -2205\,x^{10} - 37800\,x^8 - 362880\,x^6 - 1451520\,x^4\,,$$
$$R_4^{**}(x) = 245\,x^9 - 79440\,x^7 - 1406592\,x^5 - 7621632\,x^3 - 5806080\,x\,,$$
$$S_4^{**}(x) = -22050\,x^9 - 302400\,x^7 - 2177280\,x^5 - 5806080\,x^3\,,$$
$$T_4^{**}(x) = -245\,x^{10} - 4840\,x^8 - 215568\,x^6 - 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4^{**}(x) = 2205\,x^{10} - 50400\,x^8 - 544320\,x^6 - 2903040\,x^4 - 5806080\,x^2\,,$$

$$N_5^{(20,20)} = 960498\,, \quad A_5(x) = -3969\,x^{12} + 119315\,x^{10} - 6183600\,x^8 + 113915520\,x^6 - 828218880\,x^4 + 1277337600\,x^2\,,$$
$$B_5(x) = 43659\,x^{12} + 970200\,x^{10} - 16632000\,x^8 + 159667200\,x^6 - 638668800\,x^4\,,$$
$$N_5^{(20,11)} = 960498\,, \quad C_5(x) = 3969\,x^{11} - 2163350\,x^9 + 66100800\,x^7 - 843160320\,x^5 + 3951544320\,x^3 - 2554675200\,x\,,$$
$$D_5(x) = 436590\,x^{11} - 9702000\,x^9 + 133056000\,x^7 - 958003200\,x^5 + 2554675200\,x^3\,,$$
$$N_5^{(20,02)} = 960498\,, \quad E_5(x) = -3969\,x^{12} - 118825\,x^{10} + 7320800\,x^8 - 150914880\,x^6 + 1337103360\,x^4 - 3951544320\,x^2\,,$$
$$F_5(x) = 43659\,x^{12} - 1212750\,x^{10} + 22176000\,x^8 - 239500800\,x^6 + 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$A_5^*(x) = -3969\,x^{12} - 119315\,x^{10} - 6183600\,x^8 - 113915520\,x^6 - 828218880\,x^4 - 1277337600\,x^2\,,$$
$$B_5^*(x) = 43659\,x^{12} - 970200\,x^{10} - 16632000\,x^8 - 159667200\,x^6 - 638668800\,x^4\,,$$
$$C_5^*(x) = 3969\,x^{11} + 2163350\,x^9 + 66100800\,x^7 + 843160320\,x^5 + 3951544320\,x^3 + 2554675200\,x\,,$$
$$D_5^*(x) = 436590\,x^{11} + 9702000\,x^9 + 133056000\,x^7 + 958003200\,x^5 + 2554675200\,x^3\,,$$
$$E_5^*(x) = 3969\,x^{12} - 118825\,x^{10} - 7320800\,x^8 - 150914880\,x^6 - 1337103360\,x^4 - 3951544320\,x^2\,,$$
$$F_5^*(x) = -43659\,x^{12} - 1212750\,x^{10} - 22176000\,x^8 - 239500800\,x^6 - 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$A_5^{**}(x) = -3969\,x^{12} - 119315\,x^{10} - 6183600\,x^8 - 113915520\,x^6 - 828218880\,x^4 - 1277337600\,x^2\,,$$
$$B_5^{**}(x) = 43659\,x^{12} - 970200\,x^{10} - 16632000\,x^8 - 159667200\,x^6 - 638668800\,x^4\,,$$

$$C_5^{**}(x) = -3969\,x^{11} - 2163350\,x^9 - 66100800\,x^7 - 843160320\,x^5 - 3951544320\,x^3 - 2554675200\,x\,,$$
$$D_5^{**}(x) = -436590\,x^{11} - 9702000\,x^9 - 133056000\,x^7 - 958003200\,x^5 - 2554675200\,x^3\,,$$
$$E_5^{**}(x) = 3969\,x^{12} - 118825\,x^{10} - 7320800\,x^8 - 150914880\,x^6 - 1337103360\,x^4 - 3951544320\,x^2\,,$$
$$F_5^{**}(x) = -43659\,x^{12} - 1212750\,x^{10} - 22176000\,x^8 - 239500800\,x^6 - 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$N_5^{(11,20)} = 79380\,, \quad G_5(x) = -245\,x^{10} + 79440\,x^8 - 1754496\,x^6 + 13791744\,x^4 - 23224320\,x^2\,,$$
$$H_5(x) = -17640\,x^{10} + 302400\,x^8 - 2903040\,x^6 + 11612160\,x^4\,,$$
$$N_5^{(11,11)} = 39690\,, \quad I_5(x) = 10045\,x^9 - 468960\,x^7 + 6715008\,x^5 - 33389568\,x^3 + 23224320\,x\,,$$
$$K_5(x) = 88200\,x^9 - 1209600\,x^7 + 8709120\,x^5 - 23224320\,x^3\,,$$
$$N_5^{(11,02)} = 79380\,, \quad L_5(x) = -245\,x^{10} - 89120\,x^8 + 2268864\,x^6 - 21777408\,x^4 + 66779136\,x^2\,,$$
$$M_5(x) = 22050\,x^{10} - 403200\,x^8 + 4354560\,x^6 - 23224320\,x^4 + 46448640\,x^2\,,$$

$$G_5^{*}(x) = 245\,x^{10} + 79440\,x^8 + 1754496\,x^6 + 13791744\,x^4 + 23224320\,x^2\,,$$
$$H_5^{*}(x) = 17640\,x^{10} + 302400\,x^8 + 2903040\,x^6 + 11612160\,x^4\,,$$
$$I_5^{*}(x) = -10045\,x^9 - 468960\,x^7 - 6715008\,x^5 - 33389568\,x^3 - 23224320\,x\,,$$
$$K_5^{*}(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$
$$L_5^{*}(x) = -245\,x^{10} + 89120\,x^8 + 2268864\,x^6 + 21777408\,x^4 + 66779136\,x^2\,,$$
$$M_5^{*}(x) = 22050\,x^{10} + 403200\,x^8 + 4354560\,x^6 + 23224320\,x^4 + 46448640\,x^2\,,$$

$$G_5^{**}(x) = -245\,x^{10} - 79440\,x^8 - 1754496\,x^6 - 13791744\,x^4 - 23224320\,x^2\,,$$
$$H_5^{**}(x) = -17640\,x^{10} - 302400\,x^8 - 2903040\,x^6 - 11612160\,x^4\,,$$
$$I_5^{**}(x) = -10045\,x^9 - 468960\,x^7 - 6715008\,x^5 - 33389568\,x^3 - 23224320\,x\,,$$
$$K_5^{**}(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$
$$L_5^{**}(x) = 245\,x^{10} - 89120\,x^8 - 2268864\,x^6 - 21777408\,x^4 - 66779136\,x^2\,,$$
$$M_5^{**}(x) = -22050\,x^{10} - 403200\,x^8 - 4354560\,x^6 - 23224320\,x^4 - 46448640\,x^2\,,$$

$$N_5^{(02,20)} = 320166\,, \quad P_5(x) = -1323\,x^{12} - 41258\,x^{10} + 2362560\,x^8 - 44501760\,x^6 + 327029760\,x^4 - 510935040\,x^2\,,$$
$$Q_5(x) = 14553\,x^{12} - 388080\,x^{10} + 6652800\,x^8 - 63866880\,x^6 + 255467520\,x^4\,,$$
$$N_5^{(02,11)} = 320166\,, \quad R_5(x) = 1323\,x^{11} + 800660\,x^9 - 25553280\,x^7 + 330877440\,x^5 - 1563586560\,x^3 + 1021870080\,x\,,$$
$$S_5(x) = -174636\,x^{11} + 3880800\,x^9 - 53222400\,x^7 + 383201280\,x^5 - 1021870080\,x^3\,,$$
$$N_5^{(02,02)} = 320166\,, \quad T_5(x) = -1323\,x^{12} + 39445\,x^{10} - 2780480\,x^8 + 58769280\,x^6 - 526325760\,x^4 + 1563586560\,x^2\,,$$
$$U_5(x) = 14553\,x^{12} + 485100\,x^{10} - 8870400\,x^8 + 95800320\,x^6 - 510935040\,x^4 + 1021870080\,x^2\,,$$

$$P_5^{*}(x) = 1323\,x^{12} - 41258\,x^{10} - 2362560\,x^8 - 44501760\,x^6 - 327029760\,x^4 - 510935040\,x^2\,,$$
$$Q_5^{*}(x) = -14553\,x^{12} - 388080\,x^{10} - 6652800\,x^8 - 63866880\,x^6 - 255467520\,x^4\,,$$
$$R_5^{*}(x) = -1323\,x^{11} + 800660\,x^9 + 25553280\,x^7 + 330877440\,x^5 + 1563586560\,x^3 + 1021870080\,x\,,$$
$$S_5^{*}(x) = 174636\,x^{11} + 3880800\,x^9 + 53222400\,x^7 + 383201280\,x^5 + 1021870080\,x^3\,,$$
$$T_5^{*}(x) = -1323\,x^{12} - 39445\,x^{10} - 2780480\,x^8 - 58769280\,x^6 - 526325760\,x^4 - 1563586560\,x^2\,,$$

$$U_5^*(x) = 14553\,x^{12} - 485100\,x^{10} - 8870400\,x^8 - 95800320\,x^6 - 510935040\,x^4 - 1021870080\,x^2\,,$$

$$P_5^{**}(x) = 1323\,x^{12} - 41258\,x^{10} - 2362560\,x^8 - 44501760\,x^6 - 327029760\,x^4 - 510935040\,x^2\,,$$
$$Q_5^{**}(x) = -14553\,x^{12} - 388080\,x^{10} - 6652800\,x^8 - 63866880\,x^6 - 255467520\,x^4\,,$$
$$R_5^{**}(x) = 1323\,x^{11} - 800660\,x^9 - 25553280\,x^7 - 330877440\,x^5 - 1563586560\,x^3 - 1021870080\,x\,,$$
$$S_5^{**}(x) = -174636\,x^{11} - 3880800\,x^9 - 53222400\,x^7 - 383201280\,x^5 - 1021870080\,x^3\,,$$
$$T_5^{**}(x) = -1323\,x^{12} - 39445\,x^{10} - 2780480\,x^8 - 58769280\,x^6 - 526325760\,x^4 - 1563586560\,x^2\,,$$
$$U_5^{**}(x) = 14553\,x^{12} - 485100\,x^{10} - 8870400\,x^8 - 95800320\,x^6 - 510935040\,x^4 - 1021870080\,x^2\,,$$

Recurrence relations:

$$\int x^{2n+1}\,\ln x\,J_0^2(x)\,dx =$$
$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + n\left(4\,n^2+n-2\right) - \left[(2n+1)x^2 + 2\,n^2\left(4\,n^2+n-2\right)\right]\,\ln x\right\}\,J_0^2(x) -$$
$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + (4\,n+3)\,n^2 - \left[(2n+1)x^2 + 2\,(4\,n+3)\,(n-1)\,n^2\right]\,\ln x\right\}\,J_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right] J_0(x)\,J_1(x) -$$
$$-\frac{2\,\left(4\,n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx - \frac{2\,(4\,n+3)\,(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\,J_0(x)\,J_1(x)\,dx = \frac{x^{2n}}{4}\,\left\{(1-2n\ln x)\,J_0^2(x) + \left[1-2(n-1)\ln x\right]J_1^2(x)\right\} +$$
$$+n^2\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx + (n-1)^2\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,J_1^2(x)\,dx =$$
$$= \frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + 4\,n^3 + 3\,n^2 - 1 + \left[(2n+1)x^2 - 2\,n\left(4\,n^3+3\,n^2-1\right)\right]\,\ln x\right\}\,J_0^2(x) +$$
$$+\frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + \left(4\,n^2+5\,n+2\right)n + \left[(2n+1)x^2 - 2\,(n-1)\left(4\,n^2+5\,n+2\right)n\right]\,\ln x\right\}\,J_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 - 2\,(n+1)\,(2\,n+1)\,\ln x\right] J_0(x)\,J_1(x) +$$
$$+\frac{2\,\left(4\,n^3+3\,n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx + \frac{2\,\left(4\,n^2+5\,n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,I_0^2(x)\,dx =$$
$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - n\left(4\,n^2+n-2\right) + \left[-(2n+1)x^2 + 2\,n^2\left(4\,n^2+n-2\right)\right]\,\ln x\right\}\,I_0^2(x) +$$
$$+\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - (4\,n+3)\,n^2 - \left[(2n+1)x^2 - 2\,(4\,n+3)\,(n-1)\,n^2\right]\,\ln x\right\}\,I_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right] I_0(x)\,I_1(x) +$$

$$+\frac{2\left(4n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\ln x\, I_0^2(x)\,dx-\frac{2\,(4n+3)\,(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n}\ln x\, I_0(x)\,I_1(x)\,dx=-\frac{x^{2n}}{4}\left\{(1-2n\ln x)\,I_0^2(x)-[1-2(n-1)\ln x]\,I_1^2(x)\right\}+$$

$$-n^2\int x^{2n-1}\ln x\, I_0^2(x)\,dx+(n-1)^2\int x^{2n-1}\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n+1}\ln x\, I_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+4n^3+3n^2-1-\left[(2n+1)x^2+2n\left(4n^3+3n^2-1\right)\right]\ln x\right\}\,I_0^2(x)-$$

$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+\left(4n^2+5n+2\right)n-\left[(2n+1)x^2+2\,(n-1)\left(4n^2+5n+2\right)n\right]\ln x\right\}\,I_1^2(x)+$$

$$-\frac{x^{2n+1}}{2(2n+1)^2}\left[1-2\,(n+1)\,(2n+1)\,\ln x\right]I_0(x)\,I_1(x)+$$

$$+\frac{2\left(4n^3+3n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\ln x\, I_0^2(x)\,dx-\frac{2\left(4n^2+5n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n+1}\ln x\, K_0^2(x)\,dx=$$

$$=-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2-n\left(4n^2+n-2\right)-\left[(2n+1)x^2-2n^2\left(4n^2+n-2\right)\right]\ln x\right\}\,K_0^2(x)+$$

$$+\frac{x^{2n}}{2(2n+1)^2}\left\{x^2-(4n+3)\,n^2-\left[(2n+1)x^2-2\,(4n+3)\,(n-1)\,n^2\right]\ln x\right\}\,K_1^2(x)+$$

$$-\frac{x^{2n+1}}{2(2n+1)^2}\left[1+2n(2n+1)\,\ln x\right]K_0(x)\,K_1(x)+$$

$$+\frac{2\left(4n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\ln x\, K_0^2(x)\,dx-\frac{2\,(4n+3)\,(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\ln x\, K_1^2(x)\,dx$$

$$\int x^{2n}\ln x\, K_0(x)\,K_1(x)\,dx=\frac{x^{2n}}{4}\left\{(1-2n\ln x)\,K_0^2(x)-[1-2(n-1)\ln x]\,K_1^2(x)\right\}+$$

$$+n^2\int x^{2n-1}\ln x\, K_0^2(x)\,dx-(n-1)^2\int x^{2n-1}\ln x\, K_1^2(x)\,dx$$

$$\int x^{2n+1}\ln x\, K_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+4n^3+3n^2-1-\left[(2n+1)x^2+2n\left(4n^3+3n^2-1\right)\right]\ln x\right\}\,K_0^2(x)-$$

$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+\left(4n^2+5n+2\right)n-\left[(2n+1)x^2+2\,(n-1)\left(4n^2+5n+2\right)n\right]\ln x\right\}\,K_1^2(x)+$$

$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1-2\,(n+1)\,(2n+1)\,\ln x\right]K_0(x)\,K_1(x)+$$

$$+\frac{2\left(4n^3+3n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\ln x\, K_0^2(x)\,dx-\frac{2\left(4n^2+5n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\ln x\, K_1^2(x)\,dx$$

2.4.2. Integrals of the Type $\int x^n \, \ln x \, Z_\mu(x)\, Z_\nu^*(x)\, dx$:

Integrals were found in the following cases:

n	J_0I_0	J_0K_0	J_1I_1	J_1K_1	I_0K_0	I_0K_1	I_1K_0	I_1K_1
1			*	*	*			*
2						*	*	
3	*	*			*			*
4						*	*	
5			*	*	*			*
6						*	*	
7	*	*			*			*
8						*	*	
9			*	*	*			*
10						*	*	

Therefore one can expect, that integrals of the types

$$\int x^{4n+3-2\nu} \, \ln x \, J_\nu(x)\, I_\nu(x)\, dx \quad \text{and} \quad \int x^{4n+3-2\nu} \, \ln x \, J_\nu(x)\, K_\nu(x)\, dx$$

may be expressed.
For the integrals $\int x^{2n+1} \, \ln x \, I_\nu(x)\, K_\nu(x)\, dx$ and $\int x^{2n} \, \ln x \, I_\nu(x)\, K_{1-\nu}(x)\, dx$ recurrence relations were found. Holds $x[I_0(x)K_1(x) + I_1(x)K_0(x)] = 1$ ([5], XIII B. 2.), so any multiple of this expression may be added to these antiderivatives.

a) Integrals with $J_0(x)\, Z_0(x)$:

$$\int x^3 \, \ln x \, J_0(x)\, I_0(x)\, dx = \frac{x^2 \, \ln x}{2} \left[x\, J_0(x)I_1(x) + x\, J_1(x)I_0(x) - 2\, J_1(x)I_1(x) \right] -$$
$$- \frac{x}{2} \left[J_0(x)I_1(x) - J_1(x)I_0(x) + x\, J_1(x)I_1(x) \right]$$

$$\int x^3 \, \ln x \, J_0(x)\, K_0(x)\, dx = \frac{x^2 \, \ln x}{2} \left[-x\, J_0(x)K_1(x) + x\, J_1(x)K_0(x) + 2\, J_1(x)K_1(x) \right] +$$
$$+ \frac{x}{2} \left[J_0(x)K_1(x) + J_1(x)K_0(x) + x\, J_1(x)K_1(x) \right]$$

$$\int x^7 \, \ln x \, J_0(x)\, I_0(x)\, dx = \frac{x^2 \, \ln x}{2} \Big[48x^2\, J_0(x)\, I_0(x) + x \left(x^4 - 12\, x^2 - 96 \right) J_0(x)\, I_1(x) +$$
$$+ x \left(x^4 + 12\, x^2 - 96 \right) J_1(x)\, I_0(x) + (192 - 6\, x^4)\, J_1(x)\, I_1(x) \Big] + \frac{x}{2} \Big[32x^3\, J_0(x)\, I_0(x) +$$
$$+ (96 - 88\, x^2 - 5\, x^4)\, J_0(x)\, I_1(x) + (5\, x^4 - 96 - 88\, x^2)\, J_1(x)\, I_0(x) + x \left(272 - x^4 \right) J_1(x)\, I_1(x) \Big]$$

$$\int x^7 \, \ln x \, J_0(x)\, K_0(x)\, dx = \frac{x^2 \, \ln x}{2} \Big[48x^2\, J_0(x)\, K_0(x) - x \left(x^4 - 12\, x^2 - 96 \right) J_0(x)\, K_1(x) +$$
$$+ x \left(x^4 + 12\, x^2 - 96 \right) J_1(x)\, K_0(x) + (6\, x^4 - 192)\, J_1(x)\, K_1(x) \Big] + \frac{x}{2} \Big[32x^3\, J_0(x)\, K_0(x) +$$
$$+ (88\, x^2 + 5\, x^4 - 96)\, J_0(x)\, K_1(x) + (5\, x^4 - 96 - 88\, x^2)\, J_1(x)\, K_0(x) + x \left(x^4 - 272 \right) J_1(x)\, K_1(x) \Big]$$

b) Integrals with $J_1(x)\, Z_1(x)$:

$$\int x \, \ln x \, J_1(x)\, I_1(x)\, dx = \frac{x \, \ln x}{2} \left[J_1(x)I_0(x) - J_0(x)I_1(x) \right] + \frac{J_0(x)I_0(x)}{2}$$

$$\int x\,\ln x\,J_1(x)\,K_1(x)\,dx = -\,\frac{x\,\ln x}{2}\,[J_1(x)K_0(x)+J_0(x)K_1(x)] - \frac{J_0(x)K_0(x)}{2}$$

$$\int x^5\,\ln x\,J_1(x)\,I_1(x)\,dx =$$

$$= \frac{x^2\,\ln x}{2}\,\left[4x^2\,J_0(x)\,I_0(x) - x(x^2+8)\,J_0(x)\,I_1(x) + x(x^2-8)\,J_1(x)\,I_1(x) + 16\,J_1(x)\,I_1(x)\right] +$$

$$+\frac{x}{2}\,\left[x^3\,J_0(x)\,I_0(x) + (8-4x^2)\,J_0(x)\,I_1(x) - (4x^2+8)\,J_1(x)\,I_0(x) + 16x\,J_1(x)\,I_1(x)\right]$$

$$\int x^5\,\ln x\,J_1(x)\,K_1(x)\,dx =$$

$$= \frac{x^2\,\ln x}{2}\,\left[-4x^2\,J_0(x)\,K_0(x) - x(x^2+8)\,J_0(x)\,K_1(x) - x(x^2-8)\,J_1(x)\,K_0(x) + 16\,J_1(x)\,K_1(x)\right] +$$

$$+\frac{x}{2}\,\left[-x^3\,J_0(x)\,K_0(x) + (8-4x^2)\,J_0(x)\,K_1(x) + (8+4x^2)\,J_1(x)\,K_0(x) + 16x\,J_1(x)\,K_1(x)\right]$$

$$\int x^9\,\ln x\,J_1(x)\,I_1(x)\,dx = \frac{x^2\,\ln x}{2}\,\left[8\,x^2\left(x^4-192\right)\,J_0(x)\,I_0(x) + \right.$$

$$+x\left(3072+384\,x^2-32\,x^4-x^6\right)\,J_0(x)\,I_1(x) + x\left(3072-384\,x^2-32\,x^4+x^6\right)\,J_1(x)\,I_0(x) -$$

$$\left. -(6144-192\,x^4)\,J_1(x)\,I_1(x)\right] + \frac{x}{2}\,\left[x^3\left(x^4-1408\right)\,J_0(x)\,I_0(x) + \right.$$

$$+(-3072+3584\,x^2+256\,x^4-8\,x^6)\,J_0(x)\,I_1(x) + (3072+3584\,x^2-256\,x^4-8\,x^6)\,J_1(x)\,I_0(x) +$$

$$\left. +80\,x\left(x^4-128\right)\,J_1(x)\,K_1(x)\right]$$

$$\int x^9\,\ln x\,J_1(x)\,K_1(x)\,dx = \frac{x^2\,\ln x}{2}\,\left[8\,x^2\left(192-x^4\right)\,J_0(x)\,K_0(x) + \right.$$

$$+x\left(3072+384\,x^2-32\,x^4-x^6\right)\,J_0(x)\,K_1(x) - x\left(3072-384\,x^2-32\,x^4+x^6\right)\,J_1(x)\,K_0(x) -$$

$$\left. -(6144-192\,x^4)\,J_1(x)\,K_1(x)\right] + \frac{x}{2}\,\left[x^3\left(1408-x^4\right)\,J_0(x)\,K_0(x) + \right.$$

$$+(-3072+3584\,x^2+256\,x^4-8\,x^6)\,J_0(x)\,K_1(x) + (-3072-3584\,x^2+256\,x^4+8\,x^6)\,J_1(x)\,K_0(x) +$$

$$\left. +80\,x\left(x^4-128\right)\,J_1(x)\,K_1(x)\right]$$

c) Integrals with $I_\nu(x)\,K_\nu(x)$:

$$\int x\,\ln x\,I_0(x)\,K_0(x)\,dx =$$

$$= \frac{x^2\,\ln x}{2}\,[I_0(x)K_0(x) + I_1(x)K_1(x)] - \frac{x}{2}\,[x\,I_0(x)K_0(x) + I_0(x)K_1(x) + x\,I_1(x)K_1(x)]$$

$$\int x\,\ln x\,I_1(x)\,K_1(x)\,dx = \frac{x\,\ln x}{2}\,[x\,I_0(x)K_0(x) + I_0(x)K_1(x) - I_1(x)K_0(x) + x\,I_1(x)K_1(x)] +$$

$$+\frac{1}{2}\,\left[(1-x^2)\,I_0(x)K_0(x) - x\,I_0(x)K_1(x) - x^2\,I_1(x)K_1(x)\right]$$

$$\int x^3\,\ln x\,I_0(x)\,K_0(x)\,dx =$$

$$= \frac{x^2\,\ln x}{6}\,\left[\,x^2\,I_0(x)K_0(x) - x\,I_0(x)K_1(x) + x\,I_1(x)K_0(x) + (x^2+2)\,I_1(x)K_1(x)\right] -$$

$$-\frac{x}{36}\,\left[2x(x^2+3)\,I_0(x)K_0(x) + (x^2-4)\,I_0(x)K_1(x) - (x^2+16)\,I_1(x)K_0(x) + 2x(x^2-1)\,I_1(x)K_1(x)\right]$$

$$\int x^3 \ln x\, I_1(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \ln x}{6} \left[x^2\, I_0(x)K_0(x) + 2x\, I_0(x)K_1(x) - 2x\, I_1(x)K_0(x) + (x^2-4)\, I_1(x)K_1(x) \right] -$$

$$- \frac{x}{36} \left[2x(x^2-6)\, I_0(x)K_0(x) + (x^2-4)\, I_0(x)K_1(x) - (x^2-20)\, I_1(x)K_0(x) + 2x(x^2-1)\, I_1(x)K_1(x) \right]$$

$$\int x^5 \ln x\, I_0(x)\, K_0(x)\, dx =$$

$$= \frac{x^2 \ln x}{30} \left[x^2 \left(3\,x^2 - 8\right) I_0(x)\, K_0(x) - 2\,x \left(8 + 3\,x^2\right) I_0(x)\, K_1(x) + \right.$$

$$\left. +2\,x \left(8 + 3\,x^2\right) I_1(x)\, K_0(x) + (32 + 16\,x^2 + 3\,x^4)\, I_1(x)\, K_1(x) \right] +$$

$$+ \frac{x}{900} \left[-2\,x \left(240 + 56\,x^2 + 9\,x^4\right) I_0(x)\, K_0(x) + (-9\,x^4 - 344\,x^2 + 1376)\, I_1(x)\, K_0(x) + \right.$$

$$\left. +(9\,x^4 + 344\,x^2 + 2336)\, I_0(x)\, K_1(x) - 2\,x \left(9\,x^4 - 52\,x^2 - 344\right) I_1(x)\, K_1(x) \right]$$

$$\int x^5 \ln x\, I_1(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \ln x}{10} \left[x^2(x^2+4)\, I_0(x)\, K_0(x) + x(3x^2+8)\, I_0(x)\, K_1(x) - x(3x^2+8)\, I_1(x)\, K_0(x) + \right.$$

$$\left. +(x^4 - 8x^2 - 16)\, I_1(x)\, K_1(x) \right] + \frac{x}{300} \left[2\,x \left(120 + 23\,x^2 - 3\,x^4\right) I_0(x)\, K_0(x) - \right.$$

$$\left. -(3\,x^4 + 608 - 152\,x^2)\, I_0(x)\, K_1(x) + (3\,x^4 - 152\,x^2 - 1088)\, I_1(x)\, K_0(x) - 2x \left(152 + 16\,x^2 + 3x^4\right) I_1(x)\, K_1(x) \right]$$

$$\int x^7 \ln x\, I_0(x)\, K_0(x)\, dx = \frac{x^2 \ln x}{70} \left[x^2 \left(5\,x^4 - 144 - 36\,x^2\right) I_0(x)\, K_0(x) - \right.$$

$$-3\,x \left(96 + 36\,x^2 + 5\,x^4\right) I_0(x)\, K_1(x) + 3\,x \left(96 + 36\,x^2 + 5\,x^4\right) I_1(x)\, K_0(x) +$$

$$\left. +(576 + 288\,x^2 + 54\,x^4 + 5\,x^6)\, I_1(x)\, K_1(x) \right] + \frac{x}{4900} \left[-2\,x \left(10080 + 4152\,x^2 + 303\,x^4 + 25\,x^6\right) I_0(x)\, K_0(x) - \right.$$

$$-(86592 + 21648\,x^2 + 3078\,x^4 + 25\,x^6)\, I_0(x)\, K_1(x) + \left(126912 + 21648x^2 + 3078\,x^+ 25\,x^6\right) I_1(x)\, K_0(x) +$$

$$\left. +2\,x \left(21648 + 5784\,x^2 + 297\,x^4 - 25\,x^6\right) I_1(x)\, K_1(x) \right]$$

$$\int x^7 \ln x\, I_1(x)\, K_1(x)\, dx = \frac{x^2 \ln x}{70} \left[x^2 \left(192 + 48\,x^2 + 5\,x^4\right) I_0(x)\, K_0(x) + \right.$$

$$+4\,x \left(96 + 36\,x^2 + 5\,x^4\right) I_0(x)\, K_1(x) - 4\,x \left(96 + 36\,x^2 + 5\,x^4\right) I_1(x)\, K_0(x) +$$

$$\left. +(5\,x^6 - 768 - 384\,x^2 - 72\,x^4)\, I_1(x)\, K_1(x) \right] + \frac{x}{4900} \left[2\,x \left(13440 + 5256\,x^2 + 334\,x^4 - 25\,x^6\right) I_0(x)\, K_0(x) + \right.$$

$$+(27744\,x^2 + 3684\,x^4 - 25\,x^6 - 110976)\, I_0(x)\, K_1(x) + \left(25\,x^6 - 3684\,x^4 - 27744x^2 - 164736\right) I_1(x)\, K_0(x) -$$

$$\left. -2\,x \left(27744 + 7152\,x^2 + 291\,x^4 + 25\,x^6\right) I_1(x)\, K_1(x) \right]$$

$$\int x^9 \ln x\, I_0(x)\, K_0(x)\, dx = \frac{x^2 \ln x}{630} \left[x^2 \left(35\,x^6 - 18432 - 4608\,x^2 - 480\,x^4\right) I_0(x)\, K_0(x) - \right.$$

$$-4\,x \left(9216 + 3456\,x^2 + 480\,x^4 + 35\,x^6\right) I_0(x)\, K_1(x) + 4\,x \left(9216 + 3456\,x^2 + 480\,x^4 + 35\,x^6\right) I_1(x)\, K_0(x) +$$

$$\left. +(73728 + 36864\,x^2 + 6912\,x^4 + 640\,x^6 + 35\,x^8)\, I_1(x)\, K_1(x) \right] +$$

$$+ \frac{x}{396900} \left[-2\,x \left(11612160 + 6315264\,x^2 + 732096\,x^4 + 24600\,x^6 + 1225\,x^8\right) I_0(x)\, K_0(x) - \right.$$

$$-(31067136\,x^2 + 5844096\,x^4 + 348000\,x^6 + 1225\,x^8 - 124268544)\, I_0(x)\, K_1(x) +$$

$$+\left(170717184+31067136x^2+5844096\,x^4+348000\,x^6+1225\,x^8\right)\,I_1(x)\,K_0(x)+$$
$$+2\,x\left(31067136+9727488\,x^2+916704\,x^4+24400\,x^6-1225\,x^8\right)\,I_1(x)\,K_1(x)\Big]$$

$$\int x^9\,\ln x\,I_1(x)\,K_1(x)\,dx=\frac{x^2\,\ln x}{126}\,\Big[x^2\left(4608+1152\,x^2+120\,x^4+7\,x^6\right)\,I_0(x)\,K_0(x)+$$
$$+x\left(9216+3456\,x^2+480\,x^4+35\,x^6\right)\,I_0(x)\,K_1(x)-x\left(9216+3456\,x^2+480\,x^4+35\,x^6\right)\,I_1(x)\,K_0(x)+$$
$$+(7\,x^8-18432-9216\,x^2-1728\,x^4-160\,x^6)\,I_1(x)\,K_1(x)\Big]+$$
$$+\frac{x}{79380}\,\Big[2\,x\left(2903040+1542528\,x^2+173952\,x^4+5205\,x^6-245\,x^8\right)\,I_0(x)\,K_0(x)+$$
$$+(7621632\,x^2+1406592\,x^4+79440\,x^6-245\,x^8-30486528)\,I_0(x)\,K_1(x)+$$
$$+\left(245\,x^8-7621632x^2-1406592\,x^4-79440\,x^6-42098688\right)\,I_1(x)\,K_0(x)-$$
$$-2\,x\left(7621632+2359296\,x^2+215568\,x^4+4840\,x^6+245\,x^8\right)\,I_1(x)\,K_1(x)\Big]$$

About recurrence relations see page 308.

d) Integrals with $I_\nu(x)\,K_{1-\nu}(x)$:

$$\int x^2\,\ln x\,I_0(x)\,K_1(x)\,dx=\frac{x^2\,\ln x}{4}\,[x\,I_0(x)K_1(x)+x\,I_1(x)K_0(x)+2\,I_1(x)K_1(x)]-$$
$$-\frac{x}{8}\,\big[2x\,I_0(x)K_0(x)+(x^2+4)\,I_0(x)K_1(x)+x^2\,I_1(x)K_0(x)+2x\,I_1(x)K_1(x)\big]$$

$$\int x^2\,\ln x\,I_1(x)\,K_0(x)\,dx=\frac{x^2\,\ln x}{4}\,[x\,I_1(x)K_0(x)+x\,I_0(x)K_1(x)-2\,I_1(x)K_1(x)]+$$
$$+\frac{x}{8}\,\big[2x\,I_0(x)K_0(x)-(x^2+4)\,I_1(x)K_0(x)-x^2\,I_0(x)K_1(x)+2x\,I_1(x)K_1(x)\big]$$

$$\int x^4\,\ln x\,I_0(x)\,K_1(x)\,dx=$$
$$=\frac{x^2\,\ln x}{24}\,\big[-4x^2\,I_0(x)K_0(x)+x(3x^2-8)\,I_0(x)K_1(x)+x(3x^2+8)\,I_1(x)K_0(x)+8(x^2+2)\,I_1(x)K_1(x)\big]-$$
$$-\frac{x}{288}\,\big[8x(x^2+12)\,I_0(x)K_0(x)+(9x^4+40x^2-160)\,I_0(x)K_1(x)+$$
$$+(9x^4-40x^2-352)\,I_1(x)K_0(x)+8x(x^2-10)\,I_1(x)K_1(x)\big]$$

$$\int x^4\,\ln x\,I_1(x)\,K_0(x)\,dx=$$
$$=\frac{x^2\,\ln x}{24}\,\big[4x^2\,I_0(x)K_0(x)+x(3x^2+8)\,I_0(x)K_1(x)+x(3x^2-8)\,I_1(x)K_0(x)-8(x^2+2)\,I_1(x)K_1(x)\big]+$$
$$+\frac{x}{288}\,\big[8x(x^2+12)\,I_0(x)K_0(x)-(160-40x^2+9x^4)\,I_0(x)K_1(x)-$$
$$-(9x^4+40x^2+352)\,I_1(x)K_0(x)+8x(x^2-10)\,I_1(x)K_1(x)\big]$$

$$\int x^6\,\ln x\,I_0(x)\,K_1(x)\,dx=$$
$$=\frac{x^2\,\ln x}{60}\,\Big[-12\,x^2\left(4+x^2\right)\,I_0(x)\,K_0(x)+x\left(-96-36\,x^2+5\,x^4\right)\,I_0(x)\,K_1(x)+$$
$$+x\left(96+36\,x^2+5\,x^4\right)\,I_1(x)\,K_0(x)+(192+96\,x^2+18\,x^4)\,I_1(x)\,K_1(x)\Big]+$$
$$+\frac{x}{1800}\,\Big[-6\,x\left(480+152\,x^2+3\,x^4\right)\,I_0(x)\,K_0(x)-(25\,x^6+234\,x^4+2544\,x^2-10176)\,I_1(x)\,K_0(x)-$$
$$-(25\,x^6-234\,x^4-2544\,x^2-15936)\,I_0(x)\,K_1(x)+6\,x\left(848+184\,x^2-3\,x^4\right)\,I_1(x)\,K_1(x)\Big]$$

$$\int x^6 \, \ln x \, I_1(x) \, K_0(x) \, dx =$$

$$= \frac{x^2 \, \ln x}{60} \, \left[12 \, x^2 \left(4 + x^2\right) \, I_0(x) \, K_0(x) + x \left(5 \, x^4 + 36 \, x^2 + 96\right) \, I_0(x) \, K_1(x) + \right.$$
$$\left. + x \left(5 \, x^4 - 36 \, x^2 - 96\right) \, I_1(x) \, K_0(x) - (192 + 96 \, x^2 + 18 \, x^4) \, I_1(x) \, K_1(x) \right] +$$
$$+ \frac{x}{1800} \, \left[6x \left(480 + 152 \, x^2 + 3 \, x^4\right) \, I_0(x) \, K_0(x) + (-10176 + 2544 \, x^2 + 234 \, x^4 - 25 \, x^6) \, I_0(x) \, K_1(x) - \right.$$
$$\left. -(15936 + 2544 \, x^2 + 234 \, x^4 + 25 \, x^6) \, I_1(x) \, K_0(x) + 6 \, x \left(3 \, x^4 - 848 - 184 \, x^2\right) \, I_1(x) \, K_1(x) \right]$$

$$\int x^8 \, \ln x \, I_0(x) \, K_1(x) \, dx = \frac{x^2 \, \ln x}{560} \, \left[-24 \, x^2 \left(192 + 48 \, x^2 + 5 \, x^4\right) \, I_0(x) \, K_0(x) + \right.$$
$$+ x \left(35 \, x^6 - 9216 - 3456 \, x^2 - 480 \, x^4\right) \, I_0(x) \, K_1(x) + x \left(9216 + 3456 \, x^2 + 480 \, x^4 + 35 \, x^6\right) \, I_1(x) \, K_0(x) +$$
$$\left. + (18432 + 9216 \, x^2 + 1728 \, x^4 + 160 \, x^6) \, I_1(x) \, K_1(x) \right] +$$
$$+ \frac{x}{156800} \, \left[-32 \, x \left(80640 + 38256 \, x^2 + 3684 \, x^4 + 25 \, x^6\right) \, I_0(x) \, K_0(x) - \right.$$
$$-(3093504 \, x^2 + 514944 \, x^4 + 20000 \, x^6 + 1225 \, x^8 - 12374016) \, I_0(x) \, K_1(x) +$$
$$+ \left(17534976 + 3093504 x^2 + 514944 \, x^4 + 20000 \, x^6 - 1225 \, x^6\right) \, I_1(x) \, K_0(x) +$$
$$\left. + 32 \, x \left(193344 + 56352 \, x^2 + 4266 \, x^4 - 25 \, x^6\right) \, I_1(x) \, K_1(x) \right]$$

$$\int x^8 \, \ln x \, I_1(x) \, K_0(x) \, dx = \frac{x^2 \, \ln x}{560} \, \left[24 \, x^2 \left(192 + 48 \, x^2 + 5 \, x^4\right) \, I_0(x) \, K_0(x) + \right.$$
$$+ x \left(9216 + 3456 \, x^2 + 480 \, x^4 + 35 \, x^6\right) \, I_0(x) \, K_1(x) + x \left(35 \, x^6 - 9216 - 3456 \, x^2 - 480 \, x^4\right) \, I_1(x) \, K_0(x) -$$
$$\left. - (18432 + 9216 \, x^2 + 1728 \, x^4 + 160 \, x^6) \, I_1(x) \, K_1(x) \right] +$$
$$+ \frac{x}{156800} \, \left[32 \, x \left(80640 + 38256 \, x^2 + 3684 \, x^4 + 25 \, x^6\right) \, I_0(x) \, K_0(x) + \right.$$
$$+ (3093504 \, x^2 + 514944 \, x^4 + 20000 \, x^6 - 1225 \, x^8 - 12374016) \, I_0(x) \, K_1(x) -$$
$$- \left(17534976 + 3093504 x^2 + 514944 \, x^4 + 20000 \, x^6 + 1225 \, x^8\right) \, I_1(x) \, K_0(x) +$$
$$\left. + +32 \, x \left(-193344 - 56352 \, x^2 - 4266 \, x^4 + 25 \, x^6\right) \, I_1(x) \, K_1(x) \right]$$

$$\int x^{10} \, \ln x \, I_0(x) \, K_1(x) \, dx = \frac{x^2 \, \ln x}{1260} \, \left[-40 \, x^2 \left(4608 + 1152 \, x^2 + 120 \, x^4 + 7 \, x^6\right) \, I_0(x) \, K_0(x) + \right.$$
$$+ x \left(63 \, x^8 - 368640 - 138240 \, x^2 - 19200 \, x^4 - 1400 \, x^6\right) \, I_0(x) \, K_1(x) +$$
$$+ x \left(368640 + 138240 \, x^2 + 19200 \, x^4 + 1400 \, x^6 + 63 \, x^8\right) \, I_1(x) \, K_0(x) +$$
$$\left. + (737280 + 368640 \, x^2 + 69120 \, x^4 + 6400 \, x^6 + 350 \, x^8) \, I_1(x) \, K_1(x) \right] +$$
$$+ \frac{x}{793800} \, \left[-10 \, x \left(23224320 + 13791744 \, x^2 + 1754496 \, x^4 + 79440 \, x^6 + 245 \, x^8\right) \, I_0(x) \, K_0(x) - \right.$$
$$-(333895680 \, x^2 + 67150080 \, x^4 + 4689600 \, x^6 + 100450 \, x^8 + 3969 \, x^{10} - 1335582720) \, I_0(x) \, K_1(x) -$$
$$- \left(3969 \, x^{10} - 1800069120 - 333895680 x^2 - 67150080 \, x^4 - 4689600 \, x^6 - 100450 \, x^8\right) \, I_1(x) \, K_0(x) -$$
$$\left. -10 \, x \left(245 \, x^8 - 66779136 - 21777408 \, x^2 - 2268864 \, x^4 - 89120 \, x^6\right) \, I_1(x) \, K_1(x) \right]$$

$$\int x^{10} \, \ln x \, I_1(x) \, K_0(x) \, dx = \frac{x^2 \, \ln x}{1260} \, \left[40 \, x^2 \left(4608 + 1152 \, x^2 + 120 \, x^4 + 7 \, x^6\right) \, I_0(x) \, K_0(x) + \right.$$
$$+ x \left(368640 + 138240 \, x^2 + 19200 \, x^4 + 1400 \, x^6 + 63 \, x^8\right) \, I_0(x) \, K_1(x) +$$
$$+ x \left(63 \, x^8 - 368640 - 138240 \, x^2 - 19200 \, x^4 - 1400 \, x^6\right) \, I_1(x) \, K_0(x) -$$

$$-(737280+368640\,x^2+69120\,x^4+6400\,x^6+350\,x^8)\,I_1(x)\,K_1(x)\big]+$$
$$+\frac{x}{793800}\,\big[10\,x\left(23224320+13791744\,x^2+1754496\,x^4+79440\,x^6+245\,x^8\right)\,I_0(x)\,K_0(x)+$$
$$+(333895680\,x^2+67150080\,x^4+4689600\,x^6+100450\,x^8-3969\,x^{10}-1335582720)\,I_0(x)\,K_1(x)-$$
$$-\left(1800069120+333895680x^2+67150080\,x^4+4689600\,x^6+100450\,x^8+3969\,x^{10}\right)\,I_1(x)\,K_0(x)+$$
$$+10\,x\left(245\,x^8-66779136-21777408\,x^2-2268864\,x^4-89120\,x^6\right)\,I_1(x)\,K_1(x)\big]$$

Recurrence Relations:

$$\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\Big[((2n+1)x^3+4n^2(n+1)x)\,I_0(x)\,K_0(x)-2n^2x^2\,I_0(x)\,K_1(x)+\right.$$
$$+2n(n+1)x^2\,I_1(x)\,K_0(x)+(2n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x-x\,[x^2+2n(n+1)]\,I_0(x)\,K_0(x)-x^2\,I_0(x)\,K_1(x)-$$
$$\left.-\,x^3\,I_1(x)\,K_1(x)\right\}-\frac{4n^3(n+1)}{(2n-1)^2}\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx+\frac{2n^2}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx-$$
$$-\frac{4n^2(n+1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,I_1(x)\,K_1(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\Big[((2n+1)x^3-2n(2n^2+2n+1)x)\,I_0(x)\,K_0(x)+\right.$$
$$+(2n^2+4n+1)x^2\,I_0(x)\,K_1(x)-(2n^2+2n+1)x^2\,I_1(x)\,K_0(x)+(2n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x-$$
$$\left.-[x^3-(2n^2+2n+1)x]\,I_0(x)\,K_0(x)\,-x^2\,I_0(x)\,K_1(x)-x^3\,I_1(x)\,K_1(x)\right\}+$$
$$+\frac{2n^2(2n^2+2n+1)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx-\frac{2n(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx+$$
$$+\frac{2n(2n^2+2n+1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_0(x)\,K_1(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\Big[(2n^2(4n^2+5n+2)x-n(2n+1)x^3)\,I_0(x)\,K_0(x)+\right.$$
$$+\left(\frac{(2n+1)^2x^2}{2(n+1)}-n^2\right)\,x^2\,I_0(x)\,K_1(x)+\left(\frac{(2n+1)^2x^2}{2(n+1)}+n(4n^2+5n+2)\right)\,x^2\,I_1(x)\,K_0(x)+$$
$$+\,(2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x-\frac{x^3+2n(4n^2+5n+2)x}{2}\,I_0(x)\,K_0(x)-$$
$$\left.-\left(\frac{(2n+1)^2\,x^2}{2(n+1)^2}+2n^2+2n+1\right)\,x^2\,\;I_0(x)\,K_1(x)-\frac{(2n+1)^2x^4}{4(n+1)^2}\,I_1(x)\,K_0(x)-\frac{x^3}{2}\,I_1(x)\,K_1(x)\right\}-$$
$$-\frac{2n^3(4n^2+5n+2)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx+$$
$$+\frac{2n^2(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx-\frac{2n^2(4n^2+5n+2)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_1(x)\,K_0(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\Big[((2n+1)nx^3-2n^2(4n^2+5n+2)x)\,I_0(x)\,K_0(x)+\right.$$
$$+\left(\frac{(2n+1)^2}{2(n+1)}\,x^2+n^2\right)\,x^2\,I_0(x)\,K_1(x)+\left(\frac{(2n+1)^2}{2(n+1)}\,x^2-n(4n^2+5n+2)\right)\,x^2\,I_1(x)\,K_0(x)-$$
$$-\,(2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x+\left(\frac{x^3}{2}+n(4n^2+5n+2)x\right)\,I_0(x)\,K_0(x)+$$

$$\left(2n^2+2n+1-\frac{(2n+1)^2\,x^2}{4(n+1)^2}\right)\,x^2\,I_0(x)\,K_1(x)-\frac{(2n+1)^2x^4}{(2n+2)^2}\,I_1(x)\,K_0(x)+$$

$$+\;\frac{x^3}{2}\,I_1(x)\,K_1(x)\Bigg\}+\frac{2n^3(4n^2+5n+2)}{(2n+1)^2}\,\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx-$$

$$-\frac{2n^2(n+1)}{2n+1}\,\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx+\frac{2n^2(4n^2+5n+2)}{(2n+1)^2}\,\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

2.4.3. Some Cases of $\int x^n \,\ln x\, Z_\mu(x)\, Z_\nu^*(\alpha x)\, dx$:

Some integrals of the type $\int x^n\, Z_0(\alpha x)\; Z_1^*(\beta x)\, dx$ are already found. Then the integral

$$\int x^n\, \ln x\, Z_\mu\left(\frac{x}{\alpha}\right)\, Z_\nu^*(x)\, dx \quad \text{may be expressed by} \quad \int x^n\, \ln x\, Z_\mu(x)\, Z_\nu^*(\alpha x)\, dx\,,$$

but nevertheless both integrals are given in the following.

Numerical values of some α :

$(\sqrt{3}+1)/\sqrt{2} = \sqrt{2+\sqrt{3}} = 1.93185\;16526,\; (\sqrt{3}-1)/\sqrt{2} = \sqrt{2-\sqrt{3}} = 0.51763\;80902$

$\sqrt{3+\sqrt{6}} = 2.33441\;42183,\; \sqrt{3-\sqrt{6}} = 0.74196\;37843$

<u>**n = 4**</u>

$$\int x^4 \ln x\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right)\, dx =$$

$$= \frac{x^2\,\ln x}{3}\left[-2x^2\, J_0(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,x\, J_0(x)\, I_1\left(\frac{x}{\sqrt{2}}\right) + 4x\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\,(x^2-8)\, J_1(x)\, I_1\left(\frac{x}{\sqrt{2}}\right)\right] -$$

$$-\frac{x}{81}\left[24x\, J_0(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(32-9x^2)\, J_0(x)\, I_1\left(\frac{x}{\sqrt{2}}\right) - (18x^2+176)\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + \right.$$

$$\left. +168\sqrt{2}\,x\, J_1(x)\, I_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right)\, dx =$$

$$= -\frac{x^2\,\ln x}{3}\left[2x^2\, J_0(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,x\, J_0(x)\, K_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) + \right.$$

$$\left. +\sqrt{2}\,(x^2-8)\, J_1(x)\, K_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\left[24x\, J_0(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(9x^2-32)\, J_0(x)\, K_1\left(\frac{x}{\sqrt{2}}\right) - \right.$$

$$\left. -(18x^2+176)\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\,x\, J_1(x)\, K_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, J_0\left(\frac{x}{\sqrt{2}}\right)\, I_1(x)\, dx =$$

$$= \frac{x^2\,\ln x}{3}\left[2x^2\, I_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) - 4\sqrt{2}\,x\, I_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) - 4x\, I_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) + \right.$$

$$\left. +\sqrt{2}\,(x^2+8)\, I_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\left[24x\, I_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(32+9x^2)\, I_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + \right.$$

$$\left. +(18x^2-176)\, I_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\,x\, I_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right)\, dx = \frac{x^2\,\ln x}{3}\cdot$$

$$\cdot\left[-2x^2\, K_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,x\, K_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\,(x^2+8)\, K_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right] +$$

$$+\frac{x}{81}\left[24x\, K_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(9x^2+32)\, K_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) + (176-18x^2)\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + \right.$$

$$\left. +\, 168\sqrt{2}\,x\, K_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4\, \ln x\, J_0(x)\, I_1\left(\sqrt{2}\,x\right)\, dx =$$

$$= \frac{x^2\,\ln x}{3}\left[\sqrt{2}\,x^2\, J_0(x)\, I_0\left(\sqrt{2}\,x\right) - 2x\, J_0(x)\, I_1\left(\sqrt{2}\,x\right) - 2\sqrt{2}\,x\, J_1(x)\, I_0\left(\sqrt{2}\,x\right) + (x^2+4)\, J_1(x)\, I_1\left(\sqrt{2}\,x\right)\right] -$$

$$-\frac{81}{x}\left[6\sqrt{2}\,x\,J_0(x)\,I_0\left(\sqrt{2}\,x\right)+(9x^2-44)\,J_0(x)\,I_1\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+2\sqrt{2}\,(9x^2+16)\,J_1(x)\,I_0\left(\sqrt{2}\,x\right)-84x\,J_1(x)\,I_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-\sqrt{2}\,x^2\,J_0(x)\,K_0\left(\sqrt{2}\,x\right)-2x\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+\,2\sqrt{2}\,x\,J_1(x)\,K_0\left(\sqrt{2}\,x\right)+(x^2+4)\,J_1(x)\,K_1\left(\sqrt{2}\,x\right)\right]+\frac{x}{81}\left[6\sqrt{2}\,x\,J_0(x)\,K_0\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+\,(44-9x^2)\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)+2\sqrt{2}\,(9x^2+16)\,J_1(x)\,K_0\left(\sqrt{2}\,x\right)+84x\,J_1(x)\,K_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,I_0(x)\,J_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-2\sqrt{2}\,x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+\,2x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+2\sqrt{2}\,x\,I_1(x)\,J_0\left(\sqrt{2}\,x\right)+(x^2-4)\,I_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]+\frac{x}{81}\left[-6\sqrt{2}\,x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+\,(9x^2+44)\,I_0(x)\,J_1\left(\sqrt{2}\,x\right)+2\sqrt{2}\,(9x^2-16)\,I_1(x)\,J_0\left(\sqrt{2}\,x\right)-84x\,I_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-\sqrt{2}\,x^2\,K_0(x)\,J_0\left(\sqrt{2}\,x\right)+2x\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)-\right.$$
$$\left.-2\sqrt{2}\,x\,K_1(x)\,J_0\left(\sqrt{2}\,x\right)-(x^2-4)\,K_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]+\frac{x}{81}\left[-6\sqrt{2}\,x\,K_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$
$$\left.+(9x^2+44)\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)-2\sqrt{2}\,(9x^2-16)\,K_1(x)\,J_0\left(\sqrt{2}\,x\right)+84x\,K_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]$$

$\underline{\mathbf{m=5}}$

$$\int x^5\;\ln x\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\,\ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$
$$+3\,x\sqrt{2}\left(-8\,\sqrt{3}+16+\sqrt{3}x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$
$$-3\,x\left(8-8\,\sqrt{3}-3\,x^2+\sqrt{3}x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$
$$\left.-12\,\sqrt{2}\left(-4\,\sqrt{3}+8+\sqrt{3}x^2-x^2\right)\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]+$$
$$+\frac{x}{594}\left[33\,\left(\sqrt{3}-1\right)x\left(4\,\sqrt{3}-4-3\,x^2\right)\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$
$$-2\,\sqrt{2}\left(8\,\sqrt{3}-15\right)\left(4\,\sqrt{3}-64+33\,x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$
$$-2\,\left(-3+5\,\sqrt{3}\right)\left(-116+56\,\sqrt{3}-33\,x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$
$$\left.-33\,\sqrt{2}\left(\sqrt{3}-1\right)x\left(26\,\sqrt{3}-26+3\,x^2\right)\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\;\ln x\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\,\ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$
$$-\sqrt{2}\sqrt{3}x\left(16\,\sqrt{3}-24+3\,x^2\right)\,J_0(x){\cdot}K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\left(3-\sqrt{3}\right)x\left(8\,\sqrt{3}+3\,x^2\right)\,J_1(x){\cdot}K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$+12\sqrt{2}\left(\sqrt{3}-1\right)\left(x^2+2\sqrt{3}-2\right)\,J_1(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\Bigg]\,\ln x+$$

$$+\frac{x}{594}\left[(33\,\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4-3\,x^2\right))\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$

$$+2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64+33\,x^2\right)\,J_0(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-2\,\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}-33\,x^2\right)\,J_1(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$\left.+33\sqrt{2}\left(\sqrt{3}-1\right)x\left(-26+26\sqrt{3}+3\,x^2\right)\,J_1(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,\ln x\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\,\ln x}{18}\left[12\,x^2\left(1+\sqrt{3}\right)\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+3\,x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$+3\,x\left(-8-8\sqrt{3}+3\,x^2+\sqrt{3}x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\,\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3\,x^2\right)x\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33\,x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33\,x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$\left.+33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}-3\,x^2\right)x\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,\ln x\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\,\ln x}{18}\left[12\,x^2\left(1+\sqrt{3}\right)\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-\right.$$

$$-3\,x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)\,J_0(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$+3\,x\left(-8-8\sqrt{3}+3\,x^2+\sqrt{3}x^2\right)\,J_1(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$\left.+12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)\,J_1(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]\,\ln x+$$

$$+\frac{x}{594}\left[33\,\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3\,x^2\right)x\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-\right.$$

$$-2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33\,x^2\right)\,J_0(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33\,x^2\right)\,J_1(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$-33\sqrt{2}\left(1+\sqrt{3}\right)x\left(26+26\sqrt{3}-3\,x^2\right)\,J_1(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\Bigg]$$

$$\int x^5\;\ln x\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\;\ln x}{6}\left[4\left(\sqrt{3}-1\right)\,x^2\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$

$$+\sqrt{2}\left(8\sqrt{3}-16+\sqrt{3}x^2\right)\,x\,I_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\left(-8+8\sqrt{3}-3\,x^2+\sqrt{3}x^2\right)\,x\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$\left.-4\sqrt{2}\left(4\sqrt{3}-8+\sqrt{3}x^2-x^2\right)\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\,\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4+3\,x^2\right)I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-2\sqrt{2}\left(8\sqrt{3}-15\right)\left(4\sqrt{3}-64-33\,x^2\right)\,I_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-2\,\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33\,x^2\right)\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$\left.+33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3\,x^2\right)x\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\;\ln x\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\;\ln x}{18}\left[-12\,x^2\left(1+\sqrt{3}\right)\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+\sqrt{6}\,x\left(24+16\sqrt{3}+3\,x^2\right)\,I_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\left(3+\sqrt{3}\right)\left(8\sqrt{3}+3\,x^2\right)x\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-\,12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right)\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\,\left(1+\sqrt{3}\right)x\left(4\sqrt{3}+4-3\,x^2\right)\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+2\sqrt{2}\left(8\sqrt{3}+15\right)\left(4\sqrt{3}+64+33\,x^2\right)\,I_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33\,x^2\right)\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}+3\,x^2\right)x\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\;\ln x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\;\frac{x^2\;\ln x}{18}\left[12\left(\sqrt{3}-1\right)\,x^2\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-\sqrt{6}\left(-24+16\sqrt{3}-3\,x^2\right)x\,K_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\left(-3+\sqrt{3}\right)\left(8\sqrt{3}-3\,x^2\right)x\,K_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$\left.-12\sqrt{2}\left(\sqrt{3}-1\right)\left(-2+2\sqrt{3}-x^2\right)\,K_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\,\left(\sqrt{3}-1\right)\left(4\sqrt{3}-4+3x^2\right)x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64-33x^2\right)\,K_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$+2\,\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33x^2\right)\,K_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$\left.-33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3x^2\right)x\,K_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\ \ln x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\,dx=\ \frac{x^2\,\ln x}{18}\left[-12\,x^2\left(1+\sqrt{3}\right)\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+\sqrt{6}\left(24+16\sqrt{3}+3x^2\right)x\,K_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-\left(3+\sqrt{3}\right)x\left(8\sqrt{3}+3x^2\right)\,K_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$\left.+12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right)\,K_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\,\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4-3x^2\right)x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64+33x^2\right)\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$+2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33x^2\right)\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+$$

$$\left.+33\sqrt{2}\left(1+\sqrt{3}\right)x\left(26\sqrt{3}+26+3x^2\right)\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]$$

$\underline{\mathbf{n=6}}$

$$\int x^6\ \ln x\,J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70+30\sqrt{6}}\,\left[5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+10\,\left(5+2\sqrt{6}\right)x\left(-40-x^2+16\sqrt{6}\right)\,J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-2\sqrt{3+\sqrt{6}}\left(1+\sqrt{6}\right)x\left(-8+8\sqrt{6}+5x^2\right)\,J_1(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$\left.+5\,\left(2+\sqrt{6}\right)\left(-32+8x^2+x^4+16\sqrt{6}\right)\,J_1(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{625\left(4+\sqrt{6}\right)^3\left(2+\sqrt{6}\right)\left(11+4\sqrt{6}\right)}\cdot$$

$$\cdot\left[-\frac{100}{73}\sqrt{3+\sqrt{6}}\left(331+134\sqrt{6}\right)x\left(-1825x^2+436\sqrt{6}+176\right)J_0(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+2\,\left(664+271\sqrt{6}\right)\left(-18552\sqrt{6}+47168+11850x^2\sqrt{6}-30400x^2-625x^4\right)\,J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-4\sqrt{3+\sqrt{6}}\left(149+61\sqrt{6}\right)\left(5472\sqrt{6}-11448-2850x^2+3650x^2\sqrt{6}+625x^4\right)\,J_1(x){\cdot}I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+\frac{100}{67}\left(1672+683\sqrt{6}\right)x\left(1675\,x^2-21804+10706\sqrt{6}\right)J_1(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\,\ln x}{70-30\sqrt{6}}\left[5\sqrt{3-\sqrt{6}}\left(2-\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-\right.$$

$$-10\left(5-2\sqrt{6}\right)x\left(40+x^2+16\sqrt{6}\right)J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(1-\sqrt{6}\right)x\left(8+8\sqrt{6}-5\,x^2\right)J_1(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$\left.+5\left(2-\sqrt{6}\right)\left(-32+8\,x^2+x^4-16\sqrt{6}\right)J_1(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{625\,(4-\sqrt{6})^4\,(-2+\sqrt{6})}\cdot$$

$$\cdot\left[\frac{100}{73}\sqrt{3-\sqrt{6}}\left(-104+41\sqrt{6}\right)x\left(1825\,x^2-176+436\sqrt{6}\right)J_0(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-\right.$$

$$-4\left(-103+42\sqrt{6}\right)\left(-18552\sqrt{6}-47168+11850\,x^2\sqrt{6}+30400\,x^2+625\,x^4\right)J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4\sqrt{3-\sqrt{6}}\left(-46+19\sqrt{6}\right)\left(11448+5472\sqrt{6}+3650\,x^2\sqrt{6}+2850\,x^2-625\,x^4\right)J_1(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$-\frac{200}{67}\left(-259+106\sqrt{6}\right)x\left(-1675\,x^2+10706\sqrt{6}+21804\right)J_1(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\,\ln x}{70+30\sqrt{6}}\left[-5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+10\left(5+2\sqrt{6}\right)x\left(-40-x^2+16\sqrt{6}\right)J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3+\sqrt{6}}\left(\sqrt{6}+1\right)x\left(8\sqrt{6}-8+5\,x^2\right)J_1(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$\left.+5\left(2+\sqrt{6}\right)\left(-32+8\,x^2+x^4+16\sqrt{6}\right)J_1(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{152843750}\cdot\left[670\sqrt{3+\sqrt{6}}\left(-337+143\sqrt{6}\right)x\left(176+436\sqrt{6}-1825\,x^2\right)J_0(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+4891\left(3\sqrt{6}-2\right)\left(47168-18552\sqrt{6}+11850\,x^2\sqrt{6}-30400\,x^2-625\,x^4\right)J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3+\sqrt{6}}\left(4\sqrt{6}-11\right)\left(-11448+5472\sqrt{6}-2850\,x^2+3650\,x^2\sqrt{6}+625\,x^4\right)J_1(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$\left.-730\left(37\sqrt{6}-158\right)x\left(-21804+10706\sqrt{6}+1675\,x^2\right)J_1(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\,\ln x}{70-30\sqrt{6}}\left[-5\sqrt{3-\sqrt{6}}\left(-2+\sqrt{6}\right)\left(8-x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+\right.$$

$$+10\,\left(-5+2\sqrt{6}\right)\left(40+x^2+16\sqrt{6}\right)x\,J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(-1+\sqrt{6}\right)\left(8-5\,x^2+8\sqrt{6}\right)x\,J_1(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+5\,\left(-2+\sqrt{6}\right)\left(32-8\,x^2-x^4+16\sqrt{6}\right)\,J_1(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]+\frac{x}{61180296250-24977725625\sqrt{6}}\cdot$$

$$\cdot\Bigg[1675\sqrt{3-\sqrt{6}}\left(867\sqrt{6}-2128\right)\left(436\sqrt{6}-176+1825\,x^2\right)x\,J_0(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891\,\left(874\sqrt{6}-2141\right)\left(-47168-18552\sqrt{6}+30400\,x^2+11850\,x^2\sqrt{6}+625\,x^4\right)\,J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891\sqrt{3-\sqrt{6}}\left(393\sqrt{6}-962\right)\left(11448+5472\sqrt{6}+3650\,x^2\sqrt{6}+2850\,x^2-625\,x^4\right)\,J_1(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+3650\,\left(2202\sqrt{6}-5393\right)\left(10706\sqrt{6}+21804-1675\,x^2\right)x\,J_1(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70+30\sqrt{6}}\,\Bigg[5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8-x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-10\,\left(5+2\sqrt{6}\right)\left(x-4+2\sqrt{6}\right)\left(-x-4+2\sqrt{6}\right)x\,K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-2\sqrt{3+\sqrt{6}}\left(1+\sqrt{6}\right)\left(8\sqrt{6}-8-5\,x^2\right)x\,K_1(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-5\,\left(2+\sqrt{6}\right)\left(-32-8\,x^2+x^4+16\sqrt{6}\right)\,K_1(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]+$$

$$+\frac{x}{152843750}\Bigg[670\sqrt{3+\sqrt{6}}\left(-337+143\sqrt{6}\right)\left(176+436\sqrt{6}+1825\,x^2\right)x\,K_0(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+4891\,\left(3\sqrt{6}-2\right)\left(-47168+18552\sqrt{6}-30400\,x^2+11850\,x^2\sqrt{6}+625\,x^4\right)\,K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3+\sqrt{6}}\left(-11+4\sqrt{6}\right)\left(-5472\sqrt{6}+11448-2850x^2+3650x^2\sqrt{6}-625x^4\right)\,K_1(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+730\,\left(37\sqrt{6}-158\right)\left(-21804+10706\sqrt{6}-1675\,x^2\right)x\,K_1(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70-30\sqrt{6}}\,\Bigg[5\sqrt{3-\sqrt{6}}\left(-2+\sqrt{6}\right)\left(8+x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+10\,\left(-5+2\sqrt{6}\right)\left(-x+4+2\sqrt{6}\right)x\left(x+4+2\sqrt{6}\right)\,K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(-1+\sqrt{6}\right)x\left(8+5\,x^2+8\sqrt{6}\right)\,K_1(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$-5\,\left(-2+\sqrt{6}\right)\left(32+8\,x^2-x^4+16\sqrt{6}\right)\,K_1(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]+$$

$$+\frac{x}{152843750}\,\Bigg[670\sqrt{3-\sqrt{6}}\left(337+143\sqrt{6}\right)\left(436\sqrt{6}-176-1825\,x^2\right)x\,K_0(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891\,\left(3\sqrt{6}+2\right)\left(47168+18552\sqrt{6}+30400\,x^2+11850\,x^2\sqrt{6}-625\,x^4\right)\,K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3-\sqrt{6}}\left(11+4\sqrt{6}\right)\left(-11448-5472\sqrt{6}+3650x^2\sqrt{6}+2850x^2+625\,x^4\right)\,K_1(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+730\,\left(158+37\sqrt{6}\right)x\left(10706\sqrt{6}+21804+1675\,x^2\right)\,K_1(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Big]$$

2.4.4. Integrals of the type $\int x^{-1}\cdot \exp/\sin/\cos(2x)\,Z_\nu(x)\,Z_1(x)\,dx$

$$\int \frac{e^{2x}\,I_0(x)\,I_1(x)\,dx}{x} = e^{2x}\left[(1-x)\,I_0^2(x)+(2x-1)I_0(x)I_1(x)-x\,I_1^2(x)\right]$$

$$\int \frac{e^{2x}\,K_0(x)\,K_1(x)\,dx}{x} = e^{2x}\left[(x-1)\,K_0^2(x)+(2x-1)K_0(x)K_1(x)+x\,K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\,I_0(x)\,I_1(x)\,dx}{x} = -\,e^{-2x}\left[(1+x)\,I_0^2(x)+(2x+1)I_0(x)I_1(x)+x\,I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\,K_0(x)\,K_1(x)\,dx}{x} = e^{-2x}\left[(1+x)\,K_0^2(x)-(2x+1)K_0(x)K_1(x)+x\,K_1^2(x)\right]$$

$$\int \frac{e^{2x}\,I_1^2(x)\,dx}{x} = \frac{e^{2x}}{2}\,\left[(1-2x)\,I_0^2(x)+4xI_0(x)I_1(x)-(2x+1)\,I_1^2(x)\right]$$

$$\int \frac{e^{2x}\,K_1^2(x)\,dx}{x} = \frac{e^{2x}}{2}\,\left[(1-2x)\,K_0^2(x)-4xK_0(x)K_1(x)-(2x+1)\,K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\,I_1^2(x)\,dx}{x} = \frac{e^{-2x}}{2}\,\left[(1+2x)\,I_0^2(x)+4xI_0(x)I_1(x)+(2x-1)\,I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\,K_1^2(x)\,dx}{x} = \frac{e^{-2x}}{2}\,\left[(1+2x)\,K_0^2(x)-4xK_0(x)K_1(x)+(2x-1)\,K_1^2(x)\right]$$

$$\int \frac{\sin 2x\,J_0(x)\,J_1(x)\,dx}{x} = \sin 2x\,\left[-x\,J_0^2(x)-J_0(x)J_1(x)+x\,J_1^2(x)\right]+\cos 2x\,\left[2x\,J_0(x)J_1(x)-J_0^2(x)\right]$$

$$\int \frac{\cos 2x\,J_0(x)\,J_1(x)\,dx}{x} = \sin 2x\,\left[J_0^2(x)-2x\,J_0(x)J_1(x)\right]-\cos 2x\,\left[x\,J_0^2(x)+J_0(x)J_1(x)-x\,J_1^2(x)\right]$$

$$\int \frac{\sin 2x\,J_1^2(x)\,dx}{x} = \frac{\sin 2x}{2}\,\left[-J_0^2(x)+4x\,J_0(x)J_1(x)-J_1^2(x)\right]+x\,\cos 2x\,\left[J_0^2(x)-J_1^2(x)\right]$$

$$\int \frac{\cos 2x\,J_1^2(x)\,dx}{x} = \frac{\cos 2x}{2}\,\left[-J_0^2(x)+4x\,J_0(x)J_1(x)-J_1^2(x)\right]-x\,\sin 2x\,\left[J_0^2(x)-J_1^2(x)\right]$$

From this:

$$\int \frac{\sin^2 x\,J_0(x)\,J_1(x)\,dx}{x} =$$

$$=\frac{1}{2}\left[xJ_0^2(x)-J_0(x)J_1(x)+xJ_1^2(x)\right]+\frac{\sin 2x}{2}\,\left[J_0^2(x)-2x\,J_0(x)J_1(x)\right]+$$

$$+\frac{\cos 2x}{2}\,\left[x\,J_0^2(x)+J_0(x)J_1(x)-x\,J_1^2(x)\right]$$

$$\int \frac{\cos^2 x\,J_0(x)\,J_1(x)\,dx}{x} =$$

$$=\frac{1}{2}\left[xJ_0^2(x)-J_0(x)J_1(x)+xJ_1^2(x)\right]+\frac{\sin 2x}{2}\,\left[J_0^2(x)-2x\,J_0(x)J_1(x)\right]-$$

$$-\frac{\cos 2x}{2}\,\left[x\,J_0^2(x)+J_0(x)J_1(x)-x\,J_0^2(x)\right]$$

$$\int \frac{\sin^2 x\,J_1^2(x)\,dx}{x} =$$

$$=-\frac{J_0^2(x)+J_1^2(x)}{4}+\frac{\cos 2x}{4}\,\left[J_0^2(x)-4x\,J_0(x)J_1(x)+J_1^2(x)\right]+\frac{x\,\sin 2x}{2}\,\left[J_0^2(x)-J_1^2(x)\right]$$

$$\int \frac{\cos^2 x\,J_1^2(x)\,dx}{x} =$$

$$=-\frac{J_0^2(x)+J_1^2(x)}{4}-\frac{\cos 2x}{4}\,\left[J_0^2(x)-4x\,J_0(x)J_1(x)+J_1^2(x)\right]-\frac{x\,\sin 2x}{2}\,\left[J_0^2(x)-J_1^2(x)\right]$$

2.3.5. Some Cases of $\int x^n \cdot \exp(\alpha x) \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

<u>**n = 1 :**</u>

$$\int x\, e^{4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, I_0(x)\, I_0(3x) + (4x+3)\, I_0(x)\, I_1(3x) + (4x-1)\, I_0(x)\, I_1(3x) - 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, I_0(x)\, I_0(3x) + (4x-3)\, I_0(x)\, I_1(3x) + (4x+1)\, I_0(x)\, I_1(3x) + 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, K_0(x)\, K_0(3x) + (4x+3)\, K_0(x)\, K_1(3x) + (4x-1)\, K_0(x)\, K_1(3x) + 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, K_0(x)\, K_0(3x) + (4x-3)\, K_0(x)\, K_1(3x) + (4x+1)\, K_0(x)\, K_1(3x) - 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, I_0(x)\, K_0(3x) + (4x+3)\, I_0(x)\, K_1(3x) - (4x-1)\, I_0(x)\, K_1(3x) - 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, I_0(x)\, K_0(3x) + (4x-3)\, I_0(x)\, K_1(3x) - (4x+1)\, I_0(x)\, K_1(3x) + 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, K_0(x)\, I_0(3x) + (4x+3)\, K_0(x)\, I_1(3x) - (4x-1)\, K_0(x)\, I_1(3x) + 4x\, K_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, K_0(x)\, I_0(3x) + (4x-3)\, K_0(x)\, I_1(3x) - (4x+1)\, K_0(x)\, I_1(3x) - 4x\, K_1(x)\, I_1(3x)\right]$$

<u>**n = 2 :**</u>

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{5x}{3}\right)\,\left[(-64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. + (64\,x^2 + 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 - 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (-64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\left[(64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. + (64\,x^2 - 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 + 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, K_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{8x}{3}\right)\,\left[(64\,x^2 - 24\,x)\, I_0(x)\, K_0\left(\frac{5x}{3}\right) + \right.$$

$$+(64\,x^2+120\,x-45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right)-(64\,x^2-72\,x+27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right)+(-64\,x^2+24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot I_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[-(64\,x^2+24\,x)\,I_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right)-(64\,x^2+72\,x+27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right)+(64\,x^2+24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(\frac{8x}{3}\right)\,\Bigg[(-64\,x^2+24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2+120\,x-45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right)-(64\,x^2-72\,x+27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right)+(64\,x^2-24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[(64\,x^2+24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right)-(64\,x^2+72\,x+27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right)-(64\,x^2+24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(\frac{8x}{3}\right)\,\Bigg[(64\,x^2-24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2+120\,x-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+(64\,x^2-72\,x+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[-(64\,x^2+24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+(64\,x^2+72\,x+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)-$$

$$-(64\,x^2+24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

<u>**n = 3 :**</u> $8/\sqrt{51}=1.12022\;40672,\;\;\sqrt{35/51}=0.82841\;68696$

$$\int x^3\exp\left(\frac{8x}{\sqrt{51}}\right)\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=$$

$$=\frac{x}{20480}\exp\left(\frac{8x}{\sqrt{51}}\right)\,\Bigg[\sqrt{35}\,(-64\,\sqrt{51}x^2+408\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$+(1600\,\sqrt{51}x^2-14280\,x+1785\,\sqrt{51})\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$+\sqrt{35}\,(-1088\,x^2+408\,\sqrt{51}x-2601\,)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+(5440\,x^2-680\,\sqrt{51}x)\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\Bigg]$$

<u>**n = 4 :**</u> $2\sqrt{3}\,/5=0.69282\;03230,\;\;\sqrt{13}\,/5=0.72111\;02551$

$$\int x^4\exp\left(\frac{2\sqrt{3}\,x}{5}\right)\cdot J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)\,dx=$$

$$=\frac{x}{96}\exp\left(\frac{2\sqrt{3}\,x}{5}\right)\,\Bigg[(40\,\sqrt{3}x^3-180\,x^2+150\,\sqrt{3}x)\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+$$

$$+\sqrt{13}\,(20\,\sqrt{3}x^2-150\,x+125\,\sqrt{3})\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)+$$

$$+(48\,x^3-140\,\sqrt{3}x^2+750\,x-625\,\sqrt{3})\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+$$

$$+\sqrt{13}\,(8\,\sqrt{3}x^3-60\,x^2+50\,\sqrt{3}x)\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]$$

$4/\sqrt{5} = 1.78885\ 43820$, $\sqrt{7/15} = 0.68313\ 00511$

$$\int x^4 \exp\left(\frac{4x}{\sqrt{5}}\right)\cdot J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx =$$

$$=\frac{x}{7168}\exp\left(\frac{4x}{\sqrt{5}}\right)\Bigg[\sqrt{21}\,(-64\,\sqrt{5}x^3+240\,x^2-60\,\sqrt{5}x)\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(1344\,\sqrt{5}x^3-3360\,x^2+1260\,\sqrt{5}x-1575)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+\sqrt{21}\,(-192\,x^3+288\,\sqrt{5}x^2-900\,x+225\,\sqrt{5})\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(1344\,x^3-1008\,\sqrt{5}x^2+1260\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]$$

2.4.6. Some Cases of $\int x^n \cdot \left\{ \begin{array}{c} \sin\ /\ \cos \\ \sinh / \cosh \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

Some integrals, where α and β are roots of cubic equations, are left out.
With the integral $\int w(\alpha x)\, Z_\nu(x)\, Z_\nu(\beta x)\, dx$ the integral

$$\int w\left(\frac{\alpha}{\beta}\, x\right)\, Z_\nu(x)\, Z_\nu\left(\frac{x}{\beta}\right)\, dx$$

may be found. So in the following tables only one of both integrals is given.

Numerical values of the coefficients:

α		β		α/β	$1/\beta$
$8/\sqrt{51}$	1.12022 40672	$\sqrt{35/51}$	0.82841 68696	1.35224 68076	1.20712 17242
$2\sqrt{3/13}$	0.96076 89228	$5/\sqrt{13}$	1.38675 04906	0.69282 03230	0.72111 02551
$4/\sqrt{5}$	1.78885 43820	$\sqrt{7/15}$	0.68313 00511	2.61861 46828	1.46385 01094
$2/\sqrt{7}$	0.75592 89460	$\sqrt{3/7}$	0.65465 36707	1.15470 05384	1.52752 52317
$2\sqrt{3}/5$	0.69282 03230	$\sqrt{13}/5$	0.72111 02551	.96076 89228	1.38675 04906
$4/\sqrt{11}$	1.20604 53783	$\sqrt{3/11}$	0.52223 29679	2.30940 10768	1.91485 42155

2.4.6 a) $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$:

n = 1 :

$$\int x \sin 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \sin 4x}{4} \left[J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)\right] +$$

$$+\frac{x \cos 4x}{16} \left[4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)\right]$$

$$\int x \cos 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \cos 4x}{4} \left[J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)\right] +$$

$$-\frac{x \sin 4x}{16} \left[4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)\right]$$

n = 2 :

$$\int x^2 \sin \frac{8x}{3} \cdot J_0(x)\, J_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512} \cdot \sin \frac{8x}{3} \cdot$$

$$\cdot \left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] +$$

$$+\frac{x^2}{64} \cdot \cos \frac{8x}{3} \left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \cos \frac{8x}{3} \cdot J_0(x)\, J_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512} \cdot \cos \frac{8x}{3} \cdot$$

$$\cdot \left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] -$$

$$-\frac{x^2}{64} \cdot \sin \frac{8x}{3} \left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

n = 3 :

$$\int x^3 \sin\left(\frac{8x}{\sqrt{51}}\right) \cdot I_0(x)\, I_1\left(\sqrt{\frac{35}{51}}\, x\right)\, dx = \frac{x^2}{2560} \left[8\sqrt{1785}\, x\, I_0(x)\, I_0\left(\sqrt{\frac{35}{51}}\, x\right) + 1785\, I_0(x)\, I_1\left(\sqrt{\frac{35}{51}}\, x\right) -\right.$$

$$\left. -51\sqrt{1785}\, I_1(x)\, I_0\left(\sqrt{\frac{35}{51}}\, x\right) + 680x\, I_1(x)\, I_1\left(\sqrt{\frac{35}{51}}\, x\right)\right] \sin\left(\frac{8x}{\sqrt{51}}\right) +$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{51}\,(1785-1600\,x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\sqrt{51}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x}{20480}\left[-408\sqrt{35}x\;I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{35}\,(2601-1088\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-$$

$$\left.-680\sqrt{51}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)+\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+1785\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)-51\sqrt{1785}\,I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$\left.-1785\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\sqrt{51}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x}{20480}\left[408\sqrt{35}x\;K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{35}\,(2601-1088\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$\left.+680\sqrt{51}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$\left.-1785\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+1785\,I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\Bigg]\cos\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\,\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+1785\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\sqrt{51}\,x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\,\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+1785\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[-408\sqrt{35}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)$$

<u>n = 4 :</u>

$$\int x^4\,\sin\left(2\sqrt{\frac{3}{13}}\,x\right)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)\,dx=\frac{x^2}{80}\left[78x\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$\left.+169\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-26\sqrt{13}x\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\sin\left(2\sqrt{\frac{3}{13}}\,x\right)+\frac{x}{480}\left[\sqrt{39}\,x(78-40\,x^2)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$+\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+$$

$$\left.+\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\,x\right)$$

$$\int x^4\,\cos\left(2\sqrt{\frac{3}{13}}\,x\right)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)\,dx=\frac{x^2}{80}\left[78x\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$\left.+169\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-26\sqrt{13}x\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\,x\right)+\frac{x}{480}\left[\sqrt{39}\,x(40\,x^2-78)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-\right.$$

$$-\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)-\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-$$

$$-\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\Bigg]\,\sin\left(2\sqrt{\frac{3}{13}}\,x\right)$$

$$\int x^4\,\sin\left(2\sqrt{\frac{3}{13}}\,x\right)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)\,dx=\frac{x^2}{80}\,\Bigg[78x\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+$$

$$+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-169\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-26\sqrt{13}\,x\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\Bigg]\,\sin\left(2\sqrt{\frac{3}{13}}\right)+$$

$$+\frac{x}{480}\,\Bigg[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)+$$

$$-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\Bigg]\,\cos\left(2\sqrt{\frac{3}{13}}\right)$$

$$\int x^4\,\cos\left(2\sqrt{\frac{3}{13}}\,x\right)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)\,dx=\frac{x^2}{80}\,\Bigg[78x\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+$$

$$+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-169\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-26\sqrt{13}\,x\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\Bigg]\,\cos\left(2\sqrt{\frac{3}{13}}\right)-$$

$$-\frac{x}{480}\,\Bigg[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-$$

$$-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\Bigg]\,\sin\left(2\sqrt{\frac{3}{13}}\right)$$

$$\int x^4\,\sin\frac{4x}{\sqrt{5}}\cdot I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx=\frac{x}{7168}\,\Bigg[\sqrt{105}\,(64\,x^3-60\,x)\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(3360\,x^2-1575)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(1344\,x^3-1260\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\,\sin\frac{4x}{\sqrt{5}}+\frac{x^2}{1792}\,\Bigg[60\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+\sqrt{5}\,(315-336\,x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{21}\,(48\,x^2-225)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+252\sqrt{5}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\,\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4\,\cos\frac{4x}{\sqrt{5}}\cdot I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx=\frac{x^2}{1792}\,\Bigg[-60\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+\sqrt{5}\,(336\,x^2-315)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{21}\,(225-48\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)-$$

$$-252\sqrt{5}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\,\sin\frac{4x}{\sqrt{5}}+\frac{x}{7168}\,\Bigg[\sqrt{105}\,x(64x^2-60)\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(3360\,x^2-1575)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$+(1344\,x^3-1260\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\,\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4 \sin\frac{4x}{\sqrt{5}} \cdot K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\,x\right) dx = \frac{x}{7168}\left[-\sqrt{105}\,(64\,x^3-60\,x)\,K_0(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+(3360\,x^2-1575)\,K_0(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.-(1344\,x^3-1260\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\sin\frac{4x}{\sqrt{5}}-\frac{x^2}{1792}\left[60\sqrt{21}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$-\sqrt{5}\,(315-336\,x^2)\,K_0(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)-\sqrt{21}\,(48\,x^2-225)\,K_1(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+252\sqrt{5}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4 \cos\frac{4x}{\sqrt{5}} \cdot K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\,x\right) dx = \frac{x^2}{1792}\left[60\sqrt{21}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+\sqrt{5}\,(336\,x^2-315)\,K_0(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{21}\,(225-48\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+252\sqrt{5}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\sin\frac{4x}{\sqrt{5}}+\frac{x}{7168}\left[-\sqrt{105}\,x(64x^2-60)\,K_0(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+(3360\,x^2-1575)\,K_0(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{7}{15}}\,x\right)-$$

$$\left.-(1344\,x^3-1260\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4 \sin\frac{2x}{\sqrt{7}} \cdot I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\,x\right) dx = \frac{x^2}{16}\left[-6\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+(8\,x^2-63)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)+\right.$$

$$\left.+21\sqrt{21}\,I_1(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+14\,x\,I_1(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\sin\frac{2x}{\sqrt{7}}+\frac{x}{32}\left[\sqrt{3}\,(8\,x^3-42x)\,I_0(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+\right.$$

$$+\sqrt{7}\,(20\,x^2-63)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{3}\,(147-28\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$\left.+\sqrt{7}\,(14x-8\,x^3)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\cos\frac{2x}{\sqrt{7}}$$

$$\int x^4 \cos\frac{2x}{\sqrt{7}} \cdot I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\,x\right) dx = \frac{x}{32}\left[\sqrt{3}\,(42x-8x^3)\,I_0(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+\right.$$

$$+\sqrt{7}\,(63-20x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{3}\,(28x^2-147)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$\left.+\sqrt{7}\,(8x^3-14x)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\sin\frac{2x}{\sqrt{7}}+\frac{x^2}{16}\left[-6\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+\right.$$

$$\left.+(8x^2-63)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)+21\sqrt{21}\,I_1(x)\,I_0\left(\sqrt{\frac{3}{7}}\,x\right)+14x\,I_1(x)\,I_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\cos\frac{2x}{\sqrt{7}}$$

$$\int x^4 \sin\frac{2x}{\sqrt{7}} \cdot K_1(x)\, K_1\left(\sqrt{\frac{3}{7}}\,x\right) dx = \frac{x^2}{16}\left[-6\sqrt{21}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)-(8\,x^2-63)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$-21\sqrt{21}\,K_1(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)+14\,x\,K_1(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)\Bigg]\,\sin\,\frac{2x}{\sqrt{7}}+\frac{x}{32}\,\Bigg[\sqrt{3}\,(8\,x^3-42x)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)-$$

$$-\sqrt{7}\,(20\,x^2-63)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)-\sqrt{3}\,(147-28\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$+\sqrt{7}\,(14x-8\,x^3)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)\Bigg]\,\cos\,\frac{2x}{\sqrt{7}}$$

$$\int x^4\,\cos\,\frac{2x}{\sqrt{7}}\cdot K_1(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)\,dx=\frac{x}{32}\,\Bigg[\sqrt{3}\,(42x-8x^3)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$-\sqrt{7}\,(63-20x^2)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)-\sqrt{3}\,(28x^2-147)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$+\sqrt{7}\,(8x^3-14x)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)\Bigg]\,\sin\,\frac{2x}{\sqrt{7}}-\frac{x^2}{16}\,\Bigg[6\sqrt{21}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)+$$

$$+(8x^2-63)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)+21\sqrt{21}\,K_1(x)\,K_0\left(\sqrt{\frac{3}{7}}\,x\right)-14x\,K_1(x)\,K_1\left(\sqrt{\frac{3}{7}}\,x\right)\Bigg]\,\cos\,\frac{2x}{\sqrt{7}}$$

<u>**n = 5 :**</u>

$$\int x^5\,\sin\,\frac{4x}{\sqrt{11}}\cdot I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x}{24576}\,\Bigg[\sqrt{33}\,x(7260-4224\,x^2)\,I_0(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\sin\,\frac{4x}{\sqrt{11}}+\frac{x^2}{6144}\,\Bigg[\sqrt{3}\,(704\,x^3-7260x)\,I_0(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(2112\,x^2-5445)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{3}\,(19965-1408\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\cos\,\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\cos\,\frac{4x}{\sqrt{11}}\cdot I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x^2}{6144}\,\Bigg[\sqrt{3}\,(7260x-704\,x^3)I_0(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(5445-2112\,x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{3}\,(1408\,x^2-19965)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(960\,x^3-396x)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\sin\,\frac{4x}{\sqrt{11}}+\frac{x}{24576}\,\Bigg[\sqrt{33}\,(7260x-4224\,x^3)I_0(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{33}\,(256x^4+11088x^2-19965)\,I_1(x)\,I_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\cos\,\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\sin\,\frac{4x}{\sqrt{11}}\cdot K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x}{24576}\,\Bigg[-\sqrt{33}\,x(4224\,x^2-7260)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\sin\frac{4x}{\sqrt{11}}+\frac{x^2}{6144}\left[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$\left.+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\cos\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\cos\frac{4x}{\sqrt{11}}\cdot K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x}{24576}\left[\sqrt{33}\,x(7260-4224\,x^2)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\,\cos\frac{4x}{\sqrt{11}}-\frac{x^2}{6144}\left[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$\left.+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\sin\frac{4x}{\sqrt{11}}$$

2.4.6 b) $\int x^n\cdot\left\{\begin{array}{c}\sinh\\ \cosh\end{array}\right\}\alpha x\cdot Z_\mu(x)\,Z_\nu(\beta x)\,dx$:

<u>n = 1 :</u>

$$\int x\,\sinh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\sinh 4x-$$

$$-\frac{x}{16}\,[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\cosh 4x-$$

$$-\frac{x}{16}\,[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)]\,\sinh 4x$$

$$\int x\,\sinh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)]\,\sinh 4x+$$

$$+\frac{x}{16}\,[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)]\,\cosh 4x+$$

$$+\frac{x}{16}\,[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)]\,\sinh 4x$$

<u>n = 2 :</u>

$$\int x^2\,\sinh\frac{8x}{3}\cdot I_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\left[24x\,I_0(x)\,I_0\left(\frac{5x}{3}\right)+(64\,x^2-45)\,I_0(x)\,I_1\left(\frac{5x}{3}\right)+\right.$$

$$\left.+(64\,x^2+27)\,I_1(x)\,I_0\left(\frac{5x}{3}\right)+24x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\,\sinh\frac{8x}{3}-$$

$$-\frac{x^2}{64}\left[8x\,I_0(x)\,I_0\left(\frac{5x}{3}\right)-15\,I_0(x)\,I_1\left(\frac{5x}{3}\right)+9\,I_1(x)\,I_0\left(\frac{5x}{3}\right)+8x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}$$

$$\int x^2\cosh\frac{8x}{3}\cdot I_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\left[24x\,I_0(x)\,I_0\left(\frac{5x}{3}\right)+(64\,x^2-45)\,J_0(x)\,J_1\left(\frac{5x}{3}\right)+\right.$$

$$\left.+(64\,x^2+27)\,I_1(x)\,I_0\left(\frac{5x}{3}\right)+24x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}-$$

$$-\frac{x^2}{64}\left[8x\,I_0(x)\,I_0\left(\frac{5x}{3}\right)-15\,I_0(x)\,I_1\left(\frac{5x}{3}\right)+9\,I_1(x)\,I_0\left(\frac{5x}{3}\right)+8x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3}$$

$$\int x^2\sinh\frac{8x}{3}\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\left[-24x\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+(64\,x^2-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+\right.$$

$$\left.+(64\,x^2+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)-24x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3}+$$

$$+\frac{x^2}{64}\left[8x\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+15\,K_0(x)\,K_1\left(\frac{5x}{3}\right)-9\,K_1(x)\,K_0\left(\frac{5x}{3}\right)+8x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}$$

$$\int x^2\cosh\frac{8x}{3}\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\left[-24x\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+(64\,x^2-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+\right.$$

$$\left.+(64\,x^2+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)-24x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}+$$

$$+\frac{x^2}{64}\left[8x\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+15\,K_0(x)\,K_1\left(\frac{5x}{3}\right)-9\,K_1(x)\,K_0\left(\frac{5x}{3}\right)+8x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3}$$

n = 3 :

$$\int x^3\sinh\frac{8x}{\sqrt{51}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)-1785\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+51\sqrt{1785}\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sinh\frac{8x}{\sqrt{51}}+$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{51}\,(1600x^2+1785)\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$\left.-\sqrt{35}\,(1088x^2+2601)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\sqrt{51}\,x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cosh\frac{8x}{\sqrt{51}}$$

$$\int x^3\cosh\frac{8x}{\sqrt{51}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x}{20480}\left[408\sqrt{35}\,x\;J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+\sqrt{51}\,(1600\,x^2+1785)\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{35}\,(1088\,x^2+2601)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)-$$

$$\left.-680\sqrt{51}\,x\;J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sinh\frac{8x}{\sqrt{51}}+\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)-1785\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+51\sqrt{1785}\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cosh\frac{8x}{\sqrt{51}}$$

n = 4 :

$$\int x^4\sinh\frac{2\sqrt{3}\,x}{5}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)\,dx=\left[-\frac{15x^3}{8}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)-\frac{25\sqrt{13}\,x^2}{16}\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)+\right.$$

$$+\frac{8x^4+125x^2}{16}\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)-\frac{5\sqrt{13}\,x^3}{8}\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]\,\sinh\frac{2\sqrt{3}\,x}{5}+$$

$$+\left[\frac{5\sqrt{3}\,(4x^4+15x^2)}{48}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+\frac{5\sqrt{39}\,(4x^3+25x)}{96}\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)-\right.$$

$$\left.-\frac{5\sqrt{3}\,(28x^3+125x}{96}\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+\frac{\sqrt{39}\,(4x^4+25x^2)}{48}\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\right]\,\cosh\frac{2\sqrt{3}\,x}{5}$$

$$\int x^4\,\cosh\frac{2\sqrt{3}\,x}{5}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)\,dx=\frac{\sqrt{3}\,x}{96}\,\left[(40x^3+150x)\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+\right.$$

$$+\sqrt{13}\,(20x^2+125)\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)-(140\,x^2+625)\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+$$

$$\left.+\sqrt{13}\,(8\,x^3+50\,x)\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\right]\,\sinh\frac{2\sqrt{3}\,x}{5}-\frac{x^2}{16}\,\left[30\,x\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+25\,\sqrt{13}\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)-\right.$$

$$\left.-(8\,x^2+125)\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+10\,\sqrt{13}x\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\right]\,\cosh\frac{2\sqrt{3}\,x}{5}$$

$$\int x^4\,\sinh\frac{4x}{\sqrt{5}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx=\frac{x}{7168}\,\left[-\sqrt{105}\,(64x^3+60\,x)J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)-\right.$$

$$-(3360\,x^2+1575)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(288x^2+225)\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+(1344\,x^3+1260\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\,\sinh\frac{4x}{\sqrt{5}}+\frac{x^2}{1792}\,\left[60\,\sqrt{21}x\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+\sqrt{5}\,(336\,x^2+315)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)-\sqrt{21}\,(48\,x^2+225)\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)-$$

$$\left.-252\,\sqrt{5}x\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\,\cosh\frac{4x}{\sqrt{5}}$$

$$\int x^4\,\cosh\frac{4x}{\sqrt{5}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx=\frac{x^2}{1792}\,\left[60\,\sqrt{21}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+\sqrt{5}\,(336\,x^2+315)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)-\sqrt{21}\,(48\,x^2+225)\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)-$$

$$\left.-252\,\sqrt{5}\,x\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\,\sinh\frac{4x}{\sqrt{5}}+\frac{x}{7168}\,\left[-\sqrt{105}\,(64\,x^3+60x)\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)-\right.$$

$$-(3360\,x^2+1575)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(288\,x^2+225)\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+(1344\,x^3+1260\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\,\cosh\frac{4x}{\sqrt{5}}$$

$$\int x^4\,\sinh\frac{2x}{\sqrt{7}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\,dx=\frac{x^2}{16}\,\left[-6\,\sqrt{21}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-(8\,x^2+63)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)+21\,\sqrt{21}\,J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-14\,x\,J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\,\sinh\frac{2x}{\sqrt{7}}+$$

$$+\frac{x}{32}\left[\sqrt{3}\,(8\,x^3+42\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\,(20x^2+63\,)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\,(28\,x^2+147\,)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\,(8\,x^3+14\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\,\cosh\frac{2x}{\sqrt{7}}$$

$$\int x^4\,\cosh\frac{2x}{\sqrt{7}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\,dx=\frac{x^2}{16}\left[-6\,\sqrt{21}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-(8\,x^2+63)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)+21\,\sqrt{21}\,J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-14\,x\,J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\,\cosh\frac{2x}{\sqrt{7}}+$$

$$+\frac{x}{32}\left[\sqrt{3}\,(8\,x^3+42\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\,(20\,x^2+63\,)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\,(28x^2+147)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\,(8x^3+14\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\,\sinh\frac{2x}{\sqrt{7}}$$

<u>n = 5 :</u>

$$\int x^5\,\sinh\frac{4x}{\sqrt{11}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x}{24\,576}\left[-\sqrt{33}\,(4224\,x^3+7260x)\;J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$-(8448\,x^4+52272\,x^2+59895)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{33}\,(-256\,x^4+11088\,x^2+19965)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$\left.-(12672\,x^3+4356\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\sinh\frac{4x}{\sqrt{11}}+$$

$$+\frac{x^2}{6144}\left[\sqrt{3}\,(704\,x^3+7260\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{11}\,(2112\,x^2+5445)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\,(1408\,x^2+19965)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{11}\,(960\,x^3+396\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\cosh\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\cosh\frac{4x}{\sqrt{11}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\,dx=\frac{x^2}{6144}\left[\sqrt{3}\,(704\,x^3+7260\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\right.$$

$$+\sqrt{11}\,(2112\,x^2+5445)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\,(1408\,x^2+19965)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$\left.+\sqrt{11}\,(960\,x^3+396\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\sinh\frac{4x}{\sqrt{11}}-$$

$$-\frac{x}{24576}\left[\sqrt{33}\,(4224\,x^3+7260\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+(8448\,x^4+52272\,x^2+59895)\,J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$\left.-\sqrt{33}\,(-256\,x^4+11088\,x^2+19965)\,J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+(12672\,x^3+4356\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\,\cosh\frac{4x}{\sqrt{11}}$$

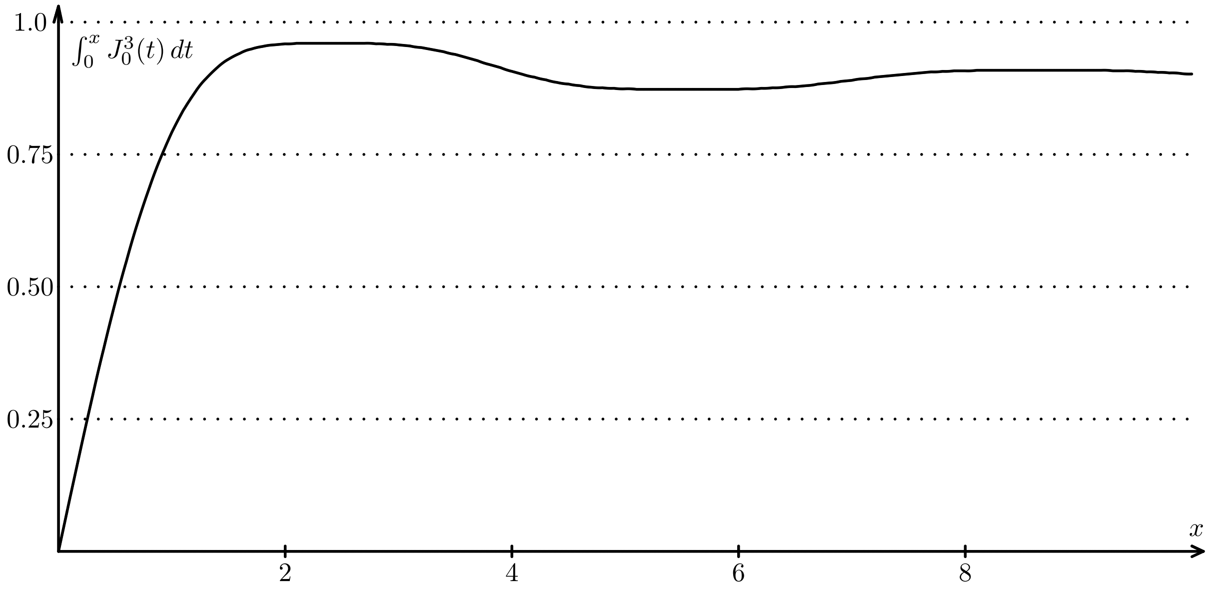
3. Products of three Bessel Functions

3.1. Integrals of the type $\int x^n \, Z_0^m(x) \, Z_1^{3-m}(x) \, dx$

These integrals are expressed by three basic integrals with $m = 0, 1, 3$. One has obviously

$$\int J_0^2(x) \, J_1(x) \, dx \;=\; -\frac{1}{3} \, J_0^3(x) \quad , \qquad \int I_0^2(x) \, I_1(x) \, dx \;=\; \frac{1}{3} \, I_0^3(x) \, .$$

a) Basic integral $\int Z_0^3(x) \, dx$:



$$\int_0^\infty J_0^3(x) \, dx \;=\; \frac{\Gamma\left(\frac{1}{6}\right)}{3\,\Gamma\left(\frac{5}{6}\right) \cdot \Gamma^2\left(\frac{2}{3}\right)} \;=\; \frac{2\pi}{3\,\Gamma^2\left(\frac{5}{6}\right) \cdot \Gamma^2\left(\frac{2}{3}\right)} \;=\; 0.89644\,07887\,76762\,86423\,\ldots$$

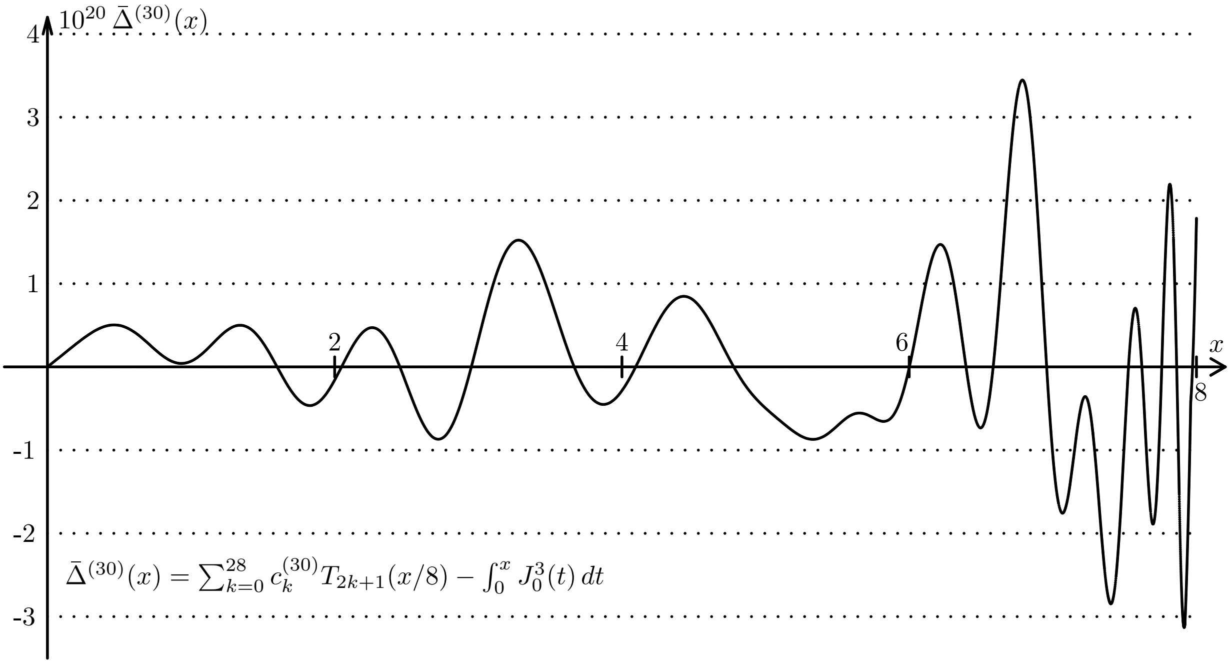
Formula 2.12.42.4 from [4] gives $2\sqrt{3}/3\pi = 0.36755\ldots$. This does not fit to the result of computations. The formula 2.12.42.18 offers $2\,\Gamma(1/6)/[3\,\Gamma(5/6) \cdot \Gamma^2(2/3)] = 2 \cdot 0.896\ldots$.
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0^3(t) \, dt = \sum_{k=0}^\infty c_k^{(30)} \, T_{2k+1}\left(\frac{x}{8}\right) \, .$$

The first coefficients are

k	$c_k^{(30)}$	k	$c_k^{(30)}$
0	1.14145 15823 43066 65430	15	-0.00001 19647 92899 04163
1	-0.37698 69057 03625 95863	16	0.00000 19134 34477 76782
2	0.24193 82520 26895 89401	17	-0.00000 02652 90698 65709
3	-0.15672 12348 70401 19757	18	0.00000 00322 34936 59711
4	0.09593 25955 24494 24624	19	-0.00000 00034 64486 68661
5	-0.06183 39745 85568 16834	20	0.00000 00003 31986 30964
6	0.04061 90786 50027 57290	21	-0.00000 00000 28561 98591
7	-0.02860 28881 71908 58991	22	0.00000 00000 02219 71843
8	0.02158 63885 61883 43325	23	-0.00000 00000 00156 67592
9	-0.01457 50343 38917 46000	24	0.00000 00000 00010 09261
10	0.00781 65287 61390 63536	25	-0.00000 00000 00000 59593
11	-0.00326 40051 49421 93383	26	0.00000 00000 00000 03238
12	0.00107 89280 58820 94395	27	-0.00000 00000 00000 00162
13	-0.00028 89222 57879 12574	28	0.00000 00000 00000 00008
14	0.00006 40382 21736 47959	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_0^x J_0^3(t)\,dt \sim 0.89644\;07887\;76762\;86423\;\ldots +$$

$$+\sqrt{\frac{2}{\pi^3\,x}}\,\sum_{k=1}^{\infty}\frac{1}{x^k}\,\left[a_k^{(30)}\,\sin\left(3x+\frac{7-2k}{4}\,\pi\right)+b_k^{(30)}\,\sin\left(x+\frac{1-2k}{4}\,\pi\right)\right]$$

with the first values

k	$a_k^{(30)}$	$a_k^{(30)}$
1	1/6	0.16666 66666 66666 66667
2	7/48	0.14583 33333 33333 33333
3	379/2304	0.16449 65277 77777 77778
4	13141/55296	0.23764 82928 24074 07407
5	250513/589824	0.42472 50027 12673 61111
6	12913841/14155776	0.91226 65546 55852 14120
7	1565082415/679477248	2.30336 25034 63839 30724
8	36535718855/5435817984	6.72129 18023 63631 16606
9	23344744269635/1043677052928	22.36778 53260 66262 13087
10	2103860629922855/25048249270272	83.99232 24662 16381 92796
k	$b_k^{(30)}$	$b_k^{(30)}$
1	3/2	1.50000 00000 00000 00000
2	39/16	2.43750 00000 00000 00000
3	1635/256	6.38671 87500 00000 00000
4	46053/2048	22.48681 64062 50000 00000
5	6664257/65536	101.68849 18212 89062 5000
6	293433849/524288	559.68065 07110 595703 125
7	30538511055/8388608	3640.47420 68052 2918 70
8	1832502818925/67108864	27306.41989 29816 48445 1
9	996997642437465/4294967296	2 32131.60281 01055 41721
10	75773171001327165/34359738368	22 05289.52199 17166 1050

The first consecutive maxima and minima of

$$\Delta_n^{(30)}(x) = 0.896\ldots + \sqrt{\frac{2}{\pi^3\,x}}\,\sum_{k=1}^{n} \frac{1}{x^k}\,\left[a_k^{(30)}\,\sin\left(3x + \frac{7-2k}{4}\,\pi\right) + b_k^{(30)}\,\sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] - \int_0^x J_0^3(t)\,dt\ :$$

i	x_i	$\Delta_1^{(30)}(x_i)$	x_i	$\Delta_2^{(30)}(x_i)$	x_i	$\Delta_3^{(30)}(x_i)$	x_i	$\Delta_4^{(30)}(x_i)$	x_i	$\Delta_5^{(30)}(x_i)$
1	3.953	-1.551E-02	2.356	3.014E-02	3.933	5.971E-03	2.356	-4.857E-02	3.928	-6.275E-03
2	7.084	4.314E-03	5.498	-2.889E-03	7.072	-6.255E-04	5.498	1.167E-03	7.069	2.440E-04
3	10.221	-1.832E-03	8.639	7.036E-04	10.213	1.380E-04	8.639	-1.302E-04	10.211	-2.818E-05
4	13.360	9.631E-04	11.781	-2.547E-04	13.354	-4.417E-05	11.781	2.690E-05	13.352	5.533E-06
5	16.500	-5.762E-04	14.923	1.153E-04	16.495	1.771E-05	14.923	-7.843E-06	16.494	-1.496E-06
6	19.641	3.758E-04	18.064	-6.022E-05	19.636	-8.262E-06	18.064	2.852E-06	19.635	5.014E-07
7	22.781	-2.607E-04	21.206	3.478E-05	22.778	4.297E-06	21.206	-1.211E-06	22.777	-1.962E-07
8	25.922	1.894E-04	24.347	-2.161E-05	25.919	-2.426E-06	24.347	5.757E-07	25.918	8.626E-08
9	29.064	-1.427E-04	27.489	1.421E-05	29.061	1.460E-06	27.489	-2.988E-07	29.060	-4.154E-08
10	32.205	1.106E-04	30.631	-9.769E-06	32.202	-9.244E-07	30.631	1.662E-07	32.202	2.152E-08

i	x_i	$\Delta_6^{(30)}(x_i)$	x_i	$\Delta_7^{(30)}(x_i)$	x_i	$\Delta_8^{(30)}(x_i)$	x_i	$\Delta_9^{(30)}(x_i)$	x_i	$\Delta_{10}^{(30)}(x_i)$
1	5.498	-1.017E-03	3.927	1.332E-02	5.498	1.579E-03	3.927	-4.858E-02	5.498	-3.900E-03
2	8.639	5.162E-05	7.069	-1.853E-04	8.639	-3.588E-05	7.069	2.333E-04	8.639	3.891E-05
3	11.781	-6.116E-06	10.210	1.115E-05	11.781	2.431E-06	10.210	-7.235E-06	11.781	-1.498E-06
4	14.923	1.154E-06	13.352	-1.345E-06	14.923	-2.973E-07	13.352	5.351E-07	14.923	1.186E-07
5	18.064	-2.934E-07	16.493	2.458E-07	18.064	5.293E-08	16.493	-6.616E-08	18.064	-1.479E-08
6	21.206	9.182E-08	19.635	-5.934E-08	21.206	-1.224E-08	19.635	1.152E-08	21.206	2.528E-09
7	24.347	-3.349E-08	22.777	1.750E-08	24.347	3.429E-09	22.777	-2.565E-09	24.347	-5.448E-10
8	27.489	1.374E-08	25.918	-6.004E-09	27.489	-1.114E-09	25.918	6.874E-10	27.489	1.403E-10
9	30.631	-6.192E-09	29.060	2.317E-09	30.631	4.071E-10	29.060	-2.129E-10	30.631	-4.162E-11
10	33.772	3.009E-09	32.201	-9.833E-10	33.772	-1.636E-10	32.201	7.404E-11	33.772	1.384E-11

In the case $x \geq 8$ one has $g_n^{(30)} \leq \Delta_n^{(30)}(x) \leq G_n^{(30)}$ with such values:

n	$g_n^{(30)}$	$G_n^{(30)}$	n	$g_n^{(30)}$	$G_n^{(30)}$
1	-1.832E-03	2.384E-03	6	5.162E-05	-6.116E-06
2	-2.547E-04	7.036E-04	7	-9.887E-05	1.115E-05
3	-3.729E-04	1.380E-04	8	-3.588E-05	2.431E-06
4	-1.302E-04	2.690E-05	9	1.129E-04	-7.235E-06
5	1.404E-04	-2.818E-05	10	3.891E-05	-1.498E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{30}(x) = 0.896\ldots + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}}\,\left[\tilde{a}_k^{(30)}\,\sin\left(3x + \frac{7-2k}{4}\,\pi\right) + \tilde{b}_k^{(30)}\,\sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(30)}$	$\tilde{b}_k^{(30)}$
1	0.042328861175	0.380959694283
2	0.037041314549	0.619046345961
3	0.041506694202	1.619134748282
4	0.063280735888	5.687020985444
5	0.009657085591	24.391624641402
6	0.955932046063	129.687141434225
7	-12.477953789465	581.699380578201
8	57.157092072696	3774.819577903598
9	-530.170189213242	9530.689591084210
10	649.179657827779	65374.322605554143

With $8 \leq x \leq 30$ holds

$$-1.8\cdot 10^{-9} \ \leq\ F_{30}(x) - \int_0^x J_0^3(t)\,dt \ \leq\ 1.1\cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \;=\; \sum_{k=0}^{\infty} d_k^{(30)}\,x^{2k+1} = x+\frac{1}{4}\,x^3+\frac{3}{64}\,x^5+\frac{31}{5376}\,x^7+\frac{71}{147456}\,x^9+\frac{47}{1638400}\,x^{11}+\frac{11723}{9201254400}\,x^{13}+\ldots$$

With $n \ge 1$ the following recurrence relation holds:

$$d_{n+1}^{(30)} = \frac{4\,\sigma_1^{(30)}(n,d) \;+\; 9\,\sigma_2^{(30)}(n,d)}{12(2n+3)(n+1)^2}$$

with

$$\sigma_1^{(30)}(n,d) = \sum_{k=1}^{n}(2k+1)k(2n-2k+3)(2n-5k+2)\cdot d_k^{(30)}\cdot d_{n-k+1}^{(30)}$$

and

$$\sigma_2^{(30)}(n,d) = \sum_{k=0}^{n}(2k+1)(2n-2k+1)\cdot d_k^{(30)}\cdot d_{n-k}^{(30)}\,.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k^{(30)}}{x^k}$$

with the first values

k	$c_k^{(30)}$	$c_k^{(30)}$
1	1/12	0.08333 33333 33333 3
2	7/96	0.07291 66666 66666 7
3	379/4608	0.08224 82638 88888 9
4	13141/110592	0.11882 41464 12037
5	250513/1179648	0.21236 25013 56337
6	12913841/28311552	0.45613 32773 27926
7	1565082415/1358954496	1.15168 12517 3192
8	36535718855/10871635968	3.36064 59011 8182
9	23344744269635/2087354105856	11.18389 26630 331
10	2103860629922855/50096498540544	41.99616 12331 082

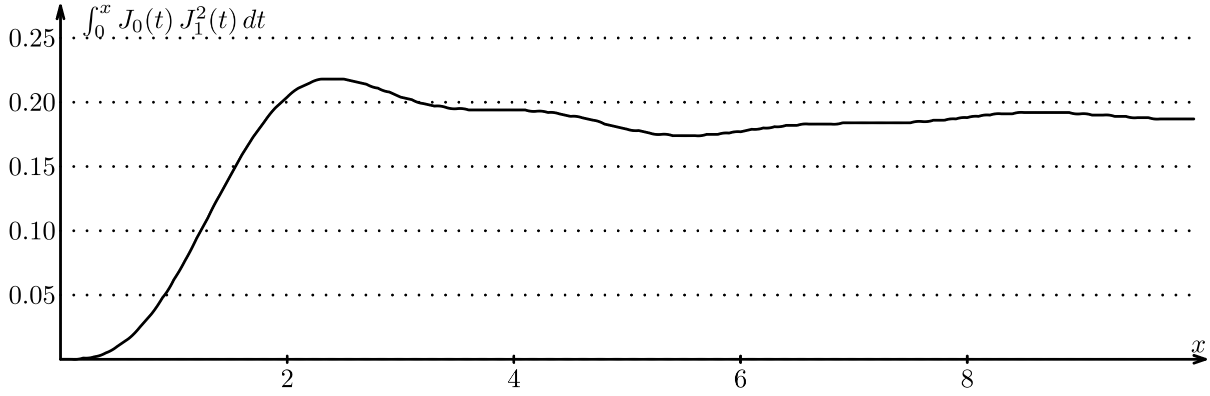
Let

$$\delta_n^{(30)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k^{(30)}}{x^k}\right]\cdot\left[\int_0^x I_0^3(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(30)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-1.897E-01	-9.019E-02	-5.944E-02	-4.435E-02	-3.538E-02
2	-4.794E-02	-1.058E-02	-4.579E-03	-2.545E-03	-1.618E-03
3	-1.595E-02	-1.597E-03	-4.529E-04	-1.874E-04	-9.492E-05
4	-6.704E-03	-3.000E-04	-5.555E-05	-1.710E-05	-6.896E-06
5	-3.400E-03	-6.813E-05	-8.209E-06	-1.877E-06	-6.026E-07
6	-1.981E-03	-1.833E-05	-1.430E-06	-2.427E-07	-6.198E-08
7	-1.264E-03	-5.756E-06	-2.887E-07	-3.632E-08	-7.376E-09
8	-8.462E-04	-2.087E-06	-6.672E-08	-6.211E-09	-1.002E-09
9	-5.678E-04	-8.660E-07	-1.747E-08	-1.201E-09	-1.539E-10
10	-3.588E-04	-4.075E-07	-5.140E-09	-2.603E-10	-2.647E-11

b) Basic integral $\int Z_0(x)\, Z_1^2(x)\, dx$:



$$\int_0^\infty J_0(x)\, J_1^2(x)\, dx \quad = \quad 0.18578\ 75214\ 63065\ 82253\ \ldots$$

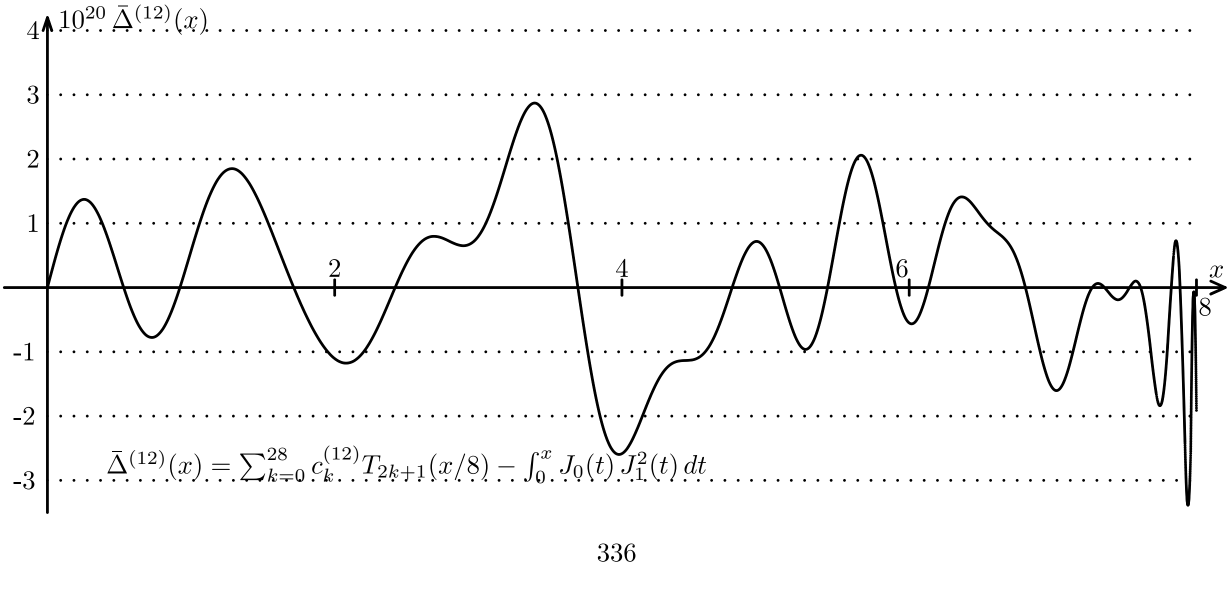
It differs from formula 2.12.42.4 in [4].
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0(t)\, J_1^2(t)\, dt = \sum_{k=0}^{\infty} c_k^{(12)}\, T_{2k+1}\left(\frac{x}{8}\right) .$$

The first coefficients are

k	$c_k^{(12)}$	k	$c_k^{(12)}$
0	0.23402 35970 33006 35445	15	0.00001 04741 31410 16206
1	-0.07444 03661 88704 59538	16	-0.00000 16973 09596 00078
2	0.04383 45295 19423 05343	17	0.00000 02378 37605 87389
3	-0.02088 20217 52543 47233	18	-0.00000 00291 52869 89193
4	0.00418 98619 72133 53415	19	0.00000 00031 56305 17549
5	0.00544 25004 57533 63387	20	-0.00000 00003 04352 15694
6	-0.00771 94350 97286 08113	21	0.00000 00000 26326 39770
7	0.00909 15741 08442 81722	22	-0.00000 00000 02055 66180
8	-0.01063 96487 30041 15062	23	0.00000 00000 00145 70197
9	0.00922 86127 36398 55548	24	-0.00000 00000 00009 42056
10	-0.00564 70826 05794 70935	25	0.00000 00000 00000 55809
11	0.00254 11445 11845 57857	26	-0.00000 00000 00000 03042
12	-0.00087 99927 23272 67696	27	0.00000 00000 00000 00153
13	0.00024 31559 72294 70777	28	-0.00000 00000 00000 00007
14	-0.00005 51197 58270 08008	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_0^x J_0(t)\, J_1^2(t)\, dt \sim 0.18578\ 75214\ 63065\ 82253\ \ldots + \sqrt{\frac{2}{\pi^3\, x}} \left\{ \frac{\left[a_0^{(12)}\ \sin\left(3x+\frac{\pi}{4}\right) + b_0^{(12)}\ \sin\left(x+\frac{7\pi}{4}\right)\right]}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty} \frac{1}{x^k} \left[a_k^{(12)}\ \sin\left(3x + \frac{7-2k}{4}\,\pi\right) + b_k^{(12)}\ \sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \right\}$$

with the first values

k	$a_k^{(12)}$	$a_k^{(12)}$
1	1/6	0.16666 66666 66666 66667
2	1/48	0.02083 33333 33333 33333
3	85/2304	0.03689 23611 11111 11111
4	3379/55296	0.06110 74942 12962 96296
5	69967/589824	0.11862 35215 92881 94444
6	3833063/14155776	0.27077 73137 97562 21065
7	487468417/679477248	0.71741 68354 19925 64260
8	11835711665/5435817984	2.17735 61402 97136 1890
9	7811427811325/1043677052928	7.48452 57825 78345 0049
10	722951995177505/25048249270272	28.86237 62633 79242 299
k	$b_k^{(12)}$	$b_k^{(12)}$
1	1/2	0.50000 00000 00000 00000
2	21/16	1.31250 00000 00000 00000
3	753/256	2.94140 62500 00000 00000
4	21375/2048	10.43701 17187 50000 0000
5	3061323/65536	46.71208 19091 79687 5000
6	134966187/524288	257.42757 22503 66210 938
7	14029169013/8388608	1672.40727 10275 65002 44
8	842027324535/67108864	12547.18489 25203 08494 6
9	458031686444595/4294967296	1 06643.81236 87624 46567
10	34812460139616855/34359738368	10 13175.93768 52090 6451

The first consecutive maxima and minima of

$$\Delta_n^{(12)}(x) = 0.185\ldots + \sqrt{\frac{2}{\pi^3\, x}} \sum_{k=1}^{n} \frac{1}{x^k} \left[a_k^{(12)}\ \sin(3x+\ldots) + b_k^{(12)}\ \sin(x+\ldots)\right] - \int_0^x J_0(t)\, J_1^2(t)\, dt \quad :$$

i	x_i	$\Delta_1^{(12)}(x_i)$	x_i	$\Delta_2^{(12)}(x_i)$	x_i	$\Delta_3^{(12)}(x_i)$	x_i	$\Delta_4^{(12)}(x_i)$	x_i	$\Delta_5^{(12)}(x_i)$
1	3.880	-8.324E-03	2.365	1.432E-02	3.923	2.771E-03	2.358	-2.242E-02	3.927	-2.887E-03
2	7.042	2.261E-03	5.503	-1.356E-03	7.066	-2.895E-04	5.499	5.375E-04	7.068	1.122E-04
3	10.192	-9.530E-04	8.643	3.293E-04	10.209	6.382E-05	8.640	-5.993E-05	10.210	-1.296E-05
4	13.338	4.995E-04	11.784	-1.191E-04	13.351	-2.042E-05	11.782	1.238E-05	13.352	2.544E-06
5	16.482	-2.984E-04	14.925	5.386E-05	16.492	8.184E-06	14.923	-3.609E-06	16.493	-6.876E-07
6	19.625	1.944E-04	18.066	-2.813E-05	19.634	-3.818E-06	18.065	1.313E-06	19.635	2.305E-07
7	22.768	-1.348E-04	21.207	1.624E-05	22.776	1.986E-06	21.206	-5.571E-07	22.776	-9.018E-08
8	25.911	9.790E-05	24.349	-1.009E-05	25.918	-1.121E-06	24.348	2.649E-07	25.918	3.965E-08
9	29.053	-7.371E-05	27.490	6.636E-06	29.059	6.745E-07	27.489	-1.375E-07	29.060	-1.910E-08
10	32.196	5.712E-05	30.632	-4.561E-06	32.201	-4.271E-07	30.631	7.645E-08	32.201	9.893E-09

i	x_i	$\Delta_6^{(12)}(x_i)$	x_i	$\Delta_7^{(12)}(x_i)$	x_i	$\Delta_8^{(12)}(x_i)$	x_i	$\Delta_9^{(12)}(x_i)$	x_i	$\Delta_{10}^{(12)}(x_i)$
1	2.356	8.374E-02	3.927	6.119E-03	2.356	-6.023E-01	3.927	-2.232E-02	2.356	7.281E+00
2	5.498	-4.673E-04	7.069	-8.515E-05	5.498	7.255E-04	7.069	1.072E-04	5.498	-1.792E-03
3	8.640	2.372E-05	10.210	5.122E-06	8.639	-1.649E-05	10.210	-3.324E-06	8.639	1.788E-05
4	11.781	-2.810E-06	13.352	-6.178E-07	11.781	1.117E-06	13.352	2.458E-07	11.781	-6.882E-07
5	14.923	5.304E-07	16.493	1.129E-07	14.923	-1.366E-07	16.493	-3.040E-08	14.923	5.448E-08
6	18.064	-1.348E-07	19.635	-2.726E-08	18.064	2.432E-08	19.635	5.293E-09	18.064	-6.794E-09
7	21.206	4.219E-08	22.777	8.042E-09	21.206	-5.621E-09	22.777	-1.179E-09	21.206	1.161E-09
8	24.347	-1.539E-08	25.918	-2.759E-09	24.347	1.575E-09	25.918	3.158E-10	24.347	-2.503E-10
9	27.489	6.316E-09	29.060	1.065E-09	27.489	-5.119E-10	29.060	-9.780E-11	27.489	6.446E-11
10	30.631	-2.845E-09	32.201	-4.518E-10	30.631	1.871E-10	32.201	3.402E-11	30.631	-1.912E-11

In the case $x \geq 8$ one has $g_n^{(12)} \leq \Delta_n^{(12)}(x) \leq G_n^{(12)}$ with such values:

n	$g_n^{(12)}$	$G_n^{(12)}$	n	$g_n^{(12)}$	$G_n^{(12)}$
1	-9.530E-04	1.314E-03	6	-2.810E-06	2.372E-05
2	-1.191E-04	3.293E-04	7	-4.543E-05	5.122E-06
3	-1.735E-04	6.382E-05	8	-1.649E-05	1.117E-06
4	-5.993E-05	1.238E-05	9	-3.324E-06	5.189E-05
5	6.457E-05	1.238E-05	10	1.788E-05	-6.882E-07

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{12}(x) = 0.18578\ 75214\ 63 + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(12)} \sin\left(3x + \frac{7-2k+4\delta_{1k}}{4}\,\pi\right) + \tilde{b}_k^{(12)} \sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.
The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329196006	0.126986297337
2	0.005292679875	0.333333960255
3	0.009245148401	0.745696621275
4	0.016862928651	2.639682546435
5	-0.014976122776	11.204590967086
6	0.401440149083	59.656792209975
7	-5.815950738680	267.223468914520
8	26.023951772804	1734.742283706764
9	-244.011181624276	4378.452764670006
10	296.167068374575	30038.714778003193

With $8 \leq x \leq 30$ holds

$$-8.1 \cdot 10^{-10} \;\leq\; F_{12}(x) - \int_0^x J_0(t)\, J_1^2(t)\, dx \;\leq\; 5.5 \cdot 10^{-10} \quad .$$

Power series for the modified Bessel functions:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \;=\; \sum_{k=1}^{\infty} d_k^{(12)}\, x^{2k+1} = \frac{x^3}{12} - \frac{x^5}{40} + \frac{5}{1344}\,x^7 - \frac{19}{55296}\,x^9 + \frac{707}{32440320}\,x^{11} - \frac{581}{575078400}\,x^{13} + \ldots$$

With $n \geq 1$ the coefficients $d_k^{(12)}$ are represented bei $d_k^{(03)}$ (see page 342):

$$d_k^{(12)} \quad = \quad \frac{4(k+1)(k+2)}{6k+3}\, d_{k+1}^{(03)} \quad .$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \sim \frac{\sqrt{2}\, e^{3x}}{\sqrt{\pi^3}\, x} \sum_{k=1}^{\infty} \frac{c_k^{(12)}}{x^k}$$

with the first values

k	$c_k^{(12)}$	$c_k^{(12)}$
1	1/12	0.08333 33333 33333 3
2	-1/96	-0.010416666666667
3	-85/4608	-0.018446180555556
4	-3379/110592	-0.030553747106481
5	-69967/1179648	-0.059311760796441
6	-3833063/28311552	-0.135388656898781
7	-487468417/1358954496	-0.358708417709963
8	-11835711665/10871635968	-1.088678070148568
9	-7811427811325/2087354105856	-3.742262891289173
10	-722951995177505/50096498540544	-14.431188131689621

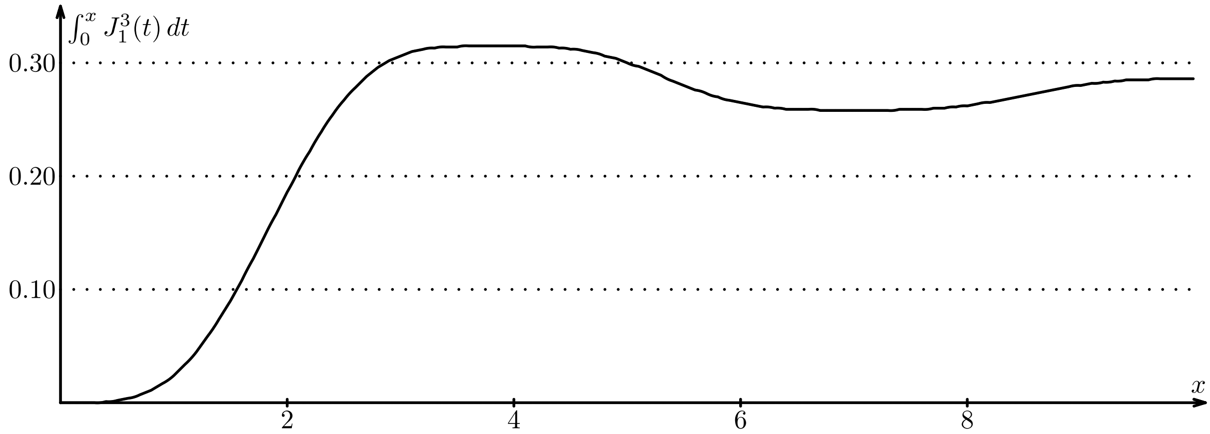
Let

$$\delta_n^{(12)}(x) = \left[\frac{\sqrt{2}\, e^{3x}}{\sqrt{\pi^3}\, x} \sum_{k=1}^{n} \frac{c_k^{(12)}}{x^k} \right] \cdot \left[\int_0^x I_0(t)\, I_1^2(t)\, dt \right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(12)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	4.085E-02	1.541E-02	9.532E-03	6.902E-03	5.409E-03
2	1.483E-02	2.715E-03	1.120E-03	6.084E-04	3.817E-04
3	5.617E-03	4.676E-04	1.265E-04	5.122E-05	2.561E-05
4	2.564E-03	9.526E-05	1.684E-05	5.070E-06	2.019E-06
5	1.379E-03	2.299E-05	2.643E-06	5.914E-07	1.875E-07
6	8.376E-04	6.492E-06	4.829E-07	8.020E-08	2.023E-08
7	5.508E-04	2.121E-06	1.014E-07	1.248E-08	2.503E-09
8	3.768E-04	7.943E-07	2.419E-08	2.204E-09	3.513E-10
9	2.571E-04	3.383E-07	6.500E-09	4.374E-10	5.537E-11
10	1.648E-04	1.624E-07	1.952E-09	9.679E-11	9.724E-12

c) Basic integral $\int Z_1^3(x)\, dx$:

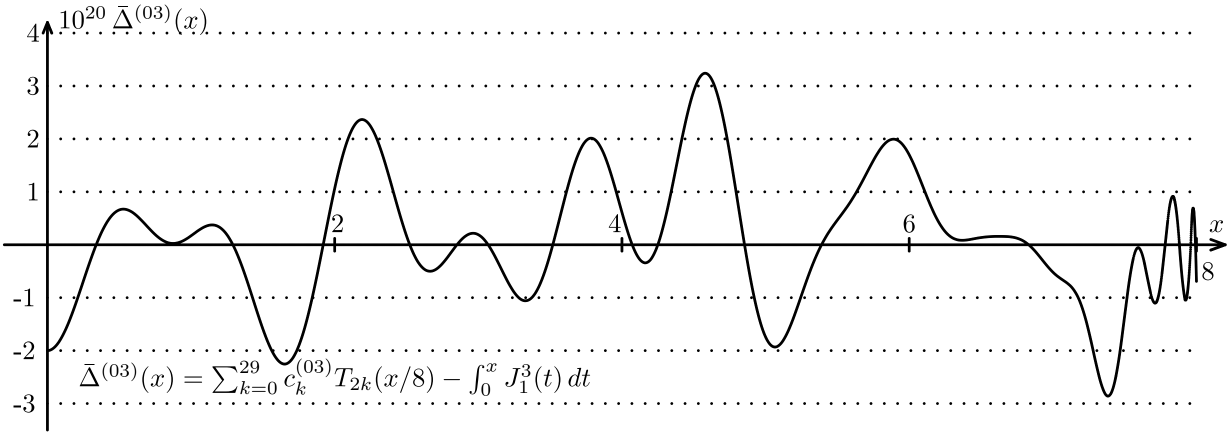


$$\int_0^\infty J_1^3(x)\, dx \quad = \quad \frac{\sqrt{3}}{2\pi} \quad = \quad 0.27566\ 44477\ 10896\ 02476\ \ldots$$

Formula 2.12.42.18 from [4] gives $2 \cdot 0.27566\ldots$.
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_1^3(t)\, dt = \sum_{k=0}^{\infty} c_k^{(03)}\, T_{2k+1}\left(\frac{x}{8}\right) .$$

Using 30 items of the series, the given approximation differs from the true function as shown in the following figure:



The first coefficients of the series are

k	$c_k^{(03)}$	k	$c_k^{(03)}$
0	0.23821 24114 31419 64234	15	-0.00002 28518 02498 15054
1	0.05565 99710 46458 43436	16	0.00000 40330 04736 70912
2	-0.07298 97702 38741 38530	17	-0.00000 06104 34965 75114
3	0.06987 40694 66707 70829	18	0.00000 00802 87294 38422
4	-0.04171 62455 49522 66601	19	-0.00000 00092 76147 24624
5	0.01717 16060 47354 59698	20	0.00000 00009 50125 56893
6	-0.00484 82605 99791 99458	21	-0.00000 00000 86955 13371
7	-0.00029 03839 52557 94438	22	0.00000 00000 07159 24024
8	0.00492 11828 87493 16152	23	-0.00000 00000 00533 44393
9	-0.00739 18550 17718 86749	24	0.00000 00000 00036 16224
10	0.00611 65128 72946 06394	25	-0.00000 00000 00002 24086
11	-0.00338 52693 70801 23544	26	0.00000 00000 00000 12747
12	0.00137 35573 01229 79344	27	-0.00000 00000 00000 00668
13	-0.00043 18988 45664 76700	28	0.00000 00000 00000 00032
14	0.00010 92654 18291 14430	29	-0.00000 00000 00000 00001

Asymptotic formula:

$$\int_0^x J_1^3(t)\,dt \sim \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3\,x}}\left\{\frac{a_1^{(03)}\sin(3x-\frac{\pi}{4}) + b_1^{(03)}\sin(x+\frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty}\frac{1}{x^k}\left[a_k^{(03)}\,\sin\left(3x+\frac{5-2k}{4}\,\pi\right) + b_k^{(03)}\,\sin\left(x+\frac{7-2k}{4}\,\pi\right)\right]\right\}$$

with $\sqrt{3}/2\pi = 0.27566\ 44477\ 10896\ 02476\ \ldots$ (see [8], 13.46) and the first values

k	$a_k^{(03)}$	$a_k^{(03)}$
1	$1/6$	0.16666 66666 66666 66667
2	$5/48$	0.10416 66666 66666 66667
3	$173/2304$	0.07508 68055 55555 55556
4	$5735/55296$	0.10371 45543 98148 14815
5	$112415/589824$	0.19059 07524 95659 72222
6	$6113875/14155776$	0.43189 96712 01352 71991
7	$790059305/679477248$	1.16274 57833 58444 40225
8	$19738125085/5435817984$	3.63112 32537 76703 35181
9	$13496143234525/13496143234525$	12.93134 03956 34871 2675
10	$1298437733131525/25048249270272$	51.83746 45318 04589 8562
k	$b_k^{(03)}$	$b_k^{(03)}$
1	$3/2$	1.50000 00000 00000 00000
2	$27/16$	1.68750 00000 00000 00000
3	$891/256$	3.48046 87500 00000 00000
4	$25065/2048$	12.23876 95312 50000 0000
5	$3564945/65536$	54.39674 37744 14062 5000
6	$156773205/156773205$	299.02115 82183 83789 063
7	$16277745015/8388608$	1940.45841 87269 21081 54
8	$976536193185/67108864$	14551.52322 62760 40077 2
9	$531096920069625/4294967296$	1 23655.63774 23985 39558
10	$40362305845577625/34359738368$	11 74697.70617 25228 8503

The first consecutive maxima and minima of

$$\Delta_n^{(03)}(x) = \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3\,x}}\,\left\{ \frac{a_1^{(03)}\sin(3x-\frac{\pi}{4}) + b_1^{(03)}\sin(x+\frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{1}{x^k}\left[a_k^{(03)}\,\sin\left(3x+\frac{7-2k}{4}\,\pi\right) + b_k^{(03)}\,\sin\left(x+\frac{1-2k}{4}\,\pi\right)\right]\right\} - \int_0^x J_0^3(t)\,dt \quad :$$

i	x_i	$\Delta_1^{(03)}(x_i)$	x_i	$\Delta_2^{(03)}(x_i)$	x_i	$\Delta_3^{(03)}(x_i)$	x_i	$\Delta_4^{(03)}(x_i)$	x_i	$\Delta_5^{(03)}(x_i)$
1	2.023	2.977E-02	3.933	4.556E-03	2.338	-1.901E-02	3.927	-2.969E-03	2.354	4.614E-02
2	5.336	-4.810E-03	7.072	-7.718E-04	5.487	9.013E-04	7.069	1.899E-04	5.496	-5.376E-04
3	8.533	1.690E-03	10.213	2.352E-04	8.632	-1.477E-04	10.210	-3.029E-05	8.638	4.048E-05
4	11.702	-8.051E-04	13.354	-9.624E-05	11.775	4.028E-05	13.352	7.586E-06	11.780	-6.330E-06
5	14.860	4.533E-04	16.495	4.706E-05	14.918	-1.461E-05	16.493	-2.496E-06	14.922	1.484E-06
6	18.013	-2.838E-04	19.636	-2.593E-05	18.060	6.359E-06	19.635	9.863E-07	18.063	-4.510E-07
7	21.162	1.911E-04	22.778	1.557E-05	21.203	-3.145E-06	22.777	-4.448E-07	21.205	1.643E-07
8	24.309	-1.358E-04	25.919	-9.966E-06	24.345	1.709E-06	25.918	2.215E-07	24.347	-6.841E-08
9	27.455	1.005E-04	29.061	6.707E-06	27.486	-9.979E-07	29.060	-1.192E-07	27.488	3.156E-08
10	30.600	-7.684E-05	32.202	-4.698E-06	30.628	6.168E-07	32.201	6.828E-08	30.630	-1.579E-08

i	x_i	$\Delta_6^{(03)}(x_i)$	x_i	$\Delta_7^{(03)}(x_i)$	x_i	$\Delta_8^{(03)}(x_i)$	x_i	$\Delta_9^{(03)}(x_i)$	x_i	$\Delta_{10}^{(03)}(x_i)$
1	3.927	4.522E-03	2.356	-2.426E-01	3.927	-1.276E-02	2.356	2.291E+00	3.927	5.861E-02
2	7.069	-1.056E-04	5.497	6.337E-04	7.069	1.047E-04	5.498	-1.254E-03	7.069	-1.634E-04
3	10.210	8.813E-06	8.639	-2.157E-05	10.210	-4.529E-06	8.639	1.893E-05	10.210	3.623E-06
4	13.352	-1.356E-06	11.781	1.933E-06	13.352	4.279E-07	11.781	-9.676E-07	13.352	-2.095E-07
5	16.493	3.014E-07	14.922	-2.937E-07	16.493	-6.433E-08	14.923	9.523E-08	16.493	2.130E-08
6	19.635	-8.572E-08	18.064	6.248E-08	19.635	1.319E-08	18.064	-1.419E-08	19.635	-3.150E-09
7	22.777	2.911E-08	21.206	-1.680E-08	22.777	-3.380E-09	21.206	2.820E-09	22.777	6.096E-10
8	25.918	-1.130E-08	24.347	5.371E-09	25.918	1.025E-09	24.347	-6.931E-10	25.918	-1.444E-10
9	29.060	4.873E-09	27.489	-1.961E-09	29.060	-3.542E-10	27.489	2.005E-10	29.060	4.007E-11
10	32.201	-2.284E-09	30.630	7.955E-10	32.201	1.361E-10	30.631	-6.600E-11	32.201	-1.262E-11

In the case $x \geq 8$ one has $g_n^{(03)} \leq \Delta_n^{(03)}(x) \leq G_n^{(03)}$ with such values:

n	g_n	G_n	n	g_n	G_n
1	-8.051E-04	1.690E-03	6	-5.870E-05	8.813E-06
2	-4.559E-04	2.352E-04	7	-2.157E-05	1.933E-06
3	-1.477E-04	4.028E-05	8	-4.529E-06	5.333E-05
4	-3.029E-05	1.120E-04	9	-9.676E-07	1.893E-05
5	-6.330E-06	4.048E-05	10	-7.485E-05	3.623E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{03}(x) = \frac{\sqrt{3}}{2\pi} + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}}\left[\tilde{a}_k^{(03)}\,\sin\left(3x+\frac{5-2k+4\delta_{1k}}{4}\,\pi\right) + \tilde{b}_k^{(03)}\,\sin\left(x+\frac{7-2k}{4}\,\pi\right)\right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.
The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329125987	0.380960729687
2	0.026455529094	0.428583324556
3	0.019031312665	0.882368927951
4	0.026093503230	3.106884244014
5	0.036754295203	13.006466163819
6	-0.015099663650	73.274117576600
7	-0.742020898226	297.966084182852
8	-21.408004618587	2526.040556452494
9	-6.362414618305	4245.043677286622
10	-1177.354351044844	55725.065225312970

With $8 \leq x \leq 30$ holds

$$-1.0\cdot 10^{-9} \;\leq\; F_{03}(x) \;\leq\; 4.5\cdot 10^{-10} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \;=\; \sum_{k=2}^{\infty} d_k^{(03)}\,x^{2k} = \frac{x^4}{32} + \frac{x^6}{128} + \frac{x^8}{1024} + \frac{19}{245760}\,x^{10} + \frac{101}{23592960}\,x^{12} + \frac{83}{471859200}\,x^{14} + \ldots$$

With $n \geq 3$ the following recurrence relation holds:

$$d_n^{(03)} = \frac{16\,\sigma_1^{(03)}(n,d) \;+\; 36\,\sigma_2^{(03)}(n,d)}{3n(n-1)(n-2)}$$

with

$$\sigma_1^{(03)}(n,d) = \sum_{k=3}^{n-1} k(n-k+2)(2nk-n+7k-5k^2)\cdot d_k^{(03)}\cdot d_{n-k+2}^{(03)}$$

and

$$\sigma_2^{(03)}(n,d) = \sum_{k=2}^{n-1} k(n-k+1)\cdot d_k^{(03)}\cdot d_{n-k+1}^{(03)}\,.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k^{(03)}}{x^k}$$

with the first values

k	$c_k^{(03)}$	$c_k^{(03)}$
1	1/12	0.08333 33333 33333 3
2	-5/96	-0.052083333333333
3	-173/4608	-0.037543402777778
4	-5735/110592	-0.051857277199074
5	-112415/1179648	-0.095295376247830
6	-6113875/28311552	-0.215949835600676
7	-790059305/1358954496	-0.581372891679222
8	-19738125085/10871635968	-1.815561626888352
9	-13496143234525/2087354105856	-6.465670197817436
10	-1298437733131525/50096498540544	-25.918732265902295

Let

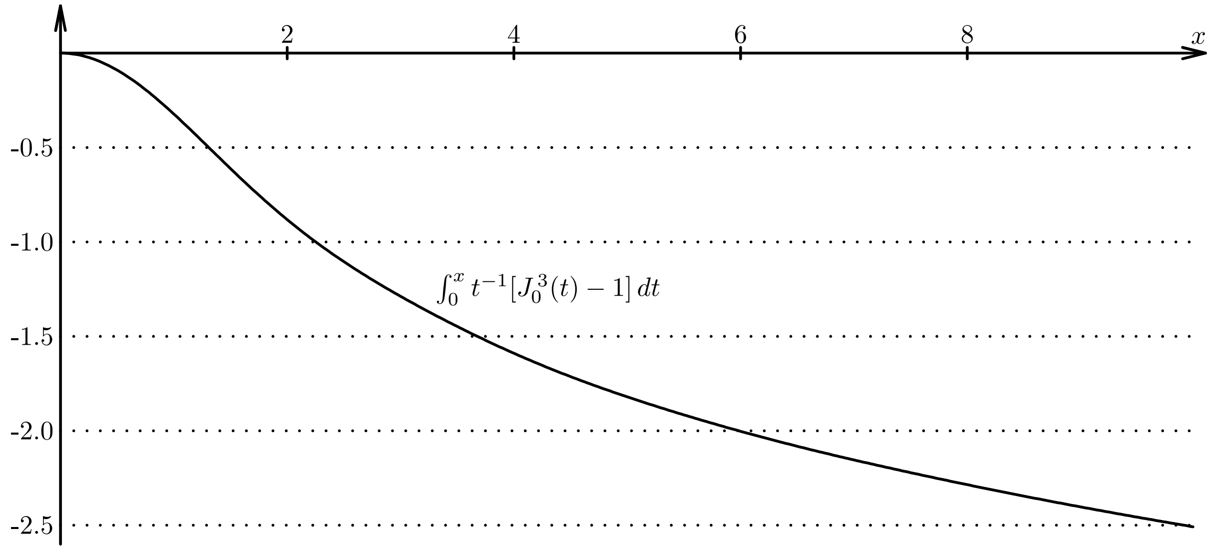
$$\delta_n^{(03)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k^{(03)}}{x^k}\right]\cdot\left[\int_0^x I_1^3(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(03)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	1.791E-01	7.271E-02	4.589E-02	3.355E-02	2.645E-02
2	3.170E-02	5.662E-03	2.315E-03	1.253E-03	7.841E-04
3	1.046E-02	8.293E-04	2.209E-04	8.875E-05	4.419E-05
4	4.587E-03	1.618E-04	2.801E-05	8.358E-06	3.311E-06
5	2.429E-03	3.912E-05	4.387E-06	9.712E-07	3.060E-07
6	1.451E-03	1.132E-05	8.179E-07	1.342E-07	3.362E-08
7	9.250E-04	3.840E-06	1.774E-07	2.154E-08	4.288E-09
8	5.962E-04	1.503E-06	4.399E-08	3.949E-09	6.244E-10
9	3.620E-04	6.710E-07	1.232E-08	8.161E-10	1.024E-10
10	1.742E-04	3.374E-07	3.863E-09	1.882E-10	1.874E-11

d) Basic integral $\int x^{-1} \cdot Z_0^3(x)\,dx$:

$$\int \frac{Z_0^3(x)\,dx}{x} \quad = \quad \ln|x| + \int_0^x \frac{Z_0^3(t)-1}{t}\,dt$$



$$\int_0^x \frac{J_0^3(t)-1}{t}\,dt = -\frac{1}{4}\,x^3 + \frac{3}{64}\,x^5 - \frac{31}{5376}\,x^7 + \frac{71}{147456}\,x^9 - \frac{47}{1638400}\,x^{11} + \frac{11723}{9201254400}\,x^{13} - \frac{2021}{46242201600}\,x^{15} +$$

$$+\frac{1567}{1315333734400}\,x^{17} - \frac{5773279}{218624250662092800}\,x^{19} + \frac{3125957}{6443662124777472000}\,x^{21} - \frac{1114457}{148511069923442688000}\,x^{23} + \dots$$

$$= -0.25x^3 + 0.046875x^5 - 0.00576\,63690\,47619x^7 + 0.00048\,14995\,65972x^9 -$$

$$-0.00002\,86865\,23438x^{11} + 0.00000\,12740\,65414x^{13} - 0.00000\,00437\,04667x^{15} + 0.00000\,00011\,91333x^{17} -$$

$$-0.00000\,00000\,26407x^{19} + 0.00000\,00000\,00485x^{21} - 0.00000\,00000\,00008x^{23} + \dots$$

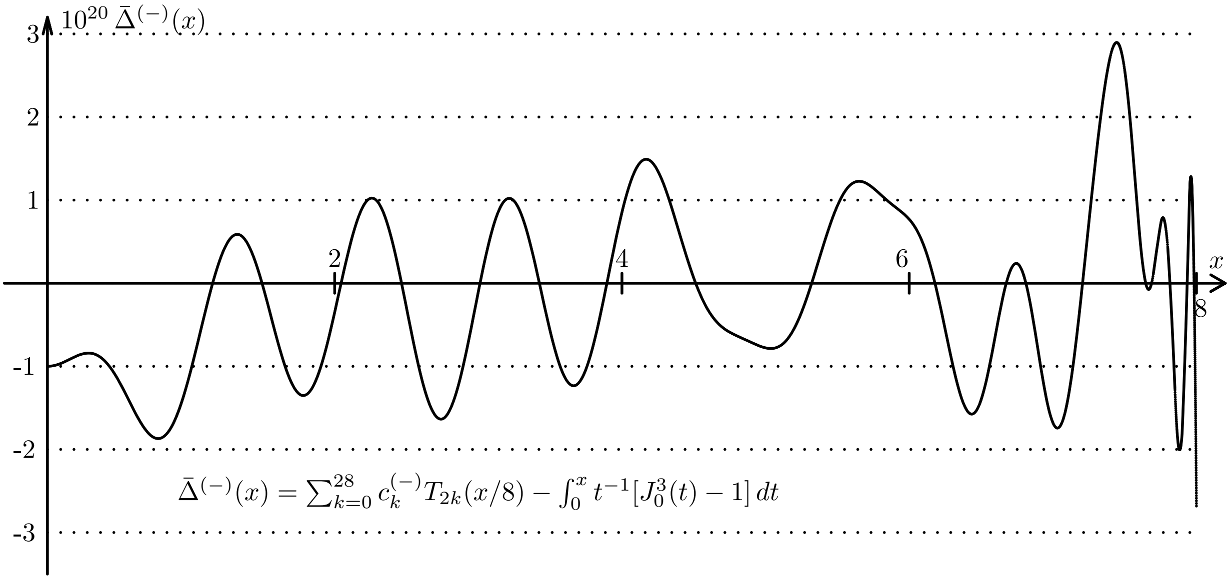
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x \frac{J_0^3(t)-1}{t}\,dt = \sum_{k=0}^{\infty} c_k^{(-)}\,T_{2k}\left(\frac{x}{8}\right) .$$

The first coefficients are

k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	-1.66487772839693011416	15	-0.00000370566539255354
1	-0.85731855220711666821	16	0.00000057633802528230
2	0.35797423128412846686	17	-0.00000007814654021909
3	-0.18824568501714899999	18	0.00000000932468761665
4	0.10690149363496148802	19	-0.00000000098725238622
5	-0.06393636091495798502	20	0.00000000009342113704
6	0.03911001491993894893	21	-0.00000000000795204305
7	-0.02409344095904841252	22	0.00000000000061237408
8	0.01437795892387628520	23	-0.00000000000004288320
9	-0.00768362174411199843	24	0.00000000000000274344
10	0.00345368891420790184	25	-0.00000000000000016101
11	-0.00127440919485714551	26	0.00000000000000000870
12	0.00038620719490337840	27	-0.00000000000000000043
13	-0.00009714124269423777	28	0.00000000000000000002
14	0.00002055168104878868	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_1^x \frac{J_0^3(t)\,dt}{t} \sim 0.11548\,77825\,09057\,98226\,\ldots +$$

$$+\sqrt{\frac{2}{\pi^3\,x}}\,\sum_{k=1}^{\infty}\frac{1}{x^{k+1}}\,\left[a_k^{(01)}\,\sin\left(3x+\frac{7-2k}{4}\,\pi\right)+b_k^{(01)}\,\sin\left(x+\frac{1-2k}{4}\,\pi\right)\right]$$

with the first values

k	$a_k^{(01)}$	$a_k^{(01)}$
1	1/6	0.16666 66666 66666 66667
2	29/144	0.20138 88888 88888 88889
3	1921/6912	0.27792245370370370370
4	8527/18432	0.46261 93576 38888 88889
5	1621523/1769472	0.91638 80524 81192 12963
6	89993003/42467328	2.11911 14967 25200 13503
7	3821763071/679477248	5.62456 37101 89160 00555
8	275582100493/16307453952	16.89914 93892 39986 24700
9	177961856737289/3131031158784	56.83809 82852 95930 27586
k	$b_k^{(01)}$	$b_k^{(01)}$
1	3/2	1.50000 00000 00000 00000
2	63/16	3.93750 00000 00000 00000
3	3603/256	14.07421 87500 00000 00000
4	129981/2048	63.46728 51562 50000 00000
5	22909281/65536	349.56788 63525 39062 5000
6	1191489153/524288	2272.58520 69854 73632 813
7	143000089119/8388608	17046.93902 95743 94226 07
8	9724198215717/67108864	1 44901.84509 33247 80464 2
9	5912428624098201/4294967296	13 76594.56210 63469 44347

The first consecutive maxima and minima of

$$\Delta_n^{(-1)}(x) = 0.115\ldots + \sqrt{\frac{2}{\pi^3\,x}}\,\sum_{k=1}^{n} \frac{1}{x^{k+1}}\,\left[a_k^{(-1)}\,\sin\left(3x+\ldots\right) + b_k^{(-1)}\,\sin\left(x+\ldots\right)\right] - \int_1^x t^{-1}\cdot J_0^3(t)\,dt \quad :$$

i	x_i	$\Delta_1^{(-1)}(x_i)$	x_i	$\Delta_2^{(-1)}(x_i)$	x_i	$\Delta_3^{(-1)}(x_i)$	x_i	$\Delta_4^{(-1)}(x_i)$	x_i	$\Delta_5^{(-1)}(x_i)$
1	3.944	-5.284E-03	2.356	2.066E-02	3.929	3.477E-03	2.356	-5.340E-02	3.927	-5.296E-03
2	7.079	8.972E-04	5.498	-1.010E-03	7.070	-2.223E-04	5.498	6.258E-04	7.069	1.237E-04
3	10.217	-2.736E-04	8.639	1.668E-04	10.211	3.547E-05	8.639	-4.719E-05	10.210	-1.032E-05
4	13.357	1.120E-04	11.781	-4.561E-05	13.353	-8.883E-06	11.781	7.385E-06	13.352	1.589E-06
5	16.498	-5.477E-05	14.923	1.656E-05	16.494	2.923E-06	14.923	-1.732E-06	16.493	-3.531E-07
6	19.639	3.018E-05	18.064	-7.216E-06	19.635	-1.155E-06	18.064	5.264E-07	19.635	1.004E-07
7	22.780	-1.812E-05	21.206	3.571E-06	22.777	5.209E-07	21.206	-1.918E-07	22.777	-3.411E-08
8	25.921	1.160E-05	24.347	-1.941E-06	25.919	-2.594E-07	24.347	7.986E-08	25.918	1.324E-08
9	29.062	-7.806E-06	27.489	1.134E-06	29.060	1.396E-07	27.489	-3.685E-08	29.060	-5.709E-09
10	32.204	5.468E-06	30.631	-7.008E-07	32.202	-7.996E-08	30.631	1.844E-08	32.201	2.676E-09

i	x_i	$\Delta_6^{(-1)}(x_i)$	x_i	$\Delta_7^{(-1)}(x_i)$	x_i	$\Delta_8^{(-1)}(x_i)$	x_i	$\Delta_9^{(-1)}(x_i)$
1	2.356	2.837E-01	3.927	1.495E-02	2.356	-2.685E+00	3.927	-6.868E-02
2	5.498	-7.418E-04	7.069	-1.227E-04	5.498	1.470E-03	7.069	1.915E-04
3	8.639	2.526E-05	10.210	5.307E-06	8.639	-2.218E-05	10.210	-4.245E-06
4	11.781	-2.264E-06	13.352	-5.014E-07	11.781	1.134E-06	13.352	2.455E-07
5	14.923	3.441E-07	16.493	7.538E-08	14.923	-1.116E-07	16.493	-2.496E-08
6	18.064	-7.318E-08	19.635	-1.546E-08	18.064	1.663E-08	19.635	3.692E-09
7	21.206	1.968E-08	22.777	3.961E-09	21.206	-3.305E-09	22.777	-7.143E-10
8	24.347	-6.291E-09	25.918	-1.201E-09	24.347	8.122E-10	25.918	1.692E-10
9	27.489	2.297E-09	29.060	4.150E-10	27.489	-2.350E-10	29.060	-4.695E-11
10	30.631	-9.319E-10	32.201	-1.594E-10	30.631	7.734E-11	32.201	1.479E-11

In the case $x \geq 8$ one has $g_n^{(-1)} \leq \Delta_n^{(-1)}(x) \leq G_n^{(-1)}$ with the following values:

n	$g_n^{(-1)}$	$G_n^{(-1)}$	n	$g_n^{(-1)}$	$G_n^{(-1)}$
1	-2.736E-04	4.984E-04	6	-2.264E-06	2.526E-05
2	-4.561E-05	1.668E-04	7	-6.249E-05	5.307E-06
3	-1.301E-04	3.547E-05	8	-2.218E-05	1.134E-06
4	-4.719E-05	7.385E-06	9	-4.245E-06	8.772E-05
5	-1.032E-05	6.871E-05	-	-	-

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{-1}(x) = 0.115\ldots + \sum_{k=1}^{9} \frac{1}{x^{k+3/2}}\,\left[\tilde{a}_k^{(-1)}\,\sin\left(3x + \frac{7-2k}{4}\,\pi\right) + \tilde{b}_k^{(-1)}\,\sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(-1)}$	$\tilde{b}_k^{(-1)}$
1	0.042328265593	0.380960857219
2	0.051124895140	0.999204230732
3	0.069839136706	3.573795779326
4	0.108172780150	15.423626931601
5	-0.035881146333	86.734624679653
6	-0.183965343668	370.667415902952
7	-39.511920554495	3145.668270195769
8	23.037919369082	5512.467112432406
9	-1996.507320270070	73698.065312438298

With $8 \leq x \leq 30$ holds

$$-2.0\cdot 10^{-9} \;\leq\; F_{-1}(x) - \int_1^x t^{-1}\cdot J_0^3(t)\,dt \;\leq\; 3.2\cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x \frac{I_0^3(t)-1}{t}\,dt = \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k} = \frac{3}{8}\,x^2+\frac{15}{256}\,x^4+\frac{31}{4608}\,x^6+\frac{71}{131072}\,x^8+\frac{517}{16384000}\,x^{10}+\frac{11723}{8493465600}\,x^{12}+\ldots$$

From this:

$$\int_1^x I_0^3(t)\,dt = \ln|x| - 0.44089\;58511\;01198\;85318 + \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k}$$

With $n \geq 1$ the following recurrence relation holds:

$$d_{n+1}^{(-1)} = \frac{1}{24(n+1)^3}\left[16\sum_{k=1}^{n-1} k(n+1-k)(3n^2+5k^2-8kn+5n-6k+2)d_k^{(-1)}d_{n+1-k}^{(-1)} - \right.$$

$$\left. -36\sum_{k=1}^{n-1} k(n-k)d_k^{(-1)}d_{n-k}^{(-1)} - 36n\,d_n^{(-1)}\right]$$

Asymptotic formula for the modified Bessel function:

$$\int_1^x \frac{I_0^3(t)\,dt}{t} \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k}{x^{k+1}}$$

with the first values

k	c_k	c_k
1	1/12	0.08333 33333 33333 3
2	29/288	0.10069 44444 44444 4
3	1921/13824	0.13896 12268 51851 9
4	8527/36864	0.2313 09678 81944 4
5	1621523/3538944	0.45819 40262 40596 1
6	89993003/84934656	1.05955 57483 62600 1
7	3821763071/1358954496	2.81228 18550 94580 0
8	275582100493/32614907904	8.44957 46946 19993 1
9	177961856737289/6262062317568	28.41904 91426 47965
10	5312592054074687/50096498540544	106.04717 31327 7115

Let

$$\delta_n(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k}{x^{k+1}} - \frac{\sqrt{2}\,e^{3}}{\sqrt{\pi^3}}\sum_{k=1}^{n} c_k\right]\cdot\left[\int_1^x t^{-1}I_0^3(t)\,dt\right]^{-1} - 1\,,$$

then one has the following values of $\delta_n(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-2.591E-01	-1.236E-01	-8.166E-02	-6.101E-02	-4.870E-02
2	-8.036E-02	-1.768E-02	-7.680E-03	-4.277E-03	-2.722E-03
3	-3.135E-02	-3.069E-03	-8.738E-04	-3.624E-04	-1.838E-04
4	-1.560E-02	-6.368E-04	-1.185E-04	-3.658E-05	-1.478E-05
5	-1.048E-02	-1.549E-04	-1.881E-05	-4.316E-06	-1.388E-06
6	-1.070E-02	-4.350E-05	-3.431E-06	-5.850E-07	-1.498E-07
7	-1.769E-02	-1.394E-05	-7.097E-07	-8.985E-08	-1.831E-08
8	-4.254E-02	-5.103E-06	-1.647E-07	-1.547E-08	-2.507E-09
9	-1.287E-01	-2.292E-06	-4.254E-08	-2.958E-09	-3.813E-10
10	-4.521E-01	-1.839E-06	-1.214E-08	-6.242E-10	-6.393E-11

e) Integrals of the type $\int x^n\, Z_0^3(x)\, dx$:

$$\int x\, J_0^3(x)\, dx = x\, J_0^2(x)\, J_1(x) + \frac{2x}{3}\, J_1^3(x) + \frac{4}{3} \int J_1^3(x)\, dx$$

$$\int x\, I_0^3(x)\, dx = x\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_1^3(x) - \frac{4}{3} \int I_1^3(x)\, dx$$

$$\int x^2\, J_0^3(x)\, dx =$$

$$= -\frac{x}{9}\, J_0^3(x) + x^2\, J_0^2(x)\, J_1(x) - \frac{2x}{3}\, J_0(x)\, J_1^2(x) + \frac{2x^2}{3}\, J_1^3(x) + \frac{1}{9} \int J_0^3(x)\, dx - \frac{2}{3} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^2\, I_0^3(x)\, dx =$$

$$= \frac{x}{9}\, I_0^3(x) + x^2\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_0(x)\, I_1^2(x) - \frac{2x^2}{3}\, I_1^3(x) - \frac{1}{9} \int I_0^3(x)\, dx - \frac{2}{3} \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^3\, J_0^3(x)\, dx = \frac{2x^2}{3}\, J_0^3(x) + \frac{3x^3 - 4x}{3}\, J_0^2(x)\, J_1(x) + \frac{6x^3 - 8x}{9}\, J_1^3(x) - \frac{16}{9} \int J_1^3(x)\, dx$$

$$\int x^3\, I_0^3(x)\, dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{3x^3 + 4x}{3}\, I_0^2(x)\, I_1(x) - \frac{6x^3 + 8x}{9}\, I_1^3(x) - \frac{16}{9} \int I_1^3(x)\, dx$$

Let

$$\int x^n\, J_0^3(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1^3(x)\, dx + \mathcal{X}_n \int \frac{J_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_0^3(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1^3(x)\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x}\, .$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_4(x) = \frac{39\, x^3 + 17\, x}{27}\, , \quad \mathcal{Q}_4(x) = \frac{3\, x^4 - 13\, x^2}{3}\, , \quad \mathcal{R}_4(x) = \frac{6\, x^3 + 28\, x}{9}\, , \quad \mathcal{S}_4(x) = \frac{6\, x^4 - 28\, x^2}{9}\, ,$$

$$\mathcal{U}_4 = -\frac{17}{24}\, , \quad \mathcal{V}_4 = \frac{28}{9}\, , \quad \mathcal{W}_4 = 0\, .$$

$$\mathcal{P}_4^*(x) = \frac{-39\, x^3 + 17\, x}{27}\, , \quad \mathcal{Q}_4^*(x) = \frac{3\, x^4 + 13\, x^2}{3}\, , \quad \mathcal{R}_4^*(x) = \frac{6\, x^3 - 28\, x}{9}\, , \quad \mathcal{S}_4^*(x) = -\frac{6\, x^4 + 28\, x^2}{9}\, ,$$

$$\mathcal{U}_4^* = -\frac{17}{27}\, , \quad \mathcal{V}_4^* = -\frac{28}{9}\, , \quad \mathcal{W}_0^* = 0$$

$$\mathcal{P}_5(x) = \frac{60\, x^4 - 160\, x^2}{27}\, , \quad \mathcal{Q}_5(x) = \frac{27\, x^5 - 240\, x^3 + 320\, x}{27}\, , \quad \mathcal{R}_5(x) = \frac{4x^4}{3}\, ,$$

$$\mathcal{S}_5(x) = \frac{54\, x^5 - 552\, x^3 + 640\, x}{81}\, , \quad \mathcal{U}_5 = 0\, , \quad \mathcal{V}_5 = 0\, , \quad \mathcal{W}_5 = \frac{1280}{81}$$

$$\mathcal{P}_5^*(x) = -\frac{60\, x^4 + 160\, x^2}{27}\, , \quad \mathcal{Q}_5^*(x) = \frac{27\, x^5 + 240\, x^3 + 320\, x}{27}\, , \quad \mathcal{R}_5^*(x) = \frac{4\, x^4}{3}\, ,$$

$$\mathcal{S}_5^*(x) = -\frac{54\, x^5 + 552\, x^3 + 640\, x}{81}\, , \quad \mathcal{U}_5^* = 0\, , \quad \mathcal{V}_5^* = 0\, , \quad \mathcal{W}_5^* = -\frac{1280}{81}$$

$$\mathcal{P}_6(x) = \frac{9\, x^5 - 69\, x^3 - 31\, x}{3}\, , \quad \mathcal{Q}_6(x) = x^6 - 15\, x^4 + 69\, x^2\, , \quad \mathcal{R}_6(x) = 2\, x^5 - 12\, x^3 - 50\, x\, ,$$

$$\mathcal{S}_6(x) = \frac{2\, x^6 - 36\, x^4 + 150\, x^2}{3}\, , \quad \mathcal{U}_6 = \frac{31}{3}\, , \quad \mathcal{V}_6 = -50\, , \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = \frac{-9\,x^5 - 69\,x^3 + 31\,x}{3}\,, \quad \mathcal{Q}_6^*(x) = x^6 + 15\,x^4 + 69\,x^2\,, \quad \mathcal{R}_6^*(x) = 2\,x^5 + 12\,x^3 - 50\,x\,,$$

$$\mathcal{S}_6^*(x) = -\frac{2\,x^6 + 36\,x^4 + 150\,x^2}{3}\,, \quad \mathcal{U}_6^* = -\frac{31}{3}\,, \quad \mathcal{V}_6^* = -50\,, \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{102\,x^6 - 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7(x) = \frac{27\,x^7 - 612\,x^5 + 5952\,x^3 - 7936\,x}{27}\,,$$

$$\mathcal{R}_7(x) = \frac{8\,x^6 - 112\,x^4}{3}\,, \quad \mathcal{S}_7(x) = \frac{54\,x^7 - 1512\,x^5 + 13920\,x^3 - 15872\,x}{81}\,,$$

$$\mathcal{U}_7 = 0\,, \quad \mathcal{V}_7 = 0\,, \quad \mathcal{W}_7 = -\frac{31744}{81}$$

$$\mathcal{P}_7^*(x) = -\frac{102\,x^6 + 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7^*(x) = \frac{27\,x^7 + 612\,x^5 + 5952\,x^3 + 7936\,x}{27}\,,$$

$$\mathcal{R}_7^*(x) = \frac{8\,x^6 + 112\,x^4}{3}\,, \quad \mathcal{S}_7^*(x) = -\frac{54\,x^7 + 1512\,x^5 + 13920\,x^3 + 15872\,x}{81}\,,$$

$$\mathcal{U}_7^* = 0\,, \quad \mathcal{V}_7^* = 0\,, \quad \mathcal{W}_7^* = -\frac{31744}{81}\,,$$

$$\mathcal{P}_8(x) = \frac{1107\,x^7 - 25947\,x^5 + 200427\,x^3 + 90373\,x}{243}\,, \quad \mathcal{Q}_8(x) = \frac{27\,x^8 - 861\,x^6 + 14415\,x^4 - 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8(x) = \frac{270\,x^7 - 6516\,x^5 + 35346\,x^3 + 145400\,x}{81}\,, \quad \mathcal{S}_8(x) = \frac{54\,x^8 - 2172\,x^6 + 35346\,x^4 - 145400\,x^2}{81}$$

$$\mathcal{U}_8 = -\frac{90373}{243}\,, \quad \mathcal{V}_8 = \frac{145400}{81}\,, \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = \frac{-1107\,x^7 - 25947\,x^5 - 200427\,x^3 + 90373\,x}{243}\,, \quad \mathcal{Q}_8^*(x) = \frac{27\,x^8 + 861\,x^6 + 14415\,x^4 + 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8^*(x) = \frac{270\,x^7 + 6516\,x^5 + 35346\,x^3 - 145400\,x}{81}\,, \quad \mathcal{S}_8^*(x) = -\frac{54\,x^8 + 2172\,x^6 + 35346\,x^4 + 145400\,x^2}{81}$$

$$\mathcal{U}_8^* = -\frac{90373}{243}\,, \quad \mathcal{V}_8^* = -\frac{145400}{81}\,, \quad \mathcal{W}_8^* = 0$$

$$\int \frac{J_0^3(x)\,dx}{x^2} = -\frac{J_0^3(x)}{x} + 3J_0^2(x)\,J_1(x) + 6\int J_0(x)\,J_1^2(x)\,dx - 3\int J_0^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^2} = -\frac{I_0^3(x)}{x} - 3I_0^2(x)\,I_1(x) + 6\int I_0(x)\,I_1^2(x)\,dx + 3\int I_0^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^3} = -\frac{x^2+1}{2x^2}\,J_0^3(x) + \frac{3}{4x}\,J_0^2(x)J_1(x) - \frac{3}{4}\,J_0(x)J_1^2(x) - \frac{3}{4}\int \frac{J_0^3(x)\,dx}{x} - \frac{3}{4}\int J_1^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^3} = -\frac{x^2-1}{2x^2}\,I_0^3(x) - \frac{3}{4x}\,I_0^2(x)I_1(x) - \frac{3}{4}\,I_0(x)i_1^2(x) + \frac{3}{4}\int \frac{I_0^3(x)\,dx}{x} + \frac{3}{4}\int I_1^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^4} = \frac{x^2-1}{3x^3}\,J_0^3(x) - \frac{13\,x^2-3}{9x^2}\,J_0^2(x)\,J_1(x) - \frac{2}{9x}\,J_0(x)\,J_1^2(x) + \frac{2}{27}\,J_1^3(x) +$$

$$+\frac{13}{9}\int J_0^3(x)\,dx - \frac{28}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^4} = -\frac{x^2+1}{3x^3}\,J_0^3(x) - \frac{13\,x^2+3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1^3(x) +$$

$$+\frac{13}{9}\int I_0^3(x)\,dx + \frac{28}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x) = \frac{23\,x^4 + 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}_{-5}(x) = \frac{-15\,x^2 + 12}{64\,x^3}\,, \quad \mathcal{R}_{-5}(x) = \frac{69\,x^2 - 24}{256\,x^2}\,, \quad \mathcal{S}_{-5}(x) = \frac{3}{128x}$$

$$\mathcal{U}_{-5} = \frac{15}{64}\,, \quad \mathcal{V}_{-5} = 0\,, \quad \mathcal{W}_{-5} = 0\,, \quad \mathcal{X}_{-5} = \frac{69}{256}$$

$$\mathcal{P}^*_{-5}(x) = \frac{23\,x^4 - 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}^*_{-5}(x) = \frac{-15\,x^2 - 12}{64\,x^3}\,, \quad \mathcal{R}^*_{-5}(x) = \frac{-69\,x^2 - 24}{256\,x^2}\,, \quad \mathcal{S}^*_{-5}(x) = \frac{-3}{128x}$$

$$\mathcal{U}^*_{-5} = \frac{15}{64}\,, \quad \mathcal{V}^*_{-5} = 0\,, \quad \mathcal{W}^*_{-5} = 0\,, \quad \mathcal{X}^*_{-5} = \frac{69}{256}$$

$$\mathcal{P}_{-6}(x) = \frac{-9\,x^4 + 5\,x^2 - 25}{125\,x^5}\,, \quad \mathcal{Q}_{-6}(x) = \frac{207\,x^4 - 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}_{-6}(x) = \frac{36\,x^2 - 30}{625\,x^3}\,,$$

$$\mathcal{S}_{-6}(x) = \frac{-12\,x^2 + 6}{625\,x^2}\,, \quad \mathcal{U}_{-6} = -\frac{207}{625}\,, \quad \mathcal{V}_{-6} = \frac{18}{25}\,, \quad \mathcal{W}_{-6} = 0$$

$$\mathcal{P}^*_{-6}(x) = -\frac{9\,x^4 + 5\,x^2 + 25}{125\,x^5}\,, \quad \mathcal{Q}^*_{-6}(x) = -\frac{207\,x^4 + 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}^*_{-6}(x) = -\frac{36\,x^2 + 30}{625\,x^3}\,,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{12\,x^2 + 6}{625\,x^2}\,, \quad \mathcal{U}^*_{-6} = \frac{207}{625}\,, \quad \mathcal{V}^*_{-6} = \frac{18}{25}\,, \quad \mathcal{W}^*_{-6} = 0$$

$$\mathcal{P}_{-7}(x) = \frac{-145\,x^6 - 68\,x^4 + 96\,x^2 - 768}{4608\,x^6}\,, \quad \mathcal{Q}_{-7}(x) = \frac{93\,x^4 - 68\,x^2 + 192}{2304\,x^5}\,,$$

$$\mathcal{R}_{-7}(x) = \frac{-435\,x^4 + 168\,x^2 - 256}{9216\,x^4}\,, \quad \mathcal{S}_{-7}(x) = \frac{-63\,x^2 + 64}{13824\,x^3}$$

$$\mathcal{U}_{-7} = -\frac{31}{768}\,, \quad \mathcal{V}_{-7} = 0\,, \quad \mathcal{W}_{-7} = 0\,, \quad \mathcal{X}_{-7} = -\frac{145}{3072}$$

$$\mathcal{P}^*_{-7}(x) = \frac{145\,x^6 - 68\,x^4 - 96\,x^2 - 768}{4608\,x^6}\,, \quad \mathcal{Q}^*_{-7}(x) = \frac{-93\,x^4 - 68\,x^2 - 192}{2304\,x^5}\,,$$

$$\mathcal{R}^*_{-7}(x) = \frac{-435\,x^4 - 168\,x^2 - 256}{9216\,x^4}\,, \quad \mathcal{S}^*_{-7}(x) = \frac{-63\,x^2 - 64}{13824\,x^3}$$

$$\mathcal{U}^*_{-7} = \frac{31}{768}\,, \quad \mathcal{V}^*_{-7} = 0\,, \quad \mathcal{W}^*_{-7} = 0\,, \quad \mathcal{X}^*_{-7} = \frac{145}{3072}$$

$$\mathcal{P}_{-8}(x) = \frac{2883\,x^6 - 1435\,x^4 + 3675\,x^2 - 42875}{300125\,x^7}\,, \quad \mathcal{Q}_{-8}(x) = \frac{-66809\,x^6 + 14415\,x^4 - 21525\,x^2 + 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}_{-8}(x) = \frac{-11782\,x^4 + 10860\,x^2 - 26250}{1500625\,x^5}\,, \quad \mathcal{S}_{-8}(x) = \frac{11782\,x^4 - 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}_{-8} = \frac{5816}{60025}\,, \quad \mathcal{W}_{-8} = 0$$

$$\mathcal{P}^*_{-8}(x) = -\frac{2883\,x^6 + 1435\,x^4 + 3675\,x^2 + 42875}{300125\,x^7}\,, \quad \mathcal{Q}^*_{-8}(x) = -\frac{66809\,x^6 + 14415\,x^4 + 21525\,x^2 + 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{11782\,x^4 + 10860\,x^2 + 26250}{1500625\,x^5}\,, \quad \mathcal{S}^*_{-8}(x) = -\frac{11782\,x^4 + 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}^*_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}^*_{-8} = -\frac{5816}{60025}\,, \quad \mathcal{W}^*_{-8} = 0$$

f) Integrals of the type $\int x^n\, Z_0^2(x)\, Z_1(x)\, dx$:

$$\int J_0^2(x)\, J_1(x)\, dx = -\frac{1}{3}\, J_0^3(x)$$

$$\int I_0^2(x)\, I_1(x)\, dx = \frac{1}{3}\, I_0^3(x)$$

$$\int x\, J_0^2(x)\, J_1(x)\, dx = -\frac{x}{3}\, J_0^3(x) + \frac{1}{3} \int J_0^3(x)\, dx$$

$$\int x\, I_0^2(x)\, I_1(x)\, dx = \frac{x}{3}\, I_0^3(x) - \frac{1}{3} \int I_0^3(x)\, dx$$

$$\int x^2\, J_0^2(x)\, J_1(x)\, dx = -\frac{x^2}{3}\, J_0^3(x) + \frac{2x}{3}\, J_0^2(x)\, J_1(x) + \frac{4x}{9}\, J_1^3(x) + \frac{8}{9} \int J_1^3(x)\, dx$$

$$\int x^2\, I_0^2(x)\, I_1(x)\, dx = \frac{x^2}{3}\, I_0^3(x) - \frac{2x}{3}\, I_0^2(x)\, I_1(x) + \frac{4x}{9}\, I_1^3(x) + \frac{8}{9} \int I_1^3(x)\, dx$$

$$\int x^3\, J_0^2(x)\, J_1(x)\, dx = -\frac{3x^3+x}{9}\, J_0^3(x) + x^2\, J_0^2(x)\, J_1(x) - \frac{2x}{3}\, J_0(x)\, J_1^2(x) + \frac{2x^2}{3}\, J_1^3(x) +$$

$$+\frac{1}{9} \int J_0^3(x)\, dx - \frac{2}{3} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^3\, I_0^2(x)\, I_1(x)\, dx = \frac{3x^3-x}{9}\, I_0^3(x) - x^2\, I_0^2(x)\, I_1(x) + \frac{2x}{3}\, I_0(x)\, I_1^2(x) + \frac{2x^2}{3}\, I_1^3(x) +$$

$$+\frac{1}{9} \int I_0^3(x)\, dx + \frac{2}{3} \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^4\, J_0^2(x)\, J_1(x)\, dx = \frac{-3x^4+8x^2}{9}\, J_0^3(x) + \frac{12x^3-16x}{9}\, J_0^2(x)\, J_1(x) + \frac{24x^3-32x}{27}\, J_1^3(x) - \frac{64}{27} \int J_1^3(x)\, dx$$

$$\int x^4\, I_0^2(x)\, I_1(x)\, dx = \frac{3x^4+8x^2}{9}\, I_0^3(x) - \frac{12x^3+16x}{9}\, I_0^2(x)\, I_1(x) + \frac{24x^3+32x}{27}\, I_1^3(x) + \frac{64}{27} \int I_1^3(x)\, dx$$

Let

$$\int x^n\, J_0^2(x)\, J_1(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1(x)^3\, dx + \mathcal{X}_n \int \frac{J_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_0^2(x)\, I_1(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1(x)^3\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x}\ .$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{-27\,x^5 + 195\,x^3 + 85\,x}{81}\ , \quad \mathcal{Q}_5(x) = \frac{15\,x^4 - 65\,x^2}{9}\ , \quad \mathcal{R}_5(x) = \frac{30\,x^3 + 140\,x}{27}\ ,$$

$$\mathcal{S}_5(x) = \frac{30\,x^4 - 140\,x^2}{27}\ , \quad \mathcal{U}_5 = -\frac{85}{81}\ , \quad \mathcal{V}_5 = \frac{140}{27}\ , \quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = \frac{27\,x^5 + 195\,x^3 - 85\,x}{81}\ , \quad \mathcal{Q}_5^*(x) = -\frac{15\,x^4 + 65\,x^2}{9}\ , \quad \mathcal{R}_5^*(x) = \frac{-30\,x^3 + 140\,x}{27}\ ,$$

$$\mathcal{S}_5^*(x) = \frac{30\,x^4 + 140\,x^2}{27}\ , \quad \mathcal{U}_5^* = \frac{85}{81}\ , \quad \mathcal{V}_5^* = \frac{140}{27}\ , \quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = \frac{-9\,x^6+120\,x^4-320\,x^2}{27}\ ,\quad \mathcal{Q}_6(x) = \frac{54\,x^5-480\,x^3+640\,x}{27}\ ,\quad \mathcal{R}_6(x) = \frac{8\,x^4}{3}\ ,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5-1104\,x^3+1280\,x}{81}\ ,\quad \mathcal{U}_6 = 0\ ,\quad \mathcal{V}_6 = 0\ ,\quad \mathcal{W}_6 = \frac{2560}{81}$$

$$\mathcal{P}_6^*(x) = \frac{9\,x^6+120\,x^4+320\,x^2}{27}\ ,\quad \mathcal{Q}_6^*(x) = -\frac{54\,x^5+480\,x^3+640\,x}{9}\ ,\quad \mathcal{R}_6^*(x) = -\frac{8x^4}{3}\ ,$$

$$\mathcal{S}_6^*(x) = \frac{108\,x^5+1104\,x^3+1280\,x}{81}\ ,\quad \mathcal{U}_6^* = 0\ ,\quad \mathcal{V}_6^* = 0\ ,\quad \mathcal{W}_6^* = \frac{2560}{81}$$

$$\mathcal{P}_7(x) = \frac{-3\,x^7+63\,x^5-483\,x^3-217\,x}{9}\ ,\quad \mathcal{Q}_7(x) = \frac{7\,x^6-105\,x^4+483\,x^2}{3}\ ,$$

$$\mathcal{R}_7(x) = \frac{14\,x^5-84\,x^3-350\,x}{3}\ ,\quad \mathcal{S}_7(x) = \frac{14\,x^6-252\,x^4+1050\,x^2}{9}\ ,$$

$$\mathcal{U}_7 = \frac{217}{9}\ ,\quad \mathcal{V}_7 = -\frac{350}{3}\ ,\quad \mathcal{W}_7 = 0$$

$$\mathcal{P}_7^*(x) = \frac{3\,x^7+63\,x^5+483\,x^3-217\,x}{9}\ ,\quad \mathcal{Q}_7^*(x) = -\frac{7\,x^6+105\,x^4+483\,x^2}{3}\ ,$$

$$\mathcal{R}_7^*(x) = \frac{-14\,x^5-84\,x^3+350\,x}{3}\ ,\quad \mathcal{S}_7^*(x) = \frac{14\,x^6+252\,x^4+1050\,x^2}{9}\ ,$$

$$\mathcal{U}_7^* = \frac{217}{9}\ ,\quad \mathcal{V}_7^* = \frac{350}{3}\ ,\quad \mathcal{W}_7^* = 0$$

$$\mathcal{P}_8(x) = \frac{-27\,x^8+816\,x^6-11904\,x^4+31744\,x^2}{81}\ ,\quad \mathcal{Q}_8(x) = \frac{216\,x^7-4896\,x^5+47616\,x^3-63488\,x}{81}\ ,$$

$$\mathcal{R}_8(x) = \frac{64\,x^6-896\,x^4}{9}\ ,\quad \mathcal{S}_8(x) = \frac{432\,x^7-12096\,x^5+111360\,x^3-126976\,x}{243}\ ,$$

$$\mathcal{U}_8 = 0\ ,\quad \mathcal{V}_8 = 0\ ,\quad \mathcal{W}_8 = -\frac{253952}{243}$$

$$\mathcal{P}_8^*(x) = \frac{27\,x^8+816\,x^6+11904\,x^4+31744\,x^2}{81}\ ,\quad \mathcal{Q}_8^*(x) = -\frac{216\,x^7+4896\,x^5+47616\,x^3+63488\,x}{81}\ ,$$

$$\mathcal{R}_8^*(x) = -\frac{64\,x^6+896\,x^4}{9}\ ,\quad \mathcal{S}_8^*(x) = \frac{432\,x^7+12096\,x^5+111360\,x^3+126976\,x}{243}\ ,$$

$$\mathcal{U}_8^* = 0\ ,\quad \mathcal{V}_8^* = 0\ ,\quad \mathcal{W}_8^* = \frac{253952}{243}$$

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x} = -J_0^2(x)\,J_1(x) + \int J_0^3(x)\,dx - 2\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x} = -I_0^2(x)\,I_1(x) + \int I_0^3(x)\,dx + 2\int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_0{}^2(x)J_1(x)\,dx}{x^2} = \frac{1}{3}\,J_0^3(x) - \frac{1}{2x}\,J_0^2(x)J_1(x) + \frac{1}{2}\,J_0(x)J_1^2(x) + \frac{1}{2}\,\int J_1{}^3(x)\,dx + \frac{1}{2}\,\int \frac{J_0{}^3(x)\,dx}{x}$$

$$\int \frac{I_0{}^2(x)I_1(x)\,dx}{x^2} = \frac{1}{3}\,I_0{}^3(x) - \frac{1}{2x}\,I_0^2(x)I_1(x) - \frac{1}{2}\,I_0(x)I_1^2(x) + \frac{1}{2}\,\int I_1{}^3(x)\,dx + \frac{1}{2}\,\int \frac{I_0{}^3(x)\,dx}{x}$$

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x^3} = -\frac{1}{3x}\,J_0^3(x) + \frac{13x^2-3}{9x^2}\,J_0^2(x)\,J_1(x) + \frac{2}{9x}\,J_0(x)\,J_1^2(x) - \frac{2}{27}\,J_1^3(x) -$$

$$-\frac{13}{9}\,\int J_0^3(x)\,dx + \frac{28}{9}\,\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x^3} = -\frac{1}{3x}\,I_0^3(x) - \frac{13x^2+3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1{}^3(x) +$$

$$+\frac{13}{9}\,\int I_0^3(x)\,dx + \frac{28}{9}\,\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-4}(x) = \frac{-23\,x^2-12}{96\,x^2}\,, \quad \mathcal{Q}_{-4}(x) = \frac{5\,x^2-4}{16\,x^3}\,, \quad \mathcal{R}_{-4}(x) = \frac{-23\,x^2+8}{64\,x^2}\,, \quad \mathcal{S}_{-4}(x) = -\frac{1}{32x}\,,$$

$$\mathcal{U}_{-4} = 0\,, \quad \mathcal{V}_{-4} = 0\,, \quad \mathcal{W}_{-4} = -\frac{23}{64}\,, \quad \mathcal{X}_{-4} = -\frac{5}{16}$$

$$\mathcal{P}^*_{-4}(x) = \frac{23\,x^2-12}{96\,x^2}\,, \quad \mathcal{Q}^*_{-4}(x) = \frac{-5\,x^2-4}{16\,x^3}\,, \quad \mathcal{R}^*_{-4}(x) = \frac{-23\,x^2-8}{64\,x^2}\,, \quad \mathcal{S}^*_{-4}(x) = -\frac{1}{32x}\,,$$

$$\mathcal{U}^*_{-4} = 0\,, \quad \mathcal{V}^*_{-4} = 0\,, \quad \mathcal{W}^*_{-4} = \frac{23}{64}\,, \quad \mathcal{X}^*_{-4} = \frac{5}{16}$$

$$\mathcal{P}_{-5}(x) = \frac{9\,x^2-5}{75\,x^3}\,, \quad \mathcal{Q}_{-5}(x) = \frac{-69\,x^4+15\,x^2-25}{125\,x^4}\,, \quad \mathcal{R}_{-5}(x) = \frac{-12\,x^2+10}{125\,x^3}\,,$$

$$\mathcal{S}_{-5}(x) = \frac{4\,x^2-2}{125\,x^2}\,, \quad \mathcal{U}_{-5} = \frac{69}{125}\,, \quad \mathcal{V}_{-5} = -\frac{6}{5}\,, \quad \mathcal{W}_{-5} = 0$$

$$\mathcal{P}^*_{-5}(x) = -\frac{9\,x^2+5}{75\,x^3}\,, \quad \mathcal{Q}^*_{-5}(x) = -\frac{69\,x^4+15\,x^2+25}{125\,x^4}\,, \quad \mathcal{R}^*_{-5}(x) = -\frac{12\,x^2+10}{125\,x^3}\,,$$

$$\mathcal{S}^*_{-5}(x) = -\frac{4\,x^2+2}{125\,x^2}\,, \quad \mathcal{U}^*_{-5} = \frac{69}{125}\,, \quad \mathcal{V}^*_{-5} = \frac{6}{5}\,, \quad \mathcal{W}^*_{-5} = 0$$

$$\mathcal{P}_{-6}(x) = \frac{145\,x^4+68\,x^2-96}{2304\,x^4}\,, \quad \mathcal{Q}_{-6}(x) = \frac{-93\,x^4+68\,x^2-192}{1152\,x^5}\,, \quad \mathcal{R}_{-6}(x) = \frac{435\,x^4-168\,x^2+256}{4608\,x^4}\,,$$

$$\mathcal{S}_{-6}(x) = \frac{63\,x^2-64}{6912\,x^3}\,, \quad \mathcal{U}_{-6} = 0\,, \quad \mathcal{V}_{-6} = 0\,, \quad \mathcal{W}_{-6} = \frac{145}{1536}\,, \quad \mathcal{X}_{-6} = \frac{31}{384}$$

$$\mathcal{P}^*_{-6}(x) = \frac{145\,x^4-68\,x^2-96}{2304\,x^4}\,, \quad \mathcal{Q}^*_{-6}(x) = \frac{-93\,x^4-68\,x^2-192}{1152\,x^5}\,, \quad \mathcal{R}^*_{-6}(x) = \frac{-435\,x^4-168\,x^2-256}{4608\,x^4}\,,$$

$$\mathcal{S}^*_{-6}(x) = \frac{-63\,x^2-64}{6912\,x^3}\,, \quad \mathcal{U}^*_{-6} = 0\,, \quad \mathcal{V}^*_{-6} = 0\,, \quad \mathcal{W}^*_{-6} = \frac{145}{1536}\,, \quad \mathcal{X}^*_{-6} = \frac{31}{384}$$

$$\mathcal{P}_{-7}(x) = \frac{-2883\,x^4+1435\,x^2-3675}{128625\,x^5}\,, \quad \mathcal{Q}_{-7}(x) = \frac{66809\,x^6-14415\,x^4+21525\,x^2-91875}{643125\,x^6}\,,$$

$$\mathcal{R}_{-7}(x) = \frac{11782\,x^4-10860\,x^2+26250}{643125\,x^5}\,, \quad \mathcal{S}_{-7}(x) = \frac{-11782\,x^4+6516\,x^2-11250}{1929375\,x^4}\,,$$

$$\mathcal{U}_{-7} = -\frac{66809}{643125}\,, \quad \mathcal{V}_{-7} = \frac{5816}{25725}\,, \quad \mathcal{W}_{-7} = 0$$

$$\mathcal{P}^*_{-7}(x) = -\frac{2883\,x^4+1435\,x^2+3675}{128625\,x^5}\,, \quad \mathcal{Q}^*_{-7}(x) = -\frac{66809\,x^6+14415\,x^4+21525\,x^2+91875}{643125\,x^6}\,,$$

$$\mathcal{R}^*_{-7}(x) = -\frac{11782\,x^4+10860\,x^2+26250}{643125\,x^5}\,, \quad \mathcal{S}^*_{-7}(x) = -\frac{11782\,x^4+6516\,x^2+11250}{1929375\,x^4}\,,$$

$$\mathcal{U}^*_{-7} = \frac{66809}{643125}\,, \quad \mathcal{V}^*_{-7} = \frac{5816}{25725}\,, \quad \mathcal{W}^*_{-7} = 0$$

$$\mathcal{P}_{-8}(x) = \frac{-1331\,x^6-616\,x^4+768\,x^2-3072}{147456\,x^6}\,, \quad \mathcal{Q}_{-8}(x) = \frac{213\,x^6-154\,x^4+384\,x^2-2304}{18432\,x^7}\,,$$

$$\mathcal{R}_{-8}(x) = \frac{-3993\,x^6+1560\,x^4-2624\,x^2+9216}{294912\,x^6}\,, \quad \mathcal{S}_{-8}(x) = \frac{-585\,x^4+656\,x^2-1728}{442368\,x^5}\,,$$

$$\mathcal{U}_{-8} = 0\,, \quad \mathcal{V}_{-8} = 0\,, \quad \mathcal{W}_{-8} = -\frac{1331}{98304}\,, \quad \mathcal{X}_{-8} = -\frac{71}{6144}$$

$$\mathcal{P}^*_{-8}(x) = \frac{1331\,x^6-616\,x^4-768\,x^2-3072}{147456\,x^6}\,, \quad \mathcal{Q}^*_{-8}(x) = \frac{-213\,x^6-154\,x^4-384\,x^2-2304}{18432\,x^7}\,,$$

$$\mathcal{R}^*_{-8}(x) = \frac{-3993\,x^6-1560\,x^4-2624\,x^2-9216}{294912\,x^6}\,, \quad \mathcal{S}^*_{-8}(x) = \frac{-585\,x^4-656\,x^2-1728}{442368\,x^5}\,,$$

$$\mathcal{U}^*_{-8} = 0\,, \quad \mathcal{V}^*_{-8} = 0\,, \quad \mathcal{W}^*_{-8} = \frac{1331}{98304}\,, \quad \mathcal{X}^*_{-8} = \frac{71}{6144}$$

g) Integrals of the type $\int x^n \, Z_0(x) \, Z_1^2(x) \, dx$:

$$\int x \, J_0(x) \, J_1^2(x) \, dx = \frac{x}{3} \, J_1^3(x) + \frac{2}{3} \int J_1^3(x) \, dx$$

$$\int x \, I_0(x) \, I_1^2(x) \, dx = \frac{x}{3} \, I_1^3(x) + \frac{2}{3} \int I_1^3(x) \, dx$$

$$\int x^2 \, J_0(x) \, J_1^2(x) \, dx = -\frac{2x}{9} \, J_0^3(x) - \frac{x}{3} \, J_0(x) \, J_1^2(x) + \frac{x^2}{3} \, J_0^3(x) + \frac{2}{9} \int J_0^3(x) \, dx - \frac{1}{3} \int J_0(x) \, J_1^2(x) \, dx$$

$$\int x^2 \, I_0(x) \, I_1^2(x) \, dx = -\frac{2x}{9} \, I_0^3(x) + \frac{x}{3} \, I_0(x) \, I_1^2(x) + \frac{x^2}{3} \, I_0^3(x) + \frac{2}{9} \int I_0^3(x) \, dx + \frac{1}{3} \int I_0(x) \, I_1^2(x) \, dx$$

$$\int x^3 \, J_0(x) \, J_1^2(x) \, dx = \frac{x^3}{3} \, J_1^3(x)$$

$$\int x^3 \, I_0(x) \, I_1^2(x) \, dx = \frac{x^3}{3} \, I_1^3(x)$$

$$\int x^4 \, J_0(x) \, J_1^2(x) \, dx = \frac{6\,x^3 + 4\,x}{27} \, J_0^3(x) - \frac{2x^2}{3} \, J_0^2(x) \, J_1(x) + \frac{3\,x^3 + 5\,x}{9} \, J_0(x) \, J_1^2(x) + \frac{3\,x^4 - 5\,x^2}{9} \, J_1^3(x) -$$
$$- \frac{4}{27} \int J_0^3(x) \, dx + \frac{5}{9} \int J_0(x) \, J_1^2(x) \, dx$$

$$\int x^4 \, I_0(x) \, I_1^2(x) \, dx = \frac{6\,x^3 - 4\,x}{27} \, I_0^3(x) - \frac{2x^2}{3} \, I_0^2(x) \, I_1(x) + \frac{-3\,x^3 + 5\,x}{9} \, I_0(x) \, I_1^2(x) + \frac{3\,x^4 + 5\,x^2}{9} \, I_1^3(x) +$$
$$+ \frac{4}{27} \int I_0^3(x) \, dx + \frac{5}{9} \int I_0(x) \, I_1^2(x) \, dx$$

Let

$$\int x^n \, J_0(x) \, J_1^2(x) \, dx = \mathcal{P}_n(x) \, J_0^3(x) + \mathcal{Q}_n(x) \, J_0^2(x) \, J_1(x) + \mathcal{R}_n(x) \, J_0(x) \, J_1^2(x) + \mathcal{S}_n(x) \, J_1^3(x) +$$
$$+ \mathcal{U}_n \int J_0^3(x) \, dx + \mathcal{V}_n \int J_0(x) \, J_1(x)^2 \, dx + \mathcal{W}_n \int J_1(x)^3 \, dx + \mathcal{X}_n \int \frac{J_0^3(x) \, dx}{x}$$

and

$$\int x^n \, I_0(x) \, I_1^2(x) \, dx = \mathcal{P}_n^*(x) \, I_0^3(x) + \mathcal{Q}_n^*(x) \, I_0^2(x) \, I_1(x) + \mathcal{R}_n^*(x) \, I_0(x) \, I_1^2(x) + \mathcal{S}_n^*(x) \, I_1^3(x) +$$
$$+ \mathcal{U}_n^* \int I_0^3(x) \, dx + \mathcal{V}_n^* \int I_0(x) \, I_1(x)^2 \, dx + \mathcal{W}_n^* \int I_1(x)^3 \, dx + \mathcal{X}_n^* \int \frac{I_0^3(x) \, dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{12\,x^4 - 32\,x^2}{27} \,, \quad \mathcal{Q}_5(x) = \frac{-48\,x^3 + 64\,x}{27} \,, \quad \mathcal{R}_5(x) = \frac{6\,x^4}{9} \,, \quad \mathcal{S}_5(x) = \frac{27\,x^5 - 132\,x^3 + 128\,x}{81} \,,$$
$$\mathcal{U}_5 = 0 \,, \quad \mathcal{V}_5 = 0 \,, \quad \mathcal{W}_5 = \frac{256}{81}$$

$$\mathcal{P}_5^*(x) = \frac{12\,x^4 + 32\,x^2}{27} \,, \quad \mathcal{Q}_5^*(x) = -\frac{48\,x^3 + 64\,x}{27} \,, \quad \mathcal{R}_5^*(x) = -\frac{6\,x^4}{9} \,, \quad \mathcal{S}_5^*(x) = \frac{27\,x^5 + 132\,x^3 + 128\,x}{81} \,,$$
$$\mathcal{U}_5^* = 0 \,, \quad \mathcal{V}_5^* = 0 \,, \quad \mathcal{W}_5^* = \frac{256}{81}$$

$$\mathcal{P}_6(x) = \frac{54\,x^5 - 444\,x^3 - 206\,x}{81} \,, \quad \mathcal{Q}_6(x) = \frac{-30\,x^4 + 148\,x^2}{9} \,, \quad \mathcal{R}_6(x) = \frac{27\,x^5 - 87\,x^3 - 325\,x}{27} \,,$$
$$\mathcal{S}_6(x) = \frac{9\,x^6 - 87\,x^4 + 325\,x^2}{27} \,, \quad \mathcal{U}_6 = \frac{206}{81} \,, \quad \mathcal{V}_6 = -\frac{325}{27} \,, \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = \frac{54\,x^5 + 444\,x^3 - 206\,x}{81} \,, \quad \mathcal{Q}_6^*(x) = -\frac{30\,x^4 + 148\,x^2}{9} \,, \quad \mathcal{R}_6^*(x) = \frac{-27\,x^5 - 87\,x^3 + 325\,x}{27} \,,$$

$$\mathcal{S}_6^*(x) = \frac{9\,x^6 + 87\,x^4 + 325\,x^2}{27}\ , \quad \mathcal{U}_6^* = \frac{206}{81}\ , \quad \mathcal{V}_6^* = \frac{325}{27}\ , \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{24\,x^6 - 384\,x^4 + 1024\,x^2}{27}\ , \quad \mathcal{Q}_7(x) = \frac{-144\,x^5 + 1536\,x^3 - 2048\,x}{27}\ , \quad \mathcal{R}_7(x) = \frac{4\,x^6 - 32\,x^4}{3}\ ,$$

$$\mathcal{S}_7(x) = \frac{27\,x^7 - 432\,x^5 + 3648\,x^3 - 4096\,x}{81}\ , \quad \mathcal{U}_7 = 0\ , \quad \mathcal{V}_7 = 0\ , \quad \mathcal{W}_7 = -\frac{8192}{81}$$

$$\mathcal{P}_7^*(x) = \frac{24\,x^6 + 384\,x^4 + 1024\,x^2}{27}\ , \quad \mathcal{Q}_7^*(x) = -\frac{144\,x^5 + 1536\,x^3 + 2048\,x}{27}\ , \quad \mathcal{R}_7^*(x) = -\frac{4\,x^6 + 32\,x^4}{3}\ ,$$

$$\mathcal{S}_7^*(x) = \frac{27\,x^7 + 432\,x^5 + 3648\,x^3 + 4096\,x}{81}\ , \quad \mathcal{U}_7^* = 0\ , \quad \mathcal{V}_7^* = 0\ , \quad \mathcal{W}_7^* = \frac{8192}{81}$$

$$\mathcal{P}_8(x) = \frac{270\,x^7 - 7020\,x^5 + 54570\,x^3 + 24680\,x}{243}\ , \quad \mathcal{Q}_8(x) = \frac{-210\,x^6 + 3900\,x^4 - 18190\,x^2}{27}\ ,$$

$$\mathcal{R}_8(x) = \frac{135\,x^7 - 1935\,x^5 + 9735\,x^3 + 39625\,x}{81}\ , \quad \mathcal{S}_8(x) = \frac{27\,x^8 - 645\,x^6 + 9735\,x^4 - 39625\,x^2}{81}\ ,$$

$$\mathcal{U}_8 = -\frac{24680}{243}\ , \quad \mathcal{V}_8 = \frac{39625}{81}\ , \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = \frac{270\,x^7 + 7020\,x^5 + 54570\,x^3 - 24680\,x}{243}\ , \quad \mathcal{Q}_8^*(x) = -\frac{210\,x^6 + 3900\,x^4 + 18190\,x^2}{27}\ ,$$

$$\mathcal{R}_8^*(x) = -\frac{135\,x^7 + 1935\,x^5 + 9735\,x^3 - 39625\,x}{81}\ , \quad \mathcal{S}_8^*(x) = \frac{27\,x^8 + 645\,x^6 + 9735\,x^4 + 39625\,x^2}{81}\ ,$$

$$\mathcal{U}_8^* = \frac{24680}{243}\ , \quad \mathcal{V}_n^* = \frac{39625}{81}\ , \quad \mathcal{W}_8^* = 0$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x} = -\frac{1}{2}\,J_0(x)\,J_1^2(x) - \frac{1}{3}J_0^3(x) - \frac{1}{2}\int J_1^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x} = -\frac{1}{2}\,I_0(x)\,I_1^2(x) + \frac{1}{3}I_0^3(x) + \frac{1}{2}\int I_1^3(x)\,dx$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,J_0^2(x)\,J_1(x) - \frac{1}{3x}\,J_0(x)\,J_1^2(x) + \frac{1}{9}\,J_1^3(x) - \frac{5}{3}\int J_0(x)\,J_1^2(x)\,dx + \frac{2}{3}\int J_0^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,I_0^2(x)\,I_1(x) - \frac{1}{3x}\,I_0(x)\,I_1^2(x) - \frac{1}{9}\,I_1^3(x) + \frac{5}{3}\int I_0(x)\,I_1^2(x)\,dx + \frac{2}{3}\int I_0^3(x)\,dx$$

$$\int \frac{J_0(x)J_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,J_0^3(x) - \frac{1}{4x}\,J_0^2(x)J_1(x) + \frac{11\,x^2 - 8}{32\,x^2}\,J_0(x)J_1^2(x) + \frac{1}{16}\,J_0(x)J_1^3(x) + \frac{11}{32}\int J_1^3(x)\,dx + \frac{1}{4}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_0(x)I_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,I_0^3(x) - \frac{1}{4x}\,I_0^2(x)I_1(x) - \frac{11\,x^2 + 8}{32x^2}\,I_0(x)I_1^2(x) - \frac{1}{16x}\,I_0(x)I_1^3(x) + \frac{11}{32}\int I_3^3(x)\,dx + \frac{1}{4}\int \frac{I_0^3(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,J_0^3(x) + \frac{148\,x^2 - 30}{225\,x^2}\,J_0^2(x)\,J_1(x) + \frac{29\,x^2 - 45}{225\,x^3}\,J_0(x)\,J_1^2(x) + \frac{-29\,x^2 + 27}{675\,x^2}\,J_1^3(x) -$$

$$-\frac{148}{225}\int J_0^3(x)\,dx + \frac{13}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,I_0^3(x) - \frac{148\,x^2 + 30}{225\,x^2}\,I_0^2(x)\,I_1(x) - \frac{29\,x^2 + 45}{225\,x^3}\,I_0(x)\,I_1^2(x) - \frac{29\,x^2 + 27}{675\,x^2}\,I_1^3(x) +$$

$$+\frac{148}{225}\int I_0^3(x)\,dx + \frac{13}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x) = \frac{-19\,x^2 - 8}{192\,x^2}\ , \quad \mathcal{Q}_{-5}(x) = \frac{3\,x^2 - 2}{24\,x^3}\ , \quad \mathcal{R}_{-5}(x) = \frac{-57\,x^4 + 24\,x^2 - 64}{384\,x^4}\ , \quad \mathcal{S}_{-5}(x) = \frac{-9\,x^2 + 16}{576\,x^3}\ ,$$

$$\mathcal{U}_{-5} = 0\,, \quad \mathcal{V}_{-5} = 0\,, \quad \mathcal{W}_{-5} = -\frac{19}{128}\,, \quad \mathcal{X}_{-5} = -\frac{1}{8}$$

$$\mathcal{P}^*_{-5}(x) = \frac{19\,x^2 - 8}{192\,x^2}\,, \quad \mathcal{Q}^*_{-5}(x) = \frac{-3\,x^2 - 2}{24\,x^3}\,, \quad \mathcal{R}^*_{-5}(x) = \frac{-57\,x^4 - 24\,x^2 - 64}{384\,x^4}\,, \quad \mathcal{S}^*_{-5}(x) = \frac{-9\,x^2 - 16}{576\,x^3}\,,$$

$$\mathcal{U}^*_{-5} = 0\,, \quad \mathcal{V}^*_{-5} = 0\,, \quad \mathcal{W}^*_{-5} = \frac{19}{128}\,, \quad \mathcal{X}^*_{-5} = \frac{1}{8}$$

$$\mathcal{P}_{-6}(x) = \frac{156\,x^2 - 70}{3675\,x^3}\,, \quad \mathcal{Q}_{-6}(x) = \frac{-3638\,x^4 + 780\,x^2 - 1050}{18375\,x^4}\,, \quad \mathcal{R}_{-6}(x) = \frac{-649\,x^4 + 645\,x^2 - 2625}{18375\,x^5}\,,$$

$$\mathcal{S}_{-6}(x) = \frac{649\,x^4 - 387\,x^2 + 1125}{55125\,x^4}\,, \quad \mathcal{U}_{-6} = \frac{3638}{18375}\,, \quad \mathcal{V}_{-6} = -\frac{317}{735}\,, \quad \mathcal{W}_{-6} = 0$$

$$\mathcal{P}^*_{-6}(x) = -\frac{156\,x^2 + 70}{3675\,x^3}\,, \quad \mathcal{Q}^*_{-6}(x) = -\frac{3638\,x^4 + 780\,x^2 + 1050}{18375\,x^4}\,, \quad \mathcal{R}^*_{-6}(x) = -\frac{649\,x^4 + 645\,x^2 + 2625}{18375\,x^5}\,,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{649\,x^4 + 387\,x^2 + 1125}{55125\,x^4}\,, \quad \mathcal{U}^*_{-6} = \frac{3638}{18375}\,, \quad \mathcal{V}^*_{-6} = \frac{317}{735}\,, \quad \mathcal{W}^*_{-6} = 0$$

$$\mathcal{P}_{-7}(x) = \frac{751\,x^4 + 344\,x^2 - 384}{36864\,x^4}\,, \quad \mathcal{Q}_{-7}(x) = \frac{-60\,x^4 + 43\,x^2 - 96}{2304\,x^5}\,,$$

$$\mathcal{R}_{-7}(x) = \frac{2253\,x^6 - 888\,x^4 + 1600\,x^2 - 9216}{73728\,x^6}\,, \quad \mathcal{S}_{-7}(x) = \frac{333\,x^4 - 400\,x^2 + 1728}{110592\,x^5}\,,$$

$$\mathcal{U}_{-7} = 0\,, \quad \mathcal{V}_{-7} = 0\,, \quad \mathcal{W}_{-7} = \frac{751}{24576}\,, \quad \mathcal{X}_{-7} = \frac{5}{192}$$

$$\mathcal{P}^*_{-7}(x) = \frac{751\,x^4 - 344\,x^2 - 384}{36864\,x^4}\,, \quad \mathcal{Q}^*_{-7}(x) = \frac{-60\,x^4 - 43\,x^2 - 96}{2304\,x^5}\,,$$

$$\mathcal{R}^*_{-7}(x) = \frac{-2253\,x^6 - 888\,x^4 - 1600\,x^2 - 9216}{73728\,x^6}\,, \quad \mathcal{S}^*_{-7}(x) = \frac{-333\,x^4 - 400\,x^2 - 1728}{110592\,x^5}\,,$$

$$\mathcal{U}^*_{-7} = 0\,, \quad \mathcal{V}^*_{-7} = 0\,, \quad \mathcal{W}^*_{-7} = \frac{751}{24576}\,, \quad \mathcal{X}^*_{-7} = \frac{5}{192}$$

$$\mathcal{P}_{-8}(x) = \frac{-22758\,x^4 + 11060\,x^2 - 22050}{3472875\,x^5}\,, \quad \mathcal{Q}_{-8}(x) = \frac{528184\,x^6 - 113790\,x^4 + 165900\,x^2 - 551250}{17364375\,x^6}\,,$$

$$\mathcal{R}_{-8}(x) = \frac{93407\,x^6 - 87735\,x^4 + 249375\,x^2 - 1929375}{17364375\,x^7}\,, \quad \mathcal{S}_{-8}(x) = \frac{-93407\,x^6 + 52641\,x^4 - 106875\,x^2 + 643125}{52093125\,x^6}\,,$$

$$\mathcal{U}_{-8} = -\frac{528184}{17364375}\,, \quad \mathcal{V}_{-8} = \frac{45991}{694575}\,, \quad \mathcal{W}_{-8} = 0$$

$$\mathcal{P}^*_{-8}(x) = -\frac{22758\,x^4 + 11060\,x^2 + 22050}{3472875\,x^5}\,, \quad \mathcal{Q}^*_{-8}(x) = -\frac{528184\,x^6 + 113790\,x^4 + 165900\,x^2 + 551250}{17364375\,x^6}\,,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{93407\,x^6 + 87735\,x^4 + 249375\,x^2 + 1929375}{17364375\,x^7}\,, \quad \mathcal{S}^*_{-8}(x) = -\frac{93407\,x^6 + 52641\,x^4 + 106875\,x^2 + 643125}{52093125\,x^6}\,,$$

$$\mathcal{U}^*_{-8} = \frac{528184}{17364375}\,, \quad \mathcal{V}^*_{-8} = \frac{45991}{694575}\,, \quad \mathcal{W}^*_{-8} = 0$$

h) Integrals of the type $\int x^n\, Z_1^3(x)\, dx$:

$$\int x\, J_1^3(x)\, dx = -\frac{2x}{3}\, J_0^3(x) - x\, J_0(x)\, J_1^2(x) + \frac{2}{3} \int J_0^3(x)\, dx - \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x\, I_1^3(x)\, dx = -\frac{2x}{3}\, I_0^3(x) + x\, I_0(x)\, I_1^2(x) + \frac{2}{3} \int I_0^3(x)\, dx + \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^2\, J_1^3(x)\, dx = -\frac{2x^2}{3}\, J_0^3(x) + \frac{4x}{3}\, J_0^2\, J_1(x) - x^2\, J_0(x)\, J_1^2(x) + \frac{8x}{9}\, J_1^3(x) + \frac{16}{9} \int J_1^3(x)\, dx$$

$$\int x^2\, I_1^3(x)\, dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{4x}{3}\, I_0^2\, I_1(x) + x^2\, I_0(x)\, I_1^2(x) - \frac{8x}{9}\, I_1^3(x) - \frac{16}{9} \int I_1^3(x)\, dx$$

$$\int x^3\, J_1^3(x)\, dx = -\frac{6\,x^3+4\,x}{9}\, J_0^3(x) + 2x^2\, J_0^2\, J_1(x) - \frac{3\,x^3+5\,x}{3}\, J_0(x)\, J_1^2(x) + \frac{5x^2}{3}\, J_1^3(x) +$$

$$+\frac{4}{9}\int J_0^3(x)\, dx - \frac{5}{3}\int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^3\, I_1^3(x)\, dx = \frac{-6\,x^3+4\,x}{9}\, I_0^3(x) + 2x^2\, I_0^2\, I_1(x) + \frac{3\,x^3-5\,x}{3}\, I_0(x)\, I_1^2(x) - \frac{5x^2}{3}\, I_1^3(x) -$$

$$-\frac{4}{9}\int I_0^3(x)\, dx - \frac{5}{3}\int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^4\, J_1^3(x)\, dx =$$

$$= \frac{-6\,x^4+16\,x^2}{9}\, J_0^3(x) + \frac{24\,x^3-32\,x}{9}\, J_0^2\, J_1(x) - x^4\, J_0(x)\, J_1^2(x) + \frac{66\,x^3-64\,x}{27}\, J_1^3(x) - \frac{128}{27}\int J_1^3(x)\, dx$$

$$\int x^4\, I_1^3(x)\, dx =$$

$$= -\frac{6\,x^4+16\,x^2}{9}\, I_0^3(x) + \frac{24\,x^3+32\,x}{9}\, I_0^2\, I_1(x) + x^4\, I_0(x)\, I_1^2(x) - \frac{66\,x^3+64\,x}{27}\, I_1^3(x) - \frac{128}{27}\int I_1^3(x)\, dx$$

Let

$$\int x^n\, J_1^3(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n\int J_0^3(x)\, dx + \mathcal{V}_n\int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n\int J_1^3(x)\, dx + \mathcal{X}_n\int \frac{I_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_1^3(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^*\int I_0^3(x)\, dx + \mathcal{V}_n^*\int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^*\int I_1^3(x)\, dx + \mathcal{X}_n^*\int \frac{I_0^3(x)\, dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{-54\,x^5+444\,x^3+206\,x}{81}\,, \quad \mathcal{Q}_5(x) = \frac{30\,x^4-148\,x^2}{9}\,, \quad \mathcal{R}_5(x) = \frac{-27\,x^5+87\,x^3+325\,x}{27}\,,$$

$$\mathcal{S}_5(x) = \frac{87\,x^4-325\,x^2}{27}\,, \quad \mathcal{U}_5 = -\frac{206}{81}\,, \quad \mathcal{V}_5 = \frac{325}{27}\,, \quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = \frac{-54\,x^5-444\,x^3+206\,x}{81}\,, \quad \mathcal{Q}_5^*(x) = \frac{30\,x^4+148\,x^2}{9}\,, \quad \mathcal{R}_5^*(x) = \frac{27\,x^5+87\,x^3-325\,x}{27}\,,$$

$$\mathcal{S}_5^*(x) = -\frac{87\,x^4+325\,x^2}{27}\,, \quad \mathcal{U}_5^* = -\frac{206}{81}\,, \quad \mathcal{V}_5^* = -\frac{325}{27}\,, \quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = \frac{-6\,x^6+96\,x^4-256\,x^2}{9}\,, \quad \mathcal{Q}_6(x) = \frac{36\,x^5-384\,x^3+512\,x}{9}\,, \quad \mathcal{R}_6(x) = -x^6+8\,x^4\,,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5-912\,x^3+1024\,x}{27}\,, \quad \mathcal{U}_6 = 0\,, \quad \mathcal{V}_6 = 0\,, \quad \mathcal{W}_6 = \frac{2048}{27}$$

$$\mathcal{P}_6^*(x) = -\frac{6\,x^6+96\,x^4+256\,x^2}{9}\,, \quad \mathcal{Q}_6^*(x) = \frac{36\,x^5+384\,x^3+512\,x}{9}\,, \quad \mathcal{R}_6^*(x) = x^6+8\,x^4\,,$$

$$\mathcal{S}_6^*(x) = -\frac{108\,x^5+912\,x^3+1024\,x}{27}\,, \quad \mathcal{U}_6^* = 0\,, \quad \mathcal{V}_6^* = 0\,, \quad \mathcal{W}_6^* = -\frac{2048}{27}$$

$$\mathcal{P}_7(x) = \frac{-54\,x^7+1404\,x^5-10914\,x^3-4936\,x}{81}\,, \quad \mathcal{Q}_7(x) = \frac{42\,x^6-780\,x^4+3638\,x^2}{9}\,,$$

$$\mathcal{R}_7(x) = \frac{-27\,x^7+387\,x^5-1947\,x^3-7925\,x}{27}\,, \quad \mathcal{S}_7(x) = \frac{129\,x^6-1947\,x^4+7925\,x^2}{27}\,,$$

$$\mathcal{U}_7 = \frac{4936}{81}\,, \quad \mathcal{V}_7 = -\frac{7925}{27}\,, \quad \mathcal{W}_7 = 0$$

$$\mathcal{P}^*_7(x) = -\frac{54\,x^7 + 1404\,x^5 + 10914\,x^3 - 4936\,x}{81}\,, \quad \mathcal{Q}^*_7(x) = \frac{42\,x^6 + 780\,x^4 + 3638\,x^2}{9}\,,$$

$$\mathcal{R}^*_7(x) = \frac{27\,x^7 + 387\,x^5 + 1947\,x^3 - 7925\,x}{27}\,, \quad \mathcal{S}^*_7(x) = -\frac{129\,x^6 + 1947\,x^4 + 7925\,x^2}{27}\,,$$

$$\mathcal{U}^*_7 = \frac{4936}{81}\,, \quad \mathcal{V}^*_7 = \frac{7925}{27}\,, \quad \mathcal{W}^*_7 = 0$$

$$\mathcal{P}_8(x) = \frac{-54\,x^8 + 2064\,x^6 - 30720\,x^4 + 81920\,x^2}{81}\,, \quad \mathcal{Q}_8(x) = \frac{432\,x^7 - 12384\,x^5 + 122880\,x^3 - 163840\,x}{81}\,,$$

$$\mathcal{R}_8(x) = \frac{-9\,x^8 + 200\,x^6 - 2368\,x^4}{9}\,, \quad \mathcal{S}_8(x) = \frac{1350\,x^7 - 31968\,x^5 + 288384\,x^3 - 327680\,x}{243}\,,$$

$$\mathcal{U}_8 = 0\,, \quad \mathcal{V}_8 = 0\,, \quad \mathcal{W}_8 = -\frac{655360}{243}$$

$$\mathcal{P}^*_8(x) = -\frac{54\,x^8 + 2064\,x^6 + 30720\,x^4 + 81920\,x^2}{81}\,, \quad \mathcal{Q}^*_8(x) = \frac{432\,x^7 + 12384\,x^5 + 122880\,x^3 + 163840\,x}{81}\,,$$

$$\mathcal{R}^*_8(x) = \frac{9\,x^8 + 200\,x^6 + 2368\,x^4}{9}\,, \quad \mathcal{S}^*_8(x) = -\frac{1350\,x^7 + 31968\,x^5 + 288384\,x^3 + 327680\,x}{243}\,,$$

$$\mathcal{U}^*_8 = 0\,, \quad \mathcal{V}^*_8 = 0\,, \quad \mathcal{W}^*_8 = -\frac{655360}{243}$$

$$\int \frac{J_1^3(x)\,dx}{x} = -\frac{1}{3}\,J_1^3(x) + \int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x} = -\frac{1}{3}\,I_1^3(x) + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^2} = -\frac{1}{8}\,J_0^3(x) - \frac{3}{8}\,J_0(x)\,J_1^2(x) - \frac{1}{4x}\,J_1^3(x) + \frac{3}{4}\int J_0(x)\,J_1^2(x)\,dx - \frac{3}{8}\int J_1^3(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x^2} = \frac{1}{4}\,I_0^3(x) - \frac{3}{8}\,I_0(x)\,I_1^2(x) - \frac{1}{4x}\,I_1^3(x) + \frac{3}{4}\int I_0(x)\,I_1^2(x)\,dx + \frac{3}{8}\int I_1^3(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,J_0^2(x)\,J_1(x) - \frac{1}{5x}\,J_0(x)\,J_1^2(x) + \frac{x^2-3}{15x^2}\,J_1^3(x) + \frac{2}{5}\int J_0(x)^3\,dx - \int J_0(x)\,J_1^2(x)\,dx$$

E :

$$\int \frac{I_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,I_0^2(x)\,I_1(x) - \frac{1}{5x}\,I_0(x)\,I_1^2(x) - \frac{x^2+3}{15x^2}\,I_1^3(x) + \frac{2}{5}\int I_0(x)^3\,dx + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,J_0^3(x) - \frac{1}{8x}\,J_0^2(x)J_1(x) + \frac{11\,x^2-8}{64\,x^2}\,J_0(x)J_1^2(x) + \frac{3\,x^2-16}{96\,x^3}\,J_0(x)J_1^3(x) + \frac{11}{64}\int J_1^3(x)\,dx + \frac{1}{8}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,I_0^3(x) - \frac{1}{8x}\,I_0^2(x)I_1(x) - \frac{11\,x^2+8}{64\,x^2}\,I_0(x)I_1^2(x) - \frac{3\,x^2+16}{96\,x^3}\,I_0(x)I_1^3(x) + \frac{11}{64}\int I_1^3(x)\,dx + \frac{1}{8}\int \frac{I_0^3(x)\,dx}{x}$$

$$\mathcal{P}_{-5}(x) = -\frac{2}{35x}\,, \quad \mathcal{Q}_{-5}(x) = \frac{148\,x^2 - 30}{525\,x^2}\,, \quad \mathcal{R}_{-5}(x) = \frac{29\,x^2 - 45}{525\,x^3}\,, \quad \mathcal{S}_{-5}(x) = \frac{-29\,x^4 + 27\,x^2 - 225}{1575\,x^4}\,,$$

$$\mathcal{U}_{-5} = -\frac{148}{525}\,, \quad \mathcal{V}_{-5} = \frac{13}{21}\,, \quad \mathcal{W}_{-5} = 0$$

$$\mathcal{P}^*_{-5}(x) = -\frac{2}{35x}\,, \quad \mathcal{Q}^*_{-5}(x) = -\frac{148\,x^2 + 30}{525x^2}\,, \quad \mathcal{R}^*_{-5}(x) = -\frac{29\,x^2 + 45}{525x^3}\,, \quad \mathcal{S}^*_{-5}(x) = -\frac{29\,x^4 + 27\,x^2 + 225}{1575\,x^4}\,,$$

$$\mathcal{U}^*_{-5} = \frac{148}{525}\,, \quad \mathcal{V}^*_{-5} = \frac{13}{21}\,, \quad \mathcal{W}^*_{-5} = 0$$

$$\mathcal{P}_{-6}(x) = \frac{-19\,x^2-8}{512\,x^2}\ , \quad \mathcal{Q}_{-6}(x) = \frac{3\,x^2-2}{64\,x^3}\ , \quad \mathcal{R}_{-6}(x) = \frac{-57\,x^4+24\,x^2-64}{1024\,x^4}\ ,$$

$$\mathcal{S}_{-6}(x) = \frac{-9\,x^4+16\,x^2-192}{1536\,x^5}\ , \quad \mathcal{U}_{-6} = 0\ , \quad \mathcal{V}_{-6} = 0\ , \quad \mathcal{W}_{-6} = -\frac{57}{1024}\ , \quad \mathcal{X}_{-6} = -\frac{3}{64}$$

$$\mathcal{P}^*_{-6}(x) = \frac{19\,x^2-8}{512\,x^2}\ , \quad \mathcal{Q}^*_{-6}(x) = \frac{-3\,x^2-2}{64\,x^3}\ , \quad \mathcal{R}^*_{-6}(x) = \frac{-57\,x^4-24\,x^2-64}{1024\,x^4}\ ,$$

$$\mathcal{S}^*_{-6}(x) = \frac{-9\,x^4-16\,x^2-192}{1536\,x^5}\ , \quad \mathcal{U}^*_{-6} = 0\ , \quad \mathcal{V}^*_{-6} = 0\ , \quad \mathcal{W}^*_{-6} = \frac{57}{1024}\ , \quad \mathcal{X}^*_{-6} = \frac{3}{64}$$

$$\mathcal{P}_{-7}(x) = \frac{156\,x^2-70}{11025\,x^3}\ , \quad \mathcal{Q}_{-7}(x) = \frac{-3638\,x^4+780\,x^2-1050}{55125\,x^4}\ , \quad \mathcal{R}_{-7}(x) = \frac{-649\,x^4+645\,x^2-2625}{55125\,x^5}\ ,$$

$$\mathcal{S}_{-7}(x) = \frac{649x^6-387x^4+1125x^2-18375}{165375\,x^6}\ , \quad \mathcal{U}_{-7} = \frac{3638}{55125}\ , \quad \mathcal{V}_{-7} = -\frac{317}{2205}\ , \quad \mathcal{W}_{-7} = 0$$

$$\mathcal{P}^*_{-7}(x) = -\frac{156\,x^2+70}{11025\,x^3}\ , \quad \mathcal{Q}^*_{-7}(x) = -\frac{3638\,x^4+780\,x^2+1050}{55125\,x^4}\ , \quad \mathcal{R}^*_{-7}(x) = -\frac{649\,x^4+645\,x^2+2625}{55125\,x^5}\ ,$$

$$\mathcal{S}^*_{-7}(x) = -\frac{649\,x^6+387\,x^4+1125\,x^2+18375}{165375\,x^6}\ , \quad \mathcal{U}^*_{-7} = \frac{3638}{55125}\ , \quad \mathcal{V}^*_{-7} = \frac{317}{2205}\ , \quad \mathcal{W}^*_{-7} = 0$$

$$\mathcal{P}_{-8}(x) = \frac{751\,x^4+344\,x^2-384}{122880\,x^4}\ , \quad \mathcal{Q}_{-8}(x) = \frac{-60\,x^4+43\,x^2-96}{7680\,x^5}\ ,$$

$$\mathcal{R}_{-8}(x) = \frac{2253\,x^6-888\,x^4+1600\,x^2-9216}{245760\,x^6}\ , \quad \mathcal{S}_{-8}(x) = \frac{333\,x^6-400\,x^4+1728\,x^2-36864}{368640\,x^7}\ ,$$

$$\mathcal{U}_{-8} = 0\ , \quad \mathcal{V}_{-8} = 0\ , \quad \mathcal{W}_{-8} = \frac{751}{81920}\ , \quad \mathcal{X}_{-8} = \frac{1}{128}$$

$$\mathcal{P}^*_{-8}(x) = \frac{751\,x^4-344\,x^2-384}{122880\,x^4}\ , \quad \mathcal{Q}^*_{-8}(x) = \frac{-60\,x^4-43\,x^2-96}{7680\,x^5}\ ,$$

$$\mathcal{R}^*_{-8}(x) = \frac{-2253\,x^6-888\,x^4-1600\,x^2-9216}{245760\,x^6}\ , \quad \mathcal{S}^*_{-8}(x) = \frac{-333\,x^6-400\,x^4-1728\,x^2-36864}{368640\,x^7}\ ,$$

$$\mathcal{U}^*_{-8} = 0\ , \quad \mathcal{V}^*_{-8} = 0\ , \quad \mathcal{W}^*_{-8} = \frac{751}{81920}\ , \quad \mathcal{X}^*_{-8} = \frac{1}{128}$$

i) Recurrence Relations:

Let
$$\mathcal{J}_n^{(kl)} = \int x^n\, J_0^k(x)\, J_1^l(x)\, dx \qquad \text{and} \qquad \mathcal{I}_n^{(kl)} = \int x^n\, I_0^k(x)\, I_1^l(x)\, dx$$
with $k+l=3,\ k,l \ge 0$. Then the following formulas hold:

Ascending recurrence:

$$\mathcal{J}_{n+1}^{(30)} = x^{n+1}\left[J_0^2(x)\, J_1(x) + \frac{2}{3}\, J_1^3(x)\right] - n\,\mathcal{J}_n^{(21)} - \frac{2}{3}(n-2)\,\mathcal{J}_n^{(03)}$$
$$\mathcal{J}_{n+1}^{(21)} = \frac{n+1}{3}\,\mathcal{J}_n^{(30)} - \frac{x^{n+1}}{3}\, J_0^3(x)$$
$$\mathcal{J}_{n+1}^{(12)} = \frac{x^{n+1}}{3}\, J_1^3(x) - \frac{n-2}{3}\,\mathcal{J}_n^{(03)}$$
$$\mathcal{J}_{n+1}^{(03)} = -x^{n+1}\left[J_0(x)\, J_1^2(x) + \frac{2}{3}\, J_0^3(x)\right] + (n-1)\mathcal{J}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{J}_n^{(30)}$$

$$\mathcal{I}_{n+1}^{(30)} = x^{n+1}\left[I_0^2(x)\, I_1(x) - \frac{2}{3}\, I_1^3(x)\right] - n\,\mathcal{I}_n^{(21)} + \frac{2}{3}(n-2)\,\mathcal{I}_n^{(03)}$$
$$\mathcal{I}_{n+1}^{(21)} = -\frac{n+1}{3}\,\mathcal{I}_n^{(30)} + \frac{x^{n+1}}{3}\, I_0^3(x)$$
$$\mathcal{I}_{n+1}^{(12)} = \frac{x^{n+1}}{3}\, I_1^3(x) - \frac{n-2}{3}\,\mathcal{I}_n^{(03)}$$
$$\mathcal{I}_{n+1}^{(03)} = x^{n+1}\left[I_0(x)\, I_1^2(x) - \frac{2}{3}\, I_0^3(x)\right] - (n-1)\mathcal{I}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{I}_n^{(30)}$$

Descending recurrence:

$$\mathcal{J}_n^{(30)} = \frac{x^{n+1} J_0^3(x) + 3\mathcal{J}_{n+1}^{(21)}}{n+1}\ , \qquad \mathcal{J}_n^{(21)} = \frac{x^{n+1} J_0^2(x)\, J_1(x) + 2\mathcal{J}_{n+1}^{(12)} - \mathcal{J}_{n+1}^{(30)}}{n}$$
$$\mathcal{J}_n^{(12)} = \frac{2\mathcal{J}_{n+1}^{(21)} - x^{n+1} J_0(x)\, J_1^2(x) - \mathcal{J}_{n+1}^{(03)}}{n-1}\ , \qquad \mathcal{J}_n^{(03)} = \frac{x^{n+1} J_1^3(x) - 3\mathcal{J}_{n+1}^{(12)}}{n-2}$$

Holds
$$J_0^3(x) + 3\mathcal{J}_0^{(21)} = xJ_0^2(x)\, J_1(x) + 2\mathcal{J}_1^{(12)} - \mathcal{J}_1^{(30)} = 2\mathcal{J}_2^{(21)} - x^2 J_0(x)\, J_1^2(x) - \mathcal{J}_2^{(03)} =$$
$$= x^3 J_1^3(x) - 3\mathcal{J}_3^{(12)} = 0\ .$$

$$\mathcal{I}_n^{(30)} = \frac{x^{n+1} I_0^3(x) - 3\mathcal{I}_{n+1}^{(21)}}{n+1}\ , \qquad \mathcal{I}_n^{(21)} = \frac{x^{n+1} I_0^2(x)\, I_1(x) - 2\mathcal{I}_{n+1}^{(12)} - \mathcal{I}_{n+1}^{(30)}}{n}$$
$$\mathcal{I}_n^{(12)} = \frac{2\mathcal{I}_{n+1}^{(21)} + \mathcal{I}_{n+1}^{(03)} - x^{n+1} I_0(x)\, I_1^2(x)}{n-1}\ , \qquad \mathcal{I}_n^{(03)} = \frac{x^{n+1} I_1^3(x) - 3\mathcal{I}_{n+1}^{(12)}}{n-2}$$

Holds
$$I_0^3(x) - 3\mathcal{I}_0^{(21)} = xI_0^2(x)\, I_1(x) - 2\mathcal{I}_1^{(12)} - \mathcal{I}_1^{(30)} = x^2 I_0(x)\, I_1^2(x) - 2\mathcal{I}_2^{(21)} - \mathcal{I}_2^{(03)} =$$
$$= x^3 I_1^3(x) - 3\mathcal{I}_3^{(12)} = 0\ .$$

j) $x^3 Z_1^2(x)\, Z_0^*(x)$

$$\int x^3\, J_1^2(x) Y_0(x)\, dx =$$

$$= \frac{1}{3}\left[2\, x^3 J_0^2(x) Y_1(x) + 4\, x^2 J_1^2(x) Y_0(x) + x^3 J_1^2(x) Y_1(x) - 2x^3\, J_0(x) J_1(x) Y_0(x) - 4x^2\, J_0(x) J_1(x) Y_1(x)\right]$$

$$\int x^3\, I_1^2(x) K_0(x)\, dx =$$

$$= \frac{1}{3}\left[-4x^2\, I_0(x) I_1(x) K_1(x) + 2\, I_0(x) I_1(x) K_0(x) x^3 - x^3 I_1^2(x) K_1(x) + 2\, x^3 I_0^2(x) K_1(x) - 4\, x^2 I_1^2(x) K_0(x)\right]$$

$$\int x^3\, K_1^2(x) I_0(x) =$$

$$= \frac{1}{3}\left[-2x^3\, I_0(x) K_0(x) K_1(x) - 4x^2\, I_1(x) K_0(x) K_1(x) + x^3 I_1(x) K_1^2(x) - 4\, x^2 I_0(x) K_1^2(x) - 2\, x^3 I_1(x) K_0^2(x)\right]$$

3.2. Integrals of the type $\int x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu(\gamma x)\, dx$

The general case is discussed in [12]. In the following some special solutions are given.

a) $x^n\, Z_\kappa(x)\, Z_\mu(x)\, Z_\nu(2x)$

With $\kappa, \mu, \nu \in \{0,1\}$ the following integrals may be expressed by functions of the same kind:

$$\int x^{2n+1}\, Z_0^2(x)\, Z_0(2x)\, dx\;, \quad \int x^{2n+1}\, Z_0(x)\, Z_1(x)\, Z_1(2x)\, dx\;, \quad \int x^{2n+1}\, Z_1^2(x)\, Z_0(2x)\, dx\;, \qquad n \geq 0,$$

and

$$\int x^{2n}\, Z_0(x)\, Z_1(x)\, Z_0(2x)\, dx\;, \quad \int x^{2n}\, Z_0^2(x)\, Z_1(2x)\, dx\;, \quad \int x^{2n}\, Z_1^2(x)\, Z_1(x)\, Z_1(2x)\, dx\;, \qquad n \geq 1.$$

$$\int J_1^2(x) J_1(2x)\, dx = \frac{x}{2}\, \left[J_0^2(x) J_1(2x) - J_1^2(x) J_1(2x) - 2 J_0(x) J_1(x) J_0(2x) \right]$$

$$\int I_1^2(x) I_1(2x)\, dx = -\frac{x}{2}\, \left[I_0^2(x) I_1(2x) + I_1^2(x) I_1(2x) - 2 I_0(x) I_1(x) I_0(2x) \right]$$

$$\int K_1^2(x) K_1(2x)\, dx = -\frac{x}{2}\, \left[K_0^2(x) K_1(2x) + K_1^2(x) K_1(2x) - 2 K_0(x) K_1(x) K_0(2x) \right]$$

$$\int x J_0^2(x) J_0(2x)\, dx = \frac{x^2}{2}\, \left[J_0^2(x) J_0(2x) - J_1^2(x) J_0(2x) + 2 J_0(x) J_1(x) J_1(2x) \right]$$

$$\int x I_0^2(x) I_0(2x)\, dx = \frac{x^2}{2}\, \left[I_0^2(x) I_0(2x) + I_1^2(x) I_0(2x) - 2 I_0(x) I_1(x) I_1(2x) \right]$$

$$\int x K_0^2(x) K_0(2x)\, dx = \frac{x^2}{2}\, \left[K_0^2(x) K_0(2x) + K_1^2(x) K_0(2x) - 2 K_0(x) K_1(x) K_1(2x) \right]$$

$$\int x J_0(x) J_1(x) J_1(2x)\, dx = \frac{x}{2}\, \left[x\, J_0^2(x) J_0(2x) - J_0^2(x) J_1(2x) - x\, J_1^2(x) J_0(2x) + 2\, x J_0(x) J_1(x) J_1(2x) \right]$$

$$\int x I_0(x) I_1(x) I_1(2x)\, dx = \frac{x}{2}\, \left[-x\, I_0^2(x) I_0(2x) + I_0^2(x) I_1(2x) - x\, I_1^2(x) I_0(2x) + 2\, x I_0(x) I_1(x) I_1(2x) \right]$$

$$\int x K_0(x) K_1(x) K_1(2x)\, dx =$$
$$= \frac{x}{2}\, \left[-x\, K_0^2(x) K_0(2x) - K_0^2(x) K_1(2x) - x\, K_1^2(x) K_0(2x) + 2\, x K_0(x) K_1(x) K_1(2x) \right]$$

$$\int x\, J_1^2(x) J_0(2x)\, dx =$$
$$= \frac{x}{2}\, \left[-x\, J_0^2(x) J_0(2x) + 2 J_0^2(x) J_1(2x) + x\, J_1^2(x) J_0(2x) - 2 J_0(x) J_1(x) J_0(2x) - 2\, x J_0(x) J_1(x) J_1(2x) \right]$$

$$\int x\, I_1^2(x) I_0(2x)\, dx =$$
$$= \frac{x}{2}\, \left[x\, I_0^2(x) I_0(2x) - 2 I_0^2(x) I_1(2x) + x\, I_1^2(x) I_0(2x) + 2 I_0(x) I_1(x) I_0(2x) - 2\, x I_0(x) I_1(x) I_1(2x) \right]$$

$$\int x\, K_1^2(x) K_0(2x)\, dx =$$
$$= \frac{x}{2}\, \left[x\, K_0^2(x) K_0(2x) + 2 K_0^2(x) K_1(2x) + x\, K_1^2(x) K_0(2x) - 2 K_0(x) K_1(x) K_0(2x) - 2\, x K_0(x) K_1(x) K_1(2x) \right]$$

$$\int x^2 J_0(x)\, J_1(x)\, J_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, J_0^2(x) J_1(2x) - J_1^2(x) J_0(2x) + x\, J_1^2(x) J_1(2x) + 2\, x J_0(x) J_1(x) J_0(2x) + 2 J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x^2 I_0(x)\, I_1(x)\, I_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, I_0^2(x) I_1(2x) - I_1^2(x) I_0(2x) - x\, I_1^2(x) I_1(2x) + 2\, x I_0(x) I_1(x) I_0(2x) + 2 I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x^2 K_0(x)\, K_1(x)\, K_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, K_0^2(x) K_1(2x) + K_1^2(x) K_0(2x) - x\, K_1^2(x) K_1(2x) + 2\, x K_0(x) K_1(x) K_0(2x) - 2 K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2\, J_0^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, J_0^2(x) J_1(2x) - 2\, J_1^2(x) J_0(2x) - x\, J_1^2(x) J_1(2x) - 2\, x J_0(x) J_1(x) J_0(2x) + 4 J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x^2\, I_0^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, I_0^2(x) I_1(2x) - 2\, I_1^2(x) I_0(2x) + x\, I_1^2(x) I_1(2x) - 2\, x I_0(x) I_1(x) I_0(2x) + 4 I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x^2\, K_0^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, K_0^2(x) K_1(2x) + 2\, K_1^2(x) K_0(2x) + x\, K_1^2(x) K_1(2x) - 2\, x K_0(x) K_1(x) K_0(2x) - 4 K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2\, J_1^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x J_0^2(x) J_1(2x) - 4\, J_1^2(x)\, J_0(2x) + x\, J_1^2(x)\, J_1(2x) + 2\, x J_0(x)\, J_1(x) J_0(2x) + 2 J_0(x)\, J_1(x) J_1(2x)\right]$$

$$\int x^2\, I_1^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x I_0^2(x) I_1(2x) + 4\, I_1^2(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x) - 2\, x I_0(x)\, I_1(x) I_0(2x) - 2 I_0(x)\, I_1(x) I_1(2x)\right]$$

$$\int x^2\, K_1^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,[x K_0^2(x) K_1(2x) - 4\, K_1^2(x)\, K_0(2x) + x\, K_1^2(x)\, K_1(2x) - 2\, x K_0(x)\, K_1(x) K_0(2x) + 2 K_0(x)\, K_1(x) K_1(2x)]$$

$$\int x^3 J_0^2(x)\, J_0(2x)\, dx = \frac{x^2}{10}[x^2\, J_0^2(x)\, J_0(2x) + 2x\, J_0(x)\, J_1(x)\, J_0(2x) + (2 - x^2)\, J_1^2(x)\, J_0(2x) + 2x\, J_0^2(x)\, J_1(2x) +$$

$$+ (2x^2 - 4)\, J_0(x)\, J_1(x)\, J_1(2x) + 2x\, J_1^2(x)\, J_1(2x)]$$

$$\int x^3 I_0^2(x)\, I_0(2x)\, dx = \frac{x^2}{10}[x^2\, I_0^2(x)\, I_0(2x) + 2x\, I_0(x)\, I_1(x)\, I_0(2x) + (x^2 + 2)\, I_1^2(x)\, I_0(2x) + 2x\, I_0^2(x)\, I_1(2x) -$$

$$- (2x^2 + 4)\, I_0(x)\, I_1(x)\, I_1(2x) - 2x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^3 K_0^2(x)\, K_0(2x)\, dx = \frac{x^2}{10}[x^2\, K_0^2(x)\, K_0(2x) - 2x\, K_0(x)\, K_1(x)\, K_0(2x) + (x^2 + 2)\, K_1^2(x)\, K_0(2x) -$$

$$- 2x\, K_0^2(x)\, K_1(2x) - (2x^2 + 4)\, K_0(x)\, K_1(x)\, K_1(2x) + 2x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^3\, J_1^2(x)\, J_0(2x)\, dx = \frac{x^2}{10}[-x^2\, J_0^2(x)\, J_0(2x) + 2x\, J_0(x)\, J_1(x)\, J_0(2x) + (2-x^2)\, J_1^2(x)\, J_0(2x) +$$
$$+2x\, J_0^2(x)\, J_1(2x) + (2x^2-4)\, J_0(x)\, J_1(x)\, J_1(2x) + 2x\, J_1^2(x)\, J_1(2x)]$$
$$\int x^3\, I_1^2(x)\, I_0(2x)\, dx = \frac{x^2}{10}[x^2\, I_0^2(x)\, I_0(2x) + 2x\, I_0(x)\, I_1(x)\, I_0(2x) + (x^2+2)\, I_1^2(x)\, I_0(2x) +$$
$$+2x\, I_0^2(x)\, I_1(2x) - (2x^2+4)\, I_0(x)\, I_1(x)\, I_1(2x) - 2x\, I_1^2(x)\, I_1(2x)]$$
$$\int x^3\, K_1^2(x)\, K_0(2x)\, dx = \frac{x^2}{10}[x^2\, K_0^2(x)\, K_0(2x) - 2x\, K_0(x)\, K_1(x)\, K_0(2x) + (x^2+2)\, K_1^2(x)\, K_0(2x) -$$
$$-2x\, K_0^2(x)\, K_1(2x) - (2x^2+4)\, K_0(x)\, K_1(x)\, K_1(2x) + 2x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^3 J_0(x)\, J_1(x)\, J_1(2x)\, dx = \frac{x^2}{30}[3x^2\, J_0^2(x)\, J_0(2x) - 4x\, J_0(x)\, J_1(x)\, J_0(2x) - (3x^2+4)\, J_1^2(x)\, J_0(2x) -$$
$$-4x\, J_0^2(x)\, J_1(2x) + (6x^2+8)\, J_0(x)\, J_1(x)\, J_1(2x) + x\, J_1^2(x)\, J_1(2x)]$$
$$\int x^3 I_0(x)\, I_1(x)\, I_1(2x)\, dx = \frac{x^2}{30}[-3x^2\, I_0^2(x)\, I_0(2x) + 4x\, I_0(x)\, I_1(x)\, I_0(2x) + (4-3x^2)\, I_1^2(x)\, I_0(2x) +$$
$$+4x\, I_0^2(x)\, I_1(2x) + (6x^2-8)\, I_0(x)\, I_1(x)\, I_1(2x) + x\, I_1^2(x)\, I_1(2x)]$$
$$\int x^3 K_0(x)\, K_1(x)\, K_1(2x)\, dx = \frac{x^2}{30}[-3x^2\, K_0^2(x)\, K_0(2x) - 4x\, K_0(x)\, K_1(x)\, K_0(2x) + (4-3x^2)\, K_1^2(x)\, K_0(2x) -$$
$$-4x\, K_0^2(x)\, K_1(2x) + (6x^2-8)\, K_0(x)\, K_1(x)\, K_1(2x) - x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_0(x)\, J_1(x)\, J_0(2x)\, dx = \frac{x^2}{42}[-3x^2\, J_0^2(x)\, J_0(2x) + (6x^3+4x)\, J_0(x)\, J_1(x)\, J_0(2x) + 4\, J_1^2(x)\, J_0(2x) +$$
$$+(4x-3x^3)\, J_0^2(x)\, J_1(2x) + (6x^2-8)\, J_0(x)\, J_1(x)\, J_1(2x) + (3x^3+2x)\, J_1^2(x)\, J_1(2x)]$$
$$\int x^4 I_0(x)\, I_1(x)\, I_0(2x)\, dx = \frac{x^2}{42}[3x^2\, I_0^2(x)\, I_0(2x) + (6x^3-4x)\, I_0(x)\, I_1(x)\, I_0(2x) - 4\, I_1^2(x)\, I_0(2x) -$$
$$-(3x^3+4x)\, I_0^2(x)\, I_1(2x) + (6x^2+8)\, I_0(x)\, I_1(x)\, I_1(2x) + (2x-3x^3)\, I_1^2(x)\, I_1(2x)]$$
$$\int x^4 K_0(x)\, K_1(x)\, K_0(2x)\, dx = \frac{x^2}{42}[-3x^2\, K_0^2(x)\, K_0(2x) + (6x^3-4x)\, K_0(x)\, K_1(x)\, K_0(2x) + 4\, K_1^2(x)\, K_0(2x) -$$
$$-(3x^3+4x)\, K_0^2(x)\, K_1(2x) - (6x^2+8)\, K_0(x)\, K_1(x)\, K_1(2x) + (2x-3x^3)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_0^2(x)\, J_1(2x)\, dx = \frac{x^2}{210}[-48x^2\, J_0^2(x)\, J_0(2x) + (-30\, x^3 + 64\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$
$$+(-42\, x^2+64)\, J_1^2(x)\, J_0(2x) + (15\, x^3+64\, x)\, J_0^2(x)\, J_1(2x) + (54\, x^2-128)\, J_0(x)\, J_1(x)\, J_1(2x) +$$
$$+(-15\, x^3+74\, x)\, J_1^2(x)\, J_1(2x)]$$
$$\int x^4 I_0^2(x)\, I_1(2x)\, dx = \frac{x^2}{210}[48x^2\, I_0^2(x)\, I_0(2x) - (30\, x^3+64\, x)\, I_0(x)\, I_1(x)\, I_0(2x) - (42\, x^2+64)\, I_1^2(x)\, I_0(2x) +$$
$$+(15\, x^3-64\, x)\, I_0^2(x)\, I_1(2x) + (54\, x^2+128)\, I_0(x)\, I_1(x)\, I_1(2x) + (15\, x^3+74\, x)\, I_1^2(x)\, I_1(2x)]$$
$$\int x^4 K_0^2(x)\, K_1(2x)\, dx = \frac{x^2}{210}[-48x^2\, K_0^2(x)\, K_0(2x) - (30\, x^3+64\, x)\, K_0(x)\, K_1(x)\, K_0(2x) +$$
$$+(42\, x^2+64)\, K_1^2(x)\, K_0(2x) + (15\, x^3-64\, x)\, K_0^2(x)\, K_1(2x) - (54\, x^2+128)\, K_0(x)\, K_1(x)\, K_1(2x) +$$
$$+(15\, x^3+74\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_1^2(x)\, J_1(2x)\, dx = \frac{x^2}{70}[\, -12x^2\, J_0^2(x)\, J_0(2x) + (10\, x^3 + 16\, x)\, J_0(x)\, J_1(x)\, J_0(2x) + (-28\, x^2 + 16)\, J_1^2(x)\, J_0(2x) +$$

$$+ (-5\, x^3 + 16\, x)\, J_0^2(x)\, J_1(2x) - (4\, x^2 + 32)\, J_0(x)\, J_1(x)\, J_1(2x) + (5\, x^3 + 36\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^4 I_1^2(x)\, I_1(2x)\, dx = \frac{x^2}{70}[\, -12x^2\, I_0^2(x)\, I_0(2x) + (-10\, x^3 + 16\, x)\, I_0(x)\, I_1(x)\, I_0(2x) + (28\, x^2 + 16)\, I_1^2(x)\, I_0(2x) +$$

$$+ (5\, x^3 + 16\, x)\, I_0^2(x)\, I_1(2x) + (4\, x^2 - 32)\, I_0(x)\, I_1(x)\, I_1(2x) + (5\, x^3 - 36\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^4 K_1^2(x)\, K_1(2x)\, dx = \frac{x^2}{70}[\, 12x^2\, K_0^2(x)\, K_0(2x) + (-10\, x^3 + 16\, x)\, K_0(x)\, K_1(x)\, K_0(2x) -$$

$$-(28\, x^2 + 16)\, K_1^2(x)\, K_0(2x) + (5\, x^3 + 16\, x)\, K_0^2(x)\, K_1(2x) + (-4\, x^2 + 32)\, K_0(x)\, K_1(x)\, K_1(2x) +$$

$$+ (5\, x^3 - 36\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^5 J_0^2(x)\, J_0(2x)\, dx = \frac{x^2}{630}[(35\, x^4 + 216\, x^2)\, J_0^2(x)\, J_0(2x) + (100\, x^3 - 288\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$

$$+ (-35\, x^4 + 224\, x^2 - 288)\, J_1^2(x)\, J_0(2x) + (160\, x^3 - 288\, x)\, J_0^2(x)\, J_1(2x) + (70\, x^4 - 208\, x^2 + 576)\, J_0(x)\, J_1(x)\, J_1(2x) +$$

$$+ (120\, x^3 - 368\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^5 I_0^2(x)\, I_0(2x)\, dx = \frac{x^2}{630}[(35\, x^4 - 216\, x^2)\, I_0^2(x)\, I_0(2x) + (100\, x^3 + 288\, x)\, I_0(x)\, I_1(x)\, I_0(2x) +$$

$$+ (35\, x^4 + 224\, x^2 + 288)\, I_1^2(x)\, I_0(2x) + (160x^3 + 288x)\, I_0^2(x)\, I_1(2x) - (70\, x^4 + 208\, x^2 + 576)\, I_0(x)\, I_1(x)\, I_1(2x) -$$

$$-(120\, x^3 + 368\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^5 K_0^2(x)\, K_0(2x)\, dx = \frac{x^2}{630}[(35\, x^4 - 216\, x^2)\, K_0^2(x)\, K_0(2x) - (100\, x^3 + 288\, x)\, K_0(x)\, K_1(x)\, K_0(2x) +$$

$$+ (35\, x^4 + 224\, x^2 + 288)\, K_1^2(x)\, K_0(2x) - (160\, x^3 + 288\, x)\, K_0^2(x)\, K_1(2x) -$$

$$-(70\, x^4 + 208\, x^2 + 576)\, K_0(x)\, K_1(x)\, K_1(2x) + (120\, x^3 + 368\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^5 J_1^2(x)\, J_0(2x)\, dx = \frac{x^2}{630}[(-35\, x^4 + 180\, x^2)\, J_0^2(x)\, J_0(2x) + (100\, x^3 - 288\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$

$$+ (-35\, x^4 + 244\, x^2 - 288)\, J_1^2(x)\, J_0(2x) + (160\, x^3 - 288\, x)\, J_0^2(x)\, J_1(2x) + (70\, x^4 - 208\, x^2 + 576)\, J_0(x)\, J_1(x)\, J_1(2x) +$$

$$+ (120\, x^3 - 368\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^5 I_1^2(x)\, I_0(2x)\, dx = \frac{x^2}{630}[(35x^4 + 180x^2)\, I_0^2(x)\, I_0(2x) + (100\, x^3 + 288\, x)\, I_0(x)\, I_1(x)\, I_0(2x) +$$

$$+ (35\, x^4 + 224\, x^2 + 288)\, I_1^2(x)\, I_0(2x) + (160\, x^3 + 288\, x)\, I_0^2(x)\, I_1(2x) - (70\, x^4 + 208\, x^2 + 576)\, I_0(x)\, I_1(x)\, I_1(2x) -$$

$$-(120\, x^3 + 368\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^5 K_1^2(x)\, K_0(2x)\, dx = \frac{x^2}{630}[(35\, x^4 + 180\, x^2)\, K_0^2(x)\, K_0(2x) - (100\, x^3 + 288\, x)\, K_0(x)\, K_1(x)\, K_0(2x) +$$

$$+ (35\, x^4 + 224\, x^2 + 288)\, K_1^2(x)\, K_0(2x) - (160\, x^3 + 288\, x)\, K_0^2(x)\, K_1(2x) -$$

$$-(70\, x^4 + 208\, x^2 + 576)\, K_0(x)\, K_1(x)\, K_1(2x) + (120\, x^3 + 368\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^5 J_0(x)\, J_1(x)\, J_1(2x)\, dx = \frac{x^2}{630}[(35\, x^4 - 72\, x^2)\, J_0^2(x)\, J_0(2x) + (-80\, x^3 + 96\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$

$$+ (-35\, x^4 - 28\, x^2 + 96)\, J_1^2(x)\, J_0(2x) + (-65\, x^3 + 96\, x)\, J_0^2(x)\, J_1(2x) + (70\, x^4 + 116\, x^2 - 192)\, J_0(x)\, J_1(x)\, J_1(2x) +$$

$$+(30\,x^3+76\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^5 I_0(x)\,I_1(x)\,I_1(2x)\,dx=\frac{x^2}{630}[-(35\,x^4+72\,x^2)\,I_0^2(x)\,I_0(2x)+(80\,x^3+96\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$
$$+(-35\,x^4+28\,x^2+96)\,I_1^2(x)\,I_0(2x)+(65\,x^3+96\,x)\,I_0^2(x)\,I_1(2x)+(70\,x^4-116\,x^2-192)\,I_0(x)\,I_1(x)\,I_1(2x)+$$
$$+(30\,x^3-76\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^5 K_0(x)\,K_1(x)\,K_1(2x)\,dx=\frac{x^2}{630}[-(35\,x^4+72\,x^2)\,K_0^2(x)\,K_0(2x)-(80\,x^3+96\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$
$$+(-35\,x^4+28\,x^2+96)\,K_1^2(x)\,K_0(2x)-(65\,x^3+96\,x)\,K_0^2(x)\,K_1(2x)+$$
$$+(70\,x^4-116\,x^2-192)\,K_0(x)\,K_1(x)\,K_1(2x)+(-30\,x^3+76\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6 J_0(x)\,J_1(x)\,J_0(2x)\,dx=$$
$$=\frac{x^2}{6930}[(-490\,x^4+1224\,x^2)\,J_0^2(x)\,J_0(2x)+(630\,x^5+940\,x^3-1632\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+$$
$$+(175\,x^4+896\,x^2-1632)\,J_1^2(x)\,J_0(2x)+(-315\,x^5+1000\,x^3-1632\,x)\,J_0^2(x)\,J_1(2x)+$$
$$+(910\,x^4-1552\,x^2+3264)\,J_0(x)\,J_1(x)\,J_1(2x)+(315\,x^5+120\,x^3-1712\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6 I_0(x)\,I_1(x)\,I_0(2x)\,dx=$$
$$=\frac{x^2}{6930}[(490\,x^4+1224\,x^2)\,I_0^2(x)\,I_0(2x)+(630\,x^5-940\,x^3-1632\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$
$$+(175\,x^4-896\,x^2-1632)\,I_1^2(x)\,I_0(2x)-(315\,x^5+1000\,x^3+1632\,x)\,I_0^2(x)\,I_1(2x)+$$
$$+(910\,x^4+1552\,x^2+3264)\,I_0(x)\,I_1(x)\,I_1(2x)+(-315\,x^5+120\,x^3+1712\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6 K_0(x)\,K_1(x)\,K_0(2x)\,dx=$$
$$=\frac{x^2}{6930}[-(490\,x^4+1224\,x^2)\,K_0^2(x)\,K_0(2x)+(630\,x^5-940\,x^3-1632\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$
$$+(-175\,x^4+896\,x^2+1632)\,K_1^2(x)\,K_0(2x)-(315\,x^5+1000\,x^3+1632\,x)\,K_0^2(x)\,K_1(2x)-$$
$$-(910\,x^4+1552\,x^2+3264)\,K_0(x)\,K_1(x)\,K_1(2x)+(-315\,x^5+120\,x^3+1712\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6 J_0^2(x)\,J_1(2x)\,dx=\frac{x^2}{6930}[(-1820\,x^4+5904\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(-630\,x^5+2360\,x^3-7872\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-1330\,x^4+6496\,x^2-7872)\,J_1^2(x)\,J_0(2x)+$$
$$+(315\,x^5+4280\,x^3-7872\,x)\,J_0^2(x)\,J_1(2x)+(1400\,x^4-5312\,x^2+15744)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(-315\,x^5+3840\,x^3-10432\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6 I_0^2(x)\,I_1(2x)\,dx=\frac{x^2}{6930}[(1820\,x^4+5904\,x^2)\,I_0^2(x)\,I_0(2x)-$$
$$-(630\,x^5+2360\,x^3+7872\,x)\,I_0(x)\,I_1(x)\,I_0(2x)-(1330\,x^4+6496\,x^2+7872)\,I_1^2(x)\,I_0(2x)+$$
$$+(315\,x^5-4280\,x^3-7872\,x)\,I_0^2(x)\,I_1(2x)+(1400\,x^4+5312\,x^2+15744)\,I_0(x)\,I_1(x)\,I_1(2x)+$$
$$+(315\,x^5+3840\,x^3+10432\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6 K_0^2(x)\,K_1(2x)\,dx=\frac{x^2}{6930}[-(1820\,x^4+5904\,x^2)\,K_0^2(x)\,K_0(2x)-$$
$$-(630\,x^5+2360\,x^3+7872\,x)\,K_0(x)\,K_1(x)\,I_0(2x)+(1330\,x^4+6496\,x^2+7872)\,K_1^2(x)\,K_0(2x)+$$
$$+(315\,x^5-4280\,x^3-7872\,x)\,K_0^2(x)\,K_1(2x)-(1400\,x^4+5312\,x^2+15744)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(315\,x^5+3840\,x^3+10432\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6\,J_1^2(x)\,J_1(2x)\,dx=\frac{x^2}{770}[(-140\,x^4+576\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(70\,x^5+80\,x^3-768\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-280\,x^4+784\,x^2-768)\,J_1^2(x)\,J_0(2x)+$$
$$+(-35\,x^5+380\,x^3-768\,x)\,J_0^2(x)\,J_1(2x)+(-70\,x^4-368\,x^2+1536)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(35\,x^5+600\,x^3-1168\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6\,I_1^2(x)\,I_1(2x)\,dx=\frac{x^2}{770}[-(140\,x^4+576\,x^2)\,I_0^2(x)\,I_0(2x)+$$
$$+(-70\,x^5+80\,x^3+768\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(280\,x^4+784\,x^2+768)\,I_1^2(x)\,I_0(2x)+$$
$$+(35\,x^5+380\,x^3+768\,x)\,I_0^2(x)\,I_1(2x)+(70\,x^4-368\,x^2-1536)\,I_0(x)\,I_1(x)\,I_1(2x)+$$
$$+(35\,x^5-600\,x^3-1168\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6\,K_1^2(x)\,K_1(2x)\,dx=\frac{x^2}{770}[(140\,x^4+576\,x^2)\,K_0^2(x)\,K_0(2x)+$$
$$+(-70\,x^5+80\,x^3+768\,x)\,K_0(x)\,K_1(x)\,K_0(2x)-(280\,x^4+784\,x^2+768)\,K_1^2(x)\,K_0(2x)+$$
$$+(35\,x^5+380\,x^3+768\,x)\,K_0^2(x)\,K_1(2x)+(-70\,x^4+368\,x^2+1536)\,K_0(x)\,K_1(x)\,K_1(2x)+$$
$$+(35\,x^5-600\,x^3-1168\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7 J_0^2(x)\,J_0(2x)\,dx=\frac{x^2}{1430}[(55\,x^6+900\,x^4-3024\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(210\,x^5-1080\,x^3+4032\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-55\,x^6+810\,x^4-3456\,x^2+4032)\,J_1^2(x)\,J_0(2x)+$$
$$+(390\,x^5-2160\,x^3+4032\,x)\,J_0^2(x)\,J_1(2x)+(110\,x^6-540\,x^4+2592\,x^2-8064)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(270\,x^5-2160\,x^3+5472\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7 I_0^2(x)\,I_0(2x)\,dx=\frac{x^2}{1430}[(55\,x^6-900\,x^4-3024\,x^2)\,I_0^2(x)\,I_0(2x)+$$
$$+(210\,x^5+1080\,x^3+4032\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(55\,x^6+810\,x^4+3456\,x^2+4032)\,I_1^2(x)\,I_0(2x)+$$
$$+(390\,x^5+2160\,x^3+4032\,x)\,I_0^2(x)\,I_1(2x)-(110\,x^6+540\,x^4+2592\,x^2+8064)\,I_0(x)\,I_1(x)\,I_1(2x)-$$
$$-(270\,x^5+2160\,x^3+5472\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7 K_0^2(x)\,K_0(2x)\,dx=\frac{x^2}{1430}[(55\,x^6-900\,x^4-3024\,x^2)\,K_0^2(x)\,K_0(2x)-$$
$$-(210\,x^5+1080\,x^3+4032\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+(55\,x^6+810\,x^4+3456\,x^2+4032)\,K_1^2(x)\,K_0(2x)-$$
$$-(390\,x^5+2160\,x^3+4032\,x)\,K_0^2(x)\,K_1(2x)-(110\,x^6+540\,x^4+2592\,x^2+8064)\,K_0(x)\,K_1(x)\,K_1(2x)+$$
$$+(270\,x^5+2160\,x^3+5472\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7\,J_1^2(x)\,J_0(2x)\,dx=\frac{x^2}{30030}[(-1155\,x^6+15680\,x^4-58176\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(4410\,x^5-22680\,x^3+84672\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-1155\,x^6+17010\,x^4-72576\,x^2+84672)\,J_1^2(x)\,J_0(2x)+$$
$$+(8190\,x^5-45360\,x^3+84672\,x)\,J_0^2(x)\,J_1(2x)+(2310\,x^6-11340\,x^4+54432\,x^2-169344)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(5670\,x^5-45360\,x^3+114912\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7\,I_1^2(x)\,I_0(2x)\,dx=\frac{x^2}{30030}[(1155\,x^6+15680\,x^4+58176\,x^2)\,I_0^2(x)\,I_0(2x)+$$
$$+(4410\,x^5+22680\,x^3+84672\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(1155\,x^6+17010\,x^4+72576\,x^2+84672)\,I_1^2(x)\,I_0(2x)+$$
$$+(8190\,x^5+45360\,x^3+84672\,x)\,I_0^2(x)\,I_1(2x)-(2310\,x^6+11340\,x^4+54432\,x^2+169344)\,I_0(x)\,I_1(x)\,I_1(2x)-$$

$$-(5670\,x^5+45360\,x^3+114912\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7\,K_1^2(x)\,K_0(2x)\,dx=\frac{x^2}{30030}[(1155\,x^6+15680\,x^4+58176\,x^2)\,K_0^2(x)\,K_0(2x)-$$
$$-(4410\,x^5+22680\,x^3+84672\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+(1155\,x^6+17010\,x^4+72576\,x^2+84672)\,K_1^2(x)\,K_0(2x)-$$
$$-(8190\,x^5+45360\,x^3+84672\,x)\,K_0^2(x)\,K_1(2x)-(2310\,x^6+11340\,x^4+54432\,x^2+169344)\,K_0(x)\,K_1(x)\,K_1(2x)+$$
$$+(5670\,x^5+45360\,x^3+114912\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7\,J_0(x)\,J_1(x)\,J_1(2x)\,dx=\frac{x^2}{30030}[(1155\,x^6-4760\,x^4+13248\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(-3780\,x^5+8000\,x^3-17664\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-1155\,x^6-280\,x^4+11872\,x^2-17664)\,J_1^2(x)\,J_0(2x)+$$
$$+(-2730\,x^5+10280\,x^3-17664\,x)\,J_0^2(x)\,J_1(2x)+(2310\,x^6+6860\,x^4-14624\,x^2+35328)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(1575\,x^5+4560\,x^3-20704\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7 I_0(x)\,I_1(x)\,I_1(2x)\,dx=\frac{x^2}{30030}[-(1155\,x^6+4760\,x^4+13248\,x^2)\,I_0^2(x)\,I_0(2x)+$$
$$+(3780\,x^5+8000\,x^3+17664\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(-1155\,x^6+280\,x^4+11872\,x^2+17664)\,I_1^2(x)\,I_0(2x)+$$
$$+(2730\,x^5+10280\,x^3+17664\,x)\,I_0^2(x)\,I_1(2x)+(2310\,x^6-6860\,x^4-14624\,x^2-35328)\,I_0(x)\,I_1(x)\,I_1(2x)+$$
$$+(1575\,x^5-4560\,x^3-20704\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7 K_0(x)\,K_1(x)\,K_1(2x)\,dx=\frac{x^2}{30030}[-(1155\,x^6+4760\,x^4+13248\,x^2)\,K_0^2(x)\,K_0(2x)-$$
$$-(3780\,x^5+8000\,x^3+17664\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+(-1155\,x^6+280\,x^4+11872\,x^2+17664)\,K_1^2(x)\,K_0(2x)-$$
$$-(2730\,x^5+10280\,x^3+17664\,x)\,K_0^2(x)\,K_1(2x)+(2310\,x^6-6860\,x^4-14624\,x^2-35328)\,K_0(x)\,K_1(x)\,K_1(2x)+$$
$$+(-1575\,x^5+4560\,x^3+20704\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^8 J_0(x)\,J_1(x)\,J_0(2x)\,dx=\frac{x^2}{30030}[(-2079\,x^6+10752\,x^4-32832\,x^2)\,J_0^2(x)\,J_0(2x)+$$
$$+(2002\,x^7+5712\,x^5-15648\,x^3+43776\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(1078\,x^6+4872\,x^4-33600\,x^2+43776)\,J_1^2(x)\,J_0(2x)+$$
$$+(-1001\,x^7+5460\,x^5-24432\,x^3+43776\,x)\,J_0^2(x)\,J_1(2x)+$$
$$+(3850\,x^6-11256\,x^4+32064\,x^2-87552)\,J_0(x)\,J_1(x)\,J_1(2x)+$$
$$+(1001\,x^7-378\,x^5-17568\,x^3+55488\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^8 I_0(x)\,I_1(x)\,I_0(2x)\,dx=\frac{x^2}{30030}[(2079\,x^6+10752\,x^4+32832\,x^2)\,I_0^2(x)\,I_0(2x)+$$
$$+(2002\,x^7-5712\,x^5-15648\,x^3-43776\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(1078\,x^6-4872\,x^4-33600\,x^2-43776)\,I_1^2(x)\,I_0(2x)-$$
$$-(1001\,x^7+5460\,x^5+24432\,x^3+43776\,x)\,I_0^2(x)\,I_1(2x)+(3850\,x^6+11256\,x^4+32064\,x^2+87552)\,I_0(x)\,I_1(x)\,I_1(2x)+$$
$$+(-1001\,x^7-378\,x^5+17568\,x^3+55488\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^8 K_0(x)\,K_1(x)\,K_0(2x)\,dx=\frac{x^2}{30030}[-(2079\,x^6+10752\,x^4+32832\,x^2)\,K_0^2(x)\,K_0(2x)+$$
$$+(2002\,x^7-5712\,x^5-15648\,x^3-43776\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$
$$+(-1078\,x^6+4872\,x^4+33600\,x^2+43776)\,K_1^2(x)\,K_0(2x)-$$
$$-(1001\,x^7+5460\,x^5+24432\,x^3+43776\,x)\,K_0^2(x)\,K_1(2x)-$$
$$-(3850\,x^6+11256\,x^4+32064\,x^2+87552)\,K_0(x)\,K_1(x)\,K_1(2x)+$$
$$+(-1001\,x^7-378\,x^5+17568\,x^3+55488\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^8 J_0^2(x)\, J_1(2x)\, dx = \frac{x^2}{30030}[(-8316\, x^6 + 64848\, x^4 - 221184\, x^2)\, J_0^2(x)\, J_0(2x) +$$
$$+(-2002\, x^7 + 11928\, x^5 - 75072\, x^3 + 294912\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$
$$+(-5698\, x^6 + 63168\, x^4 - 256704\, x^2 + 294912)\, J_1^2(x)\, J_0(2x) +$$
$$+(1001\, x^7 + 27300\, x^5 - 157008\, x^3 + 294912\, x)\, J_0^2(x)\, J_1(2x) +$$
$$+(5390\, x^6 - 34104\, x^4 + 185664\, x^2 - 589824)\, J_0(x)\, J_1(x)\, J_1(2x) +$$
$$+(-1001\, x^7 + 23058\, x^5 - 163872\, x^3 + 404160\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^8 I_0^2(x)\, I_1(2x)\, dx = \frac{x^2}{30030}[(8316\, x^6 + 64848\, x^4 + 221184\, x^2)\, I_0^2(x)\, I_0(2x) -$$
$$-(2002\, x^7 + 11928\, x^5 + 75072\, x^3 + 294912\, x)\, I_0(x)\, I_1(x)\, I_0(2x) -$$
$$-(5698\, x^6 + 63168\, x^4 + 256704\, x^2 + 294912)\, I_1^2(x)\, I_0(2x) +$$
$$+(1001\, x^7 - 27300\, x^5 - 157008\, x^3 - 294912\, x)\, I_0^2(x)\, I_1(2x) +$$
$$+(5390\, x^6 + 34104\, x^4 + 185664\, x^2 + 589824)\, I_0(x)\, I_1(x)\, I_1(2x) +$$
$$+(1001\, x^7 + 23058\, x^5 + 163872\, x^3 + 404160\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^8 K_0^2(x)\, K_1(2x)\, dx = \frac{x^2}{30030}[-(8316\, x^6 + 64848\, x^4 + 221184\, x^2)\, K_0^2(x)\, K_0(2x) -$$
$$-(2002\, x^7 + 11928\, x^5 + 75072\, x^3 + 294912\, x)\, K_0(x)\, K_1(x)\, K_0(2x) +$$
$$+(5698\, x^6 + 63168\, x^4 + 256704\, x^2 + 294912)\, K_1^2(x)\, K_0(2x) +$$
$$+(1001\, x^7 - 27300\, x^5 - 157008\, x^3 - 294912\, x)\, K_0^2(x)\, K_1(2x) -$$
$$-(5390\, x^6 + 34104\, x^4 + 185664\, x^2 + 589824)\, K_0(x)\, K_1(x)\, K_1(2x) +$$
$$+(1001\, x^7 + 23058\, x^5 + 163872\, x^3 + 404160\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^8\, J_1^2(x)\, J_1(2x)\, dx = \frac{x^2}{30030}[(-5544\, x^6 + 57792\, x^4 - 207360\, x^2)\, J_0^2(x)\, J_0(2x) +$$
$$+(2002\, x^7 + 672\, x^5 - 58368\, x^3 + 276480\, x)\, J_0(x)\, J_1(x)\, J_0(2x) +$$
$$+(-10472\, x^6 + 71232\, x^4 - 252672\, x^2 + 276480)\, J_1^2(x)\, J_0(2x) +$$
$$+(-1001\, x^7 + 21840\, x^5 - 144192\, x^3 + 276480\, x)\, J_0^2(x)\, J_1(2x) +$$
$$+(-3080\, x^6 - 15456\, x^4 + 162048\, x^2 - 552960)\, J_0(x)\, J_1(x)\, J_1(2x) +$$
$$+(1001\, x^7 + 31752\, x^5 - 171648\, x^3 + 390912\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^8\, I_1^2(x)\, I_1(2x)\, dx = \frac{x^2}{30030}[-(5544\, x^6 + 57792\, x^4 + 207360\, x^2)\, I_0^2(x)\, I_0(2x) +$$
$$+(-2002\, x^7 + 672\, x^5 + 58368\, x^3 + 276480\, x)\, I_0(x)\, I_1(x)\, I_0(2x) +$$
$$+(10472\, x^6 + 71232\, x^4 + 252672\, x^2 + 276480)\, I_1^2(x)\, I_0(2x) +$$
$$+(1001\, x^7 + 21840\, x^5 + 144192\, x^3 + 276480\, x)\, I_0^2(x)\, I_1(2x) +$$
$$+(3080\, x^6 - 15456\, x^4 - 162048\, x^2 - 552960)\, I_0(x)\, I_1(x)\, I_1(2x) +$$
$$+(1001\, x^7 - 31752\, x^5 - 171648\, x^3 - 390912\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^8\, K_1^2(x)\, K_1(2x)\, dx = \frac{x^2}{30030}[(5544\, x^6 + 57792\, x^4 + 207360\, x^2)\, K_0^2(x)\, K_0(2x) +$$
$$+(-2002\, x^7 + 672\, x^5 + 58368\, x^3 + 276480\, x)\, K_0(x)\, K_1(x)\, K_0(2x) -$$
$$-(10472\, x^6 + 71232\, x^4 + 252672\, x^2 + 276480)\, K_1^2(x)\, K_0(2x) +$$
$$+(1001\, x^7 + 21840\, x^5 + 144192\, x^3 + 276480\, x)\, K_0^2(x)\, K_1(2x) +$$
$$+(-3080\, x^6 + 15456\, x^4 + 162048\, x^2 + 552960)\, K_0(x)\, K_1(x)\, K_1(2x) +$$
$$+(1001\, x^7 - 31752\, x^5 - 171648\, x^3 - 390912\, x)\, K_1^2(x)\, K_1(2x)]$$

Recurrence relations:

$$\int x^{2n+1}\,J_0^2(x)\,J_0(2x)\,dx=-\frac{n}{4n+1}\int x^{2n}\,J_1(2x)\,\left[3n\,J_0^2(x)+(n-1)\,J_1^2(x)\right]\,dx+$$

$$+\frac{x^{2n+1}}{2(4n+1)}[xJ_0^2(x)\,J_0(2x)-xJ_1^2(x)\,J_0(2x)+3nJ_0^2(x)\,J_1(2x)+2xJ_0(x)\,J_1(x)\,J_1(2x)+nJ_1^2(x)\,J_1(2x)]$$

$$\int x^{2n+1}\,J_0(x)\,J_1(x)\,J_1(2x)\,dx=$$

$$=\frac{1}{2(4n+1)}\int x^{2n}\,\left[-n(2n-1)\,J_0^2(x)J_1(2x)+2n(4n+1)\,J_0(x)J_1(x)J_0(2x)+(n-1)(2n+1)J_1^2(x)J_1(2x)\right]\,dx+$$

$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1)J_0^2(x)\,J_1(2x)-2(4n+1)\,J_0(x)J_1(x)J_0(2x)-(2n+1)J_1^2(x)\,J_1(2x)+2xJ_0^2(x)\,J_0(2x)+$$

$$+4xJ_0(x)J_1(x)J_1(2x)-2xJ_1^2(x)J_0(2x)]$$

$$\int x^{2n+1}\,J_1^2(x)\,J_0(2x)\,dx=-\frac{1}{4n+1}\int x^{2n}\,\left[n\,(n+1)J_0^2(x)J_1(2x)+(3n+1)(n-1)\,J_1^2(x)\,J_1(2x)\right]\,dx+$$

$$+\frac{x^{2n+1}}{2(4n+1)}[(n+1)J_0^2(x)J_1(2x)+(3n+1)J_1^2(x)J_1(2x)-xJ_0^2(x)\,J_0(2x)-2xJ_0(x)J_1(x)J_1(2x)+xJ_1^2(x)J_0(2x)]$$

$$\int x^{2n+2}\,J_0(x)\,J_1(x)\,J_0(2x)\,dx=$$

$$=\frac{1}{4(4n+3)}\left\{-2(n+1)(2n+1)\int x^{2n+1}\,J_0^2(x)\,J_0(2x)\,dx-4(4n+3)\int x^{2n+1}\,J_0(x)\,J_1(x)\,J_1(2x)\,dx+\right.$$

$$+2n(2n+3)\int x^{2n+1}\,J_1^2(x)\,J_0(2x)\,dx+x^{2n+2}[(2n+1)J_0^2(x)J_0(2x)-2xJ_0^2(x)J_1(2x)+4xJ_0(x)J_1(x)J_0(2x)-$$

$$\left.-(2n+3)J_1^2(x)J_0(2x)+2(4n+3)J_0(x)J_1(x)J_1(2x)+2xJ_1^2(x)J_1(2x)]\right\}$$

$$\int x^{2n+2}\,J_0^2(x)\,J_1(2x)\,dx=$$

$$=\frac{1}{2(4n+3)}\left\{2(n+1)(3n+2)\int x^{2n+1}J_0^2(x)J_0(2x)\,dx+2n^2\int x^{2n+1}J_1^2(x)J_0(2x)\,dx+\right.$$

$$\left.+x^{2n+2}\,[-(3n+2)J_0^2(x)J_0(2x)+xJ_0^2(x)J_1-2xJ_0(x)J_1(x)J_0(2x)-nJ_1^2(x)J_0(2x)-xJ_1^2(x)J_1(2x)]\right\}$$

$$\int x^{2n+2}\,J_1^2(x)\,J_1(2x)\,dx=$$

$$=\frac{1}{2(4n+3)}\left\{2(n+1)^2\int x^{2n+1}\,J_0^2(x)\,J_0(2x)\,dx+6n(n+1)\int x^{2n+1}\,J_1^2(x)\,J_0(2x)\,dx+\right.$$

$$\left.+x^{2n+2}[-(n+1)J_0^2(x)J_0(2x)-xJ_0^2(x)J_1(2x)+2xJ_0(x)J_1(x)J_0(2x)-3(n+1)J_1^2(x)J_0(2x)+xJ_0(x)J_1(x)J_0(2x)]\right\}$$

$$\int x^{2n+1}\, I_0^2(x)\, I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n}[-3n^2\, I_0^2(x) I_1(2x) + n(n-1) I_1^2(x) I_1(2x)]\, dx +$$
$$+ \frac{x^{2n+1}}{2(4n+1)}[x I_0^2(x)\, I_0(2x) + x I_1^2(x)\, I_0(2x) - 2x I_0(x)\, I_1(x)\, I_1(2x) + 3n I_0^2(x)\, I_1(2x) - n I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+1}\, I_0(x)\, I_1(x)\, I_1(2x)\, dx =$$
$$= \frac{1}{2(4n+1)} \int x^{2n}\, [n(2n-1)\, I_0^2(x) I_1(2x) - 2n(4n+1)\, I_0(x) I_1(x) I_0(2x) + (n-1)(2n+1) I_1^2(x) I_1(2x)]\, dx +$$
$$+ \frac{x^{2n+1}}{4(4n+1)}[-(2n-1) I_0^2(x)\, I_1(2x) + 2(4n+1) I_0(x) I_1(x) I_0(2x) - (2n+1) I_1^2(x)\, I_1(2x) -$$
$$-2x I_0^2(x)\, I_0(2x) + 4x I_0(x) I_1(x) I_1(2x) - 2x I_1^2(x)\, I_0(2x)]$$

$$\int x^{2n+1}\, I_1^2(x)\, I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n}\, \left[n\,(n+1) I_0^2(x) I_1(2x) - (3n+1)(n-1)\, I_1^2(x)\, I_1(2x)\right]\, dx +$$
$$+ \frac{x^{2n+1}}{2(4n+1)}[-(n+1) I_0^2(x) I_1(2x) + (3n+1) I_1^2(x) I_1(2x) + x I_0^2(x)\, I_0(2x) - 2x I_0(x) I_1(x) I_1(2x) + x I_1^2(x) I_0(2x)]$$

$$\int x^{2n+2}\, I_0(x)\, I_1(x)\, I_0(2x)\, dx =$$
$$= \frac{1}{2(4n+3)} \int x^{2n+1}\, [(n+1)(2n+1) I_0^2(x)\, I_0(2x) - 2n(4n+3)\, I_0(x)\, I_1(x)\, I_1(2x) + n(2n+3) I_1^2(x) I_0(2x)]\, dx +$$
$$+ x^{2n+2}[-(2n+1)\, I_0^2(x)\, I_0(2x) + 2(4n+3)\, I_0(x)\, I_1(x)\, I_1(2x) - (2n+3)x\, I_1^2(x)\, I_0(2x) -$$
$$-2x\, I_0^2(x)\, I_1(2x) + 4x I_0(x)\, I_1(x)\, I_0(2x) - 3x\, I_1^2(x)\, I_1(2x)\}$$

$$\int x^{2n+2}\, I_0^2(x)\, I_1(2x)\, dx =$$
$$= \frac{1}{4n+3} \int x^{2n+1}\, [-(n+1)(3n+2) I_0^2(x) I_0(2x) + n^2 I_1^2(x) I_0(2x)]\, dx +$$
$$+ x^{2n+2}[(3n+2)\, I_0^2(x)\, I_0(2x) - n\, I_1^2(x)\, I_0(2x) + x\ I_0^2(x)\, I_1(2x) - 2x\ I_0(x) I_1(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+2}\, I_1^2(x)\, I_1(2x)\, dx =$$
$$= \frac{1}{4n+3} \int x^{2n+1}\, [(n+1)^2\, I_0^2(x)\, I_0(2x) - 3n(n+1)\, I_1^2(x)\, I_0(2x)]\, dx +$$
$$+ x^{2n+2}[-(n+1)\, I_0^2(x)\, I_0(2x) + 3(n+1)\, I_1^2(x)\, I_0(2x) + x\, I_0^2(x)\, I_1(2x) - 2x\, I_0(x)\, I_1(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+1}\,K_0^2(x)\,K_0(2x)\,dx=\frac{1}{4n+1}\int x^{2n}[3n^2\,K_0^2(x)K_1(2x)-n(n-1)K_1^2(x)K_1(2x)]\,dx+$$
$$+\frac{x^{2n+1}}{2(4n+1)}[xK_0^2(x)\,K_0(2x)+xK_1^2(x)\,K_0(2x)-2xK_0(x)\,K_1(x)\,K_1(2x)-3nK_0^2(x)\,K_1(2x)+nK_1^2(x)\,K_1(2x)]$$

$$\int x^{2n+1}\,K_0(x)\,K_1(x)\,K_1(2x)\,dx=$$
$$=\frac{1}{2(4n+1)}\int x^{2n}\,[-n(2n-1)\,K_0^2(x)K_1(2x)+2n(4n+1)\,K_0(x)K_1(x)K_0(2x)-(n-1)(2n+1)K_1^2(x)K_1(2x)]\,dx+$$
$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1)K_0^2(x)\,K_1(2x)-2(4n+1)K_0(x)K_1(x)K_0(2x)+(2n+1)K_1^2(x)\,K_1(2x)-$$
$$-2xK_0^2(x)\,K_0(2x)+4xK_0(x)K_1(x)K_1(2x)-2xK_1^2(x)\,K_0(2x)]$$

$$\int x^{2n+1}\,K_1^2(x)\,K_0(2x)\,dx=\frac{1}{4n+1}\int x^{2n}\,\left[-n\,(n+1)K_0^2(x)K_1(2x)+(3n+1)(n-1)\,K_1^2(x)\,K_1(2x)\right]\,dx+$$
$$+\frac{x^{2n+1}}{2(4n+1)}\,\Big[(n+1)K_0^2(x)K_1(2x)-(3n+1)K_1^2(x)K_1(2x)+xK_0^2(x)\,K_0(2x)-2xK_0(x)K_1(x)K_1(2x)+$$
$$+\,xK_1^2(x)K_0(2x)\Big]$$

$$\int x^{2n+2}\,K_0(x)\,K_1(x)\,K_0(2x)\,dx=\frac{1}{2(4n+3)}\cdot$$
$$\cdot\int x^{2n+1}\,[(n+1)(2n+1)K_0^2(x)\,K_0(2x)-2n(4n+3)\,K_0(x)\,K_1(x)\,K_1(2x)+n(2n+3)K_1^2(x)K_0(2x)]\,dx+$$
$$+\frac{x^{2n+2}}{4(4n+3)}[-(2n+1)\,K_0^2(x)\,K_0(2x)+2(4n+3)\,K_0(x)\,K_1(x)\,K_1(2x)-(3n+2)x\,K_1^2(x)\,K_0(2x)-$$
$$-2x\,K_0^2(x)\,K_1(2x)+4xK_0(x)\,K_1(x)\,K_0(2x)-3x\,K_1^2(x)\,K_1(2x)$$

$$\int x^{2n+2}\,K_0^2(x)\,K_1(2x)\,dx=$$
$$=\frac{1}{4n+3}\int x^{2n+1}\,[(n+1)(3n+2)K_0^2(x)K_0(2x)-n^2K_1^2(x)K_0(2x)]\,dx+$$
$$+\frac{x^{2n+2}}{2(4n+3)}\,\Big[-(3n+2)\,K_0^2(x)\,K_0(2x)+n\,K_1^2(x)\,K_0(2x)+x\;K_0^2(x)\,K_1(2x)-2x\;K_0(x)K_1(x)\,K_0(2x)+$$
$$+x\,K_1^2(x)\,K_1(2x)\Big]$$

$$\int x^{2n+2}\,K_1^2(x)\,K_1(2x)\,dx=$$
$$=\frac{1}{4n+3}\int x^{2n+1}\,[-(n+1)^2\,K_0^2(x)\,K_0(2x)+3n(n+1)\,K_1^2(x)\,K_0(2x)]\,dx+$$
$$+\frac{x^{2n+2}}{2(4n+3)}\,\Big[(n+1)\,K_0^2(x)\,K_0(2x)-3(n+1)\,K_1^2(x)\,K_0(2x)+x\,K_0^2(x)\,K_1(2x)-2x\,K_0(x)\,K_1(x)\,K_0(2x)\,+$$
$$+x\,K_1^2(x)\,K_1(2x)\Big]$$

b) $x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu((\alpha+\beta)x)$

Formulas were found for the following integrals only:

$$\int x^{2n+1}\, Z_0(\alpha x)\, Z_0(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ , \quad \int x^{2n+1}\, Z_0(\alpha x)\, Z_1(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ ,$$

$$\int x^{2n+1}\, Z_1(\alpha x)\, Z_1(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ , \quad n \geq 0,$$

and

$$\int x^{2n}\, Z_0(\alpha x)\, Z_0(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ , \quad \int x^{2n}\, Z_0(\alpha x)\, Z_1(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ ,$$

$$\int x^{2n}\, Z_1(\alpha x)\, Z_1(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ , \quad n \geq 1.$$

The integrals $\int x^n\, Z_\nu(\alpha x)\, Z_\nu(\beta x)\, Z_{1-\nu}((\alpha+\beta)x)\, dx$ and $\int x^n\, Z_{1-\nu}(\alpha x)\, Z_\nu(\beta x)\, Z_\nu((\alpha+\beta)x)\, dx$ may be expressed by each other. Nevertheless, they are both listed.

$$\int J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [\, J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) -$$
$$- J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) - J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) - J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)]$$

$$\int I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [-I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) +$$
$$+ I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) - I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)]$$

$$\int K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [-K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) +$$
$$+ K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) - K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)]$$

$$\int x\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [J_0(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) +$$
$$+ J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x) + J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) - J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)]$$

$$\int x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [\, I_0(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) -$$
$$- I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x) - I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) + I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)]$$

$$\int x\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [\, K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) -$$
$$- K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x) - K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) + K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)]$$

$$\int x\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x}{2\beta(\alpha+\beta)}\, \big[(\beta\,\alpha\, x + \beta^2 x)\, J_0(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) -$$
$$- (\alpha+\beta)\, J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) - \beta\, J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) +$$
$$+ (\beta\,\alpha\, x + \beta^2 x)\, J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x) + \alpha\, J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) +$$
$$+ (\beta\,\alpha\, x + \beta^2 x)\, J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) - (\beta\,\alpha\, x + \beta^2 x)\, J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)\big]$$

$$\int x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x}{2\beta(\alpha+\beta)}\, \big[-(\beta\,\alpha\, x + \beta^2 x)\, I_0(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) +$$
$$+ (\alpha+\beta)\, I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) + \beta\, I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) +$$

$$+(\beta\,\alpha\,x+\beta^2x)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-\alpha\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-(\beta\,\alpha\,x+\beta^2x)\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)\big]$$

$$\int x\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx=\frac{x}{2\beta(\alpha+\beta)}\,\big[-(\beta\,\alpha\,x+\beta^2x)\,K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$
$$-(\alpha+\beta)K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-\beta K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)+\alpha K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-(\beta\,\alpha\,x+\beta^2x)K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)\big]$$

$$\int x\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\,[-\alpha\beta x\,J_0(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+$$
$$+(\alpha+\beta)\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-\beta\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)-\alpha\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+\alpha\beta x\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)]$$

$$\int x\,I_1(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\,[\alpha\beta x\,I_0(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$
$$-(\alpha+\beta)\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+\beta\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)+\alpha\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+\alpha\beta x\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)]$$

$$\int x\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\,[\alpha\beta x\,K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\alpha+\beta)\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-\beta\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-\alpha\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+\alpha\beta x\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)]$$

$$\int x^2\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\,[\alpha\beta\,x\;(\alpha+\beta)\;J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$
$$-\alpha\beta\,x\;(\alpha+\beta)\;J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+\alpha\;(\alpha+3\,\beta)\;J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)-$$
$$-\alpha\beta\;(\alpha+\beta)\;x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+\beta\;(3\,\alpha+\beta)\;J_1(\alpha x)J_0(\beta x)\,J_1((\alpha+\beta)x)-$$
$$-\,(\alpha+\beta)^2\;J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)-\alpha\beta\;(\alpha+\beta)\;x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\Big]$$

$$\int x^2\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\,[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)\;+\alpha(\alpha+3\beta)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)+\beta(3\alpha+\beta)\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-(\alpha+\beta)^2\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\,[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-\alpha(\alpha+3\beta)\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) - \beta(3\alpha+\beta)\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) +$$
$$+(\alpha+\beta)^2\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)\big]$$

$$\int x^2\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha(2\alpha+3\beta)\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) - \beta^2\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) -$$
$$-(\alpha+\beta)(2\alpha-\beta)\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)]$$

$$\int x^2\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + \alpha(2\alpha+3\beta)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) - \beta^2\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) -$$
$$-(\alpha+\beta)(2\alpha-\beta)\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)]$$

$$\int x^2\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) - \alpha(2\alpha+3\beta)\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) + \beta^2\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) +$$
$$+(\alpha+\beta)(2\alpha-\beta)\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)]$$

$$\int x^2\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + 2\alpha^2\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x) +$$
$$+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) + 2\beta^2\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) -$$
$$-2(\alpha+\beta)^2\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\big]$$

$$\int x^2\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) -$$
$$-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) - 2\alpha^2\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x) -$$
$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) - 2\beta^2\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) +$$
$$+\,2(\alpha+\beta)^2\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\;[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) -$$
$$-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + 2\alpha^2\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x) -$$
$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) + 2\beta^2\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) -$$
$$-2(\alpha+\beta)^2\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)]$$

Let

$$\int x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu((\alpha+\beta)x)\, dx = {}^{\kappa\mu\nu}V_n \sum_{\kappa',\mu',\nu'=0}^{1} {}^{\kappa\mu\nu}P_{n,Z}^{\kappa'\mu'\nu'}(x)\, Z_{\kappa'}(\alpha x)\, Z_{\mu'}(\beta x)\, Z_{\nu'}((\alpha+\beta)x)\;,$$

then holds

$${}^{000}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$${}^{000}P_{3,J}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2$$

$${}^{000}P_{3,J}^{001}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\,x$$

$${}^{000}P_{3,J}^{010}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\,x$$

$${}^{000}P_{3,J}^{011}(x) = \alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2-4\,\alpha^2-10\,\alpha\,\beta-10\,\beta^2]$$

$${}^{000}P_{3,J}^{100}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$${}^{000}P_{3,J}^{101}(x) = \beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-4\,\beta^2-10\,\alpha\,\beta-10\,\alpha^2]$$

$${}^{000}P_{3,J}^{110}(x) = -\,(\alpha+\beta)^2\;[3\,\alpha^2\beta^2x^2-4\,\alpha^2+2\,\alpha\,\beta-4\,\beta^2]$$

$${}^{000}P_{3,J}^{111}(x) = 4\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)x$$

$${}^{000}P_{3,I}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2$$

$${}^{000}P_{3,I}^{001}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\,x$$

$${}^{000}P_{3,I}^{010}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\,x$$

$${}^{000}P_{3,I}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$${}^{000}P_{3,I}^{100}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$${}^{000}P_{3,I}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$${}^{000}P_{3,I}^{110}(x) = (\alpha+\beta)^2\;[3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$${}^{000}P_{3,I}^{111}(x) = -4\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)\,x$$

$${}^{000}P_{3,K}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2$$

$${}^{000}P_{3,K}^{001}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\,x$$

$${}^{000}P_{3,K}^{010}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\,x$$

$${}^{000}P_{3,K}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$${}^{000}P_{3,K}^{100}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$${}^{000}P_{3,K}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$${}^{000}P_{3,K}^{110}(x) = (\alpha+\beta)^2\;[3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$${}^{000}P_{3,K}^{111}(x) = 4\,\alpha\,\beta\;(\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)\,x$$

$${}^{011}V_3 = \frac{x^2}{30\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$${}^{011}P_{3,J}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2$$

$${}^{011}P_{3,J}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$$^{011}P_{3,J}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x$$
$$^{011}P_{3,J}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2+6\,\alpha^2+20\,\alpha\,\beta+20\,\beta^2\right]$$
$$^{011}P_{3,J}^{100}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right)\,x$$
$$^{011}P_{3,J}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\;(5\,\alpha+2\,\beta)\right]$$
$$^{011}P_{3,J}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2+2\;(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right]$$
$$^{011}P_{3,J}^{111}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$$^{011}P_{3,I}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2$$
$$^{011}P_{3,I}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\,x$$
$$^{011}P_{3,I}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x$$
$$^{011}P_{3,I}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-6\,\alpha^2-20\,\alpha\,\beta-20\,\beta^2\right]$$
$$^{011}P_{3,I}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right)$$
$$^{011}P_{3,I}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\;(5\,\alpha+2\,\beta)\right]$$
$$^{011}P_{3,I}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2-2\;(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right]$$
$$^{011}P_{3,I}^{111}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$$^{011}P_{3,K}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2$$
$$^{011}P_{3,K}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\,x$$
$$^{011}P_{3,K}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x$$
$$^{011}P_{3,K}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-6\,\alpha^2-20\,\alpha\,\beta-20\,\beta^2\right]$$
$$^{011}P_{3,K}^{100}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right)\,x$$
$$^{011}P_{3,K}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\;(5\,\alpha+2\,\beta)\right]$$
$$^{011}P_{3,K}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2-2\;(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right]$$
$$^{011}P_{3,K}^{111}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x$$

$$^{110}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{110}P_{3,J}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2$$
$$^{110}P_{3,J}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x$$
$$^{110}P_{3,J}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x$$
$$^{110}P_{3,J}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2+2\,\alpha\;(3\,\alpha+5\,\beta)\right]$$
$$^{110}P_{3,J}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x$$
$$^{110}P_{3,J}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\;(5\,\alpha+3\,\beta)\right]$$
$$^{110}P_{3,J}^{110}(x) = \left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2+6\,\alpha^2-8\,\alpha\,\beta+6\,\beta^2\right]$$
$$^{110}P_{3,J}^{111}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x$$

$$^{110}P_{3,I}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2$$

$$ {}^{110}P^{001}_{3,I}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x $$
$$ {}^{110}P^{010}_{3,I}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x $$
$$ {}^{110}P^{011}_{3,I}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\,(3\,\alpha+5\,\beta)\right] $$
$$ {}^{110}P^{100}_{3,I}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x $$
$$ {}^{110}P^{101}_{3,I}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\,(5\,\alpha+3\,\beta)\right] $$
$$ {}^{110}P^{110}_{3,I}(x) = (\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right] $$
$$ {}^{110}P^{111}_{3,I}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x $$

$$ {}^{110}P^{000}_{3,K}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{110}P^{001}_{3,K}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x $$
$$ {}^{110}P^{010}_{3,K}(x) = \alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x $$
$$ {}^{110}P^{011}_{3,K}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\,(3\,\alpha+5\,\beta)\right] $$
$$ {}^{110}P^{100}_{3,K}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x $$
$$ {}^{110}P^{101}_{3,K}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\,(5\,\alpha+3\,\beta)\right] $$
$$ {}^{110}P^{110}_{3,K}(x) = (\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right] $$
$$ {}^{110}P^{111}_{3,K}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x $$

$$ {}^{001}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha+\beta)^3} $$
$$ {}^{001}P^{000}_{4,J}(x) = -6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{001}P^{001}_{4,J}(x) = \alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+52\,\alpha^3\beta+116\,\alpha^2\beta^2+52\,\alpha\,\beta^3+18\,\beta^4\right] $$
$$ {}^{001}P^{010}_{4,J}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-58\,\alpha^2\beta^2+34\,\alpha\,\beta^3+18\,\beta^4\right] $$
$$ {}^{001}P^{011}_{4,J}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2-4\,\alpha\,\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right] $$
$$ {}^{001}P^{100}_{4,J}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+34\,\alpha^3\beta-58\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right] $$
$$ {}^{001}P^{101}_{4,J}(x) = -\beta^2\left[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2+4\,\beta\,\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)\right] $$
$$ {}^{001}P^{110}_{4,J}(x) = -(\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2-4\,\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right] $$
$$ {}^{001}P^{111}_{4,J}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-52\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right] $$

$$ {}^{001}P^{000}_{4,I}(x) = 6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{001}P^{001}_{4,I}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-52\,\alpha^3\beta-116\,\alpha^2\beta^2-52\,\alpha\,\beta^3-18\,\beta^4\right] $$
$$ {}^{001}P^{010}_{4,I}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+58\,\alpha^2\beta^2-34\,\alpha\,\beta^3-18\,\beta^4\right] $$
$$ {}^{001}P^{011}_{4,I}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+4\,\alpha\,\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right] $$
$$ {}^{001}P^{100}_{4,I}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-34\,\alpha^3\beta+58\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right] $$
$$ {}^{001}P^{101}_{4,I}(x) = -\beta^2\left[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2-4\,\beta\,\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)\right] $$
$$ {}^{001}P^{110}_{4,I}(x) = -(\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2+4\,\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right] $$
$$ {}^{001}P^{111}_{4,I}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+52\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right] $$

$$ {}^{001}P_{4,K}^{000}(x) = -6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\;x^2 $$

$$ {}^{001}P_{4,K}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 18\,\alpha^4 - 52\,\alpha^3\beta - 116\,\alpha^2\beta^2 - 52\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{001}P_{4,K}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 58\,\alpha^2\beta^2 - 34\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{001}P_{4,K}^{011}(x) = -\alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(9\,\alpha^3 + 35\,\beta\,\alpha^2 + 49\,\alpha\,\beta^2 + 35\,\beta^3\right)] $$

$$ {}^{001}P_{4,K}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 18\,\alpha^4 - 34\,\alpha^3\beta + 58\,\alpha^2\beta^2 + 86\,\alpha\,\beta^3 + 36\,\beta^4] $$

$$ {}^{001}P_{4,K}^{101}(x) = \beta^2\,[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2 - 4\,\beta\;\left(35\,\alpha^3 + 49\,\beta\,\alpha^2 + 35\,\alpha\,\beta^2 + 9\,\beta^3\right)] $$

$$ {}^{001}P_{4,K}^{110}(x) = (\alpha+\beta)^2\;[3\,\alpha^2\beta^2\left(9\,\alpha^2 + 10\,\alpha\,\beta + 9\,\beta^2\right)x^2 + 4\;\left(9\,\alpha^2 - 10\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2] $$

$$ {}^{001}P_{4,K}^{111}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 52\,\alpha^2\beta^2 + 86\,\alpha\,\beta^3 + 36\,\beta^4] $$

$$ {}^{010}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha+\beta)^3} $$

$$ {}^{010}P_{4,J}^{000}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2 $$

$$ {}^{010}P_{4,J}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 24\,\alpha^4 - 60\,\alpha^3\beta - 52\,\alpha^2\beta^2 + 38\,\alpha\,\beta^3 + 18\,\beta^4] $$

$$ {}^{010}P_{4,J}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta + 68\,\alpha^2\beta^2 + 20\,\alpha\,\beta^3 + 18\,\beta^4] $$

$$ {}^{010}P_{4,J}^{011}(x) = \alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 - 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)] $$

$$ {}^{010}P_{4,J}^{100}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 24\,\alpha^4 - 36\,\alpha^3\beta - 16\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4] $$

$$ {}^{010}P_{4,J}^{101}(x) = -\beta^2\,[3\,\alpha^2\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 4\,\beta^2\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)] $$

$$ {}^{010}P_{4,J}^{110}(x) = $$

$$ = -\,(\alpha+\beta)^2\;[9\,\alpha^2\beta^2\,(4\,\alpha+3\,\beta)\,(\alpha-\beta)\,x^2 - 4\;(\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)] $$

$$ {}^{010}P_{4,J}^{111}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4] $$

$$ {}^{010}P_{4,I}^{000}(x) = 6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2 $$

$$ {}^{010}P_{4,I}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{010}P_{4,I}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{010}P_{4,I}^{011}(x) = \alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)] $$

$$ {}^{010}P_{4,I}^{100}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 36\,\alpha^3\beta + 16\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4] $$

$$ {}^{010}P_{4,I}^{101}(x) = -\beta^2\,[3\,\alpha^2\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 4\,\beta^2\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)] $$

$$ {}^{010}P_{4,I}^{110}(x) = $$

$$ = -\,(\alpha+\beta)^2\;[9\,\alpha^2\beta^2\,(4\,\alpha+3\,\beta)\,(\alpha-\beta)\,x^2 + 4\;(\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)] $$

$$ {}^{010}P_{4,I}^{111}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta + 10\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4] $$

$$ {}^{010}P_{4,K}^{000}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2 $$

$$ {}^{010}P_{4,K}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{010}P_{4,K}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\,\beta^3 - 18\,\beta^4] $$

$$ {}^{010}P_{4,K}^{011}(x) = $$

$$ = -\alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)] $$

$$ {}^{010}P_{4,K}^{100}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[15\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+24\,\alpha^4+36\,\alpha^3\beta+16\,\alpha^2\beta^2+58\,\alpha\,\beta^3+36\,\beta^4\right] $$

$$ {}^{010}P_{4,K}^{101}(x) = \beta^2\left[3\,\alpha^2\left(8\,\alpha^2+8\,\alpha\,\beta+9\,\beta^2\right)\left(\alpha+\beta\right)^2x^2+4\,\beta^2\left(28\,\alpha^2+28\,\alpha\,\beta+9\,\beta^2\right)\right] $$

$$ {}^{010}P_{4,K}^{110}(x) = $$

$$ = \left(\alpha+\beta\right)^2\,\left[9\,\alpha^2\beta^2\left(4\,\alpha+3\,\beta\right)\left(\alpha-\beta\right)x^2+4\,\left(\alpha+\beta\right)\left(12\,\alpha^3-6\,\beta\,\alpha^2+8\,\alpha\,\beta^2-9\,\beta^3\right)\right] $$

$$ {}^{010}P_{4,K}^{111}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[-15\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+48\,\alpha^4+96\,\alpha^3\beta-10\,\alpha^2\beta^2-58\,\alpha\,\beta^3-36\,\beta^4\right] $$

$$ {}^{111}V_4 = \frac{x^2}{70\,\alpha^3\beta^3\left(\alpha+\beta\right)^3} $$

$$ {}^{111}P_{4,J}^{000}(x) = -8\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(\alpha+\beta\right)^2\,x^2 $$

$$ {}^{111}P_{4,J}^{001}(x) = -\alpha\,x\beta\,\left(\alpha+\beta\right)\,\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-8\,\alpha^4-20\,\alpha^3\beta-8\,\alpha^2\beta^2-20\,\beta^3\alpha-8\,\beta^4\right] $$

$$ {}^{111}P_{4,J}^{010}(x) = \alpha\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+16\,\alpha^4+32\,\alpha^3\beta+4\,\alpha^2\beta^2-12\,\beta^3\alpha-8\,\beta^4\right] $$

$$ {}^{111}P_{4,J}^{011}(x) = 4\alpha^2\left[\beta^2\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\left(\alpha+\beta\right)^2x^2-2\,\alpha^2\left(2\,\alpha^2+7\,\alpha\,\beta+7\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,J}^{100}(x) = \alpha\,x\beta\,\left(\alpha+\beta\right)\,\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-8\,\alpha^4-12\,\alpha^3\beta+4\,\alpha^2\beta^2+32\,\beta^3\alpha+16\,\beta^4\right] $$

$$ {}^{111}P_{4,J}^{101}(x) = -4\,\beta^2\left[\alpha^2\left(2\,\alpha^2+2\,\alpha\,\beta-3\,\beta^2\right)\left(\alpha+\beta\right)^2x^2+2\,\beta^2\left(7\,\alpha^2+7\,\alpha\,\beta+2\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,J}^{110}(x) = -4\,\left(\alpha+\beta\right)^2\,\left[\alpha^2\beta^2\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)x^2-2\,\left(2\,\alpha^2-3\,\alpha\,\beta+2\,\beta^2\right)\left(\alpha+\beta\right)^2\right] $$

$$ {}^{111}P_{4,J}^{111}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+16\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2\right] $$

$$ {}^{111}P_{4,I}^{000}(x) = -8\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(\alpha+\beta\right)^2\,x^2 $$

$$ {}^{111}P_{4,I}^{001}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+8\,\alpha^4+20\,\alpha^3\beta+8\,\alpha^2\beta^2+20\,\beta^3\alpha+8\,\beta^4\right] $$

$$ {}^{111}P_{4,I}^{010}(x) = -\alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-16\,\alpha^4-32\,\alpha^3\beta-4\,\alpha^2\beta^2+12\,\beta^3\alpha+8\,\beta^4\right] $$

$$ {}^{111}P_{4,I}^{011}(x) = -4\,\alpha^2\left[\beta^2\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\left(\alpha+\beta\right)^2x^2+2\,\alpha^2\left(2\,\alpha^2+7\,\alpha\,\beta+7\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,I}^{100}(x) = -\alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+8\,\alpha^4+12\,\alpha^3\beta-4\,\alpha^2\beta^2-32\,\beta^3\alpha-16\,\beta^4\right] $$

$$ {}^{111}P_{4,I}^{101}(x) = 4\,\beta^2\left[\alpha^2\left(2\,\alpha^2+2\,\alpha\,\beta-3\,\beta^2\right)\left(\alpha+\beta\right)^2x^2-2\,\beta^2\left(7\,\alpha^2+7\,\alpha\,\beta+2\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,I}^{110}(x) = 4\,\left(\alpha+\beta\right)^2\,\left[\alpha^2\beta^2\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)x^2+2\,\left(2\,\alpha^2-3\,\alpha\,\beta+2\,\beta^2\right)\left(\alpha+\beta\right)^2\right] $$

$$ {}^{111}P_{4,I}^{111}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-16\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2\right] $$

$$ {}^{111}P_{4,K}^{000}(x) = 8\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(\alpha+\beta\right)^2\,x^2 $$

$$ {}^{111}P_{4,K}^{001}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+8\,\alpha^4+20\,\alpha^3\beta+8\,\alpha^2\beta^2+20\,\beta^3\alpha+8\,\beta^4\right] $$

$$ {}^{111}P_{4,K}^{010}(x) = -\alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-16\,\alpha^4-32\,\alpha^3\beta-4\,\alpha^2\beta^2+12\,\beta^3\alpha+8\,\beta^4\right] $$

$$ {}^{111}P_{4,K}^{011}(x) = 4\,\alpha^2\left[\beta^2\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\left(\alpha+\beta\right)^2x^2+2\,\alpha^2\left(2\,\alpha^2+7\,\alpha\,\beta+7\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,K}^{100}(x) = -\alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2+8\,\alpha^4+12\,\alpha^3\beta-4\,\alpha^2\beta^2-32\,\beta^3\alpha-16\,\beta^4\right] $$

$$ {}^{111}P_{4,K}^{101}(x) = -4\,\beta^2\left[\alpha^2\left(2\,\alpha^2+2\,\alpha\,\beta-3\,\beta^2\right)\left(\alpha+\beta\right)^2x^2-2\,\beta^2\left(7\,\alpha^2+7\,\alpha\,\beta+2\,\beta^2\right)\right] $$

$$ {}^{111}P_{4,K}^{110}(x) = -4\,\left(\alpha+\beta\right)^2\,\left[\alpha^2\beta^2\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)x^2+2\,\left(2\,\alpha^2-3\,\alpha\,\beta+2\,\beta^2\right)\left(\alpha+\beta\right)^2\right] $$

$$ {}^{111}P_{4,K}^{111}(x) = \alpha\,\beta\,\left(\alpha+\beta\right)\,x\left[5\,\alpha^2\beta^2\left(\alpha+\beta\right)^2x^2-16\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2\right] $$

$$ {}^{000}V_5 = \frac{x^2}{630\,\alpha^4\beta^4\left(\alpha+\beta\right)^4} $$

$$ {}^{000}P_{5,J}^{000}(x) = [35\,\alpha^4\beta^4\left(\alpha+\beta\right)^4x^2 + 96\,\alpha^2\beta^2\left(\alpha+\beta\right)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2]\,x^2 $$

$$ {}^{000}P_{5,J}^{001}(x) = 4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(\alpha+3\,\beta\right)\left(3\,\alpha+\beta\right)\left(\alpha+\beta\right)^2x^2 - $$

$$ -12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(2\,\beta^4+5\,\beta^3\alpha+2\,\alpha^2\beta^2+5\,\alpha^3\beta+2\,\alpha^4\right)] $$

$$ {}^{000}P_{5,J}^{010}(x) = 4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(2\,\alpha+3\,\beta\right)\left(2\,\alpha-\beta\right)\left(\alpha+\beta\right)^2x^2 - $$

$$ -12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(-2\,\beta^4-3\,\beta^3\alpha+\alpha^2\beta^2+8\,\alpha^3\beta+4\,\alpha^4\right)] $$

$$ {}^{000}P_{5,J}^{011}(x) = $$

$$ = \alpha^2\,[35\,\beta^4\alpha^2\left(\alpha+\beta\right)^4x^4 - 8\,\beta^2\left(18\,\alpha^4+41\,\alpha^3\beta+29\,\alpha^2\beta^2-24\,\beta^3\alpha-12\,\beta^4\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +96\,\alpha^2\left(7\,\beta^2+7\,\alpha\,\beta+2\,\alpha^2\right)\left(\alpha^2+\alpha\,\beta+\beta^2\right)] $$

$$ {}^{000}P_{5,J}^{100}(x) = -4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(3\,\alpha+2\,\beta\right)\left(\alpha-2\,\beta\right)\left(\alpha+\beta\right)^2x^2 - $$

$$ -12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(-4\,\beta^4-8\,\beta^3\alpha-\alpha^2\beta^2+3\,\alpha^3\beta+2\,\alpha^4\right)] $$

$$ {}^{000}P_{5,J}^{101}(x) = $$

$$ = \beta^2\,[35\,\beta^2\alpha^4\left(\alpha+\beta\right)^4x^4 + 8\,\alpha^2\left(12\,\alpha^4+24\,\alpha^3\beta-29\,\alpha^2\beta^2-41\,\beta^3\alpha-18\,\beta^4\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +96\,\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(2\,\beta^2+7\,\alpha\,\beta+7\,\alpha^2\right)] $$

$$ {}^{000}P_{5,J}^{110}(x) = $$

$$ = -\left(\alpha+\beta\right)^2\,[35\,\alpha^4\beta^4\left(\alpha+\beta\right)^2x^4 - 8\,\alpha^2\beta^2\left(18\,\alpha^4+31\,\alpha^3\beta+14\,\alpha^2\beta^2+31\,\beta^3\alpha+18\,\beta^4\right)x^2 + $$

$$ +96\,\left(2\,\beta^2-3\,\alpha\,\beta+2\,\alpha^2\right)\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(\alpha+\beta\right)^2] $$

$$ {}^{000}P_{5,J}^{111}(x) = 16\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(\alpha+\beta\right)^2x^2 - $$

$$ -12\,\alpha^6-36\,\alpha^5\beta-37\,\beta^2\alpha^4-14\,\alpha^3\beta^3-37\,\beta^4\alpha^2-36\,\beta^5\alpha-12\,\beta^6] $$

$$ {}^{000}P_{5,I}^{000}(x) = 35\,\alpha^4\beta^4\left(\alpha+\beta\right)^4x^2 - 96\,\alpha^2\beta^2\left(\alpha+\beta\right)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2\,x^2 $$

$$ {}^{000}P_{5,I}^{001}(x) = 4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(\alpha+3\,\beta\right)\left(3\,\alpha+\beta\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(2\,\beta^4+5\,\beta^3\alpha+2\,\alpha^2\beta^2+5\,\alpha^3\beta+2\,\alpha^4\right)] $$

$$ {}^{000}P_{5,I}^{010}(x) = 4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(2\,\alpha+3\,\beta\right)\left(2\,\alpha-\beta\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(-2\,\beta^4-3\,\beta^3\alpha+\alpha^2\beta^2+8\,\alpha^3\beta+4\,\alpha^4\right)] $$

$$ {}^{000}P_{5,I}^{011}(x) = -\alpha^2\,[35\,\beta^4\alpha^2\left(\alpha+\beta\right)^4x^4 + $$

$$ +8\,\beta^2\left(18\,\alpha^4+41\,\alpha^3\beta+29\,\alpha^2\beta^2-24\,\beta^3\alpha-12\,\beta^4\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +96\,\alpha^2\left(7\,\beta^2+7\,\alpha\,\beta+2\,\alpha^2\right)\left(\alpha^2+\alpha\,\beta+\beta^2\right)] $$

$$ {}^{000}P_{5,I}^{100}(x) = -4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(3\,\alpha+2\,\beta\right)\left(\alpha-2\,\beta\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +12\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(-4\,\beta^4-8\,\beta^3\alpha-\alpha^2\beta^2+3\,\alpha^3\beta+2\,\alpha^4\right)] $$

$$ {}^{000}P_{5,I}^{101}(x) = -\beta^2\,[35\,\beta^2\alpha^4\left(\alpha+\beta\right)^4x^4 - $$

$$ -8\,\alpha^2\left(12\,\alpha^4+24\,\alpha^3\beta-29\,\alpha^2\beta^2-41\,\beta^3\alpha-18\,\beta^4\right)\left(\alpha+\beta\right)^2x^2 + $$

$$ +96\,\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(2\,\beta^2+7\,\alpha\,\beta+7\,\alpha^2\right)] $$

$$ {}^{000}P_{5,I}^{110}(x) = \left(\alpha+\beta\right)^2\,[35\,\alpha^4\beta^4\left(\alpha+\beta\right)^2x^4 + $$

$$ +8\,\alpha^2\beta^2\left(18\,\alpha^4+31\,\alpha^3\beta+14\,\alpha^2\beta^2+31\,\beta^3\alpha+18\,\beta^4\right)x^2 + $$

$$+96\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(2\,\beta^2-3\,\alpha\,\beta+2\,\alpha^2\right)(\alpha+\beta)^2]$$

$$^{000}P_{5,I}^{111}(x)=-16\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+12\,\alpha^6+36\,\alpha^5\beta+37\,\beta^2\alpha^4+14\,\alpha^3\beta^3+37\,\beta^4\alpha^2+36\,\beta^5\alpha+12\,\beta^6]$$

$$^{000}P_{5,K}^{000}(x)=35\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-96\,\alpha^2\beta^2\,(\alpha+\beta)^2\left(\beta\,\alpha+\alpha^2+\beta^2\right)^2x^2$$

$$^{000}P_{5,K}^{001}(x)=-4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\beta^2\alpha^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\,\left(\beta\,\alpha+\alpha^2+\beta^2\right)\left(2\,\beta^4+5\,\alpha\,\beta^3+2\,\beta^2\alpha^2+5\,\beta\,\alpha^3+2\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{010}(x)=-4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\beta^2\alpha^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\,\left(\beta\,\alpha+\alpha^2+\beta^2\right)\left(-2\,\beta^4-3\,\alpha\,\beta^3+\beta^2\alpha^2+8\,\beta\,\alpha^3+4\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{011}(x)=-\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+8\,\beta^2\left(18\,\alpha^4+41\,\beta\,\alpha^3+29\,\beta^2\alpha^2-24\,\alpha\,\beta^3-12\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^2\left(7\,\beta^2+7\,\beta\,\alpha+2\,\alpha^2\right)\left(\beta\,\alpha+\alpha^2+\beta^2\right)]$$

$$^{000}P_{5,K}^{100}(x)=4\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\beta^2\alpha^2\,(3\,\alpha+2\,\beta)\,(\alpha-2\,\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\,\left(\beta\,\alpha+\alpha^2+\beta^2\right)\left(-4\,\beta^4-8\,\alpha\,\beta^3-\beta^2\alpha^2+3\,\beta\,\alpha^3+2\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{101}(x)=-\beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4-$$

$$-8\,\alpha^2\left(12\,\alpha^4+24\,\beta\,\alpha^3-29\,\beta^2\alpha^2-41\,\alpha\,\beta^3-18\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\beta^2\left(2\,\beta^2+7\,\beta\,\alpha+7\,\alpha^2\right)\left(\beta\,\alpha+\alpha^2+\beta^2\right)]$$

$$^{000}P_{5,K}^{110}(x)=$$

$$=(\alpha+\beta)^2\,\left[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+8\,\beta^2\alpha^2\left(18\,\alpha^4+31\,\beta\,\alpha^3+14\,\beta^2\alpha^2+31\,\alpha\,\beta^3+18\,\beta^4\right)x^2+\right.$$

$$+96\,\left(2\,\beta^2-3\,\beta\,\alpha+2\,\alpha^2\right)\left(\beta\,\alpha+\alpha^2+\beta^2\right)(\alpha+\beta)^2]$$

$$^{000}P_{5,K}^{111}(x)=16\,\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\beta^2\alpha^2\left(\beta\,\alpha+\alpha^2+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+12\,\alpha^6+36\,\alpha^5\beta+37\,\beta^2\alpha^4+14\,\beta^3\alpha^3+37\,\beta^4\alpha^2+36\,\beta^5\alpha+12\,\beta^6]$$

$$^{011}V_5=\frac{x^2}{630\alpha^4\beta^4(\alpha+\beta)^4}$$

$$^{011}P_{5,J}^{000}(x)=\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^4-276\,\alpha^3\beta-180\,\alpha^2\beta^2+192\,\beta^3\alpha+96\,\beta^4]$$

$$^{011}P_{5,J}^{001}(x)=-\alpha\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+23\,\alpha\,\beta-12\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-456\,\alpha^5\beta-624\,\alpha^4\beta^2-384\,\beta^3\alpha^3+384\,\beta^4\alpha^2+336\,\beta^5\alpha+96\,\beta^6]$$

$$^{011}P_{5,J}^{010}(x)=-\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2+47\,\alpha\,\beta+12\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-240\,\alpha^6-792\,\alpha^5\beta-912\,\alpha^4\beta^2-480\,\beta^3\alpha^3-144\,\beta^4\alpha^2-240\,\beta^5\alpha-96\,\beta^6]$$

$$^{011}P_{5,J}^{011}(x)=\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\beta^2\left(45\,\alpha^4+116\,\alpha^3\beta+140\,\alpha^2\beta^2+48\,\beta^3\alpha+24\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^2\left(5\,\alpha^4+24\,\alpha^3\beta+45\,\alpha^2\beta^2+42\,\beta^3\alpha+21\,\beta^4\right)]$$

$$^{011}P_{5,J}^{100}(x)=$$

$$=\alpha\,\beta\,\left(\alpha+\beta\right)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+16\,\alpha\,\beta+16\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-336\,\alpha^5\beta-288\,\alpha^4\beta^2-96\,\beta^3\alpha^3-528\,\beta^4\alpha^2-576\,\beta^5\alpha-192\,\beta^6]$$

$$
\begin{gathered}
{}^{011}P_{5,J}^{101}(x) = \beta^2 \left[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4 - \right.\\
-4\,\alpha^2\left(30\,\alpha^4+69\,\alpha^3\beta+40\,\alpha^2\beta^2+82\,\beta^3\alpha+36\,\beta^4\right)(\alpha+\beta)^2\,x^2+\\
\left.+48\,\beta^3\left(4\,\beta^3+21\,\alpha^3+30\,\alpha^2\beta+18\,\beta^2\alpha\right)\right]\\
{}^{011}P_{5,J}^{110}(x) =\\
= -(\alpha+\beta)^2\,\left[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+4\,\alpha^2\beta^2\left(45\,\alpha^4+91\,\alpha^3\beta-10\,\alpha^2\beta^2-62\,\beta^3\alpha-36\,\beta^4\right)x^2-\right.\\
\left.-48\,\left(5\,\alpha^3-6\,\alpha^2\beta+6\,\beta^2\alpha-4\,\beta^3\right)(\alpha+\beta)^3\right]\\
{}^{011}P_{5,J}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^2\,x^2-\right.\\
\left.-60\,\alpha^6-198\,\alpha^5\beta-218\,\alpha^4\beta^2+8\,\beta^3\alpha^3+124\,\beta^4\alpha^2+144\,\beta^5\alpha+48\,\beta^6\right]
\end{gathered}
$$

$$
\begin{gathered}
{}^{011}P_{5,I}^{000}(x) = -\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\left[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+\right.\\
\left.+120\,\alpha^4+276\,\alpha^3\beta+180\,\alpha^2\beta^2-192\,\beta^3\alpha-96\,\beta^4\right]\\
{}^{011}P_{5,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2+23\,\alpha\,\beta-12\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+120\,\alpha^6+456\,\alpha^5\beta+624\,\beta^2\alpha^4+384\,\beta^3\alpha^3-384\,\beta^4\alpha^2-336\,\beta^5\alpha-96\,\beta^6\right]\\
{}^{011}P_{5,I}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(20\,\alpha^2+47\,\alpha\,\beta+12\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+240\,\alpha^6+792\,\alpha^5\beta+912\,\beta^2\alpha^4+480\,\beta^3\alpha^3+144\,\beta^4\alpha^2+240\,\beta^5\alpha+96\,\beta^6\right]\\
{}^{011}P_{5,I}^{011}(x) = \alpha^2\left[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-\right.\\
-4\,\beta^2\left(45\,\alpha^4+116\,\alpha^3\beta+140\,\alpha^2\beta^2+48\,\beta^3\alpha+24\,\beta^4\right)(\alpha+\beta)^2\,x^2-\\
\left.-48\,\alpha^2\left(21\,\beta^4+45\,\alpha^2\beta^2+24\,\alpha^3\beta+42\,\beta^3\alpha+5\,\alpha^4\right)\right]\\
{}^{011}P_{5,I}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2+16\,\alpha\,\beta+16\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+120\,\alpha^6+336\,\alpha^5\beta+288\,\beta^2\alpha^4+96\,\beta^3\alpha^3+528\,\beta^4\alpha^2+576\,\beta^5\alpha+192\,\beta^6\right]\\
{}^{011}P_{5,I}^{101}(x) =\\
= \beta^2\left[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4+4\,\alpha^2\left(30\,\alpha^4+69\,\alpha^3\beta+40\,\alpha^2\beta^2+82\,\beta^3\alpha+36\,\beta^4\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+48\,\beta^3\left(21\,\alpha^3+30\,\alpha^2\beta+18\,\beta^2\alpha+4\,\beta^3\right)\right]\\
{}^{011}P_{5,I}^{110}(x) =\\
= -(\alpha+\beta)^2\,\left[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-4\,\alpha^2\beta^2\left(45\,\alpha^4+91\,\alpha^3\beta-10\,\alpha^2\beta^2-62\,\beta^3\alpha-36\,\beta^4\right)x^2-\right.\\
\left.-48\,\left(5\,\alpha^3-6\,\alpha^2\beta+6\,\beta^2\alpha-4\,\beta^3\right)(\alpha+\beta)^3\right]\\
{}^{011}P_{5,I}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+60\,\alpha^6+198\,\alpha^5\beta+218\,\beta^2\alpha^4-8\,\beta^3\alpha^3-124\,\beta^4\alpha^2-144\,\beta^5\alpha-48\,\beta^6\right]
\end{gathered}
$$

$$
\begin{gathered}
{}^{011}P_{5,K}^{000}(x) =\\
= -\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\left[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+120\,\alpha^4+276\,\alpha^3\beta+180\,\alpha^2\beta^2-192\,\beta^3\alpha-96\,\beta^4\right]\\
{}^{011}P_{5,K}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2+23\,\alpha\,\beta-12\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+120\,\alpha^6+456\,\alpha^5\beta+624\,\beta^2\alpha^4+384\,\beta^3\alpha^3-384\,\beta^4\alpha^2-336\,\beta^5\alpha-96\,\beta^6\right]\\
{}^{011}P_{5,K}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(20\,\alpha^2+47\,\alpha\,\beta+12\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.\\
\left.+240\,\alpha^6+792\,\alpha^5\beta+912\,\beta^2\alpha^4+480\,\beta^3\alpha^3+144\,\beta^4\alpha^2+240\,\beta^5\alpha+96\,\beta^6\right]\\
{}^{011}P_{5,K}^{011}(x) =\\
= \alpha^2\left[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-4\,\beta^2\left(45\,\alpha^4+116\,\alpha^3\beta+140\,\alpha^2\beta^2+48\,\beta^3\alpha+24\,\beta^4\right)(\alpha+\beta)^2\,x^2-\right.
\end{gathered}
$$

$$-48\,\alpha^2\left(21\,\beta^4+45\,\alpha^2\beta^2+24\,\alpha^3\beta+42\,\beta^3\alpha+5\,\alpha^4\right)]$$

$${}^{011}P_{5,K}^{100}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+16\,\alpha\,\beta+16\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+336\,\alpha^5\beta+288\,\beta^2\alpha^4+96\,\beta^3\alpha^3+528\,\beta^4\alpha^2+576\,\beta^5\alpha+192\,\beta^6]$$

$${}^{011}P_{5,K}^{101}(x)=\beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\alpha^2\left(30\,\alpha^4+69\,\alpha^3\beta+40\,\alpha^2\beta^2+82\,\beta^3\alpha+36\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+48\,\beta^3\left(21\,\alpha^3+30\,\alpha^2\beta+18\,\beta^2\alpha+4\,\beta^3\right)]$$

$${}^{011}P_{5,K}^{110}(x)-(\alpha+\beta)^2\,\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-$$

$$-4\,\alpha^2\beta^2\left(45\,\alpha^4+91\,\alpha^3\beta-10\,\alpha^2\beta^2-62\,\beta^3\alpha-36\,\beta^4\right)x^2-$$

$$-48\,\left(5\,\alpha^3-6\,\alpha^2\beta+6\,\beta^2\alpha-4\,\beta^3\right)(\alpha+\beta)^3]$$

$${}^{011}P_{5,K}^{111}(x)=4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+60\,\alpha^6+198\,\alpha^5\beta+218\,\beta^2\alpha^4-8\,\beta^3\alpha^3-124\,\beta^4\alpha^2-144\,\beta^5\alpha-48\,\beta^6]$$

$${}^{110}V_5=\frac{x^2}{630\alpha^4\beta^4(\alpha+\beta)^4}$$

$${}^{110}P_{5,J}^{000}(x)=-\alpha^2\beta^2\,(\alpha+\beta)^2\,\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^4-204\,\beta\,\alpha^3-72\,\alpha^2\beta^2-204\,\alpha\,\beta^3-120\,\beta^4]$$

$${}^{110}P_{5,J}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+14\,\alpha\,\beta+15\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-384\,\alpha^5\beta-408\,\alpha^4\beta^2-96\,\alpha^3\beta^3-408\,\beta^4\alpha^2-384\,\beta^5\alpha-120\,\beta^6]$$

$${}^{110}P_{5,J}^{010}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2-7\,\alpha\,\beta-15\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-240\,\alpha^6-648\,\alpha^5\beta-552\,\alpha^4\beta^2-48\,\alpha^3\beta^3+144\,\beta^4\alpha^2+264\,\beta^5\alpha+120\,\beta^6]$$

$${}^{110}P_{5,J}^{011}(x)=-\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\beta^2\left(45\,\alpha^4+89\,\beta\,\alpha^3-13\,\alpha^2\beta^2-51\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^3\left(21\,\beta^3+5\,\alpha^3+21\,\beta\,\alpha^2+33\,\alpha\,\beta^2\right)]$$

$${}^{110}P_{5,J}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+7\,\alpha\,\beta-20\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-264\,\alpha^5\beta-144\,\alpha^4\beta^2+48\,\alpha^3\beta^3+552\,\beta^4\alpha^2+648\,\beta^5\alpha+240\,\beta^6]$$

$${}^{110}P_{5,J}^{101}(x)=-\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4-$$

$$-4\,\alpha^2\left(30\,\alpha^4+51\,\beta\,\alpha^3+13\,\alpha^2\beta^2-89\,\alpha\,\beta^3-45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\beta^3\left(21\,\alpha^3+33\,\beta\,\alpha^2+21\,\alpha\,\beta^2+5\,\beta^3\right)]$$

$${}^{110}P_{5,J}^{110}(x)=(\alpha+\beta)^2\,\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+$$

$$+4\,\alpha^2\beta^2\left(45\,\alpha^4+64\,\beta\,\alpha^3+62\,\alpha^2\beta^2+64\,\alpha\,\beta^3+45\,\beta^4\right)x^2-$$

$$-48\,\left(5\,\alpha^4-4\,\beta\,\alpha^3+3\,\alpha^2\beta^2-4\,\alpha\,\beta^3+5\,\beta^4\right)(\alpha+\beta)^2]$$

$${}^{110}P_{5,J}^{111}(x)=4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-60\,\alpha^6-162\,\alpha^5\beta-128\,\alpha^4\beta^2-100\,\alpha^3\beta^3-128\,\beta^4\alpha^2-162\,\beta^5\alpha-60\,\beta^6]$$

$${}^{110}P_{5,I}^{000}(x)=\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^4+204\,\beta\,\alpha^3+72\,\alpha^2\beta^2+204\,\alpha\,\beta^3+120\,\beta^4]$$

$${}^{110}P_{5,I}^{001}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+14\,\alpha\,\beta+15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+384\,\alpha^5\beta+408\,\alpha^4\beta^2+96\,\alpha^3\beta^3+408\,\beta^4\alpha^2+384\,\beta^5\alpha+120\,\beta^6]$$

$${}^{110}P_{5,I}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2-7\,\alpha\,\beta-15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+240\,\alpha^6+648\,\alpha^5\beta+552\,\alpha^4\beta^2+48\,\alpha^3\beta^3-144\,\beta^4\alpha^2-264\,\beta^5\alpha-120\,\beta^6]$$

$${}^{110}P_{5,I}^{011}(x)=-\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-$$

$$-4\,\beta^2\left(45\,\alpha^4+89\,\beta\,\alpha^3-13\,\alpha^2\beta^2-51\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^3\left(21\,\beta^3+5\,\alpha^3+21\,\beta\,\alpha^2+33\,\alpha\,\beta^2\right)]$$

$${}^{110}P_{5,I}^{100}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+7\,\alpha\,\beta-20\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+264\,\alpha^5\beta+144\,\alpha^4\beta^2-48\,\alpha^3\beta^3-552\,\beta^4\alpha^2-648\,\beta^5\alpha-240\,\beta^6]$$

$${}^{110}P_{5,I}^{101}(x)=-\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\alpha^2\left(30\,\alpha^4+51\,\beta\,\alpha^3+13\,\alpha^2\beta^2-89\,\alpha\,\beta^3-45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\beta^3\left(21\,\alpha^3+33\,\beta\,\alpha^2+21\,\alpha\,\beta^2+5\,\beta^3\right)]$$

$${}^{110}P_{5,I}^{110}(x)=(\alpha+\beta)^2\,\left[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-\right.$$

$$-4\,\alpha^2\beta^2\left(45\,\alpha^4+64\,\beta\,\alpha^3+62\,\alpha^2\beta^2+64\,\alpha\,\beta^3+45\,\beta^4\right)x^2-$$

$$-48\,\left(5\,\alpha^4-4\,\beta\,\alpha^3+3\,\alpha^2\beta^2-4\,\alpha\,\beta^3+5\,\beta^4\right)(\alpha+\beta)^2]$$

$${}^{110}P_{5,I}^{111}(x)=4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+60\,\alpha^6+162\,\alpha^5\beta+128\,\alpha^4\beta^2+100\,\alpha^3\beta^3+128\,\beta^4\alpha^2+162\,\beta^5\alpha+60\,\beta^6]$$

$${}^{110}P_{5,K}^{000}(x)=\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^4+204\,\beta\,\alpha^3+72\,\alpha^2\beta^2+204\,\alpha\,\beta^3+120\,\beta^4]$$

$${}^{110}P_{5,K}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+14\,\alpha\,\beta+15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+384\,\alpha^5\beta+408\,\alpha^4\beta^2+96\,\alpha^3\beta^3+408\,\beta^4\alpha^2+384\,\beta^5\alpha+120\,\beta^6]$$

$${}^{110}P_{5,K}^{010}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2-7\,\alpha\,\beta-15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+240\,\alpha^6+648\,\alpha^5\beta+552\,\alpha^4\beta^2+48\,\alpha^3\beta^3-144\,\beta^4\alpha^2-264\,\beta^5\alpha-120\,\beta^6]$$

$${}^{110}P_{5,K}^{011}(x)=-\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-$$

$$-4\,\beta^2\left(45\,\alpha^4+89\,\beta\,\alpha^3-13\,\alpha^2\beta^2-51\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^3\left(21\,\beta^3+5\,\alpha^3+21\,\beta\,\alpha^2+33\,\alpha\,\beta^2\right)]$$

$${}^{110}P_{5,K}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+7\,\alpha\,\beta-20\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+264\,\alpha^5\beta+144\,\alpha^4\beta^2-48\,\alpha^3\beta^3-552\,\beta^4\alpha^2-648\,\beta^5\alpha-240\,\beta^6]$$

$${}^{110}P_{5,K}^{101}(x)=-\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\alpha^2\left(30\,\alpha^4+51\,\beta\,\alpha^3+13\,\alpha^2\beta^2-89\,\alpha\,\beta^3-45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\beta^3\left(21\,\alpha^3+33\,\beta\,\alpha^2+21\,\alpha\,\beta^2+5\,\beta^3\right)]$$

$${}^{110}P_{5,K}^{110}(x)=(\alpha+\beta)^2\,\left[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-\right.$$

$$-4\,\alpha^2\beta^2\left(45\,\alpha^4+64\,\beta\,\alpha^3+62\,\alpha^2\beta^2+64\,\alpha\,\beta^3+45\,\beta^4\right)x^2-$$

$$-48\,\left(5\,\alpha^4-4\,\beta\,\alpha^3+3\,\alpha^2\beta^2-4\,\alpha\,\beta^3+5\,\beta^4\right)(\alpha+\beta)^2]$$

$${}^{110}P_{5,K}^{111}(x)=-4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+60\,\alpha^6+162\,\alpha^5\beta+128\,\alpha^4\beta^2+100\,\alpha^3\beta^3+128\,\beta^4\alpha^2+162\,\beta^5\alpha+60\,\beta^6]$$

$$^{001}V_6 = \frac{x^2}{6930\,\alpha^5\,\beta^5\,(\alpha+\beta)^5}$$

$$^{001}P_{6,J}^{000}(x) = -4\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2\,[35\,\alpha^2\beta^2\left(5\,\alpha^2+16\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$
$$-600\,\alpha^6-1932\,\alpha^5\beta-2172\,\alpha^4\beta^2-2400\,\alpha^3\beta^3-2172\,\beta^4\alpha^2-1932\,\beta^5\alpha-600\,\beta^6]$$

$$^{001}P_{6,J}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\alpha^2\beta^2\left(75\,\alpha^4+184\,\beta\,\alpha^3+338\,\alpha^2\beta^2+184\,\alpha\,\beta^3+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-11328\,\beta\,\alpha^7-20880\,\beta^2\alpha^6-18720\,\alpha^5\beta^3-19296\,\alpha^4\beta^4-18720\,\alpha^3\beta^5-20880\,\alpha^2\beta^6-$$
$$-11328\,\alpha\,\beta^7-2400\,\beta^8]$$

$$^{001}P_{6,J}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(100\,\alpha^4+222\,\beta\,\alpha^3+138\,\alpha^2\beta^2-149\,\alpha\,\beta^3-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32832\,\beta^2\alpha^6+25488\,\alpha^5\beta^3+9648\,\alpha^4\beta^4-6768\,\alpha^3\beta^5-11952\,\alpha^2\beta^6-$$
$$-8928\,\alpha\,\beta^7-2400\,\beta^8]$$

$$^{001}P_{6,J}^{011}(x) = \alpha^2\,[175\,\beta^4\alpha^2\left(5\,\alpha^2+7\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$
$$-16\,\beta^2\left(225\,\alpha^6+787\,\alpha^5\beta+972\,\alpha^4\beta^2+495\,\alpha^3\beta^3-518\,\beta^4\alpha^2-483\,\beta^5\alpha-150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^3\left(671\,\beta^2\alpha^3+286\,\beta\,\alpha^4+825\,\alpha^2\beta^3+561\,\beta^4\alpha+50\,\alpha^5+231\,\beta^5\right)]$$

$$^{001}P_{6,J}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\alpha^2\beta^2\left(75\,\alpha^4+149\,\beta\,\alpha^3-138\,\alpha^2\beta^2-222\,\alpha\,\beta^3-100\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-8928\,\beta\,\alpha^7-11952\,\beta^2\alpha^6-6768\,\alpha^5\beta^3+9648\,\alpha^4\beta^4+25488\,\alpha^3\beta^5+32832\,\alpha^2\beta^6+$$
$$+20256\,\alpha\,\beta^7+4800\,\beta^8]$$

$$^{001}P_{6,J}^{101}(x) = -\beta^2\,[175\,\alpha^4\beta^2\left(4\,\alpha^2-7\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$
$$-16\,\alpha^2\left(150\,\alpha^6+483\,\alpha^5\beta+518\,\alpha^4\beta^2-495\,\alpha^3\beta^3-972\,\beta^4\alpha^2-787\,\beta^5\alpha-225\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$
$$-96\,\beta^3\left(825\,\beta^2\alpha^3+671\,\alpha^2\beta^3+231\,\alpha^5+561\,\beta\,\alpha^4+286\,\beta^4\alpha+50\,\beta^5\right)]$$

$$^{001}P_{6,J}^{110}(x) = -\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(25\,\alpha^2+26\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^2\,x^4-$$
$$-16\,\alpha^2\beta^2\left(225\,\alpha^6+662\,\alpha^5\beta+632\,\alpha^4\beta^2+210\,\alpha^3\beta^3+632\,\beta^4\alpha^2+662\,\beta^5\alpha+225\,\beta^6\right)x^2+$$
$$+96\;\left(50\,\alpha^4-64\,\beta\,\alpha^3+69\,\alpha^2\beta^2-64\,\alpha\,\beta^3+50\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{001}P_{6,J}^{111}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-40\,\alpha^2\beta^2\left(50\,\alpha^4+111\,\beta\,\alpha^3+62\,\alpha^2\beta^2+111\,\alpha\,\beta^3+50\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32032\,\beta^2\alpha^6+22912\,\alpha^5\beta^3+6912\,\alpha^4\beta^4+22912\,\alpha^3\beta^5+32032\,\alpha^2\beta^6+$$
$$+20256\,\alpha\,\beta^7+4800\,\beta^8]$$

$$^{001}P_{6,I}^{000}(x) = 4\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2\,[35\,\alpha^2\beta^2\left(5\,\alpha^2+16\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+600\,\alpha^6+1932\,\alpha^5\beta+2172\,\alpha^4\beta^2+2400\,\beta^3\alpha^3+2172\,\beta^4\alpha^2+1932\,\alpha\,\beta^5+600\,\beta^6]$$

$$^{001}P_{6,I}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(75\,\alpha^4+184\,\alpha^3\beta+338\,\alpha^2\beta^2+184\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-11328\,\beta\,\alpha^7-20880\,\beta^2\alpha^6-18720\,\alpha^5\beta^3-19296\,\alpha^4\beta^4-18720\,\beta^5\alpha^3-20880\,\beta^6\alpha^2-$$
$$-11328\,\beta^7\alpha-2400\,\beta^8]$$

$$^{001}P_{6,I}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(100\,\alpha^4+222\,\alpha^3\beta+138\,\alpha^2\beta^2-149\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32832\,\beta^2\alpha^6+25488\,\alpha^5\beta^3+9648\,\alpha^4\beta^4-6768\,\beta^5\alpha^3-11952\,\beta^6\alpha^2-$$
$$-8928\,\beta^7\alpha-2400\,\beta^8]$$
$${}^{001}P_{6,I}^{011}(x)=\alpha^2\,[175\,\beta^4\alpha^2\left(5\,\alpha^2+7\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+16\,\beta^2\left(225\,\alpha^6+787\,\alpha^5\beta+972\,\alpha^4\beta^2+495\,\beta^3\alpha^3-518\,\beta^4\alpha^2-483\,\alpha\,\beta^5-150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^3\left(825\,\beta^3\alpha^2+231\,\beta^5+286\,\alpha^4\beta+561\,\beta^4\alpha+671\,\alpha^3\beta^2+50\,\alpha^5\right)]$$
$${}^{001}P_{6,I}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(75\,\alpha^4+149\,\alpha^3\beta-138\,\alpha^2\beta^2-222\,\beta^3\alpha-100\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-8928\,\beta\,\alpha^7-11952\,\beta^2\alpha^6-6768\,\alpha^5\beta^3+9648\,\alpha^4\beta^4+25488\,\beta^5\alpha^3+32832\,\beta^6\alpha^2+$$
$$+20256\,\beta^7\alpha+4800\,\beta^8]$$
$${}^{001}P_{6,I}^{101}(x)=-\beta^2\,[175\,\alpha^4\beta^2\left(4\,\alpha^2-7\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+16\,\alpha^2\left(150\,\alpha^6+483\,\alpha^5\beta+518\,\alpha^4\beta^2-495\,\beta^3\alpha^3-972\,\beta^4\alpha^2-787\,\alpha\,\beta^5-225\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$
$$-96\,\beta^3\left(825\,\alpha^3\beta^2+231\,\alpha^5+286\,\beta^4\alpha+561\,\alpha^4\beta+671\,\beta^3\alpha^2+50\,\beta^5\right)]$$
$${}^{001}P_{6,I}^{110}(x)=-\left(\alpha+\beta\right)^2\,[35\,\alpha^4\beta^4\left(25\,\alpha^2+26\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$
$$+16\,\alpha^2\beta^2\left(225\,\alpha^6+662\,\alpha^5\beta+632\,\alpha^4\beta^2+210\,\beta^3\alpha^3+632\,\beta^4\alpha^2+662\,\alpha\,\beta^5+225\,\beta^6\right)x^2+$$
$$+96\,\left(50\,\alpha^4-64\,\alpha^3\beta+69\,\alpha^2\beta^2-64\,\beta^3\alpha+50\,\beta^4\right)(\alpha+\beta)^4]$$
$${}^{001}P_{6,I}^{111}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+40\,\alpha^2\beta^2\left(50\,\alpha^4+111\,\alpha^3\beta+62\,\alpha^2\beta^2+111\,\beta^3\alpha+50\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32032\,\beta^2\alpha^6+22912\,\alpha^5\beta^3+6912\,\alpha^4\beta^4+22912\,\beta^5\alpha^3+32032\,\beta^6\alpha^2+$$
$$+20256\,\beta^7\alpha+4800\,\beta^8]$$

$${}^{001}P_{6,K}^{000}(x)=-4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\left(5\,\alpha^2+16\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+600\,\alpha^6+1932\,\alpha^5\beta+2172\,\alpha^4\beta^2+2400\,\beta^3\alpha^3+2172\,\beta^4\alpha^2+1932\,\alpha\,\beta^5+600\,\beta^6]$$
$${}^{001}P_{6,K}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(75\,\alpha^4+184\,\alpha^3\beta+338\,\alpha^2\beta^2+184\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-11328\,\beta\,\alpha^7-20880\,\beta^2\alpha^6-18720\,\alpha^5\beta^3-19296\,\alpha^4\beta^4-18720\,\beta^5\alpha^3-20880\,\beta^6\alpha^2-$$
$$-11328\,\beta^7\alpha-2400\,\beta^8]$$
$${}^{001}P_{6,K}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\alpha^2\beta^2\left(100\,\alpha^4+222\,\alpha^3\beta+138\,\alpha^2\beta^2-149\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32832\,\beta^2\alpha^6+25488\,\alpha^5\beta^3+9648\,\alpha^4\beta^4-6768\,\beta^5\alpha^3-11952\,\beta^6\alpha^2-$$
$$-8928\,\beta^7\alpha-2400\,\beta^8]$$
$${}^{001}P_{6,K}^{011}(x)=-\alpha^2\,[175\,\beta^4\alpha^2\left(5\,\alpha^2+7\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+16\,\beta^2\left(225\,\alpha^6+787\,\alpha^5\beta+972\,\alpha^4\beta^2+495\,\beta^3\alpha^3-518\,\beta^4\alpha^2-483\,\alpha\,\beta^5-150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^3\left(825\,\beta^3\alpha^2+231\,\beta^5+286\,\alpha^4\beta+561\,\beta^4\alpha+671\,\alpha^3\beta^2+50\,\alpha^5\right)]$$
$${}^{001}P_{6,K}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(75\,\alpha^4+149\,\alpha^3\beta-138\,\alpha^2\beta^2-222\,\beta^3\alpha-100\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-2400\,\alpha^8-8928\,\beta\,\alpha^7-11952\,\beta^2\alpha^6-6768\,\alpha^5\beta^3+9648\,\alpha^4\beta^4+25488\,\beta^5\alpha^3+32832\,\beta^6\alpha^2+$$

$$+20256\,\beta^7\alpha+4800\,\beta^8$$

$$^{001}P^{101}_{6,K}(x)=\beta^2\,[175\,\alpha^4\beta^2\left(4\,\alpha^2-7\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+16\,\alpha^2\left(150\,\alpha^6+483\,\alpha^5\beta+518\,\alpha^4\beta^2-495\,\beta^3\alpha^3-972\,\beta^4\alpha^2-787\,\alpha\,\beta^5-225\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^3\left(825\,\alpha^3\beta^2+231\,\alpha^5+286\,\beta^4\alpha+561\,\alpha^4\beta+671\,\beta^3\alpha^2+50\,\beta^5\right)]$$

$$^{001}P^{110}_{6,K}(x)=(\alpha+\beta)^2\;[35\,\alpha^4\beta^4\left(25\,\alpha^2+26\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$

$$+16\,\alpha^2\beta^2\left(225\,\alpha^6+662\,\alpha^5\beta+632\,\alpha^4\beta^2+210\,\beta^3\alpha^3+632\,\beta^4\alpha^2+662\,\alpha\,\beta^5+225\,\beta^6\right)x^2+$$

$$+96\;\left(50\,\alpha^4-64\,\alpha^3\beta+69\,\alpha^2\beta^2-64\,\beta^3\alpha+50\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{001}P^{111}_{6,K}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+40\,\alpha^2\beta^2\left(50\,\alpha^4+111\,\alpha^3\beta+62\,\alpha^2\beta^2+111\,\beta^3\alpha+50\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32032\,\beta^2\alpha^6+22912\,\alpha^5\beta^3+6912\,\alpha^4\beta^4+22912\,\beta^5\alpha^3+32032\,\beta^6\alpha^2+$$

$$+20256\,\beta^7\alpha+4800\,\beta^8]$$

$$^{010}V_6=\frac{x^2}{6930\,\alpha^5\beta^5(\alpha+\beta)^5}$$

$$^{010}P^{000}_{6,J}(x)=-4\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2\,[35\,\alpha^2\beta^2\left(6\,\alpha^2+6\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-720\,\alpha^6-2160\,\beta\,\alpha^5-2316\,\beta^2\alpha^4-1032\,\beta^3\alpha^3+1512\,\beta^4\alpha^2+1668\,\beta^5\alpha+600\,\beta^6]$$

$$^{010}P^{001}_{6,J}(x)=-\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\alpha^2\beta^2\left(90\,\alpha^4+201\,\alpha^3\beta+135\,\alpha^2\beta^2-151\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8+12960\,\alpha^7\beta+22944\,\alpha^6\beta^2+19824\,\alpha^5\beta^3+8688\,\alpha^4\beta^4-11856\,\alpha^3\beta^5-16656\,\alpha^2\beta^6-$$

$$-10272\,\beta^7\alpha-2400\,\beta^8]$$

$$^{010}P^{010}_{6,J}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(120\,\alpha^4+240\,\alpha^3\beta+236\,\alpha^2\beta^2+116\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-5760\,\alpha^8-23040\,\alpha^7\beta-35808\,\alpha^6\beta^2-26784\,\alpha^5\beta^3-10416\,\alpha^4\beta^4-3072\,\alpha^3\beta^5-$$

$$-8784\,\alpha^2\beta^6-7872\,\beta^7\alpha-2400\,\beta^8]$$

$$^{010}P^{011}_{6,J}(x)=\alpha^2\,[175\,\beta^4\alpha^2\left(6\,\alpha^2+15\,\alpha\,\beta+4\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$

$$-16\,\beta^2\left(270\,\alpha^6+885\,\beta\,\alpha^5+1041\,\beta^2\alpha^4+737\,\beta^3\alpha^3+353\,\beta^4\alpha^2+417\,\beta^5\alpha+150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^3\left(60\,\alpha^5+748\,\alpha^3\beta^2+330\,\alpha^4\beta+594\,\alpha\,\beta^4+891\,\beta^3\alpha^2+231\,\beta^5\right)]$$

$$^{010}P^{100}_{6,J}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\alpha^2\beta^2\left(90\,\alpha^4+159\,\alpha^3\beta+72\,\alpha^2\beta^2+178\,\beta^3\alpha+100\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8+10080\,\alpha^7\beta+12864\,\alpha^6\beta^2+6960\,\alpha^5\beta^3+1728\,\alpha^4\beta^4+14928\,\alpha^3\beta^5+25440\,\alpha^2\beta^6+$$

$$+18144\,\beta^7\alpha+4800\,\beta^8]$$

$$^{010}P^{101}_{6,J}(x)=-\beta^2\,[35\,\beta^2\alpha^4\left(24\,\alpha^2+24\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$

$$-16\,\alpha^2\left(180\,\alpha^6+540\,\beta\,\alpha^5+549\,\beta^2\alpha^4+198\,\beta^3\alpha^3+697\,\beta^4\alpha^2+688\,\beta^5\alpha+225\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\beta^4\left(561\,\alpha^2\beta^2+264\,\beta^3\alpha+297\,\alpha^4+594\,\alpha^3\beta+50\,\beta^4\right)]$$

$$^{010}P^{110}_{6,J}(x)=-\,(\alpha+\beta)^2\;[175\,\alpha^4\beta^4\left(6\,\alpha^2-3\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^4-$$

$$-16\,\alpha^2\beta^2\left(270\,\alpha^6+735\,\beta\,\alpha^5+666\,\beta^2\alpha^4-23\,\beta^3\alpha^3-412\,\beta^4\alpha^2-563\,\beta^5\alpha-225\,\beta^6\right)x^2+$$

$$+96\;\left(60\,\alpha^5-30\,\alpha^4\beta+28\,\alpha^3\beta^2-27\,\beta^3\alpha^2+36\,\alpha\,\beta^4-50\,\beta^5\right)(\alpha+\beta)^3]$$

$$ {}^{010}P_{6,J}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 + $$
$$ +40\,\alpha^2\beta^2\left(60\,\alpha^4+120\,\alpha^3\beta-29\,\alpha^2\beta^2-89\,\beta^3\alpha-50\,\beta^4\right)(\alpha+\beta)^2\,x^2- $$
$$ -5760\,\alpha^8-23040\,\alpha^7\beta-34848\,\alpha^6\beta^2-23904\,\alpha^5\beta^3-128\,\alpha^4\beta^4+12704\,\alpha^3\beta^5+24640\,\alpha^2\beta^6+ $$
$$ +18144\,\beta^7\alpha+4800\,\beta^8] $$

$$ {}^{010}P_{6,I}^{000}(x) = 4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\left(6\,\alpha^2+6\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^2+ $$
$$ +720\,\alpha^6+2160\,\alpha^5\beta+2316\,\alpha^4\beta^2+1032\,\beta^3\alpha^3-1512\,\beta^4\alpha^2-1668\,\beta^5\alpha-600\,\beta^6] $$
$$ {}^{010}P_{6,I}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+ $$
$$ +20\,\alpha^2\beta^2\left(90\,\alpha^4+201\,\alpha^3\beta+135\,\alpha^2\beta^2-151\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+ $$
$$ +2880\,\alpha^8+12960\,\alpha^7\beta+22944\,\beta^2\alpha^6+19824\,\alpha^5\beta^3+8688\,\alpha^4\beta^4-11856\,\beta^5\alpha^3-16656\,\beta^6\alpha^2- $$
$$ -10272\,\beta^7\alpha-2400\,\beta^8] $$
$$ {}^{010}P_{6,I}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4- $$
$$ -20\,\alpha^2\beta^2\left(120\,\alpha^4+240\,\alpha^3\beta+236\,\alpha^2\beta^2+116\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2- $$
$$ -5760\,\alpha^8-23040\,\alpha^7\beta-35808\,\beta^2\alpha^6-26784\,\alpha^5\beta^3-10416\,\alpha^4\beta^4-3072\,\beta^5\alpha^3-8784\,\beta^6\alpha^2- $$
$$ -7872\,\beta^7\alpha-2400\,\beta^8] $$
$$ {}^{010}P_{6,I}^{011}(x) = \alpha^2\,[175\,\beta^4\alpha^2\left(6\,\alpha^2+15\,\alpha\,\beta+4\,\beta^2\right)(\alpha+\beta)^4\,x^4+ $$
$$ +16\,\beta^2\left(270\,\alpha^6+885\,\alpha^5\beta+1041\,\alpha^4\beta^2+737\,\beta^3\alpha^3+353\,\beta^4\alpha^2+417\,\beta^5\alpha+150\,\beta^6\right)(\alpha+\beta)^2\,x^2+ $$
$$ +96\,\alpha^3\left(594\,\alpha\,\beta^4+330\,\alpha^4\beta+891\,\beta^3\alpha^2+748\,\alpha^3\beta^2+231\,\beta^5+60\,\alpha^5\right)] $$
$$ {}^{010}P_{6,I}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+ $$
$$ +20\,\alpha^2\beta^2\left(90\,\alpha^4+159\,\alpha^3\beta+72\,\alpha^2\beta^2+178\,\beta^3\alpha+100\,\beta^4\right)(\alpha+\beta)^2\,x^2+ $$
$$ +2880\,\alpha^8+10080\,\alpha^7\beta+12864\,\beta^2\alpha^6+6960\,\alpha^5\beta^3+1728\,\alpha^4\beta^4+14928\,\beta^5\alpha^3+25440\,\beta^6\alpha^2+ $$
$$ +18144\,\beta^7\alpha+4800\,\beta^8] $$
$$ {}^{010}P_{6,I}^{101}(x) = -\beta^2\,[35\,\alpha^4\beta^2\left(24\,\alpha^2+24\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^4\,x^4+ $$
$$ +16\,\alpha^2\left(180\,\alpha^6+540\,\alpha^5\beta+549\,\alpha^4\beta^2+198\,\beta^3\alpha^3+697\,\beta^4\alpha^2+688\,\beta^5\alpha+225\,\beta^6\right)(\alpha+\beta)^2\,x^2+ $$
$$ +96\,\beta^4\left(561\,\alpha^2\beta^2+264\,\beta^3\alpha+297\,\alpha^4+594\,\alpha^3\beta+50\,\beta^4\right)] $$
$$ {}^{010}P_{6,I}^{110}(x) = -(\alpha+\beta)^2\,[175\,\alpha^4\beta^4\left(6\,\alpha^2-3\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^4+ $$
$$ +16\,\alpha^2\beta^2\left(270\,\alpha^6+735\,\alpha^5\beta+666\,\alpha^4\beta^2-23\,\beta^3\alpha^3-412\,\beta^4\alpha^2-563\,\beta^5\alpha-225\,\beta^6\right)x^2+ $$
$$ +96\,\left(60\,\alpha^5-30\,\alpha^4\beta+28\,\alpha^3\beta^2-27\,\beta^3\alpha^2+36\,\alpha\,\beta^4-50\,\beta^5\right)(\alpha+\beta)^3] $$
$$ {}^{010}P_{6,I}^{111}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4- $$
$$ -40\,\alpha^2\beta^2\left(60\,\alpha^4+120\,\alpha^3\beta-29\,\alpha^2\beta^2-89\,\beta^3\alpha-50\,\beta^4\right)(\alpha+\beta)^2\,x^2- $$
$$ -5760\,\alpha^8-23040\,\alpha^7\beta-34848\,\beta^2\alpha^6-23904\,\alpha^5\beta^3-128\,\alpha^4\beta^4+12704\,\beta^5\alpha^3+24640\,\beta^6\alpha^2+ $$
$$ +18144\,\beta^7\alpha+4800\,\beta^8] $$

$$ {}^{010}P_{6,K}^{000}(x) = -4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\left(6\,\alpha^2+6\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^2+ $$
$$ +720\,\alpha^6+2160\,\alpha^5\beta+2316\,\alpha^4\beta^2+1032\,\beta^3\alpha^3-1512\,\beta^4\alpha^2-1668\,\beta^5\alpha-600\,\beta^6] $$
$$ {}^{010}P_{6,K}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+ $$
$$ +20\,\alpha^2\beta^2\left(90\,\alpha^4+201\,\alpha^3\beta+135\,\alpha^2\beta^2-151\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+ $$

$$+2880\,\alpha^8+12960\,\alpha^7\beta+22944\,\beta^2\alpha^6+19824\,\alpha^5\beta^3+8688\,\alpha^4\beta^4-11856\,\beta^5\alpha^3-16656\,\beta^6\alpha^2-$$
$$-10272\,\beta^7\alpha-2400\,\beta^8$$
$${}^{010}P_{6,K}^{010}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\alpha^2\beta^2\left(120\,\alpha^4+240\,\alpha^3\beta+236\,\alpha^2\beta^2+116\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-5760\,\alpha^8-23040\,\alpha^7\beta-35808\,\beta^2\alpha^6-26784\,\alpha^5\beta^3-10416\,\alpha^4\beta^4-3072\,\beta^5\alpha^3-8784\,\beta^6\alpha^2-$$
$$-7872\,\beta^7\alpha-2400\,\beta^8]$$
$${}^{010}P_{6,K}^{011}(x)=-\alpha^2\,[175\,\beta^4\alpha^2\left(6\,\alpha^2+15\,\alpha\,\beta+4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+16\,\beta^2\left(270\,\alpha^6+885\,\alpha^5\beta+1041\,\alpha^4\beta^2+737\,\beta^3\alpha^3+353\,\beta^4\alpha^2+417\,\beta^5\alpha+150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^3\left(594\,\alpha\,\beta^4+330\,\alpha^4\beta+891\,\beta^3\alpha^2+748\,\alpha^3\beta^2+231\,\beta^5+60\,\alpha^5\right)]$$
$${}^{010}P_{6,K}^{100}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\alpha^2\beta^2\left(90\,\alpha^4+159\,\alpha^3\beta+72\,\alpha^2\beta^2+178\,\beta^3\alpha+100\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+2880\,\alpha^8+10080\,\alpha^7\beta+12864\,\beta^2\alpha^6+6960\,\alpha^5\beta^3+1728\,\alpha^4\beta^4+14928\,\beta^5\alpha^3+25440\,\beta^6\alpha^2+$$
$$+18144\,\beta^7\alpha+4800\,\beta^8]$$
$${}^{010}P_{6,K}^{101}(x)=\beta^2\,[35\,\alpha^4\beta^2\left(24\,\alpha^2+24\,\alpha\,\beta+25\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+16\,\alpha^2\left(180\,\alpha^6+540\,\alpha^5\beta+549\,\alpha^4\beta^2+198\,\beta^3\alpha^3+697\,\beta^4\alpha^2+688\,\beta^5\alpha+225\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\beta^4\left(561\,\alpha^2\beta^2+264\,\beta^3\alpha+297\,\alpha^4+594\,\alpha^3\beta+50\,\beta^4\right)]$$
$${}^{010}P_{6,K}^{110}(x)=(\alpha+\beta)^2\;[175\,\alpha^4\beta^4\left(6\,\alpha^2-3\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$
$$+16\,\alpha^2\beta^2\left(270\,\alpha^6+735\,\alpha^5\beta+666\,\alpha^4\beta^2-23\,\beta^3\alpha^3-412\,\beta^4\alpha^2-563\,\beta^5\alpha-225\,\beta^6\right)x^2+$$
$$+96\;\left(60\,\alpha^5-30\,\alpha^4\beta+28\,\alpha^3\beta^2-27\,\beta^3\alpha^2+36\,\alpha\,\beta^4-50\,\beta^5\right)(\alpha+\beta)^3]$$
$${}^{010}P_{6,K}^{111}(x)=-\alpha\,\beta\;(\alpha+\beta)\;x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-40\,\alpha^2\beta^2\left(60\,\alpha^4+120\,\alpha^3\beta-29\,\alpha^2\beta^2-89\,\beta^3\alpha-50\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-5760\,\alpha^8-23040\,\alpha^7\beta-34848\,\beta^2\alpha^6-23904\,\alpha^5\beta^3-128\,\alpha^4\beta^4+12704\,\beta^5\alpha^3+24640\,\beta^6\alpha^2+$$
$$+18144\,\beta^7\alpha+4800\,\beta^8$$

$${}^{111}V_6=\frac{x^2}{2310\,\alpha^5\beta^5(\alpha+\beta)^5}$$
$${}^{111}P_{6,I}^{000}(x)=-8\,\alpha^2\beta^2\,(\alpha+\beta)^2\;x^2\,[35\,\alpha^2\beta^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2-$$
$$-120\,\alpha^6-360\,\alpha^5\beta-342\,\alpha^4\beta^2-84\,\beta^3\alpha^3-342\,\beta^4\alpha^2-360\,\beta^5\alpha-120\,\beta^6$$
$${}^{111}P_{6,J}^{001}(x)=-\alpha\,\beta\;(\alpha+\beta)\;x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\beta^2\alpha^2\left(30\,\alpha^4+67\,\alpha^3\beta+34\,\beta^2\alpha^2+67\,\alpha\,\beta^3+30\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+960\,\alpha^8+4320\,\alpha^7\beta+7296\,\alpha^6\beta^2+5376\,\alpha^5\beta^3+960\,\alpha^4\beta^4+5376\,\beta^5\alpha^3+7296\,\beta^6\alpha^2+$$
$$+4320\,\beta^7\alpha+960\,\beta^8]$$
$${}^{111}P_{6,J}^{010}(x)=\alpha\,\beta\;(\alpha+\beta)\;x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\beta^2\alpha^2\left(40\,\alpha^4+80\,\alpha^3\beta-13\,\beta^2\alpha^2-53\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-1920\,\alpha^8-7680\,\alpha^7\beta-11232\,\alpha^6\beta^2-6816\,\alpha^5\beta^3-480\,\alpha^4\beta^4+1440\,\beta^5\alpha^3+3936\,\beta^6\alpha^2+$$
$$+3360\,\beta^7\alpha+960\,\beta^8]$$
$${}^{111}P_{6,J}^{011}(x)=2\,\alpha^2\,[35\,\beta^4\alpha^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$

$$-8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$
$${}^{111}P_{6,J}^{100}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$
$$-7680\,\beta^7\alpha-1920\,\beta^8]$$
$${}^{111}P_{6,J}^{101}(x)=-2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$
$$-8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$
$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$
$${}^{111}P_{6,J}^{110}(x)=-2\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4-$$
$$-8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$
$$+96\,\left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\,\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$
$${}^{111}P_{6,J}^{111}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2x^2-$$
$$-32\,\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

$${}^{111}P_{6,I}^{000}(x)=-8\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\beta^2\alpha^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+120\,\alpha^6+360\,\alpha^5\beta+342\,\beta^2\alpha^4+84\,\alpha^3\beta^3+342\,\beta^4\alpha^2+360\,\alpha\,\beta^5+120\,\beta^6]$$
$${}^{111}P_{6,I}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+67\,\alpha^3\beta+34\,\beta^2\alpha^2+67\,\alpha\,\beta^3+30\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+960\,\alpha^8+4320\,\alpha^7\beta+7296\,\alpha^6\beta^2+5376\,\alpha^5\beta^3+960\,\alpha^4\beta^4+5376\,\beta^5\alpha^3+7296\,\beta^6\alpha^2+$$
$$+4320\,\beta^7\alpha+960\,\beta^8]$$
$${}^{111}P_{6,I}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$
$$-20\,\beta^2\alpha^2\left(40\,\alpha^4+80\,\alpha^3\beta-13\,\beta^2\alpha^2-53\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-1920\,\alpha^8-7680\,\alpha^7\beta-11232\,\alpha^6\beta^2-6816\,\alpha^5\beta^3-480\,\alpha^4\beta^4+1440\,\beta^5\alpha^3+3936\,\beta^6\alpha^2+$$
$$+3360\,\beta^7\alpha+960\,\beta^8]$$
$${}^{111}P_{6,I}^{011}(x)=-2\,\alpha^2\,[35\,\beta^4\alpha^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$
$${}^{111}P_{6,I}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$
$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$
$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$
$$-7680\,\beta^7\alpha-1920\,\beta^8]$$
$${}^{111}P_{6,I}^{101}(x)=2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$
$$+8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$
$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$
$${}^{111}P_{6,I}^{110}(x)=2\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$
$$+8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$

$$+96\ \left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\,\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{111}P^{111}_{6,I}(x)=\alpha\,\beta\ (\alpha+\beta)\ x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2x^2-$$

$$-32\ \left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

$$^{111}P^{000}_{6,K}(x)=8\,\alpha^2\beta^2\,(\alpha+\beta)^2\ x^2\,[35\,\beta^2\alpha^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+360\,\alpha^5\beta+342\,\beta^2\alpha^4+84\,\alpha^3\beta^3+342\,\beta^4\alpha^2+360\,\alpha\,\beta^5+120\,\beta^6]$$

$$^{111}P^{001}_{6,K}(x)=\alpha\,\beta\ (\alpha+\beta)\ x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+67\,\alpha^3\beta+34\,\beta^2\alpha^2+67\,\alpha\,\beta^3+30\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+4320\,\alpha^7\beta+7296\,\alpha^6\beta^2+5376\,\alpha^5\beta^3+960\,\alpha^4\beta^4+5376\,\beta^5\alpha^3+7296\,\beta^6\alpha^2+$$

$$+4320\,\beta^7\alpha+960\,\beta^8]$$

$$^{111}P^{010}_{6,K}(x)=-\alpha\,\beta\ (\alpha+\beta)\ x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\beta^2\alpha^2\left(40\,\alpha^4+80\,\alpha^3\beta-13\,\beta^2\alpha^2-53\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-1920\,\alpha^8-7680\,\alpha^7\beta-11232\,\alpha^6\beta^2-6816\,\alpha^5\beta^3-480\,\alpha^4\beta^4+1440\,\beta^5\alpha^3+3936\,\beta^6\alpha^2+$$

$$+3360\,\beta^7\alpha+960\,\beta^8]$$

$$^{111}P^{011}_{6,K}(x)=2\,\alpha^2\,[35\,\beta^4\alpha^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$

$$^{111}P^{100}_{6,K}(x)=-\alpha\,\beta\ (\alpha+\beta)\ x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$

$$-7680\,\beta^7\alpha-1920\,\beta^8]$$

$$^{111}P^{101}_{6,K}(x)=-2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$

$$^{111}P^{110}_{6,K}(x)=-2\ (\alpha+\beta)^2\ [35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$

$$+8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$

$$+96\ \left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\,\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{111}P^{111}_{6,K}(x)=\alpha\,\beta\ (\alpha+\beta)\ x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2x^2-$$

$$-32\ \left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

Recurrence relations:

$$\int x^{2n+1}\,J_0(\alpha x)\,J_0(\beta x)\,J_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Bigg\{-4n^2\beta(2\alpha+\beta)\int x^{2n}J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)\,dx\,-$$

$$-4n^2(\alpha^2-\beta^2)\int x^{2n}\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx-4n(n-1)\alpha^2\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_0(\beta x)\,J_0((\alpha+\beta)x)+2n\beta(2\alpha+\beta)\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)+$$

$$+2n(\alpha^2-\beta^2)\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)+$$

$$+\,2n\alpha^2\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)]\Bigg\}$$

$$\int x^{2n+1}\,J_0(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Bigg\{2n\beta(\alpha-2n\beta)\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)\,dx+$$

$$+2n[(2n+1)\alpha+2n\beta](\alpha+\beta)\int x^{2n}\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx+$$

$$+2(2n+1)(n-1)\alpha^2\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_0(\beta x)\,J_0((\alpha+\beta)x)+$$

$$+\beta(2n\beta-\alpha)\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)-(\alpha+\beta)((2n+1)\alpha+2n\beta)\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)-$$

$$-\,\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)-(2n+1)\alpha^2\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)]\Bigg\}$$

$$\int x^{2n+1}\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Bigg\{-2n\beta[(2n+1)\beta+\alpha]\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)\,dx-$$

$$-2n(2n+1)(\alpha^2-\beta^2)\int x^{2n}\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx-$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta]\int x^{2n+1}\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_0(\beta x)\,J_0((\alpha+\beta)x)+\beta[(2n+1)\beta+\alpha]\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)+$$

$$+(2n+1)(\alpha^2-\beta^2)\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)+$$

$$+\alpha[(4n+1)\beta+(2n+1)\alpha]\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)]\Bigg\}$$

$$\int x^{2n+2}\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{ 2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx-$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx+$$

$$+2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[-\beta[(4n+3)\alpha+(2n+1)\beta]\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)+$$

$$+(2n+1)(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)-$$

$$-\alpha[(2n+1)\alpha-\beta]\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, -\alpha\beta(\alpha+\beta)\, x\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+2}\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{ -2\beta^2(n+1)(2n+1) \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx-$$

$$-2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx+$$

$$+2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[(2n+1)\beta^2\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) + (\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha[2(n+1)\alpha+\beta)]\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+2}\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{ 4\beta^2(n+1)^2 \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx-$$

$$-4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx+$$

$$+4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[-2(n+1)\beta^2\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) + 2(n+1)(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - 2(n+1)\alpha(\alpha+2\beta)\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -4n^2\beta(2\alpha+\beta) \int x^{2n} I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx - \right.$$

$$-4n^2(\alpha^2-\beta^2) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx + 4n(n-1)\alpha^2 \int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + 2n\beta(2\alpha+\beta)\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+2n(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. - 2n\alpha^2\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -2n\beta(\alpha-2n\beta) \int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx - \right.$$

$$-2n[(2n+1)\alpha+2n\beta](\alpha+\beta) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+2(2n+1)(n-1)\alpha^2 \int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) -$$

$$-\beta(2n\beta-\alpha)\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + (\alpha+\beta)((2n+1)\alpha+2n\beta)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$\left. - \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - (2n+1)\alpha^2\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ 2n\beta[(2n+1)\beta+\alpha] \int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx + \right.$$

$$+2n(2n+1)(\alpha^2-\beta^2) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx -$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \beta[(2n+1)\beta+\alpha]\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-(2n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$\left. +\alpha[(4n+1)\beta+(2n+1)\alpha]\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{-2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx-$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx+$$

$$+2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[\beta[(4n+3)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)+$$

$$+(2n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)-$$

$$-\alpha[(2n+1)\alpha-\beta]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)\, x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{2\beta^2(n+1)(2n+1) \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx-$$

$$-2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx+$$

$$+2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[-(2n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)+(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)-\alpha[2(n+1)\alpha+\beta)]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+2}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\, \Big\{4\beta^2(n+1)^2 \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx+$$

$$+4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx-$$

$$-4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx+$$

$$+x^{2n+2}\big[-2(n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)-2(n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)+2(n+1)\alpha(\alpha+2\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\big]\Big\}$$

$$\int x^{2n+1}\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \,\Big\{ 4n^2\beta(2\alpha+\beta) \int x^{2n} K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+4n^2(\alpha^2-\beta^2) \int x^{2n}\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx - 4n(n-1)\alpha^2 \int x^{2n}\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) - 2n\beta(2\alpha+\beta)\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-2n(\alpha^2-\beta^2)\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) +$$

$$+ 2n\alpha^2\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+1}\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \,\Big\{ 2n\beta(\alpha-2n\beta) \int x^{2n}\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+2n[(2n+1)\alpha+2n\beta](\alpha+\beta) \int x^{2n}\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx -$$

$$-2(2n+1)(n-1)\alpha^2 \int x^{2n}\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) +$$

$$+\beta(2n\beta-\alpha)\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) - (\alpha+\beta)((2n+1)\alpha+2n\beta)\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) + (2n+1)\alpha^2\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \,\Big\{ -2n\beta[(2n+1)\beta+\alpha] \int x^{2n}\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x)\, dx -$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n}\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx +$$

$$+2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta] \int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) + \beta[(2n+1)\beta+\alpha]\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) -$$

$$-\alpha[(4n+1)\beta+(2n+1)\alpha]\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+2}\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\,\Big\{2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta]\int x^{2n+1}\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+2n(2n+1)(\alpha^2-\beta^2)\int x^{2n+1}\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx-$$

$$-2\alpha n[(2n+1)\alpha-\beta]\int x^{2n+1}\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+x^{2n+2}\mathbf{[}-\beta[(4n+3)\alpha+(2n+1)\beta]\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-$$

$$-(2n+1)(\alpha^2-\beta^2)\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\alpha[(2n+1)\alpha-\beta]\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)\,x\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\mathbf{]}\Big\}$$

$$\int x^{2n+2}\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\,\Big\{-2\beta^2(n+1)(2n+1)\int x^{2n+1}\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\int x^{2n+1}\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx-$$

$$-2n\alpha[2(n+1)\alpha+\beta]\int x^{2n+1}\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+x^{2n+2}\mathbf{[}(2n+1)\beta^2\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+\alpha[2(n+1)\alpha+\beta)]\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\mathbf{]}\Big\}$$

$$\int x^{2n+2}\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)}\,\Big\{-4\beta^2(n+1)^2\int x^{2n+1}\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)\,dx-$$

$$-4n(n+1)(\alpha^2-\beta^2)\int x^{2n+1}\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx+$$

$$+4n(n+1)\alpha(\alpha+2\beta)\int x^{2n+1}\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+x^{2n+2}\,\mathbf{[}2(n+1)\beta^2\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+2(n+1)(\alpha^2-\beta^2)\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)-2(n+1)\alpha(\alpha+2\beta)\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\mathbf{]}\,\Big\}$$

c) $x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu(\sqrt{\alpha^2 \pm \beta^2}\, x)$

Formulas with $\sqrt{\alpha^2+\beta^2}$ were found for the following integrals.
Replacing β by $\beta\, i$ one gets some modifications.

$$\int x\, J_1(\alpha x)\, J_1(\beta x)\, J_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta}\, \Big[\sqrt{\alpha^2+\beta^2}\, J_0(\alpha x)\, J_0(\beta x)\, J_1(\sqrt{\alpha^2+\beta^2}\, x) - \beta\, J_0(\alpha x)\, J_1(\beta x)\, J_0(\sqrt{\alpha^2+\beta^2}\, x) -$$

$$- \alpha\, J_1(\alpha x)\, J_0(\beta x)\, J_0(\sqrt{\alpha^2+\beta^2}\, x)\Big]$$

$$\int x\, I_1(\alpha x)\, I_1(\beta x)\, I_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta}\, \Big[-\sqrt{\alpha^2+\beta^2}\, I_0(\alpha x)\, I_0(\beta x)\, I_1(\sqrt{\alpha^2+\beta^2}\, x) + \beta\, I_0(\alpha x)\, I_1(\beta x)\, I_0(\sqrt{\alpha^2+\beta^2}\, x) +$$

$$+ \alpha\, I_1(\alpha x)\, I_0(\beta x)\, I_0(\sqrt{\alpha^2+\beta^2}\, x)\Big]$$

$$\int x\, K_1(\alpha x)\, K_1(\beta x)\, K_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta}\, \Big[\sqrt{\alpha^2+\beta^2}\, K_0(\alpha x)\, K_0(\beta x)\, K_1(\sqrt{\alpha^2+\beta^2}\, x) - \beta\, K_0(\alpha x)\, K_1(\beta x)\, K_0(\sqrt{\alpha^2+\beta^2}\, x) -$$

$$- \alpha\, K_1(\alpha x)\, K_0(\beta x)\, K_0(\sqrt{\alpha^2+\beta^2}\, x)\Big]$$

$$\int x\, J_1(\alpha x)\, I_1(\beta x)\, J_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta}\, \Big[-\sqrt{\alpha^2-\beta^2}\, J_0(\alpha x)\, I_0(\beta x)\, J_1(\sqrt{\alpha^2-\beta^2}\, x) - \beta\, J_0(\alpha x)\, I_1(\beta x)\, J_0(\sqrt{\alpha^2-\beta^2}\, x) +$$

$$+ \alpha\, J_1(\alpha x)\, I_0(\beta x)\, J_0(\sqrt{\alpha^2-\beta^2}\, x)\Big]$$

$$\int x\, I_1(\alpha x)\, J_1(\beta x)\, I_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta}\, \Big[\sqrt{\alpha^2-\beta^2}\, I_0(\alpha x)\, J_0(\beta x)\, I_1(\sqrt{\alpha^2-\beta^2}\, x) + \beta\, I_0(\alpha x)\, J_1(\beta x)\, I_0(\sqrt{\alpha^2-\beta^2}\, x) -$$

$$- \alpha\, I_1(\alpha x)\, J_0(\beta x)\, I_0(\sqrt{\alpha^2-\beta^2}\, x)\Big]$$

$$\int x^2\, J_0(\alpha x)\, J_0(\beta x)\, J_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\, \Big[\alpha\, J_0(\alpha\, x) J_1(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) +$$

$$+ \beta\, J_1(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) - \sqrt{\alpha^2+\beta^2} J_1(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, I_0(\alpha x)\, I_0(\beta x)\, I_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\, \Big[\alpha\, I_0(\alpha\, x) I_1(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) +$$

$$+ \beta\, I_1(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) - \sqrt{\alpha^2+\beta^2} I_1(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, K_0(\alpha x)\, K_0(\beta x)\, K_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = -\frac{x^2}{2\alpha\beta}\, \Big[\alpha\, K_0(\alpha\, x) K_1(\beta\, x) K_1(\sqrt{\alpha^2+\beta^2} x) +$$

$$+ \beta\, K_1(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2+\beta^2} x) - \sqrt{\alpha^2+\beta^2} K_1(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, J_0(\alpha x)\, I_0(\beta x)\, J_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\ \Big[\alpha\, J_0(\alpha\, x) I_1(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, J_1(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) - \sqrt{\alpha^2-\beta^2} J_1(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, I_0(\alpha x)\, J_0(\beta x)\, I_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\ \Big[\alpha\, I_0(\alpha\, x) J_1(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, I_1(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) - \sqrt{\alpha^2-\beta^2} I_1(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, J_0(\alpha x)\, J_1(\beta x)\, J_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[-\sqrt{\alpha^2-\beta^2}\, J_0(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, J_0(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x) + \alpha\, J_1(\alpha\, x) J_0(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, I_0(\alpha x)\, I_1(\beta x)\, I_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[\sqrt{\alpha^2-\beta^2}\, I_0(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, I_0(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x) - \alpha\, I_1(\alpha\, x) I_0(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, K_0(\alpha x)\, K_1(\beta x)\, K_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[-\sqrt{\alpha^2-\beta^2}\, K_0(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, K_0(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x) + \alpha\, K_1(\alpha\, x) K_0(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, J_0(\alpha x)\, I_1(\beta x)\, J_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2+\beta^2}}\ \Big[\sqrt{\alpha^2+\beta^2}\, J_0(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) -$$

$$-\beta\, J_0(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x) - \alpha\, J_1(\alpha\, x) I_0(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x\, I_0(\alpha x)\, J_1(\beta x)\, I_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2+\beta^2}}\ \Big[-\sqrt{\alpha^2+\beta^2}\, I_0(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) +$$

$$+\beta\, I_0(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x) + \alpha\, I_1(\alpha\, x) J_0(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, J_1(\alpha x)\, J_0(\beta x)\, J_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[\sqrt{\alpha^2-\beta^2}\, J_1(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, J_1(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) - \alpha\, J_0(\alpha\, x) J_1(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, I_1(\alpha x)\, I_0(\beta x)\, I_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[\sqrt{\alpha^2-\beta^2}\, I_1(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, I_1(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) - \alpha\, I_0(\alpha\, x) I_1(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, K_1(\alpha x)\, K_0(\beta x)\, K_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\ \Big[-\sqrt{\alpha^2-\beta^2}\, K_1(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, K_1(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x) + \alpha\, K_0(\alpha\, x) K_1(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, J_1(\alpha x)\, I_0(\beta x)\, J_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\beta\sqrt{\alpha^2+\beta^2}}\, \Big[\sqrt{\alpha^2+\beta^2}\, J_1(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x) +$$

$$+\beta\, J_1(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) - \alpha\, J_0(\alpha\, x) I_1(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, I_1(\alpha x)\, J_0(\beta x)\, I_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\beta\sqrt{\alpha^2+\beta^2}}\, \Big[\sqrt{\alpha^2+\beta^2}\, I_1(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x) +$$

$$+\beta\, I_1(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) - \alpha\, I_0(\alpha\, x) J_1(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x)\Big]$$

4. Products of four Bessel Functions

4.1. Integrals of the type $\int x^m Z_0^n(x) Z_1^{4-n}(x)\,dx$

4.1. a) Explicit Integrals

$$\int J_0^3(x)\,J_1(x)\,dx = -\frac{1}{4}\,J_0^4(x)$$
$$\int I_0^3(x)\,I_1(x)\,dx = \frac{1}{4}\,I_0^4(x)$$
$$\int K_0^3(x)\,K_1(x)\,dx = -\frac{1}{4}\,K_0^4(x)$$

$$\int J_0(x)J_1^3(x)\,dx = \frac{1}{4}\left[x^2J_0^4(x) - 2xJ_0^3(x)J_1(x) + 2x^2J_0^2(x)J_1^2(x) - 2xJ_0(x)J_1^3(x) + x^2J_1^4(x)\right]$$
$$\int I_0(x)I_1^3(x)\,dx = \frac{1}{4}\left[-x^2I_0^4(x) + 2xI_0^3(x)I_1(x) + 2x^2I_0^2(x)I_1^2(x) - 2xI_0(x)I_1^3(x) - x^2I_1^4(x)\right]$$
$$\int K_0(x)K_1^3(x)\,dx = \frac{1}{4}\left[x^2K_0^4(x) + 2xK_0^3(x)K_1(x) - 2x^2K_0^2(x)K_1^2(x) - 2xK_0(x)K_1^3(x) + x^2K_1^4(x)\right]$$

$$\int x^4\,J_0(x)\,J_1^3(x)\,dx = \frac{x^4}{4}\,J_1^4(x) \qquad \text{(See also p. 442.)}$$
$$\int x^4\,I_0(x)\,I_1^3(x)\,dx = \frac{x^4}{4}\,I_1^4(x)$$
$$\int x^4\,K_0(x)\,K_1^3(x)\,dx = -\frac{x^4}{4}\,K_1^4(x)$$

$$\int \frac{J_0^2(x)J_1^2(x)}{x}\,dx = -\frac{x^2+1}{4}J_0^4(x) + \frac{x}{2}J_0^3(x)J_1(x) - \frac{x^2+1}{2}J_0^2(x)J_1^2(x) + \frac{x}{2}J_0(x)J_1^3(x) - \frac{x^2}{4}J_1^4(x)$$
$$\int \frac{I_0^2(x)I_1^2(x)}{x}\,dx = -\frac{x^2-1}{4}I_0^4(x) + \frac{x}{2}I_0^3(x)I_1(x) + \frac{x^2-1}{2}I_0^2(x)I_1^2(x) - \frac{x}{2}I_0(x)I_1^3(x) - \frac{x^2}{4}I_1^4(x)$$
$$\int \frac{K_0^2(x)K_1^2(x)}{x}\,dx = -\frac{x^2-1}{4}K_0^4(x) - \frac{x}{2}K_0^3(x)K_1(x) + \frac{x^2-1}{2}K_0^2(x)K_1^2(x) + \frac{x}{2}K_0(x)K_1^3(x) - \frac{x^2}{4}K_1^4(x)$$

$$\int \frac{J_0(x)J_1^3(x)}{x^2}\,dx = -\frac{4x^2+3}{16}J_0^4(x) + \frac{x}{2}J_0^3(x)J_1(x) - \frac{4x^2+3}{8}J_0^2(x)J_1^2(x) + \frac{2x^2-1}{4x}J_0(x)J_1^3(x) - \frac{4x^2-1}{16}J_1^4(x)$$
$$\int \frac{I_0(x)I_1^3(x)}{x^2}\,dx = -\frac{4x^2-3}{16}I_0^4(x) + \frac{x}{2}I_0^3(x)I_1(x) + \frac{4x^2-3}{8}I_0^2(x)I_1^2(x) - \frac{2x^2+1}{4x}I_0(x)I_1^3(x) - \frac{4x^2+1}{16}I_1^4(x)$$
$$\int \frac{K_0(x)K_1^3(x)}{x^2}\,dx =$$
$$= \frac{4x^2-3}{16}K_0^4(x) + \frac{x}{2}K_0^3(x)K_1(x) - \frac{4x^2-3}{8}K_0^2(x)K_1^2(x) - \frac{2x^2+1}{4x}K_0(x)K_1^3(x) + \frac{4x^2+1}{16}K_1^4(x)$$

$$\int \frac{J_1^4(x)}{x}\,dx = \frac{1}{4}\left[x^2J_0^4(x) - 2xJ_0^3(x)J_1(x) + 2x^2J_0^2(x)J_1^2(x) - 2xJ_0(x)J_1^3(x) + (x^2-1)J_1^4(x)\right]$$
$$\int \frac{I_1^4(x)}{x}\,dx = \frac{1}{4}\left[-x^2I_0^4(x) + 2xJ_I^3(x)I_1(x) + 2x^2I_0^2(x)I_1^2(x) - 2xI_0(x)I_1^3(x) - (x^2+1)I_1^4(x)\right]$$

$$\int \frac{K_1^4(x)}{x}\, dx =$$

$$= \frac{1}{4}\left[-x^2 K_0^4(x) - 2x K_I^3(x) K_1(x) + 2x^2 K_0^2(x) K_1^2(x) + 2x K_0(x) K_1^3(x) - (x^2+1) K_1^4(x)\right]$$

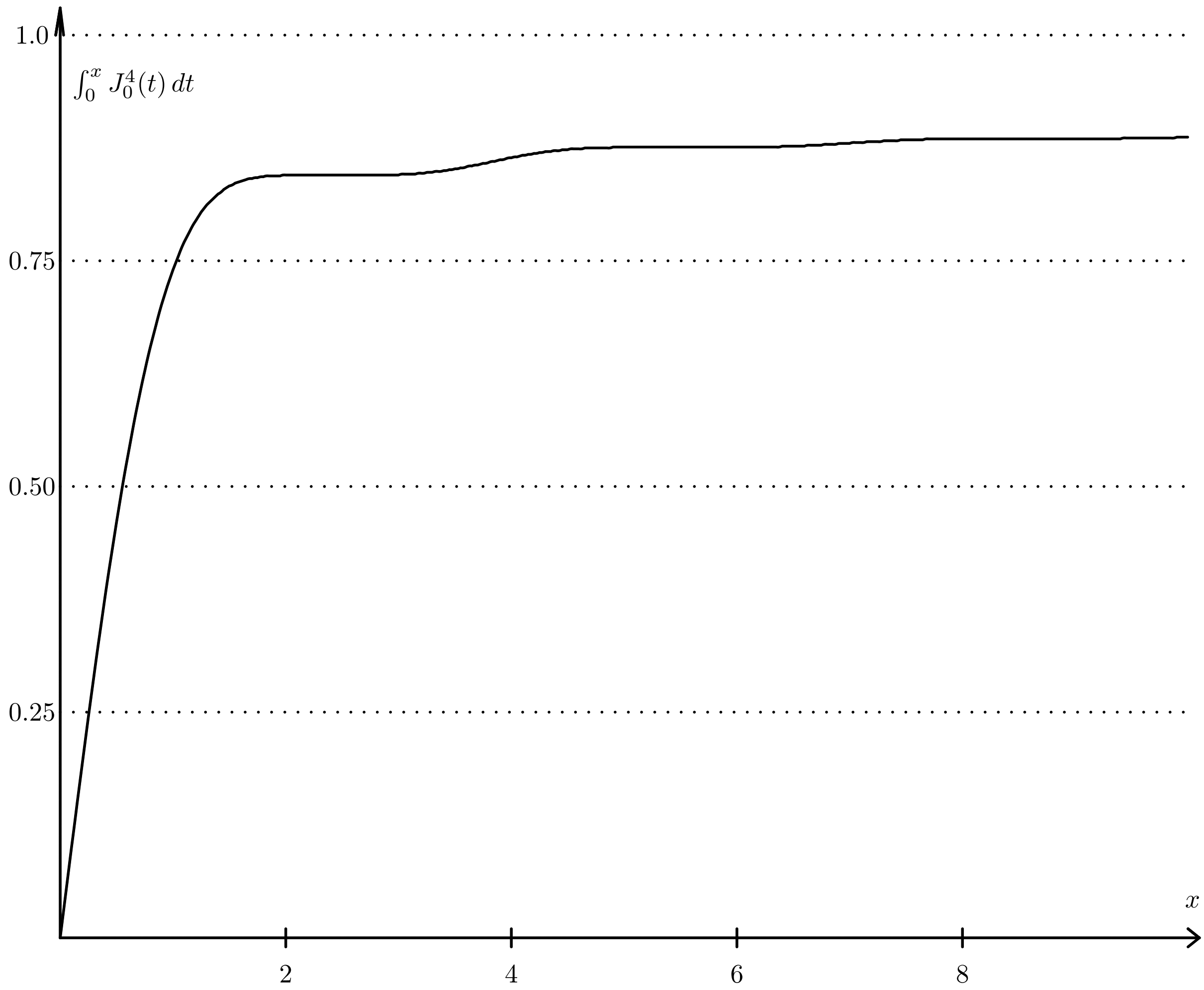
$$\int \frac{J_1^4(x)}{x^3}\, dx = -\frac{4x^2+3}{24} J_0^4(x) + \frac{x}{3} J_0^3(x) J_1(x) - \frac{4x^2+3}{12} J_0^2(x) J_1^2(x) + \frac{2x^2-1}{6x} J_0(x) J_1^3(x) - \frac{4x^4-x^2+4}{24x^2} J_1^4(x)$$

$$\int \frac{I_1^4(x)}{x^3}\, dx = -\frac{4x^2-3}{24} I_0^4(x) + \frac{x}{3} I_0^3(x) I_1(x) + \frac{4x^2-3}{12} I_0^2(x) I_1^2(x) - \frac{2x^2+1}{6x} I_0(x) I_1^3(x) - \frac{4x^4+x^2+4}{24x^2} I_1^4(x)$$

$$\int \frac{K_1^4(x)}{x^3}\, dx =$$

$$= -\frac{4x^2-3}{24} K_0^4(x) - \frac{x}{3} K_0^3(x) K_1(x) + \frac{4x^2-3}{12} K_0^2(x) K_1^2(x) + \frac{2x^2+1}{6x} K_0(x) K_1^3(x) - \frac{4x^4+x^2+4}{24x^2} K_1^4(x)$$

4.1. b) Basic Integral $Z_0^4(x)$



Power series:

$$\int_0^x J_0^4(t)\, dt = \sum_{k=0}^{\infty} (-1)^k a_k\, x^{2k+1} = x - \frac{1}{3}\, x^3 + \frac{7}{80}\, x^5 - \frac{1}{63}\, x^7 + \frac{679}{331776}\, x^9 - \frac{179}{921600}\, x^{11} + \frac{6049}{431308800}\, x^{13} - \ldots$$

$$\int_0^x I_0^4(t)\, dt = \sum_{k=0}^{\infty} a_k\, x^{2k+1} = x + \frac{1}{3}\, x^3 + \frac{7}{80}\, x^5 + \frac{1}{63}\, x^7 + \frac{679}{331776}\, x^9 + \frac{179}{921600}\, x^{11} + \frac{6049}{431308800}\, x^{13} + \ldots$$

k	a_k	a_k
0	1	1.00000 00000 00000 00000
1	$\frac{1}{3}$	0.33333 33333 33333 33333
2	$\frac{7}{80}$	0.08750 00000 00000 00000
3	$\frac{1}{63}$	0.01587 30158 73015 87302
4	$\frac{679}{331776}$	0.00204 65615 35493 82716
5	$\frac{179}{921600}$	0.00019 42274 30555 55556
6	$\frac{6049}{431308800}$	0.00001 40247 54421 88984
7	$\frac{9671}{12192768000}$	0.00000 07931 75101 83086
8	$\frac{16304551}{452803638067200}$	0.00000 00360 07994 70074
9	$\frac{7844077}{5856006714163200}$	0.00000 00013 39492 48744
10	$\frac{752932783}{18122799725936640000}$	0.00000 00000 41546 16253
11	$\frac{93524251}{85775087818506240000}$	0.00000 00000 01090 34282
12	$\frac{36868956721}{150367458297485721600000 0}$	0.00000 00000 00024 51924
13	$\frac{131084576323}{27445068488457093906432 0000}$	0.00000 00000 00000 47763
14	$\frac{134309549357}{16507700453797896482979840000}$	0.00000 00000 00000 00814
15	$\frac{242618760673}{1985193287331729792565248000000}$	0.00000 00000 00000 00012

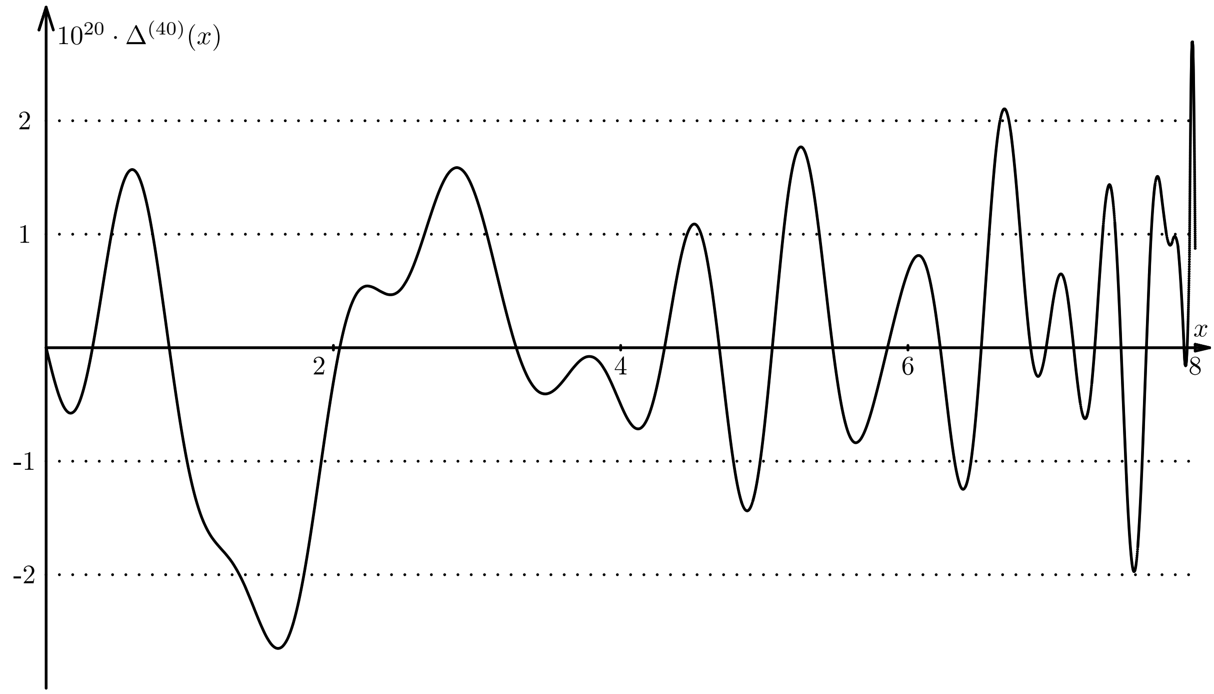
Approximation by Chebyshev polynomials: For $|x| \le 8$ holds (based on [2], 9.7)

$$\int_0^x J_0^4(t)\,dt \;=\; \sum_{k=0}^{33} \alpha_k\, T_{2n+1}\left(\frac{x}{8}\right) + \Delta^{(40)}(x)$$

with the coefficients

k	α_k	k	α_k
0	1.11216 87232 77997 70374	17	-0.00004 87217 67090 45658
1	-0.34519 17415 46385 57003	18	0.00001 20357 15738 11243
2	0.19338 82074 76973 67514	19	-0.00000 25907 87311 63049
3	-0.12647 08183 95063 85502	20	0.00000 04907 99998 13736
4	0.09004 56892 27881 33025	21	-0.00000 00825 41454 11576
5	-0.06770 17817 94869 79248	22	0.00000 00124 18007 27445
6	0.04909 46023 85233 15906	23	-0.00000 00016 82526 91798
7	-0.03372 40353 97394 50397	24	0.00000 00002 06529 29192
8	0.02266 69978 75674 80247	25	-0.00000 00000 23088 90409
9	-0.01515 53930 28431 47066	26	0.00000 00000 02361 98127
10	0.01031 44234 66796 38411	27	-0.00000 00000 00222 04627
11	-0.00711 67527 37994 13547	28	0.00000 00000 00019 25615
12	0.00463 50732 52595 22669	29	-0.00000 00000 00001 54585
13	-0.00263 71375 49146 56248	30	0.00000 00000 00000 11524
14	0.00126 22032 52082 49660	31	-0.00000 00000 00000 00800
15	-0.00050 48484 79216 82394	32	0.00000 00000 00000 00052
16	0.00016 99635 21655 61120	33	-0.00000 00000 00000 00003

The derivation $\Delta^{(40)}(x)$:



Asymptotic formula:

$$\int_0^\infty J_0^4(x)\,dx \quad = \quad 0.90272\ 85783\ 23834\ 82419\ \ldots$$

$$\int_0^x J_0^4(t)\,dt \sim 0.90272\ \ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\,\cos(2\,x)+\sin(4\,x)}{8x^2} - \frac{1-10\,\sin(2\,x)+\cos(4\,x)}{8\,x^3} + \ldots\right] =$$

$$= 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{p_k + q_k\sin 2x + r_k\sin 4x + s_k\cos 2x + t_k\cos 4x}{n_k\cdot x^k} = 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\sigma_k(x)}{x^k}$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	8	0	8
3	-1	10	0	0	-1	8
4	0	0	-9	-138	0	64
5	168	-5770	0	0	245	1280
6	0	0	644	24095	0	2048
7	-27648	2053401	0	0	-35364	57344
8	0	0	-93636	-93636	0	65536
9	1042944	-134972229	0	0	1015836	262144
10	0	0	102333888	19644534099	0	8388608
11	-1979596800	394714074735	0	0	-1482416640	33554432
12	0	0	-48926574720	-17487139338315	0	268435456

Let

$$D_n(x) = 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	1.505	3.298	4.622	6.418	7.755	9.552	10.893
$D_1(x_k)$	-3.113E-02	8.502E-03	-4.513E-03	2.446E-03	-1.678E-03	1.126E-03	-8.628E-04
x_k	2.356	3.964	5.498	7.091	8.639	10.226	11.781
$D_2(x_k)$	-6.617E-03	1.749E-03	-6.893E-04	3.361E-04	-1.879E-04	1.152E-04	-7.558E-05

x_k	3.221	4.656	6.355	7.796	9.494	10.936	12.634
$D_3(x_k)$	-1.514E-03	3.813E-04	-1.230E-04	5.450E-05	-2.601E-05	1.463E-05	-8.457E-06
x_k	3.945	5.498	7.080	8.639	10.219	11.781	13.358
$D_4(x_k)$	-3.508E-04	7.283E-05	-2.295E-05	8.509E-06	-3.893E-06	1.881E-06	-1.046E-06
x_k	3.176	4.688	6.314	7.830	9.454	10.971	12.595
$D_5(x_k)$	6.526E-04	-7.890E-05	1.534E-05	-4.427E-06	1.511E-06	-6.268E-07	2.819E-07
x_k	3.937	5.498	7.076	8.639	10.216	11.781	13.356
$D_6(x_k)$	1.452E-04	-1.689E-05	3.288E-06	-8.541E-07	2.789E-07	-1.044E-07	4.469E-08
x_k	3.157	4.702	6.297	7.844	9.438	10.985	12.579
$D_7(x_k)$	-5.521E-04	3.208E-05	-3.752E-06	7.106E-07	-1.734E-07	5.326E-08	-1.868E-08
x_k	3.933	5.498	7.074	8.639	10.214	11.781	13.355
$D_8(x_k)$	-1.077E-04	6.921E-06	-8.432E-07	1.521E-07	-3.594E-08	1.033E-08	-3.490E-09
x_k	3.150	4.708	6.290	7.849	9.431	10.991	12.573
$D_9(x_k)$	7.860E-04	-2.183E-05	1.525E-06	-1.904E-07	3.320E-08	-7.650E-09	2.080E-09
x_k	3.931	5.498	7.072	8.639	10.213	11.781	13.354
$D_{10}(x_k)$	1.271E-04	-4.450E-06	3.409E-07	**-4.254E-08**	7.270E-09	-1.600E-09	4.100E-10

Holds $D_9(8) =$ -1.798E-7 and

$$\min\ \{D_n(x_k)\,|\,8 \le x\ \} = |D_{10}(8.639)| = 4.254E-8\ .$$

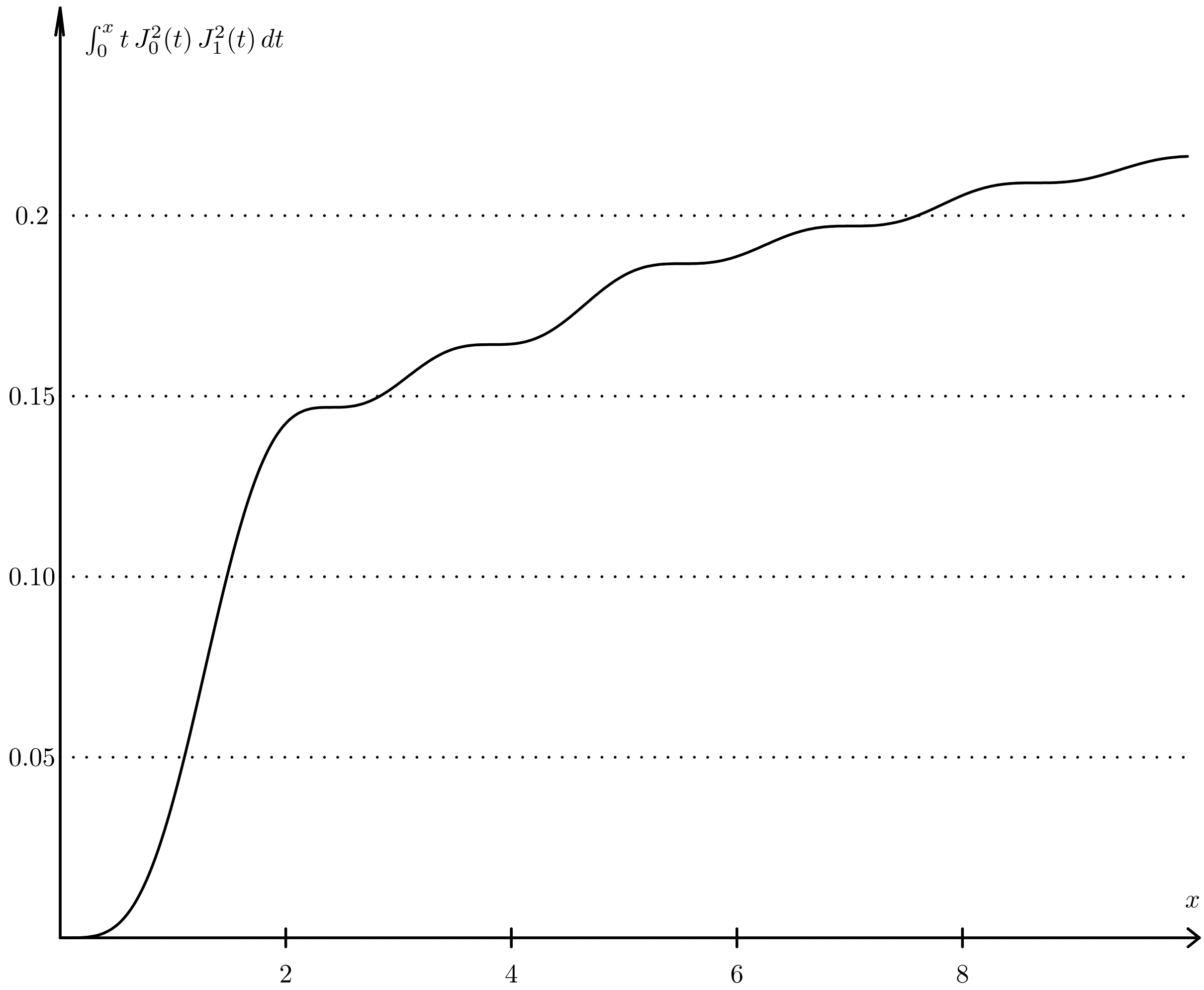
Therefore using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \ge 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_0^4(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x^2}\left[1+\frac{1}{x}+\frac{9}{8x^2}+\frac{49}{32\,x^3}+\frac{161}{64\,x^4}+\frac{1263}{256\,x^5}+\frac{23409}{2048\,x^6}+\ldots\right] = \frac{e^{4x}}{16\pi^2\,x^2}\sum_{k=0}^{\infty}\frac{\mu_k}{x^k}$$

k	μ_k	μ_k	μ_k/μ_{k-1}
0	1	1.000 000 000 000	-
1	1	1.000 000 000 000	1.000
2	$\frac{9}{8}$	1.125 000 000 000	1.125
3	$\frac{49}{32}$	1.531 250 000 000	1.361
4	$\frac{161}{64}$	2.515 625 000 000	1.643
5	$\frac{1263}{256}$	4.933 593 750 000	1.961
6	$\frac{23409}{2048}$	11.430 175 781 25	2.317
7	$\frac{253959}{8192}$	31.000 854 492 19	2.712
8	$\frac{1598967}{16384}$	97.593 200 683 59	3.148
9	$\frac{2895345}{8192}$	353.435 668 945 3	3.622
10	$\frac{382238865}{262144}$	1 458.125 553 131	4.126
11	$\frac{7110791145}{1048576}$	6 781.378 884 315	4.651
12	$\frac{295087625775}{8388608}$	35 177.186 223 864	5.187

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \le x$.

4.1. c) Basic Integral $x\,Z_0^2(x)\,Z_1^2(x)$



Power series:

$$\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt = \sum_{k=2}^{\infty}(-1)^k b_k\,x^{2k} = \frac{x^4}{16} - \frac{x^6}{32} + \frac{47}{6144}\,x^8 - \frac{43}{36864}\,x^{10} + \frac{17}{138240}\,x^{12} - \frac{211}{22118400}\,x^{14} + \dots$$

$$\int_0^x t\,I_0^2(t)\,I_1^2(t)\,dt = \sum_{k=2}^{\infty} b_k\,x^{2k} = \frac{x^4}{16} + \frac{x^6}{32} + \frac{47}{6144}\,x^8 + \frac{43}{36864}\,x^{10} + \frac{17}{138240}\,x^{12} + \frac{211}{22118400}\,x^{14} + \dots$$

k	b_k	b_k
2	$\frac{1}{4}$	0.25000 00000 00000 00000
3	$\frac{1}{32}$	0.03125 00000 00000 00000
4	$\frac{47}{6144}$	0.00764 97395 83333 33333
5	$\frac{43}{36864}$	0.00116 64496 52777 77778
6	$\frac{17}{138240}$	0.00012 29745 37037 03704
7	$\frac{211}{22118400}$	0.00000 95395 68865 74074
8	$\frac{540619}{951268147200}$	0.00000 05683 13993 89465

k	b_k	b_k
9	$\dfrac{1072333}{39953262182400}$	0.00000 00268 39685 70838
10	$\dfrac{19751801}{19177565847552000}$	0.00000 00010 29943 06770
11	$\dfrac{11307553}{345196185255936000}$	0.00000 00000 32756 88864
12	$\dfrac{88869497}{101257547675074560000}$	0.00000 00000 00877 65800
13	$\dfrac{402630853}{20048994439664762880000}$	0.00000 00000 00020 08235
14	$\dfrac{17384556227}{43787003856227842129920000}$	0.00000 00000 00000 39703
15	$\dfrac{16710855809}{2439561643418408347238400000}$	0.00000 00000 00000 00685
16	$\dfrac{58219427293829}{559576891520740833056155238400000}$	0.00000 00000 00000 00010

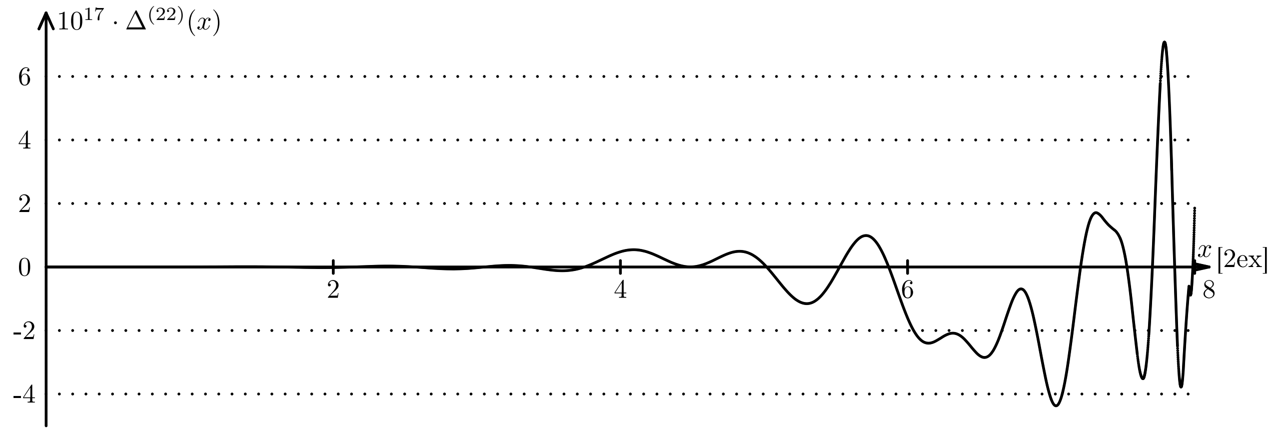
Approximation by Chebyshev polynomials: For $|x| \leq 8$ holds (based on [2], 9.7)

$$\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt \;=\; x^4 \sum_{k=0}^{33} \beta_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(22)}(x)$$

with the coefficients

k	β_k	k	β_k
0	0.00654 92504 91232 31929	16	0.00000 13488 21251 71102
1	-0.01239 97369 78264 31272	17	-0.00000 03066 11870 00840
2	0.01094 53569 42769 10753	18	0.00000 00617 07706 64541
3	-0.00914 38524 55194 87209	19	-0.00000 00110 63931 41290
4	0.00727 27632 79880 71248	20	0.00000 00017 77850 51057
5	-0.00551 50237 23344 53306	21	-0.00000 00002 57472 65556
6	0.00398 42908 14902 62113	22	0.00000 00000 33780 93775
7	-0.00273 36991 93851 90098	23	-0.00000 00000 04034 52317
8	0.00177 13047 44123 60885	24	0.00000 00000 00440 55264
9	-0.00107 36173 62206 65294	25	-0.00000 00000 00044 16078
10	0.00059 89651 07157 49244	26	0.00000 00000 00004 07863
11	-0.00030 12533 67887 99958	27	-0.00000 00000 00000 34826
12	0.00013 40719 51669 75647	28	0.00000 00000 00000 02758
13	-0.00005 21809 55261 86499	29	-0.00000 00000 00000 00203
14	0.00001 76814 63326 68133	30	0.00000 00000 00000 00014
15	-0.00000 52208 90537 98925	31	-0.00000 00000 00000 00001

The derivation $\Delta^{(22)(x)}$:



Asymptotic formula:

$$\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt \sim 0.09947\ 25799\ 65044\ 03230\ldots +$$

$$+\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\frac{\sin 4x}{8x}+\frac{\cos 4x-16\sin 2x-6}{8x^2}+\frac{\sin 4x+12\cos 2x}{32\,x^3}+\ldots\right]=$$

$$=0.09947\ \ldots+\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\sum_{k=1}^{\infty}\frac{p_k+q_k\sin 2x+r_k\sin 4x+s_k\cos 2x+t_k\cos 4x}{n_k\cdot x^k}\right]=$$

$$=0.09947\ \ldots+\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{\infty}\frac{\sigma_k(x)}{x^k}\right]$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	0	0	1	0	0	8
2	-6	-16	0	0	1	32
3	0	0	1	12	0	32
4	18	180	0	0	-9	256
5	0	0	-51	-1356	0	1024
6	-576	-15483	0	0	357	4096
7	0	0	3015	182385	0	16384
8	99360	99360	0	0	-60300	131072
9	0	0	-703530	-88668135	0	524288
10	-137687040	-13633922835	0	0	75666960	16777216
11	0	0	1159159680	271539997785	0	67108864
12	67108864	12571439587875	0	0	-40044715200	536870912

Let

$$D_n(x)=0.09947\ \ldots+\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k}\right]-\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	3.391	4.211	5.147	5.931	6.602	7.398	8.273
$D_1(x_k)$	-5.002E-04	6.375E-03	-2.955E-03	1.615E-03	-8.607E-04	2.793E-03	-1.744E-03
x_k	3.870	5.505	7.037	8.644	10.188	11.784	13.335
$D_2(x_k)$	4.475E-03	-8.864E-04	1.430E-03	-3.717E-04	6.919E-04	-2.023E-04	4.062E-04
x_k	3.248	4.676	6.384	7.800	9.520	10.934	12.658
$D_3(x_k)$	-1.021E-03	3.135E-04	-1.471E-04	7.402E-05	-4.504E-05	2.779E-05	-1.922E-05
x_k	3.901	5.502	7.053	8.642	10.199	11.783	13.343
$D_4(x_k)$	-2.742E-04	5.650E-05	-3.011E-05	1.011E-05	-7.214E-06	3.019E-06	-2.511E-06
x_k	3.201	4.711	6.340	7.840	9.477	10.975	12.616
$D_5(x_k)$	2.732E-04	-4.339E-05	1.169E-05	-3.970E-06	1.674E-06	-7.807E-07	4.107E-07
x_k	3.175	4.724	6.316	7.857	9.455	10.994	12.594
$D_6(x_k)$	-1.747E-04	1.386E-05	-2.198E-06	4.983E-07	-1.481E-07	5.176E-08	-2.095E-08
x_k	3.921	5.499	7.064	8.640	10.207	11.782	13.349
$D_7(x_k)$	-3.990E-05	3.253E-06	-5.342E-07	1.093E-07	-3.219E-08	9.988E-09	-3.994E-09
x_k	3.161	4.726	6.305	7.863	9.444	11.001	12.584
$D_8(x_k)$	2.032E-04	-7.880E-06	7.394E-07	-1.115E-07	2.348E-08	-6.142E-09	1.922E-09
x_k	3.924	5.499	7.066	8.640	10.208	11.782	13.350
$D_9(x_k)$	3.864E-05	-1.772E-06	1.785E-07	**-2.587E-08**	5.416E-09	-1.316E-09	4.030E-10
x_k	3.154	4.725	6.299	7.864	9.439	11.003	12.579
$D_{10}(x_k)$	-3.766E-04	7.014E-06	-3.886E-07	3.885E-08	-5.815E-09	1.138E-09	-2.759E-10

Holds

$$\min\ \{D_n(x_k)\,|\,8 \le x\ \} = |D_9(9.439)| = 2.587E-08\ .$$

Therefore using the partial sum of the asymptotic series with $n = 9$ means to get the best uniform approximation of the integral with $x \ge 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

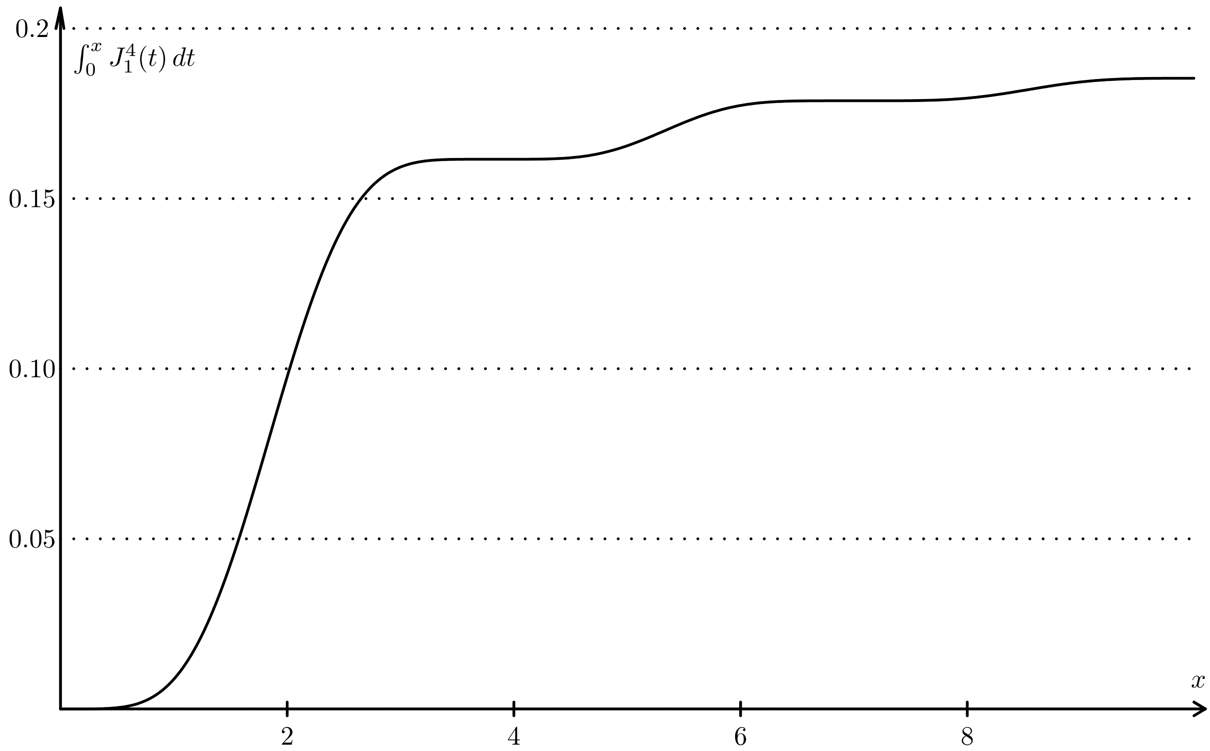
$$\int_0^x t\, I_0^2(t)\, I_1^2(t)\, dt \sim \frac{e^{4x}}{16\pi^2\, x}\left[1 - \frac{1}{4x} - \frac{1}{4x^2} - \frac{9}{32\, x^3} - \frac{51}{128\, x^4} - \frac{357}{512\, x^5} - \frac{3015}{2048\, x^6} - \ldots\right] =$$

$$= \frac{e^{4x}}{16\pi^2\, x}\left(1 - \sum_{k=1}^{\infty} \frac{\mu_k}{x^k}\right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	$\frac{1}{4}$	0.250 000 000 000	-
2	$\frac{1}{4}$	0.250 000 000 000	1.000
3	$\frac{9}{32}$	0.281 250 000 000	1.125
4	$\frac{51}{128}$	0.398 437 500 000	1.417
5	$\frac{357}{512}$	0.697 265 625 000	1.750
6	$\frac{3015}{2048}$	1.472 167 968 750	2.111
7	$\frac{15075}{4096}$	3.680 419 921 875	2.500
8	$\frac{351765}{32768}$	10.735 015 869 14	2.917
9	$\frac{4729185}{131072}$	36.080 818 176 27	3.361
10	$\frac{9055935}{65536}$	138.182 601 928 7	3.830
11	$\frac{625698675}{1048576}$	596.712 756 156 9	4.318
12	$\frac{24131137275}{8388608}$	2 876.655 730 605	4.821

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \le x$.

4.1. d) Basic Integral $Z_1^4(x)$



Power series:

$$\int_0^x J_1^4(t)\,dt = \sum_{k=2}^{\infty}(-1)^k c_k\, x^{2k+1} = \frac{1}{80}\,x^5 - \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 - \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} - \frac{1223}{2654208000}\,x^{15} + \ldots$$

$$\int_0^x I_1^4(t)\,dt = \sum_{k=2}^{\infty} c_k\, x^{2k+1} = \frac{1}{80}\,x^5 + \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 + \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} + \frac{1223}{2654208000}\,x^{15} + \ldots$$

k	c_k	c_k
2	$\frac{1}{80}$	0.01250 00000 00000 00000
3	$\frac{1}{224}$	0.00446 42857 14285 71429
4	$\frac{11}{13824}$	0.00079 57175 92592 59259
5	$\frac{37}{405504}$	0.00009 12444 76010 10101
6	$\frac{11}{1474560}$	0.00000 74598 52430 55556
7	$\frac{1223}{2654208000}$	0.00000 04607 77753 66512
8	$\frac{45173}{2021444812800}$	0.00000 00223 46887 58949
9	$\frac{221467}{253037327155200}$	0.00000 00008 75234 50587
10	$\frac{2278819}{80545776559718400}$	0.00000 00000 28292 22210
11	$\frac{1524667}{1984878065221632000}$	0.00000 00000 00768 14139

k	c_k	c_k
12	$\frac{33739889}{1898579018907648000000}$	0.00000 00000 00017 77113
13	$\frac{191964463}{541322849870948597760000}$	0.00000 00000 00000 35462
14	$\frac{639303779}{103659029537192442593280000}$	0.00000 00000 00000 00617
15	$\frac{1383459431}{14666940304673097457336320000}$	0.00000 00000 00000 00009

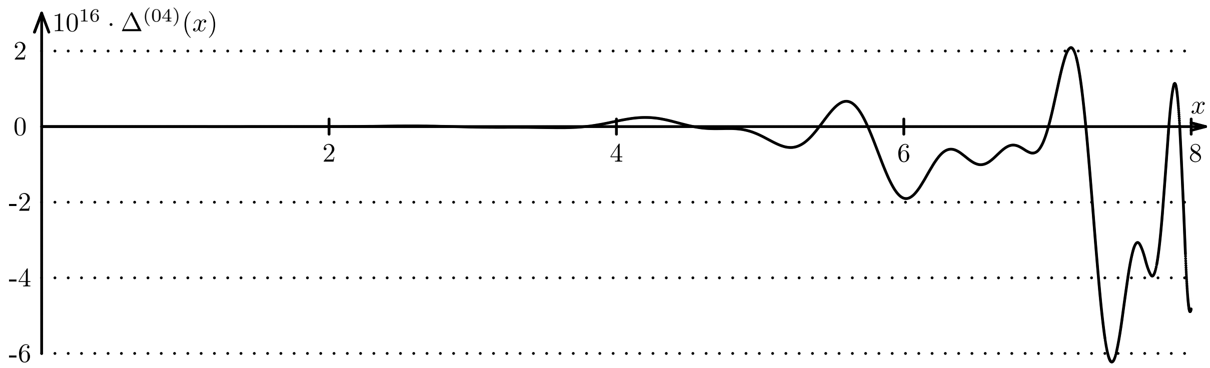
Approximation by Chebyshev polynomials: For $|x| \le 8$ holds (based on [2], 9.7)

$$\int_0^x J_1^4(t)\,dt \;=\; x^5 \sum_{k=0}^{33} \gamma_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(04)}(x)$$

with the coefficients

k	γ_k	k	γ_k
0	0.00153 17807 42787 63788	15	-0.00000 01214 49955 80695
1	-0.00288 55180 23979 94997	16	0.00000 00263 11808 30629
2	0.00246 63655 58097 67279	17	-0.00000 00050 87634 73551
3	-0.00194 01230 51814 45392	18	0.00000 00008 82115 64716
4	0.00141 41566 95709 63740	19	-0.00000 00001 37795 20459
5	-0.00095 77666 19168 38344	20	0.00000 00000 19482 53784
6	0.00060 29805 69397 05353	21	-0.00000 00000 02504 22692
7	-0.00035 21217 05640 59655	22	0.00000 00000 00293 84879
8	0.00018 96461 12672 32841	23	-0.00000 00000 00031 59998
9	-0.00009 33179 14008 29393	24	0.00000 00000 00003 12564
10	0.00004 14766 56943 42490	25	-0.00000 00000 00000 28533
11	-0.00001 64773 77688 70038	26	0.00000 00000 00000 02411
12	0.00000 58075 26406 73172	27	-0.00000 00000 00000 00189
13	-0.00000 18093 92101 58874	28	0.00000 00000 00000 00014
14	0.00000 04981 61958 19526	29	-0.00000 00000 00000 00001

The derivation $\Delta^{(04)(x)}$:



Asymptotic formula:

$$\int_0^\infty J_1^4(x)\,dx \quad = \quad 0.20025\;27575\;82806\;70455\;\ldots$$

$$\int_0^x J_1^4(t)\,dt \sim 0.20025\;\ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\,\cos(2\,x) - \sin(4\,x)}{8x^2} - \frac{3 - 2\,\sin(2\,x) + \cos(4\,x)}{8\,x^3} + \ldots\right] =$$

$$= 0.20025\;\ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{p_k + q_k \sin 2x + r_k \sin 4x + s_k \cos 2x + t_k \cos 4x}{n_k \cdot x^k} = 0.20025\;\ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{\sigma_k(x)}{x^k}$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	-8	0	8
3	3	-2	0	0	1	8
4	0	0	3	-6	0	64
5	-216	-30	0	0	-75	1280
6	0	0	-204	1173	0	2048
7	34560	65536	0	0	12348	57344
8	0	0	37116	-586503	0	65536
9	-1267200	-9916623	0	0	-9916623	262144
10	0	0	-52716096	1650714957	0	8388608
11	25778995200	376332256575	0	0	9405918720	369098752
12	0	0	369098752	-1633144646925	0	-1633144646925

Let

$$D_n(x) = 0.20025\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	2.930	4.738	6.115	7.917	9.274	11.073	12.424
$D_1(x_k)$	-9.815E-03	5.061E-03	-2.533E-03	1.764E-03	-1.140E-03	8.883E-04	-6.457E-04
x_k	3.927	4.804	7.069	8.003	10.210	11.962	11.962
$D_2(x_k)$	-6.708E-06	5.006E-04	-5.760E-07	1.111E-04	-1.053E-07	3.014E-05	3.014E-05
x_k	4.118	4.967	5.948	7.057	7.182	8.102	9.122
$D_3(x_k)$	-7.413E-06	1.140E-05	-1.107E-05	-5.756E-07	-5.905E-07	2.135E-06	-1.971E-06
x_k	4.567	5.842	6.168	7.055	7.737	8.858	9.382
$D_4(x_k)$	-1.253E-05	-1.396E-06	-1.515E-06	-5.756E-07	-9.571E-07	-1.679E-07	-2.281E-07
x_k	4.932	6.114	8.037	9.272	11.165	12.418	14.300
$D_5(x_k)$	-2.582E-06	9.433E-07	-1.787E-07	8.754E-08	-2.737E-08	1.592E-08	-6.480E-09
x_k	3.932	5.630	7.072	8.735	10.212	11.855	13.353
$D_6(x_k)$	7.787E-06	-6.017E-07	1.910E-07	-3.239E-08	1.673E-08	-4.050E-09	2.700E-09
x_k	3.044	4.823	6.215	7.943	9.364	11.075	12.507
$D_7(x_k)$	-4.681E-05	1.729E-06	-2.866E-07	4.340E-08	-1.287E-08	3.440E-09	-1.340E-09
x_k	3.927	5.571	7.069	8.696	10.210	11.826	13.352
$D_8(x_k)$	-7.839E-06	4.320E-07	-6.420E-08	9.910E-09	-2.750E-09	7.500E-10	-2.800E-10
x_k	3.092	4.775	6.248	7.905	9.394	11.040	12.537
$D_9(x_k)$	7.406E-05	-1.525E-06	1.346E-07	-1.447E-08	**2.910E-09**	-5.800E-10	2.100E-10
x_k	3.927	5.542	7.069	8.677	10.210	11.812	13.352
$D_{10}(x_k)$	1.082E-05	-3.483E-07	2.989E-08	-3.330E-09	6.900E-10	-1.500E-10	2.000E-11

Note that $D_2(x)$, $D_3(x)$ and $D_4(x)$ do not have the regular behaviour of the other functions.
Holds

$$\max\{|D_{10}(x)| \mid x \geq 8\} = 4.00E-9\ .$$

Using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_1^4(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x^2}\left[1 - \frac{1}{x} - \frac{3}{8x^2} - \frac{15}{32\,x^3} - \frac{51}{64\,x^4} - \frac{441}{256\,x^5} - \frac{9279}{2048\,x^6} - \ldots\right] = \frac{e^{4x}}{16\pi^2\,x^2}\left(1 - \sum_{k=1}^{\infty}\frac{\mu_k}{x^k}\right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	1	1.000 000 000 000	-
2	$\frac{3}{8}$	0.375 000 000 000	0.375
3	$\frac{15}{32}$	0.468 750 000 000	1.250
4	$\frac{51}{64}$	0.796 875 000 000	1.700
5	$\frac{441}{256}$	1.722 656 250 000	2.162
6	$\frac{9279}{2048}$	4.530 761 718 750	2.630
7	$\frac{115137}{8192}$	14.054 809 570 31	3.102
8	$\frac{823689}{16384}$	50.273 986 816 41	3.577
9	$\frac{1670085}{8192}$	203.867 797 851 6	4.055
10	$\frac{242464455}{262144}$	924.928 493 499 8	4.537
11	$\frac{4871010735}{1048576}$	4 645.357 832 909	5.022
12	$\frac{214767448785}{8388608}$	25 602.274 988 294 5.511	

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \leq x$.

4.1. e) Integrals of $x^m\, Z_0^4(x)$

With the basic integrals

$$\mathcal{I}_0^{(40)}(x) = \int J_0^4(x)\, dx\,, \quad \mathcal{I}_1^{(22)}(x) = \int x\, J_0^2(x)\, {J_1}^2(x) dx\,, \quad \mathcal{I}_0^{(04)}(x) = \int J_1^4(x)\, dx$$

and

$$\mathcal{I}_0^{*(40)}(x) = \int I_0^4(x)\, dx\,, \quad \mathcal{I}_1^{*(22)}(x) = \int x\, I_0^2(x)\, {I_1}^2(x) dx\,, \quad \mathcal{I}_0^{*(04)}(x) = \int I_1^4(x)\, dx$$

holds

$$\int x\, J_0^4(x)\, dx = x\, J_0^3(x)\, J_1(x) + 3\mathcal{I}_1^{(22)}(x)$$

$$\int x\, I_0^4(x)\, dx = x\, I_0^3(x)\, I_1(x) - 3\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\, J_0^4(x)\, dx = \frac{12x^3 - x}{32}\, J_0^4(x) - \frac{x^2}{8}\, J_0^3(x)\, J_1(x) + \frac{3x^3}{4}\, J_0^2(x)\, J_1^2(x) - \frac{3x^2}{8}\, J_0(x)\, J_1^3(x) +$$
$$+ \frac{12x^3 - 3x}{32}\, J_1^4(x) + \frac{1}{32}\, \mathcal{I}_0^{(40)}(x) - \frac{9}{32}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^2\, I_0^4(x)\, dx = \frac{12x^3 + x}{32}\, I_0^4(x) - \frac{x^2}{8}\, I_0^3(x)\, I_1(x) - \frac{3x^3}{4}\, I_0^2(x)\, I_1^2(x) + \frac{3x^2}{8}\, I_0(x)\, I_1^3(x) +$$
$$+ \frac{12x^3 + 3x}{32}\, I_1^4(x) - \frac{1}{32}\, \mathcal{I}_0^{*(40)}(x) + \frac{9}{32}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\, J_0^4(x)\, dx = \frac{3x^4 + 2x^2}{16}\, J_0^4(x) + \frac{x^3 - x}{4}\, J_0^3(x)\, J_1(x) + \frac{3x^4}{8}\, J_0^2(x)\, J_1^2(x) + \frac{3x^4}{16}\, J_1^4(x) - \frac{3}{4}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^3\, I_0^4(x)\, dx = \frac{3\,x^4 - 2\,x^2}{16}\, I_0^4(x) + \frac{x^3 + x}{4}\, I_0^3(x)\, I_1(x) - \frac{3x^4}{8}\, I_0^2(x)\, I_1^2(x) + \frac{3x^4}{16}\, I_1^4(x) - \frac{3}{4}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\ J_0^4(x)\,dx = \frac{32\,x^5-12\,x^3+9\,x}{256}\ J_0^4(x)+\frac{24\,x^4+9\,x^2}{64}\ J_0^3(x)\,J_1(x)+\frac{8\,x^5-21\,x^3}{32}\ J_0^2(x)\,J_1^2(x)+$$
$$+\frac{8\,x^4+23\,x^2}{64}\ J_0(x)\,J_1^3(x)+\frac{32\,x^5-92\,x^3+23\,x}{256}\ J_1^4(x)-\frac{9}{256}\,\mathcal{I}_0^{(40)}(x)+\frac{69}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\ I_0^4(x)\,dx = \frac{32\,x^5+12\,x^3+9\,x}{256}\ I_0^4(x)+\frac{24\,x^4-9\,x^2}{64}\ I_0^3(x)\,I_1(x)-\frac{8\,x^5+21\,x^3}{32}\ I_0^2(x)\,I_1^2(x)-$$
$$-\frac{8\,x^4-23\,x^2}{64}\ I_0(x)\,I_1^3(x)+\frac{32\,x^5+92\,x^3+23\,x}{256}\ I_1^4(x)-\frac{9}{256}\,\mathcal{I}_0^{*(40)}(x)+\frac{69}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\ J_0^4(x)\,dx = \frac{6\,x^6+7\,x^4-14\,x^2}{64}\ J_0^4(x)+\frac{7\,x^5-7\,x^3+7\,x}{16}\ J_0^3(x)\,J_1(x)+\frac{6\,x^6-21\,x^4}{32}\ J_0^2(x)\,J_1^2(x)+$$
$$+\frac{3x^5}{16}\ J_0(x)\,J_1^3(x)+\frac{6\,x^6-27\,x^4}{64}\ J_1^4(x)+\frac{21}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\ I_0^4(x)\,dx = \frac{6\,x^6-7\,x^4-14\,x^2}{64}\ I_0^4(x)+\frac{7\,x^5+7\,x^3+7\,x}{16}\ I_0^3(x)\,I_1(x)-\frac{6\,x^6+21\,x^4}{32}\ I_0^2(x)\,I_1^2(x)-$$
$$-\frac{3x^5}{16}\ I_0(x)\,I_1^3(x)+\frac{6\,x^6+27\,x^4}{64}\ I_1^4(x)-\frac{21}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\ J_0^4(x)\,dx = \frac{768\,x^7+2496\,x^5+1668\,x^3-1251\,x}{10240}\ J_0^4(x)+\frac{1216\,x^6-3120\,x^4-1251\,x^2}{2560}\ J_0^3(x)\,J_1(x)+$$
$$+\frac{192\,x^7-896\,x^5+2757\,x^3}{1280}\ J_0^2(x)\,J_1^2(x)+\frac{576\,x^6-1328\,x^4-3089\,x^2}{2560}\ J_0(x)\,J_1^3(x)+$$
$$+\frac{768\,x^7-5312\,x^5+12356\,x^3-3089\,x}{10240}\ J_1^4(x)+\frac{1251}{10240}\,\mathcal{I}_0^{(40)}(x)-\frac{9267}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\ I_0^4(x)\,dx = \frac{768\,x^7-2496\,x^5+1668\,x^3+1251\,x}{10240}\ I_0^4(x)+\frac{1216\,x^6+3120\,x^4-1251\,x^2}{2560}\ I_0^3(x)\,I_1(x)-$$
$$-\frac{192\,x^7+896\,x^5+2757\,x^3}{1280}\ I_0^2(x)\,I_1^2(x)-\frac{576\,x^6+1328\,x^4-3089\,x^2}{2560}\ I_0(x)\,I_1^3(x)+$$
$$+\frac{768\,x^7+5312\,x^5+12356\,x^3+3089\,x}{10240}\ I_1^4(x)-\frac{1251}{10240}\,\mathcal{I}_0^{*(40)}(x)+\frac{9267}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\ J_0^4(x)\,dx = \frac{x^8+6\,x^6-9\,x^4+18\,x^2}{16}\ J_0^4(x)+\frac{2\,x^7-9\,x^5+9\,x^3-9\,x}{4}\ J_0^3(x)\,J_1(x)+$$
$$+\frac{x^8-6\,x^6+27\,x^4}{8}\ J_0^2(x)\,J_1^2(x)+\frac{x^7-5\,x^5}{4}\ J_0(x)\,J_1^3(x)+\frac{x^8-10\,x^6+37\,x^4}{16}\ J_1^4(x)-\frac{27}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^7\ I_0^4(x)\,dx = \frac{x^8-6\,x^6-9\,x^4-18\,x^2}{16}\ I_0^4(x)+\frac{2\,x^7+9\,x^5+9\,x^3+9\,x}{4}\ I_0^3(x)\,I_1(x)-$$
$$-\frac{x^8+6\,x^6+27\,x^4}{8}\ I_0^2(x)\,I_1^2(x)-\frac{x^7+5\,x^5}{4}\ I_0(x)\,I_1^3(x)+\frac{x^8+10\,x^6+37\,x^4}{16}\ I_1^4(x)-\frac{27}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\ J_0^4(x)\,dx = \frac{15360\,x^9+145024\,x^7-535152\,x^5-364116\,x^3+273087\,x}{286720}\ J_0^4(x)+$$

$$+\frac{37120\,x^8-253792\,x^6+668940\,x^4+273087\,x^2}{71680}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{1920\,x^9-14352\,x^7+91726\,x^5-296367\,x^3}{17920}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{19200\,x^8-158112\,x^6+302036\,x^4+668243\,x^2}{71860}\,J_0(x)\,J_1^3(x)+$$
$$+\frac{15360\,x^9-210816\,x^7+1208144\,x^5-2672972\,x^3+668243\,x}{286720}\,J_1^4(x)-\frac{273087}{286720}\,\mathcal{I}_0^{(40)}(x)+\frac{2004729}{286720}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\;I_0^4(x)\,dx=\frac{15360\,x^9-145024\,x^7-535152\,x^5+364116\,x^3+273087\,x}{286720}\,I_0^4(x)+$$
$$+\frac{37120\,x^8+253792\,x^6+668940\,x^4-273087\,x^2}{71680}\,I_0^3(x)\,I_1(x)-$$
$$-\frac{1920\,x^9+14352\,x^7+91726\,x^5+296367\,x^3}{17920}\,I_0^2(x)\,I_1^2(x)-$$
$$-\frac{19200\,x^8+158112\,x^6+302036\,x^4-668243\,x^2}{71680}\,I_0(x)\,I_1^3(x)+$$
$$+\frac{15360\,x^9+210816\,x^7+1208144\,x^5+2672972\,x^3+668243\,x}{286720}\,I_1^4(x)-$$
$$-\frac{273087}{286720}\,\mathcal{I}_0^{*(40)}(x)+\frac{2004729}{286720}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\;J_0^4(x)\,dx=\frac{48\,x^{10}+652\,x^8-4074\,x^6+6111\,x^4-12222\,x^2}{1024}\,J_0^4(x)+$$
$$+\frac{136\,x^9-1304\,x^7+6111\,x^5-6111\,x^3+6111\,x}{256}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{48\,x^{10}-436\,x^8+3750\,x^6-18333\,x^4}{512}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{72\,x^9-868\,x^7+3611\,x^5}{256}\,J_0(x)\,J_1^3(x)+\frac{48\,x^{10}-868\,x^8+7222\,x^6-25555\,x^4}{1024}\,J_1^4(x)+\frac{18333}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\;I_0^4(x)\,dx=\frac{48\,x^{10}-652\,x^8-4074\,x^6-6111\,x^4-12222\,x^2}{1024}\,I_0^4(x)+$$
$$+\frac{136\,x^9+1304\,x^7+6111\,x^5+6111\,x^3+6111\,x}{256}\,I_0^3(x)\,I_1(x)-$$
$$-\frac{48\,x^{10}+436\,x^8+3750\,x^6+18333\,x^4}{512}\,I_0^2(x)\,I_1^2(x)-$$
$$-\frac{72\,x^9+868\,x^7+3611\,x^5}{256}\,I_0(x)\,I_1^3(x)+\frac{48\,x^{10}+868\,x^8+7222\,x^6+25555\,x^4}{1024}\,I_1^4(x)-\frac{18333}{256}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\;J_0^4(x)\,dx=$$
$$=\frac{573440\,x^{11}+10567680\,x^9-97664256\,x^7+363060288\,x^5+248212404\,x^3-186159303\,x}{13762560}\,J_0^4(x)+$$
$$+\frac{1863680\,x^{10}-23777280\,x^8+170912448\,x^6-453825360\,x^4-186159303\,x^2}{3440640}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{143360\,x^{11}-1551360\,x^9+17194176\,x^7-122875488\,x^5+402422121\,x^3}{1720320}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{1003520\,x^{10}-16537600\,x^8+113598528\,x^6-208074384\,x^4-454440717\,x^2}{3440640}\,J_0(x)\,J_1^3(x)+$$
$$+\frac{573440\,x^{11}-13230080\,x^9+151464704\,x^7-832297536\,x^5+1817762868\,x^3-454440717\,x}{13762560}\,J_1^4(x)-$$

$$+\frac{62053101}{4587520}\,\mathcal{I}_0^{(40)}(x)-\frac{454440717}{4587520}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\,I_0^4(x)\,dx=$$

$$=\frac{573440\,x^{11}-10567680\,x^9-97664256\,x^7-363060288\,x^5+248212404\,x^3+186159303\,x}{13762560}\,I_0^4(x)+$$

$$+\frac{1863680\,x^{10}+23777280\,x^8+170912448\,x^6+453825360\,x4-186159303\,x^2}{3440640}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{143360\,x^{11}+1551360\,x^9+17194176\,x^7+122875488\,x^5+402422121\,x^3}{1720320}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{1003520\,x^{10}+16537600\,x^8+113598528\,x^6+208074384\,x^4-454440717\,x^2}{3440640}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{573440\,x^{11}+13230080\,x^9+151464704\,x^7+832297536\,x^5+1817762868\,x^3+454440717\,x}{13762560}\,I_1^4(x)+$$

$$-\frac{62053101}{4587520}\,\mathcal{I}_0^{*(40)}(x)+\frac{454440717}{4587520}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0^4(x)\,dx}{x^2}=-\frac{6x^2+1}{x}\,J_0^4(x)+4\,J_0^3(x)\,J_1(x)-12x\,J_0^2(x)\,J_1^2(x)-6x\,J_1^4(x)+2\mathcal{I}_0^{(40)}(x)-18\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0^4(x)\,dx}{x^2}=\frac{6x^2-1}{x}\,I_0^4(x)-4\,I_0^3(x)\,I_1(x)-12x\,I_0^2(x)\,I_1^2(x)+6x\,I_1^4(x)-2\mathcal{I}_0^{*(40)}(x)+18\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0^4(x)\,dx}{x^4}=\frac{40\,x^4+4\,x^2-3}{9x^3}\,J_0^4(x)-\frac{24\,x^2-4}{9x^2}\,J_0^3(x)\,J_1(x)+\frac{80\,x^2-4}{9x}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{8}{27}\,J_0(x)\,J_1^3(x)+\frac{40x}{9}\,J_1^4(x)-\frac{16}{9}\,\mathcal{I}_0^{(40)}(x)+\frac{368}{27}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0^4(x)\,dx}{x^4}=\frac{40\,x^4-4\,x^2-3}{9x^3}\,I_0^4(x)-\frac{24\,x^2+4}{9x^2}\,I_0^3(x)\,I_1(x)-\frac{80\,x^2+4}{9x}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{8}{27}\,I_0(x)\,I_1^3(x)+\frac{40x}{9}\,I_1^4(x)-\frac{16}{9}\,\mathcal{I}_0^{*(40)}(x)+\frac{368}{27}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0^4(x)\,dx}{x^6}=-\frac{42752\,x^6+3800\,x^4-1500\,x^2+5625}{28125\,x^5}\,J_0^4(x)+\frac{4992\,x^4-760\,x^2+900}{5625\,x^4}\,J_0^3(x)\,J_1(x)-$$

$$-\frac{85504\,x^4-4880\,x^2+2700}{28125\,x^3}\,J_0^2(x)\,J_1^2(x)-\frac{10624\,x^2-3240}{84375\,x^2}\,J_0(x)\,J_1^3(x)-\frac{42752\,x^2+216}{28125\,x}\,J_1^4(x)+$$

$$+\frac{17792}{28125}\,\mathcal{I}_0^{(40)}(x)-\frac{395392}{84375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0^4(x)\,dx}{x^6}=\frac{42752\,x^6-3800\,x^4-1500\,x^2-5625}{28125\,x^5}\,I_0^4(x)-\frac{4992\,x^4+760\,x^2+900}{5625\,x^4}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{85504\,x^4+4880\,x^2+2700}{28125\,x^3}\,I_0^2(x)\,I_1^2(x)-\frac{10624\,x^2+3240}{84375\,x^2}\,I_0(x)\,I_1^3(x)+\frac{42752\,x^2-216}{28125\,x}\,I_1^4(x)+$$

$$+\frac{17792}{28125}\,\mathcal{I}_0^{*(40)}(x)-\frac{395392}{84375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. f) Integrals of $x^m\,Z_0^3(x)\,Z_1(x)$

Explicit and basic integrals are omitted.

With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 413 holds

$$\int x\, J_0^3(x)\, J_1(x)\, dx = -\frac{x}{4}\, J_0^4(x) + \frac{1}{4}\, \mathcal{I}_0^{(40)}(x)$$

$$\int x\, I_0^3(x)\, I_1(x)\, dx = \frac{x}{4}\, I_0^4(x) - \frac{1}{4}\, \mathcal{I}_0^{*(40)}(x)$$

$$\int x^2\, J_0^3(x)\, J_1(x)\, dx = -\frac{x^2}{4}\, J_0^4(x) + \frac{x}{2}\, J_0^3(x)\, J_1(x) + \frac{3}{2}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^2\, I_0^3(x)\, I_1(x)\, dx = \frac{x^2}{4}\, I_0^4(x) - \frac{x}{2}\, I_0^3(x)\, I_1(x) + \frac{3}{2}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^3\, J_0^3(x)\, J_1(x)\, dx = \frac{4\,x^3 - 3\,x}{128}\, J_0^4(x) - \frac{3x^2}{32}\, J_0^3(x)\, J_1(x) + \frac{9x^3}{16}\, J_0^2(x)\, J_1^2(x) - \frac{9x^2}{32}\, J_0(x)\, J_1^3(x) +$$
$$+ \frac{36\,x^3 - 9\,x}{128}\, J_1^4(x) + \frac{3}{128}\, \mathcal{I}_0^{(40)}(x) - \frac{27}{128}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^3\, I_0^3(x)\, I_1(x)\, dx = -\frac{4\,x^3 + 3\,x}{128}\, I_0^4(x) + \frac{3x^2}{32}\, I_0^3(x)\, I_1(x) + \frac{9x^3}{16}\, I_0^2(x)\, I_1^2(x) - \frac{9x^2}{32}\, I_0(x)\, I_1^3(x) -$$
$$- \frac{36\,x^3 + 9\,x}{128}\, I_1^4(x) + \frac{3}{128}\, \mathcal{I}_0^{*(40)}(x) - \frac{27}{128}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^4\, J_0^3(x)\, J_1(x)\, dx = -\frac{x^4 - 2\,x^2}{16}\, J_0^4(x) + \frac{x^3 - x}{4}\, J_0^3(x)\, J_1(x) + \frac{3x^4}{8}\, J_0^2(x)\, J_1^2(x) + \frac{3x^4}{16}\, J_1^4(x) - \frac{3}{4}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^4\, I_0^3(x)\, I_1(x)\, dx = \frac{x^4 + 2\,x^2}{16}\, I_0^4(x) - \frac{x^3 + x}{4}\, I_0^3(x)\, I_1(x) + \frac{3x^4}{8}\, I_0^2(x)\, I_1^2(x) - \frac{3x^4}{16}\, I_1^4(x) + \frac{3}{4}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^5\, J_0^3(x)\, J_1(x)\, dx = -\frac{96\,x^5 + 60\,x^3 - 45\,x}{1024}\, J_0^4(x) + \frac{120\,x^4 + 45\,x^2}{256}\, J_0^3(x)\, J_1(x) +$$
$$+ \frac{40\,x^5 - 105\,x^3}{128}\, J_0^2(x)\, J_1^2(x) + \frac{40\,x^4 + 115\,x^2}{256}\, J_0(x)\, J_1^3(x) + \frac{160\,x^5 - 460\,x^3 + 115\,x}{1024}\, J_1^4(x) -$$
$$- \frac{45}{1024}\, \mathcal{I}_0^{(40)}(x) + \frac{345}{1024}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^5\, I_0^3(x)\, I_1(x)\, dx = \frac{96\,x^5 - 60\,x^3 - 45\,x}{1024}\, I_0^4(x) - \frac{120\,x^4 - 45\,x^2}{256}\, I_0^3(x)\, I_1(x) +$$
$$+ \frac{40\,x^5 + 105\,x^3}{128}\, I_0^2(x)\, I_1^2(x) + \frac{40\,x^4 - 115\,x^2}{256}\, I_0(x)\, I_1^3(x) - \frac{160\,x^5 + 460\,x^3 + 115\,x}{1024}\, I_1^4(x) +$$
$$+ \frac{45}{1024}\, \mathcal{I}_0^{*(40)}(x) - \frac{345}{1024}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^6\, J_0^3(x)\, J_1(x)\, dx = -\frac{14\,x^6 - 21\,x^4 + 42\,x^2}{128}\, J_0^4(x) + \frac{21\,x^5 - 21\,x^3 + 21\,x}{32}\, J_0^3(x)\, J_1(x) +$$
$$+ \frac{18\,x^6 - 63\,x^4}{64}\, J_0^2(x)\, J_1^2(x) + \frac{9x^5}{32}\, J_0(x)\, J_1^3(x) + \frac{18\,x^6 - 81\,x^4}{128}\, J_1^4(x) + \frac{63}{32}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^6\, I_0^3(x)\, I_1(x)\, dx = \frac{14\,x^6 + 21\,x^4 + 42\,x^2}{128}\, I_0^4(x) - \frac{21\,x^5 + 21\,x^3 + 21\,x}{32}\, I_0^3(x)\, I_1(x) +$$

$$+\frac{18\,x^6+63\,x^4}{64}\,I_0^2(x)\,I_1^2(x)+\frac{9x^5}{32}\,I_0(x)\,I_1^3(x)-\frac{18\,x^6+81\,x^4}{128}\,I_1^4(x)+\frac{63}{32}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^7\;J_0^3(x)\,J_1(x)\,dx=-\frac{4864\,x^7-17472\,x^5-11676\,x^3+8757\,x}{40960}\,J_0^4(x)+$$
$$+\frac{8512\,x^6-21840\,x^4-8757\,x^2}{10240}\,J_0^3(x)\,J_1(x)+\frac{1344\,x^7-6272\,x^5+19299\,x^3}{5120}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{4032\,x^6-9296\,x^4-21623\,x^2}{10240}\,J_0(x)\,J_1^3(x)+\frac{5376\,x^7-37184\,x^5+86492\,x^3-21623\,x}{40960}\,J_1^4(x)+$$
$$+\frac{8757}{40960}\,\mathcal{I}_0^{(40)}(x)-\frac{64869}{40960}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^7\;I_0^3(x)\,I_1(x)\,dx=\frac{4864\,x^7+17472\,x^5-11676\,x^3-8757\,x}{40960}\,I_0^4(x)-$$
$$-\frac{8512\,x^6+21840\,x^4-8757\,x^2}{10240}\,I_0^3(x)\,I_1(x)+\frac{1344\,x^7+6272\,x^5+19299\,x^3}{5120}\,I_0^2(x)\,I_1^2(x)+$$
$$+\frac{4032\,x^6+9296\,x^4-21623\,x^2}{10240}\,I_0(x)\,I_1^3(x)-\frac{5376\,x^7+37184\,x^5+86492\,x^3+21623\,x}{40960}\,I_1^4(x)+$$
$$+\frac{8757}{40960}\,\mathcal{I}_0^{*(40)}(x)-\frac{64869}{40690}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^8\;J_0^3(x)\,J_1(x)\,dx=-\frac{x^8-6\,x^6+9\,x^4-18\,x^2}{8}\,J_0^4(x)+\frac{2\,x^7-9\,x^5+9\,x^3-9\,x}{2}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{x^8-6\,x^6+27\,x^4}{4}\,J_0^2(x)\,J_1^2(x)+\frac{x^7-5\,x^5}{2}\,J_0(x)\,J_1^3(x)+\frac{x^8-10\,x^6+37\,x^4}{8}\,J_1^4(x)-\frac{27}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^8\;I_0^3(x)\,I_1(x)\,dx=\frac{x^8+6\,x^6+9\,x^4+18\,x^2}{8}\,I_0^4(x)-\frac{2\,x^7+9\,x^5+9\,x^3+9\,x}{2}\,I_0^3(x)\,I_1(x)+$$
$$+\frac{x^8+6\,x^6+27\,x^4}{4}\,I_0^2(x)\,I_1^2(x)+\frac{x^7+5\,x^5}{2}\,I_0(x)\,I_1^3(x)-\frac{x^8+10\,x^6+37\,x^4}{8}\,I_1^4(x)+\frac{27}{2}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^9\;J_0^3(x)\,J_1(x)\,dx=-\frac{148480\,x^9-1305216\,x^7+4816368\,x^5+3277044\,x^3-2457783\,x}{1146880}\,J_0^4(x)+$$
$$+\frac{334080\,x^8-2284128\,x^6+6020460\,x^4+2457783\,x^2}{286720}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{17280\,x^9-129168\,x^7+825534\,x^5-2667303\,x^3}{71680}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{172800\,x^8-1423008\,x^6+2718324\,x^4+6014187\,x^2}{286720}\,J_0(x)\,J_1^3(x)+$$
$$+\frac{138240\,x^9-1897344\,x^7+10873296\,x^5-24056748\,x^3+6014187\,x}{1146880}\,J_1^4(x)-$$
$$-\frac{2457783}{1146880}\,\mathcal{I}_0^{(40)}(x)+\frac{18042561}{1146880}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^9\;I_0^3(x)\,I_1(x)\,dx=\frac{148480\,x^9+1305216\,x^7+4816368\,x^5-3277044\,x^3-2457783\,x}{1146880}\,I_0^4(x)-$$
$$-\frac{334080\,x^8+2284128\,x^6+6020460\,x^4-2457783\,x^2}{286720}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{17280\,x^9+129168\,x^7+825534\,x^5+2667303\,x^3}{71680}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{172800\,x^8+1423008\,x^6+2718324\,x^4-6014187\,x^2}{286720}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{138240\,x^9+1897344\,x^7+10873296\,x^5+24056748\,x^3+6014187\,x}{1146880}\,I_1^4(x)+$$

$$+\frac{2457783}{1146880}\,\mathcal{I}_0^{*(40)}(x)-\frac{18042561}{1146880}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^{10}\,J_0^3(x)\,J_1(x)\,dx=-\frac{272\,x^{10}-3260\,x^8+20370\,x^6-30555\,x^4+61110\,x^2}{2048}\,J_0^4(x)+$$

$$+\frac{680\,x^9-6520\,x^7+30555\,x^5-30555\,x^3+30555\,x}{512}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{240\,x^{10}-2180\,x^8+18750\,x^6-91665\,x^4}{1024}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{360\,x^9-4340\,x^7+18055\,x^5}{512}\,J_0(x)\,J_1^3(x)+\frac{240\,x^{10}-4340\,x^8+36110\,x^6-127775\,x^4}{2048}\,J_1^4(x)+\frac{91665}{512}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^{10}\,I_0^3(x)\,I_1(x)\,dx=\frac{272\,x^{10}+3260\,x^8+20370\,x^6+30555\,x^4+61110\,x^2}{2048}\,I_0^4(x)-$$

$$-\frac{680\,x^9+6520\,x^7+30555\,x^5+30555\,x^3+30555\,x}{512}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{240\,x^{10}+2180\,x^8+18750\,x^6+91665\,x^4}{1024}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{360\,x^9+4340\,x^7+18055\,x^5}{512}\,I_0(x)\,I_1^3(x)-\frac{240\,x^{10}+4340\,x^8+36110\,x^6+127775\,x^4}{2048}\,I_1^4(x)+\frac{91665}{512}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int\frac{J_0^3(x)\,J_1(x)\,dx}{x}=\frac{3x}{2}\,J_0^4(x)-J_0^3(x)\,I_1(x)+3x\,J_0^2(x)\,J_1^2(x)+\frac{3x}{2}\,J_1^4(x)-\frac{1}{2}\,\mathcal{I}_0^{(40)}(x)+\frac{9}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0^3(x)\,I_1(x)\,dx}{x}=\frac{3x}{2}\,I_0^4(x)-I_0^3(x)\,I_1(x)-3x\,I_0^2(x)\,I_1^2(x)+\frac{3x}{2}\,I_1^4(x)-\frac{1}{2}\,\mathcal{I}_0^{*(40)}(x)+\frac{9}{2}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0^3(x)\,J_1(x)\,dx}{x^3}=-\frac{10\,x^2+1}{3x}\,J_0^4(x)+\frac{6\,x^2-1}{3x^2}\,J_0^3(x)\,I_1(x)-\frac{20\,x^2-1}{3x}\,J_0^2(x)\,J_1^2(x)-\frac{2}{9}\,J_0(x)\,J_1^3(x)-$$

$$-\frac{10x}{3}\,J_1^4(x)+\frac{4}{3}\,\mathcal{I}_0^{(40)}(x)-\frac{92}{9}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0^3(x)\,I_1(x)\,dx}{x^3}=\frac{10\,x^2-1}{3x}\,I_0^4(x)-\frac{6\,x^2+1}{3x^2}\,I_0^3(x)\,I_1(x)-\frac{20\,x^2+1}{3x}\,I_0^2(x)\,I_1^2(x)-\frac{2}{9}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{10x}{3}\,I_1^4(x)-\frac{4}{3}\,\mathcal{I}_0^{*(40)}(x)+\frac{92}{9}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0^3(x)\,J_1(x)\,dx}{x^5}=\frac{10688\,x^4+950\,x^2-375}{5625\,x^3}\,J_0^4(x)-\frac{1248\,x^4-190\,x^2+225}{1125\,x^4}\,J_0^3(x)\,I_1(x)+$$

$$+\frac{21376\,x^4-1220\,x^2+675}{5625\,x^3}\,J_0^2(x)\,J_1^2(x)+\frac{2656\,x^2-810}{16875\,x^2}\,J_0(x)\,J_1^3(x)+\frac{10688\,x^2+54}{5625\,x}\,J_1^4(x)-$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{(40)}(x)+\frac{98848}{16875}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x^5} = \frac{10688\,x^4-950\,x^2-375}{5625\,x^3}\,I_0^4(x) - \frac{1248\,x^4+190\,x^2+225}{1125\,x^4}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{21376\,x^4+1220\,x^2+675}{5625\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{2656\,x^2+810}{16875\,x^2}\,I_0(x)\,I_1^3(x) + \frac{10688\,x^2-54}{5625\,x}\,I_1^4(x) -$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{*(40)}(x) + \frac{98848}{16875}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. g) Integrals of $x^m\,Z_0^2(x)\,Z_1^2(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 413 holds

$$\int J_0^2(x)\,J_1^2(x)\,dx = -\frac{x}{2}\,J_0^4(x) - x\,J_0^2(x)\,J_1^2(x) - \frac{x}{2}\,J_1^4(x) + \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int I_0^2(x)\,I_1^2(x)\,dx = \frac{x}{2}\,I_0^4(x) - x\,I_0^2(x)\,I_1^2(x) + \frac{x}{2}\,I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{3}{2}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^2\,J_0^2(x)\,J_1^2(x)\,dx = \frac{4\,x^3-3\,x}{32}\,J_0^4(x) - \frac{3x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{x^2}{8}\,J_0(x)\,J_1^3(x) +$$

$$+\frac{4\,x^3-x}{32}\,J_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{4\,x^3+3\,x}{32}\,I_0^4(x) + \frac{3x^2}{8}\,I_0^3(x)\,I_1(x) + \frac{x^3}{4}\,I_0^2(x)\,I_1^2(x) - \frac{x^2}{8}\,I_0(x)\,I_1^3(x) -$$

$$-\frac{4\,x^3+x}{32}\,I_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{*(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\,J_0^2(x)\,J_1^2(x)\,dx = \frac{x^4-2\,x^2}{16}\,J_0^4(x) - \frac{x^3-x}{4}\,J_0^3(x)\,J_1(x) + \frac{x^4}{8}\,J_0^2(x)\,J_1^2(x) + \frac{x^4}{16}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{x^4+2\,x^2}{16}\,I_0^4(x) + \frac{x^3+x}{4}\,I_0^3(x)\,I_1(x) + \frac{x^4}{8}\,I_0^2(x)\,I_1^2(x) - \frac{x^4}{16}\,I_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\,J_0^2(x)\,J_1^2(x)\,dx = \frac{32\,x^5+12\,x^3-9\,x}{768}\,J_0^4(x) - \frac{40\,x^4+9\,x^2}{192}\,J_0^3(x)\,J_1(x) + \frac{8\,x^5+33\,x^3}{96}\,J_0^2(x)\,J_1^2(x) +$$

$$+\frac{8\,x^4-31\,x^2}{192}\,J_0(x)\,J_1^3(x) + \frac{32\,x^5+124\,x^3-31\,x}{768}\,J_1^4(x) + \frac{3}{256}\,\mathcal{I}_0^{(40)}(x) - \frac{31}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{32\,x^5-12\,x^3-9\,x}{768}\,I_0^4(x) + \frac{40\,x^4-9\,x^2}{192}\,I_0^3(x)\,I_1(x) + \frac{8\,x^5-33\,x^3}{96}\,I_0^2(x)\,I_1^2(x) +$$

$$+\frac{8\,x^4+31\,x^2}{192}\,I_0(x)\,I_1^3(x) - \frac{32\,x^5-124\,x^3-31\,x}{768}\,I_1^4(x) - \frac{3}{256}\,\mathcal{I}_0^{*(40)}(x) + \frac{31}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\,J_0^2(x)\,J_1^2(x)\,dx = \frac{2\,x^6-3\,x^4+6\,x^2}{64}\,J_0^4(x) - \frac{3\,x^5-3\,x^3+3\,x}{16}\,J_0^3(x)\,J_1(x) + \frac{2\,x^6+9\,x^4}{32}\,J_0^2(x)\,J_1^2(x) +$$

$$+\frac{x^5}{16}\,J_0(x)\,J_1^3(x)+\frac{2\,x^6+7\,x^4}{64}\,J_1^4(x)-\frac{9}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\;I_0^2(x)\,I_1^2(x)\,dx=-\frac{2\,x^6+3\,x^4+6\,x^2}{64}\,I_0^4(x)+\frac{3\,x^5+3\,x^3+3\,x}{16}\,I_0^3(x)\,I_1(x)+\frac{2\,x^6-9\,x^4}{32}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{x^5}{16}\,I_0(x)\,I_1^3(x)-\frac{2\,x^6-7\,x^4}{64}\,I_1^4(x)-\frac{9}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\;J_0^2(x)\,J_1^2(x)\,dx=\frac{256\,x^7-768\,x^5-444\,x^3+333\,x}{10240}\,J_0^4(x)-\frac{448\,x^6-960\,x^4-333\,x^2}{2560}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{64\,x^7+368\,x^5-831\,x^3}{1280}\,J_0^2(x)\,J_1^2(x)+\frac{192\,x^6+224\,x^4+887\,x^2}{2560}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{256\,x^7+896\,x^5-3548\,x^3+887\,x}{10240}\,J_1^4(x)-\frac{333}{10240}\,\mathcal{I}_0^{(40)}(x)+\frac{2661}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\;I_0^2(x)\,I_1^2(x)\,dx=-\frac{256\,x^7+768\,x^5-444\,x^3-333\,x}{10240}\,I_0^4(x)+\frac{448\,x^6+960\,x^4-333\,x^2}{2560}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{64\,x^7-368\,x^5-831\,x^3}{1280}\,I_0^2(x)\,I_1^2(x)+\frac{192\,x^6-224\,x^4+887\,x^2}{2560}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{256\,x^7-896\,x^5-3548\,x^3-887\,x}{10240}\,I_1^4(x)-\frac{333}{10240}\,\mathcal{I}_0^{*(40)}(x)+\frac{2661}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\;J_0^2(x)\,J_1^2(x)\,dx=\frac{4\,x^8-18\,x^6+27\,x^4-54\,x^2}{192}\,J_0^4(x)-\frac{8\,x^7-27\,x^5+27\,x^3-27\,x}{48}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{4\,x^8+30\,x^6-81\,x^4}{96}\,J_0^2(x)\,J_1^2(x)+\frac{4\,x^7+7\,x^5}{48}\,J_0(x)\,J_1^3(x)\frac{4\,x^8+14\,x^6-95\,x^4}{192}\,J_1^4(x)+\frac{27}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^7\;I_0^2(x)\,I_1^2(x)\,dx=-\frac{4\,x^8+18\,x^6+27\,x^4+54\,x^2}{192}\,I_0^4(x)+\frac{8\,x^7+27\,x^5+27\,x^3+27\,x}{48}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{4\,x^8-30\,x^6-81\,x^4}{96}\,I_0^2(x)\,I_1^2(x)+\frac{4\,x^7-7\,x^5}{48}\,I_0(x)\,I_1^3(x)-\frac{4\,x^8-14\,x^6-95\,x^4}{192}\,I_1^4(x)-\frac{27}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\;J_0^2(x)\,J_1^2(x)\,dx=\frac{2560\,x^9-15552\,x^7+53496\,x^5+34668\,x^3-26001\,x}{143360}\,J_0^4(x)-$$

$$-\frac{5760\,x^8-27216\,x^6+66870\,x^4+26001\,x^2}{35840}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{1280\,x^9+12384\,x^7-41292\,x^5+117639\,x^3}{35840}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{1600\,x^8+3288\,x^6-12789\,x^4-32607\,x^2}{17920}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{1280\,x^9+4384\,x^7-51156\,x^5+130428\,x^3-32607\,x}{71680}\,J_1^4(x)-$$

$$+\frac{26001}{143360}\,\mathcal{I}_0^{(40)}(x)-\frac{97821}{71680}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\;I_0^2(x)\,I_1^2(x)\,dx=-\frac{2560\,x^9+15552\,x^7+53496\,x^5-34668\,x^3-26001\,x}{143360}\,I_0^4(x)+$$

$$+\frac{5760\,x^8+27216\,x^6+66870\,x^4-26001\,x^2}{35840}\,I_0^3(x)\,I_1(x)+$$
$$+\frac{1280\,x^9-12384\,x^7-41292\,x^5-117639\,x^3}{35840}\,I_0^2(x)\,I_1^2(x)+$$
$$+\frac{1600\,x^8-3288\,x^6-12789\,x^4+32607\,x^2}{17920}\,I_0(x)\,I_1^3(x)-$$
$$-\frac{1280\,x^9-4384\,x^7-51156\,x^5-130428\,x^3-32607\,x}{71680}\,I_1^4(x)-\frac{26001}{143360}\,\mathcal{I}_0^{*(40)}(x)+\frac{97821}{71680}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\;J_0^2(x)\,J_1^2(x)\,dx=\frac{16\,x^{10}-124\,x^8+690\,x^6-1035\,x^4+2070\,x^2}{1024}\,J_0^4(x)-$$
$$-\frac{40\,x^9-248\,x^7+1035\,x^5-1035\,x^3+1035\,x}{256}\,J_0^3(x)\,J_1(x)+\frac{16\,x^{10}+196\,x^8-798\,x^6+3105\,x^4}{512}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{24\,x^9+52\,x^7-503\,x^5}{256}\,J_0(x)\,J_1^3(x)+\frac{16\,x^{10}+52\,x^8-1006\,x^6+4111\,x^4}{1024}\,J_1^4(x)-\frac{3105}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\;I_0^2(x)\,I_1^2(x)\,dx=-\frac{16\,x^{10}+124\,x^8+690\,x^6+1035\,x^4+2070\,x^2}{1024}\,I_0^4(x)+$$
$$+\frac{40\,x^9+248\,x^7+1035\,x^5+1035\,x^3+1035\,x}{256}\,I_0^3(x)\,I_1(x)+\frac{16\,x^{10}-196\,x^8-798\,x^6-3105\,x^4}{512}\,I_0^2(x)\,I_1^2(x)+$$
$$+\frac{24\,x^9-52\,x^7-503\,x^5}{256}\,I_0(x)\,I_1^3(x)-\frac{16\,x^{10}-52\,x^8-1006\,x^6-4111\,x^4}{1024}\,I_1^4(x)-\frac{3105}{256}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\;J_0^2(x)\,J_1^2(x)\,dx=$$
$$=\frac{573440\,x^{11}-5468160\,x^9+43299072\,x^7-157107456\,x^5-105708348\,x^3+79281261\,x}{41287680}\,J_0^4(x)-$$
$$-\frac{1576960\,x^{10}-12303360\,x^8+75773376\,x^6-196384320\,x^4-79281261\,x^2}{10321920}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{143360\,x^{11}+2181120\,x^9-10706112\,x^7+55439856\,x^5-173715327\,x^3}{5160960}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{1003520\,x^{10}+2124800\,x^8-40086336\,x^6+85504608\,x^4+195091479\,x^2}{10321920}\,J_0(x)\,J_1^3(x)+$$
$$+\frac{573440\,x^{11}+1699840\,x^9-53448448\,x^7+342018432\,x^5-780365916\,x^3+195091479\,x}{41287680}\,J_1^4(x)-$$
$$-\frac{8809029}{4587520}\,\mathcal{I}_0^{(40)}(x)+\frac{65030493}{4587520}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\;I_0^2(x)\,I_1^2(x)\,dx=$$
$$=-\frac{573440\,x^{11}+5468160\,x^9+43299072\,x^7+157107456\,x^5-105708348\,x^3-79281261\,x}{41287680}\,I_0^4(x)+$$
$$+\frac{1576960\,x^{10}+12303360\,x^8+75773376\,x^6+196384320\,x^4-79281261\,x^2}{10321920}\,I_0^3(x)\,I_1(x)+$$
$$+\frac{143360\,x^{11}-2181120\,x^9-10706112\,x^7-55439856\,x^5-173715327\,x^3}{5160960}\,I_0^2(x)\,I_1^2(x)+$$
$$+\frac{1003520\,x^{10}-2124800\,x^8-40086336\,x^6-85504608\,x^4+195091479\,x^2}{10321920}\,I_0(x)\,I_1^3(x)-$$
$$-\frac{573440\,x^{11}-1699840\,x^9-53448448\,x^7-342018432\,x^5-780365916\,x^3-195091479\,x}{41287680}\,I_1^4(x)-$$
$$-\frac{8809029}{4587520}\,\mathcal{I}_0^{*(40)}(x)+\frac{65030493}{4587520}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^2} = \frac{4x}{3}\,J_0^4(x) - \frac{2}{3}\,J_0^3(x)\,J_1(x) + \frac{8\,x^2-1}{3x}\,J_0^2(x)\,J_1^2(x) + \frac{2}{9}\,J_0(x)\,J_1^3(x) + \frac{4x}{3}\,J_1^4(x) -$$
$$-\frac{2}{3}\,\mathcal{I}_0^{(40)}(x) + \frac{38}{9}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^2} = \frac{4x}{3}\,I_0^4(x) - \frac{2}{3}\,I_0^3(x)\,I_1(x) - \frac{8\,x^2+1}{3x}\,I_0^2(x)\,I_1^2(x) - \frac{2}{9}\,I_0(x)\,I_1^3(x) + \frac{4x}{3}\,I_1^4(x) -$$
$$-\frac{2}{3}\,\mathcal{I}_0^{*(40)}(x) + \frac{38}{9}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^4} = -\frac{632\,x^2+50}{375\,x}\,J_0^4(x) + \frac{72\,x^2-10}{75\,x^2}\,J_0^3(x)\,J_1(x) - \frac{1264\,x^4-80\,x^2+75}{375\,x^3}\,J_0^2(x)\,J_1^2(x) -$$
$$-\frac{184\,x^2-90}{1125x^2}\,J_0(x)\,J_1^3(x) - \frac{632\,x^2+6}{375\,x}\,J_1^4(x) + \frac{272}{375}\,\mathcal{I}_0^{(40)}(x) - \frac{5872}{1125}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^4} = \frac{632\,x^2-50}{375\,x}\,I_0^4(x) - \frac{72\,x^2+10}{75x^2}\,I_0^3(x)\,I_1(x) - \frac{1264\,x^4+80\,x^2+75}{375x^3}\,I_0^2(x)\,I_1^2(x) -$$
$$-\frac{184\,x^2+90}{1125\,x^2}\,I_0(x)\,I_1^3(x) + \frac{632\,x^2-6}{375\,x}\,I_1^4(x) - \frac{272}{375}\,\mathcal{I}_0^{*(40)}(x) + \frac{5872}{1125}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4+124600\,x^2-36750}{1929375\,x^3}\,J_0^4(x) - \frac{171264\,x^4-24920\,x^2+22050}{385875\,x^4}\,J_0^3(x)\,J_1(x) +$$
$$+\frac{2970368\,x^6-178960\,x^4+113400\,x^2-275625}{1929375\,x^5}\,J_0^2(x)\,J_1^2(x) + \frac{401408\,x^4-163080\,x^2+236250}{5788125\,x^4}\,J_0(x)\,J_1^3(x) +$$
$$+\frac{1485184\,x^4+10872\,x^2-11250}{1929375\,x^3}\,J_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4-124600\,x^2-36750}{1929375\,x^3}\,I_0^4(x) - \frac{171264\,x^4+24920\,x^2+22050}{385875\,x^4}\,I_0^3(x)\,I_1(x) -$$
$$-\frac{2970368\,x^6+178960\,x^4+113400\,x^2+275625}{1929375\,x^5}\,I_0^2(x)\,I_1^2(x) \frac{401408\,x^4+163080\,x^2+236250}{5788125\,x^4}\,I_0(x)\,I_1^3(x) +$$
$$+\frac{1485184\,x^4-10872\,x^2-11250}{1929375\,x^3}\,I_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{*(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. h) Integrals of $x^m\,Z_0(x)\,Z_1^3(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 413 holds

$$\int x\,J_0(x)\,J_1^3(x)\,dx = \frac{x}{4}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x\,J_0(x)\,I_1^3(x)\,dx = \frac{x}{4}\,I_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^2\,J_0(x)\,J_1^3(x)\,dx = -\frac{x^2}{4}\,J_0^4(x) + \frac{x}{2}\,J_0^3(x)\,J_1(x) - \frac{x^2}{2}\,J_0^2(x)\,J_1^2(x) + \frac{3}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^2\, I_0(x)\, I_1^3(x)\, dx = -\frac{x^2}{4}\, I_0^4(x) + \frac{x}{2}\, I_0^3(x)\, I_1(x) + \frac{x^2}{2}\, I_0^2(x)\, I_1^2(x) - \frac{3}{2}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^3\, J_0(x)\, J_1^3(x)\, dx = \frac{12\,x^3 - 9\,x}{128}\, J_0^4(x) - \frac{9x^2}{32}\, J_0^3(x)\, J_1(x) + \frac{3x^3}{16}\, J_0^2(x)\, J_1^2(x) - \frac{11x^2}{32}\, J_0(x)\, J_1^3(x) +$$

$$+ \frac{44\,x^3 - 11\,x}{128}\, J_1^4(x) + \frac{9}{128}\, \mathcal{I}_0^{(40)}(x) - \frac{33}{128}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^3\, I_0(x)\, I_1^3(x)\, dx = \frac{12\,x^3 + 9\,x}{128}\, I_0^4(x) - \frac{9x^2}{32}\, I_0^3(x)\, I_1(x) - \frac{3x^3}{16}\, I_0^2(x)\, I_1^2(x) + \frac{11x^2}{32}\, I_0(x)\, I_1^3(x) +$$

$$+ \frac{44\,x^3 + 11\,x}{128}\, I_1^4(x) - \frac{9}{128}\, \mathcal{I}_0^{*(40)}(x) + \frac{33}{128}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\, J_0(x)\, J_1^3(x)\, dx = -\frac{32\,x^5 + 36\,x^3 - 27\,x}{1024}\, J_0^4(x) + \frac{40\,x^4 + 27\,x^2}{256}\, J_0^3(x)\, J_1(x) - \frac{8\,x^5 + 39\,x^3}{128}\, J_0^2(x)\, J_1^2(x) +$$

$$+ \frac{56\,x^4 + 53\,x^2}{256}\, J_0(x)\, J_1^3(x) + \frac{224\,x^5 - 212\,x^3 + 53\,x}{1024}\, J_1^4(x) - \frac{27}{1024}\, \mathcal{I}_0^{(40)}(x) + \frac{159}{1024}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^5\, I_0(x)\, I_1^3(x)\, dx = -\frac{32\,x^5 - 36\,x^3 - 27\,x}{1024}\, I_0^4(x) + \frac{40\,x^4 - 27\,x^2}{256}\, I_0^3(x)\, I_1(x) + \frac{8\,x^5 - 39\,x^3}{128}\, I_0^2(x)\, I_1^2(x) -$$

$$- \frac{56\,x^4 - 53\,x^2}{256}\, I_0(x)\, I_1^3(x) + \frac{224\,x^5 + 212\,x^3 + 53\,x}{1024}\, I_1^4(x) - \frac{27}{1024}\, \mathcal{I}_0^{*(40)}(x) + \frac{159}{1024}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^6\, J_0(x)\, J_1^3(x)\, dx = -\frac{6\,x^6 - 9\,x^4 + 18\,x^2}{128}\, J_0^4(x) + \frac{9\,x^5 - 9\,x^3 + 9\,x}{32}\, J_0^3(x)\, J_1(x) -$$

$$- \frac{6\,x^6 + 27\,x^4}{64}\, J_0^2(x)\, J_1^2(x) + \frac{13x^5}{32}\, J_0(x)\, J_1^3(x) + \frac{26\,x^6 - 53\,x^4}{128}\, J_1^4(x) + \frac{27}{32}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^6\, I_0(x)\, I_1^3(x)\, dx = -\frac{6\,x^6 + 9\,x^4 + 18\,x^2}{128}\, I_0^4(x) + \frac{9\,x^5 + 9\,x^3 + 9\,x}{32}\, I_0^3(x)\, I_1(x) +$$

$$+ \frac{6\,x^6 - 27\,x^4}{64}\, I_0^2(x)\, I_1^2(x) - \frac{13x^5}{32}\, I_0(x)\, I_1^3(x) + \frac{26\,x^6 + 53\,x^4}{128}\, I_1^4(x) - \frac{27}{32}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^7\, J_0(x)\, J_1^3(x)\, dx = -\frac{2304\,x^7 - 9792\,x^5 - 7236\,x^3 + 5427\,x}{40960}\, J_0^4(x) +$$

$$+ \frac{4032\,x^6 - 12240\,x^4 - 5427\,x^2}{10240}\, J_0^3(x)\, J_1(x) - \frac{576\,x^7 + 2592\,x^5 - 10989\,x^3}{5120}\, J_0^2(x)\, J_1^2(x) +$$

$$+ \frac{5952\,x^6 - 7056\,x^4 - 12753\,x^2}{10240}\, J_0(x)\, J_1^3(x) + \frac{7936\,x^7 - 28224\,x^5 + 51012\,x^3 - 12753\,x}{40960}\, J_1^4(x) +$$

$$+ \frac{5427}{40960}\, \mathcal{I}_0^{(40)}(x) - \frac{38259}{40960}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^7\, I_0(x)\, I_1^3(x)\, dx = -\frac{2304\,x^7 + 9792\,x^5 - 7236\,x^3 - 5427\,x}{40960}\, I_0^4(x) +$$

$$+ \frac{4032\,x^6 + 12240\,x^4 - 5427\,x^2}{10240}\, I_0^3(x)\, I_1(x) + \frac{576\,x^7 - 2592\,x^5 - 10989\,x^3}{5120}\, I_0^2(x)\, I_1^2(x) -$$

$$- \frac{5952\,x^6 + 7056\,x^4 - 12753\,x^2}{10240}\, I_0(x)\, I_1^3(x) + \frac{7936\,x^7 + 28224\,x^5 + 51012\,x^3 + 12753\,x}{40960}\, I_1^4(x) -$$

$$-\frac{5427}{40960}\,\mathcal{I}_0^{*(40)}(x)+\frac{38259}{40960}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^8\;J_0(x)\,J_1^3(x)\,dx=-\frac{4\,x^8-30\,x^6+45\,x^4-90\,x^2}{64}\;J_0^4(x)+\frac{8\,x^7-45\,x^5+45\,x^3-45\,x}{16}\;J_0^3(x)\,J_1(x)-$$
$$-\frac{4\,x^8+18\,x^6-135\,x^4}{32}\;J_0^2(x)\,J_1^2(x)+\frac{12\,x^7-33\,x^5}{16}\;J_0(x)\,J_1^3(x)+\frac{12\,x^8-66\,x^6+201\,x^4}{64}\;J_1^4(x)-\frac{135}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^8\;I_0(x)\,I_1^3(x)\,dx=-\frac{4\,x^8+30\,x^6+45\,x^4+90\,x^2}{64}\;I_0^4(x)+\frac{8\,x^7+45\,x^5+45\,x^3+45\,x}{16}\;I_0^3(x)\,I_1(x)+$$
$$+\frac{4\,x^8-18\,x^6-135\,x^4}{32}\;I_0^2(x)\,I_1^2(x)-\frac{12\,x^7+33\,x^5}{16}\;I_0(x)\,I_1^3(x)+\frac{12\,x^8+66\,x^6+201\,x^4}{64}\;I_1^4(x)-\frac{135}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^9\;J_0(x)\,J_1^3(x)\,dx=-\frac{15360\,x^9-173952\,x^7+663696\,x^5+461268\,x^3-345951\,x}{229376}\;J_0^4(x)+$$
$$+\frac{34560\,x^8-304416\,x^6+829620\,x^4+345951\,x^2}{57344}\;J_0^3(x)\,J_1(x)-$$
$$-\frac{960\,x^9+4248\,x^7-53649\,x^5+184383\,x^3}{7168}\;J_0^2(x)\,J_1^2(x)+$$
$$+\frac{52480\,x^8-247776\,x^6+400428\,x^4+837639\,x^2}{57344}\;J_0(x)\,J_1^3(x)+$$
$$+\frac{41984\,x^9-330368\,x^7+1601712\,x^5-3350556\,x^3+837639\,x}{229376}\;J_1^4(x)-\frac{345951}{229376}\,\mathcal{I}_0^{(40)}(x)+\frac{2512917}{229376}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^9\;I_0(x)\,I_1^3(x)\,dx=-\frac{15360\,x^9+173952\,x^7+663696\,x^5-461268\,x^3-345951\,x}{229376}\;I_0^4(x)+$$
$$+\frac{34560\,x^8+304416\,x^6+829620\,x^4-345951\,x^2}{57344}\;I_0^3(x)\,I_1(x)+$$
$$+\frac{3840\,x^9-16992\,x^7-214596\,x^5-737532\,x^3}{28762}\;I_0^2(x)\,I_1^2(x)-$$
$$-\frac{52480\,x^8+247776\,x^6+400428\,x^4-837639\,x^2}{57344}\;I_0(x)\,I_1^3(x)+$$
$$+\frac{41984\,x^9+330368\,x^7+1601712\,x^5+3350556\,x^3+837639\,x}{229376}\;I_1^4(x)-\frac{345951}{229376}\,\mathcal{I}_0^{*(40)}(x)+\frac{2512917}{229376}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^{10}\;J_0(x)\,J_1^3(x)\,dx=-\frac{144\,x^{10}-2268\,x^8+14850\,x^6-22275\,x^4+44550\,x^2}{2048}\;J_0^4(x)+$$
$$+\frac{360\,x^9-4536\,x^7+22275\,x^5-22275\,x^3+22275\,x}{512}\;J_0^3(x)\,J_1(x)-$$
$$-\frac{144\,x^{10}+612\,x^8-12366\,x^6+66825\,x^4}{1024}\;J_0^2(x)\,J_1^2(x)+\frac{552\,x^9-3924\,x^7+14031\,x^5}{512}\;J_0(x)\,J_1^3(x)+$$
$$+\frac{368\,x^{10}-3924\,x^8+28062\,x^6-94887\,x^4}{2048}\;J_1^4(x)+\frac{66825}{512}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^{10}\;I_0(x)\,I_1^3(x)\,dx=-\frac{144\,x^{10}+2268\,x^8+14850\,x^6+22275\,x^4+44550\,x^2}{2048}\;I_0^4(x)+$$
$$+\frac{360\,x^9+4536\,x^7+22275\,x^5+22275\,x^3+22275\,x}{512}\;I_0^3(x)\,I_1(x)+$$

$$+\frac{144\,x^{10}-612\,x^8-12366\,x^6-66825\,x^4}{1024}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{552\,x^9+3924\,x^7+14031\,x^5}{512}\,I_0(x)\,I_1^3(x)+\frac{368\,x^{10}+3924\,x^8+28062\,x^6+94887\,x^4}{2048}\,I_1^4(x)-\frac{66825}{512}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int\frac{J_0(x)\,J_1^3(x)\,dx}{x}=-\frac{x}{2}\,J_0^4(x)-x\,J_0^2(x)\,J_1^2(x)-\frac{1}{3}\,J_0(x)\,J_1^3(x)-\frac{x}{2}\,J_1^4(x)+\frac{1}{2}\,\mathcal{I}_0^{(40)}(x)-\frac{11}{6}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0(x)\,I_1^3(x)\,dx}{x}=\frac{x}{2}\,I_0^4(x)-x\,I_0^2(x)\,I_1^2(x)-\frac{1}{3}\,I_0(x)\,I_1^3(x)+\frac{x}{2}\,I_1^4(x)-\frac{1}{2}\,\mathcal{I}_0^{*(40)}(x)+\frac{11}{6}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0(x)\,J_1^3(x)\,dx}{x^3}=\frac{22x}{25}\,J_0^4(x)-\frac{2}{5}\,J_0^3(x)\,J_1(x)+\frac{44\,x^2-5}{25x}\,J_0^2(x)\,J_1^2(x)+\frac{14\,x^2-15}{75x^2}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{22\,x^2+1}{25x}\,J_1^4(x)-\frac{12}{25}\,\mathcal{I}_0^{(40)}(x)+\frac{212}{75}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0(x)\,I_1^3(x)\,dx}{x^3}=\frac{22x}{25}\,I_0^4(x)-\frac{2}{5}\,I_0^3(x)\,I_1(x)-\frac{44\,x^2+5}{25x}\,I_0^2(x)\,I_1^2(x)-\frac{14\,x^2+15}{75x^2}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{22\,x^2-1}{25x}\,I_1^4(x)-\frac{12}{25}\,\mathcal{I}_0^{*(40)}(x)+\frac{212}{75}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_0(x)\,J_1^3(x)\,dx}{x^5}=-\frac{4864\,x^2+350}{6125\,x}\,J_0^4(x)+\frac{544\,x^2-70}{1225\,x^2}\,J_0^3(x)\,J_1(x)-\frac{9728\,x^4-660\,x^2+525}{6125\,x^3}\,J_0^2(x)\,J_1^2(x)-$$

$$-\frac{1568\,x^4-930\,x^2+2625}{18375\,x^4}\,J_0(x)\,J_1^3(x)-\frac{4864\,x^4+62\,x^2-125}{6125\,x^3}\,J_1^4(x)+\frac{2144}{6125}\,\mathcal{I}_0^{(40)}(x)-\frac{45344}{18375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_0(x)\,I_1^3(x)\,dx}{x^5}=\frac{4864\,x^2-350}{6125\,x}\,I_0^4(x)-\frac{544\,x^2+70}{1225\,x^2}\,I_0^3(x)\,I_1(x)-\frac{9728\,x^4+660\,x^2+525}{6125\,x^3}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{1568\,x^4+930\,x^2+2625}{18375\,x^4}\,I_0(x)\,I_1^3(x)+\frac{4864\,x^4-62\,x^2-125}{6125\,x^3}\,I_1^4(x)-\frac{2144}{6125}\,\mathcal{I}_0^{*(40)}(x)+\frac{45344}{18375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. i) Integrals of $x^m\,Z_1^4(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 413 holds

$$\int x\,J_1^4(x)\,dx=-\frac{x^2}{2}\,J_0^4(x)+x\,J_0^3(x)\,J_1(x)-x^2\,J_0^2(x)\,J_1^2(x)-\frac{x^2}{2}\,J_1^4(x)+3\mathcal{I}_1^{(22)}(x)$$

$$\int x\,I_1^4(x)\,dx=-\frac{x^2}{2}\,I_0^4(x)+x\,I_0^3(x)\,I_1(x)+x^2\,I_0^2(x)\,I_1^2(x)-\frac{x^2}{2}\,I_1^4(x)-3\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\,J_1^4(x)\,dx=\frac{12\,x^3-9\,x}{32}\,J_0^4(x)-\frac{9x^2}{8}\,J_0^3(x)\,J_1(x)+\frac{3x^3}{4}\,J_0^2(x)\,J_1^2(x)-\frac{11x^2}{8}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{12\,x^3-11\,x}{32}\,J_1^4(x)+\frac{9}{32}\,\mathcal{I}_0^{(40)}(x)-\frac{33}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\ I_1^4(x)\,dx = \frac{12\,x^3+9\,x}{32}\,I_0^4(x) - \frac{9x^2}{8}\,I_0^3(x)\,I_1(x) - \frac{3x^3}{4}\,I_0^2(x)\,I_1^2(x) + \frac{11x^2}{8}\,I_0(x)\,I_1^3(x) +$$

$$+\frac{12\,x^3+11\,x}{32}\,I_1^4(x) - \frac{9}{32}\,\mathcal{I}_0^{*(40)}(x) + \frac{33}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\ J_1^4(x)\,dx =$$

$$= \frac{3\,x^4-6\,x^2}{16}\,J_0^4(x) - \frac{3\,x^3-3\,x}{4}\,J_0^3(x)\,J_1(x) + \frac{3x^4}{8}\,J_0^2(x)\,J_1^2(x) - x^3\,J_0(x)\,J_1^3(x) + \frac{3x^4}{16}\,J_1^4(x) + \frac{9}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\ I_1^4(x)\,dx =$$

$$= \frac{3\,x^4+6\,x^2}{16}\,I_0^4(x) - \frac{3\,x^3+3\,x}{4}\,I_0^3(x)\,I_1(x) - \frac{3x^4}{8}\,I_0^2(x)\,I_1^2(x) + x^3\,I_0(x)\,I_1^3(x) + \frac{3x^4}{16}\,I_1^4(x) + \frac{9}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\ J_1^4(x)\,dx = \frac{32\,x^5+36\,x^3-27\,x}{256}\,J_0^4(x) - \frac{40\,x^4+27\,x^2}{64}\,J_0^3(x)\,J_1(x) + \frac{8\,x^5+39\,x^3}{32}\,J_0^2(x)\,J_1^2(x) -$$

$$-\frac{56\,x^4+53\,x^2}{64}\,J_0(x)\,J_1^3(x) + \frac{32\,x^5+212\,x^3-53\,x}{256}\,J_1^4(x) + \frac{27}{256}\,\mathcal{I}_0^{(40)}(x) - \frac{159}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\ I_1^4(x)\,dx = \frac{32\,x^5-36\,x^3-27\,x}{256}\,I_0^4(x) - \frac{40\,x^4-27\,x^2}{64}\,I_0^3(x)\,I_1(x) - \frac{8\,x^5-39\,x^3}{32}\,I_0^2(x)\,I_1^2(x) +$$

$$+\frac{56\,x^4-53\,x^2}{64}\,I_0(x)\,I_1^3(x) + \frac{32\,x^5-212\,x^3-53\,x}{256}\,I_1^4(x) + \frac{27}{256}\,\mathcal{I}_0^{*(40)}(x) - \frac{159}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\ J_1^4(x)\,dx = \frac{6\,x^6-9\,x^4+18\,x^2}{64}\,J_0^4(x) - \frac{9\,x^5-9\,x^3+9\,x}{16}\,J_0^3(x)\,J_1(x) + \frac{6\,x^6+27\,x^4}{32}\,J_0^2(x)\,J_1^2(x) -$$

$$-\frac{13x^5}{16}\,J_0(x)\,J_1^3(x) + \frac{6\,x^6+53\,x^4}{64}\,J_1^4(x) - \frac{27}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\ I_1^4(x)\,dx = \frac{6\,x^6+9\,x^4+18\,x^2}{64}\,I_0^4(x) - \frac{9\,x^5+9\,x^3+9\,x}{16}\,I_0^3(x)\,I_1(x) - \frac{6\,x^6-27\,x^4}{32}\,I_0^2(x)\,I_1^2(x) +$$

$$+\frac{13x^5}{16}\,I_0(x)\,I_1^3(x) + \frac{6\,x^6-53\,x^4}{64}\,I_1^4(x) + \frac{27}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\ J_1^4(x)\,dx = \frac{768\,x^7-3264\,x^5-2412\,x^3+1809\,x}{10240}\,J_0^4(x) - \frac{1344\,x^6-4080\,x^4-1809\,x^2}{2560}\,J_0^3(x)\,J_1(x) +$$

$$+\frac{192\,x^7+864\,x^5-3663\,x^3}{1280}\,J_0^2(x)\,J_1^2(x) - \frac{1984\,x^6-2352\,x^4-4251\,x^2}{2560}\,J_0(x)\,J_1^3(x) +$$

$$+\frac{768\,x^7+9408\,x^5-17004\,x^3+4251\,x}{10240}\,J_1^4(x) - \frac{1809}{10240}\,\mathcal{I}_0^{(40)}(x) + \frac{12753}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\ I_1^4(x)\,dx = \frac{768\,x^7+3264\,x^5-2412\,x^3-1809\,x}{10240}\,I_0^4(x) - \frac{1344\,x^6+4080\,x^4-1809\,x^2}{2560}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{192\,x^7-864\,x^5-3663\,x^3}{1280}\,I_0^2(x)\,I_1^2(x) + \frac{1984\,x^6+2352\,x^4-4251\,x^2}{2560}\,I_0(x)\,I_1^3(x) +$$

$$+\frac{768\,x^7-9408\,x^5-17004\,x^3-4251\,x}{10240}\,I_1^4(x)+\frac{1809}{10240}\,\mathcal{I}_0^{*(40)}(x)-\frac{12753}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\ J_1^4(x)\,dx=\frac{4\,x^8-30\,x^6+45\,x^4-90\,x^2}{64}\,J_0^4(x)-\frac{8\,x^7-45\,x^5+45\,x^3-45\,x}{16}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{4\,x^8+18\,x^6-135\,x^4}{32}\,J_0^2(x)\,J_1^2(x)-\frac{12\,x^7-33\,x^5}{16}\,J_0(x)\,J_1^3(x)+\frac{4\,x^8+66\,x^6-201\,x^4}{64}\,J_1^4(x)+\frac{135}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^7\ I_1^4(x)\,dx=\frac{4\,x^8+30\,x^6+45\,x^4+90\,x^2}{64}\,I_0^4(x)-\frac{8\,x^7+45\,x^5+45\,x^3+45\,x}{16}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{4\,x^8-18\,x^6-135\,x^4}{32}\,I_0^2(x)\,I_1^2(x)+\frac{12\,x^7+33\,x^5}{16}\,I_0(x)\,I_1^3(x)+\frac{4\,x^8-66\,x^6-201\,x^4}{64}\,I_1^4(x)\frac{135}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\ J_1^4(x)\,dx=\frac{15360\,x^9-173952\,x^7+663696\,x^5+461268\,x^3-345951\,x}{286720}\,J_0^4(x)-$$

$$-\frac{34560\,x^8-304416\,x^6+829620\,x^4+345951\,x^2}{71680}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{960\,x^9+4248\,x^7-53649\,x^5+184383\,x^3}{8960}\,J_0^2(x)\,J_1^2(x)-$$

$$-\frac{52480\,x^8-247776\,x^6+400428\,x^4+837639\,x^2}{71680}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{15360\,x^9+330368\,x^7-1601712\,x^5+3350556\,x^3-837639\,x}{286720}\,J_1^4(x)+$$

$$+\frac{345951}{286720}\,\mathcal{I}_0^{(40)}(x)-\frac{2512917}{286720}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\ I_1^4(x)\,dx=\frac{15360\,x^9+173952\,x^7+663696\,x^5-461268\,x^3-345951\,x}{286720}\,I_0^4(x)-$$

$$-\frac{34560\,x^8+304416\,x^6+829620\,x^4-345951\,x^2}{71680}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{960\,x^9-4248\,x^7-53649\,x^5-184383\,x^3}{8960}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{52480\,x^8+247776\,x^6+400428\,x^4-837639\,x^2}{71680}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{15360\,x^9-330368\,x^7-1601712\,x^5-3350556\,x^3-837639\,x}{286720}\,I_1^4(x)+$$

$$+\frac{345951}{286720}\,\mathcal{I}_0^{*(40)}(x)-\frac{2512917}{286720}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\ J_1^4(x)\,dx=\frac{48\,x^{10}-756\,x^8+4950\,x^6-7425\,x^4+14850\,x^2}{1024}\,J_0^4(x)-$$

$$-\frac{120\,x^9-1512\,x^7+7425\,x^5-7425\,x^3+7425\,x}{256}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{48\,x^{10}+204\,x^8-4122\,x^6+22275\,x^4}{512}\,J_0^2(x)\,J_1^2(x)-\frac{184\,x^9-1308\,x^7+4677\,x^5}{256}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{48\,x^{10}+1308\,x^8-9354\,x^6+31629\,x^4}{1024}\,J_1^4(x)-\frac{22275}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\ I_1^4(x)\,dx=\frac{48\,x^{10}+756\,x^8+4950\,x^6+7425\,x^4+14850\,x^2}{1024}\,I_0^4(x)-$$

$$-\frac{120\,x^9+1512\,x^7+7425\,x^5+7425\,x^3+7425\,x}{256}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{48\,x^{10}-204\,x^8-4122\,x^6-22275\,x^4}{512}\,I_0^2(x)\,I_1^2(x)+\frac{184\,x^9+1308\,x^7+4677\,x^5}{256}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{48\,x^{10}-1308\,x^8-9354\,x^6-31629\,x^4}{1024}\,I_1^4(x)+\frac{22275}{256}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\;J_1^4(x)\,dx=$$

$$=\frac{573440\,x^{11}-11919360\,x^9+116358912\,x^7-435859776\,x^5-299440908\,x^3+224580681\,x}{13762560}\,J_0^4(x)-$$

$$-\frac{1576960\,x^{10}-26818560\,x^8+203628096\,x^6-544824720\,x^4-224580681\,x^2}{3440640}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{143360\,x^{11}+568320\,x^9-17842752\,x^7+145570176\,x^5-483478767\,x^3}{1720320}\,J_0^2(x)\,J_1^2(x)-$$

$$-\frac{2437120\,x^{10}-24166400\,x^8+144152256\,x^6-253684368\,x^4-546899859\,x^2}{3440640}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{573440\,x^{11}+19333120\,x^9-192203008\,x^7+1014737472\,x^5-2187599436\,x^3+546899859\,x}{13762560}\,J_1^4(x)-$$

$$-\frac{74860227}{4587520}\,\mathcal{I}_0^{(40)}(x)+\frac{546899859}{4587520}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\;I_1^4(x)\,dx=\frac{573440\,x^{11}+11919360\,x^9+116358912\,x^7+435859776\,x^5-299440908\,x^3-224580681\,x}{13762560}\,I_0^4(x)-$$

$$-\frac{1576960\,x^{10}+26818560\,x^8+203628096\,x^6+544824720\,x^4-224580681\,x^2}{3440640}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{143360\,x^{11}-568320\,x^9-17842752\,x^7-145570176\,x^5-483478767\,x^3}{1720320}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{2437120\,x^{10}+24166400\,x^8+144152256\,x^6+253684368\,x^4-546899859\,x^2}{3440640}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{573440\,x^{11}-19333120\,x^9-192203008\,x^7-1014737472\,x^5-2187599436\,x^3-546899859\,x}{13762560}\,I_1^4(x)+$$

$$+\frac{74860227}{4587520}\,\mathcal{I}_0^{*(40)}(x)-\frac{546899859}{4587520}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_1^4(x)\,dx}{x^2}=$$

$$=-\frac{2x}{5}\,J_0^4(x)-\frac{4x}{5}\,J_0^2(x)\,J_1^2(x)-\frac{4}{15}\,J_0(x)\,J_1^3(x)-\frac{2\,x^2+1}{5x}\,J_1^4(x)+\frac{2}{5}\,\mathcal{I}_0^{(40)}(x)-\frac{22}{15}\,\mathcal{I}_0^{(04)}(x)$$

$$\int\frac{I_1^4(x)\,dx}{x^2}=$$

$$=\frac{2x}{5}\,I_0^4(x)-\frac{4x}{5}\,I_0^2(x)\,I_1^2(x)-\frac{4}{15}\,I_0(x)\,I_1^3(x)+\frac{2\,x^2-1}{5x}\,I_1^4(x)-\frac{2}{5}\,\mathcal{I}_0^{*(40)}(x)+\frac{22}{15}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int\frac{J_1^4(x)\,dx}{x^4}=\frac{88x}{175}\,J_0^4(x)-\frac{8}{35}\,J_0^3(x)\,J_1(x)+\frac{176\,x^2-20}{175x}\,J_0^2(x)\,J_1^2(x)+\frac{56\,x^2-60}{525\,x^2}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{88\,x^4+4\,x^2-25}{175\,x^3}\,J_1^4(x)-\frac{48}{175}\,\mathcal{I}_0^{(40)}(x)+\frac{848}{525}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^4} = \frac{88x}{175}\,I_0^4(x) - \frac{8}{35}\,I_0^3(x)\,I_1(x) - \frac{176\,x^2+20}{175x}\,I_0^2(x)\,I_1^2(x) - \frac{56\,x^2+60}{525\,x^2}\,I_0(x)\,I_1^3(x) +$$

$$+\frac{88\,x^4-4\,x^2-25}{175\,x^3}\,I_1^4(x) - \frac{48}{175}\,\mathcal{I}_0^{*(40)}(x) + \frac{848}{525}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\,dx}{x^6} = -\frac{19456\,x^2+1400}{55125\,x}\,J_0^4(x) + \frac{2176\,x^2-280}{11025\,x^2}\,J_0^3(x)\,J_1(x) -$$

$$-\frac{38912\,x^4-2640\,x^2+2100}{55125\,x^3}\,J_0^2(x)\,J_1^2(x) - \frac{6272\,x^4-3720\,x^2+10500}{165375\,x^4}\,J_0(x)\,J_1^3(x) -$$

$$-\frac{19456\,x^6+248\,x^4-500\,x^2+6125}{55125\,x^5}\,J_1^4(x) + \frac{8576}{55125}\,\mathcal{I}_0^{(40)}(x) - \frac{181376}{165375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^6} = \frac{19456\,x^2-1400}{55125\,x}\,I_0^4(x) - \frac{2176\,x^2+280}{11025\,x^2}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{38912\,x^4+2640\,x^2+2100}{55125\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{6272\,x^4+3720\,x^2+10500}{165375\,x^4}\,I_0(x)\,I_1^3(x) +$$

$$+\frac{19456\,x^6-248\,x^4-500\,x^2-6125}{55125\,x^5}\,I_1^4(x) - \frac{8576}{55125}\,\mathcal{I}_0^{*(40)}(x) + \frac{181376}{165375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. j) Recurrence relations

With the integrals

$$\mathcal{I}_n^{(pq)}(x) \quad = \quad \int x^n\,J_0^p(x)\,J_1^q(x)\,dx\,, \qquad p+q=4$$

and

$$\mathcal{I}_n^{*(pq)}(x) \quad = \quad \int x^n\,I_0^p(x)\,I_1^q(x)\,dx\,, \qquad p+q=4$$

holds

Ascending recurrence relations:

$$\mathcal{I}_{n+1}^{(40)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) + (5n-6)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) + 3(n-2)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right] +$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{(31)}(x) - \frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(31)}(x) = -\frac{x^{n+1}}{4}\,J_0^4(x) + \frac{n+1}{4}\,\mathcal{I}_n^{(40)}(x)$$

$$\mathcal{I}_{n+1}^{(22)}(x) = \frac{x^{n+1}}{8n}\left[x\,J_0^4(x) - (n+2)\,J_0^3(x)\,J_1(x) + 2x\,J_0^2(x)\,J_1^2(x) + (n-2)\,J_0(x)\,J_1^3(x) + x\,J_1^4(x)\right] +$$

$$+\frac{n+2}{8}\,\mathcal{I}_{n+1}^{(31)}(x) - \frac{(n-2)^2}{8n}\mathcal{I}_{n+1}^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(13)}(x) = -\frac{x^{n+1}}{4}\left[J_0^4(x) + 2\,J_0^2(x)\,J_1^2(x)\right] + \frac{n+1}{4}\,\mathcal{I}_{n+1}^{(40)}(x) + \frac{n-1}{2}\,\mathcal{I}_{n+1}^{(22)}(x)$$

$$\mathcal{I}_{n+1}^{(04)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) - 3(n+2)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) - (5n+6)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right] +$$

$$+\frac{3(n+2)}{8}\,\mathcal{I}_{n+1}^{(31)}(x)+\frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_{n+1}^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(40)}(x)=\frac{x^{n+1}}{8n}\,\Big[3x\,I_0^4(x)+(5n-6)\,I_0^3(x)\,I_1(x)-6x\,I_0^2(x)\,I_1^2(x)-3(n-2)\,I_0(x)\,I_1^3(x)+3x\,I_1^4(x)\Big]-$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{*(31)}(x)+\frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(31)}(x)=\frac{x^{n+1}}{4}\,I_0^4(x)-\frac{n+1}{4}\,\mathcal{I}_n^{*(40)}(x)$$

$$\mathcal{I}_{n+1}^{*(22)}(x)=\frac{x^{n+1}}{8n}\,\Big[-x\,I_0^4(x)+(n+2)\,I_0^3(x)\,I_1(x)+2x\,I_0^2(x)\,I_1^2(x)+(n-2)\,I_0(x)\,I_1^3(x)-x\,I_1^4(x)\Big]-$$

$$-\frac{n+2}{8}\,\mathcal{I}_{n+1}^{*(31)}(x)-\frac{(n-2)^2}{8n}\mathcal{I}_{n+1}^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(13)}(x)=\frac{x^{n+1}}{4}\,\Big[-I_0^4(x)+2\,I_0^2(x)\,I_1^2(x)\Big]+\frac{n+1}{4}\,\mathcal{I}_{n+1}^{*(40)}(x)-\frac{n-1}{2}\,\mathcal{I}_{n+1}^{*(22)}(x)$$

$$\mathcal{I}_{n+1}^{*(04)}(x)=\frac{x^{n+1}}{8n}\,\Big[3x\,I_0^4(x)-3(n+2)\,I_0^3(x)\,I_1(x)-6x\,I_0^2(x)\,I_1^2(x)+(5n+6)\,I_0(x)\,I_1^3(x)+3x\,I_1^4(x)\Big]+$$

$$+\frac{3(n+2)}{8}\,\mathcal{I}_{n+1}^{*(31)}(x)-\frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_{n+1}^{*(13)}(x)$$

5. Quotients

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p=1,2$.
Integrals are omitted, when $f(x)$ turns out to be of a very special kind or when the antiderivative is expressed by Whittaker or other hypergeometric functions.

5.1. Denominator $p\,(x)\,Z_0(x)+q(x)\,Z_1(x)$

a) Typ $f(x)\,Z_\nu(x)\,/\,[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]$:

$$\int\frac{J_1(x)\,dx}{J_0(x)}=-\ln|J_0(x)|\,,\quad\int\frac{I_1(x)\,dx}{I_0(x)}=\ln|I_0(x)|\,,\quad\int\frac{K_1(x)\,dx}{K_0(x)}=-\ln|K_0(x)|$$

$$\int\frac{J_0(x)\,dx}{J_1(x)}=\ln|x\,J_1(x)|\,,\quad\int\frac{I_0(x)\,dx}{I_1(x)}=\ln|x\,I_1(x)|\,,\quad\int\frac{K_0(x)\,dx}{K_1(x)}=-\ln|x\,K_1(x)|$$

$$\int\frac{(x^2+a^2-2a)\,J_1(x)\,dx}{x[x\,J_0(x)-a\,J_1(x)]}=-\ln|x^a\,J_0(x)-ax^{a-1}\,J_1(x)|$$

$$\int\frac{(x^2-a^2+2a)\,I_1(x)\,dx}{x[x\,I_0(x)-a\,I_1(x)]}=\ln|x^a\,I_0(x)-ax^{a-1}\,I_1(x)|$$

$$\int\frac{(x^2-a^2+2a)\,K_1(x)\,dx}{x[x\,K_0(x)+a\,K_1(x)]}=-\ln|x^a\,K_0(x)+ax^{a-1}\,K_1(x)|$$

$$\int\frac{\cos x\,J_1(x)\,dx}{x[\sin x\,J_0(x)-\cos x\,J_1(x)]}=\ln|\sin x\,J_0(x)-\cos x\,J_1(x)|$$

$$\int\frac{(2x\,\sin x+\cos x)\,I_1(x)\,dx}{x[\sin x\,I_0(x)-\cos x\,I_1(x)]}=\ln|\sin x\,I_0(x)-\cos x\,I_1(x)|$$

$$\int\frac{(2x\,\sin x+\cos x)\,K_1(x)\,dx}{x[\sin x\,K_0(x)+\cos x\,K_1(x)]}=-\ln|\sin x\,K_0(x)+\cos x\,K_1(x)|$$

$$\int\frac{\sin x\,J_1(x)\,dx}{x[\cos x\,J_0(x)+\sin x\,J_1(x)]}=-\ln|\cos x\,J_0(x)+\sin x\,J_1(x)|$$

$$\int\frac{(2x\,\cos x-\sin x)\,I_1(x)\,dx}{x[\cos x\,I_0(x)+\sin x\,I_1(x)]}=\ln|\cos\,I_0(x)+\sin x\,I_1(x)|$$

$$\int\frac{(2x\,\cos x-\sin x)\,K_1(x)\,dx}{x[\cos x\,K_0(x)-\sin x\,K_1(x)]}=-\ln|\cos\,K_0(x)-\sin x\,K_1(x)|$$

5.2. Denominator $[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^2$

a) Typ $f(x)\,Z_\mu(x)/[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^2$:

$$\int\frac{[(a^2+b^2)x+ab]\exp(-\frac{ax}{b})\,J_0(x)\,dx}{x^2\,[a\,J_0(x)+b\,J_1(x)]^2}=-\frac{b\exp(-\frac{ax}{b})}{x\,[a\,J_0(x)+b\,J_1(x)]}$$

$$\int\frac{[(a^2-b^2)x-ab]\exp(\frac{ax}{b})\,I_0(x)\,dx}{x^2\,[a\,I_0(x)+b\,I_1(x)]^2}=\frac{b\exp(\frac{ax}{b})}{x\,[a\,I_0(x)+b\,I_1(x)]}$$

$$\int\frac{[(a^2-b^2)x+ab]\exp(-\frac{ax}{b})\,K_0(x)\,dx}{x^2\,[a\,K_0(x)+b\,K_1(x)]^2}=-\frac{b\exp(-\frac{ax}{b})}{x\,[a\,I_0(x)+b\,I_1(x)]}$$

$$\int\frac{[(a^2+b^2)x+ab]\exp(\frac{bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x)+b\,J_1(x)]^2}=\frac{a\exp(\frac{bx}{a})}{a\,J_0(x)+b\,J_1(x)}$$

$$\int\frac{[(a^2-b^2)x-ab]\exp(\frac{bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x)+b\,I_I(x)]^2}=-\frac{a\exp(\frac{bx}{a})}{a\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{bx}{a})\, K_1(x)\, dx}{x\,[a\,K_0(x)+b\,K_I(x)]^2} = \frac{a\exp(-\frac{bx}{a})}{a\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{ax^2}{2b})\, J_0(x)\, dx}{x[ax\,J_0(x)+b\,J_1(x)]^2} = -\,\frac{b\exp(\frac{-ax^2}{2b})}{x\,[ax\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{ax^2}{2b})\, I_0(x)\, dx}{x[ax\,I_0(x)+b\,I_1(x)]^2} = \frac{b\exp(\frac{ax^2}{2b})}{x\,[ax\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{ax^2}{2b})\, K_0(x)\, dx}{x[ax\,K_0(x)+b\,K_1(x)]^2} = -\,\frac{b\exp(\frac{-ax^2}{2b})}{x\,[ax\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{x^{b/a}\,[a^2x^2+b^2+2ab]\, J_1(x)\, dx}{[ax\,J_0(x)+b\,J_1(x)]^2} = \frac{a\,x^{1+b/a}}{ax\,J_0(x)+b\,J_1(x)}$$

$$\int \frac{x^{b/a}\,[a^2x^2-b^2-2ab]\, I_1(x)\, dx}{[ax\,I_0(x)+b\,I_1(x)]^2} = -\,\frac{a\,x^{1+b/a}}{ax\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{x^{-b/a}\,[a^2x^2-b^2+2ab]\, K_1(x)\, dx}{[ax\,K_0(x)+b\,K_1(x)]^2} = \frac{a\,x^{1-b/a}}{ax\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{x^{-a/b}\,[a^2+b^2x^2]\, J_0(x)\, dx}{x\,[a\,J_0(x)+bx\,J_1(x)]^2} = -\,\frac{b\,x^{-a/b}}{a\,J_0(x)+bx\,J_1(x)}$$

$$\int \frac{x^{a/b}\,[a^2-b^2x^2]\, I_0(x)\, dx}{x\,[a\,I_0(x)+bx\,I_1(x)]^2} = \frac{b\,x^{a/b}}{a\,I_0(x)+bx\,I_1(x)}$$

$$\int \frac{x^{-a/b}\,[a^2-b^2x^2]\, K_0(x)\, dx}{x\,[a\,K_0(x)+bx\,K_1(x)]^2} = -\,\frac{b\,x^{-a/b}}{a\,K_0(x)+bx\,K_1(x)}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{bx^2}{2a})\, J_1(x)\, dx}{[a\,J_0(x)+bx\,J_1(x)]^2} = \frac{a\exp(\frac{bx^2}{2a})}{a\,J_0(x)+bx\,J_1(x)}$$

$$\int \frac{[b^2x^2-a^2]\exp(\frac{bx^2}{2a})\, I_1(x)\, dx}{[a\,I_0(x)+bx\,I_1(x)]^2} = \frac{a\exp(\frac{bx^2}{2a})}{a\,I_0(x)+bx\,I_1(x)}$$

$$\int \frac{[b^2x^2-a^2]\exp(-\frac{bx^2}{2a})\, K_1(x)\, dx}{[a\,K_0(x)+bx\,K_1(x)]^2} = -\,\frac{a\exp(-\frac{bx^2}{2a})}{a\,K_0(x)+bx\,K_1(x)}$$

$$\int \frac{[(a^2+c^2)x^3+(2ab+ac+2cd)x^2+(b^2+2ad+d^2)x+bd]\,(cx+d)^{(da-cb)/c^2}\exp(-\frac{ax}{c})\, J_0(x)\, dx}{x^2\,[(ax+b)\,J_0(x)+(cx+d)\,J_1(x)]^2} =$$

$$= -\,\frac{(cx+d)^{1+(da-cb)/c^2}\exp(-\frac{ax}{c})}{x\,[(ax+b)\,J_0(x)+(cx+d)\,J_1(x)]}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab-ac-2cd)x^2+(b^2-2ad-d^2)x-bd]\,(cx+d)^{(cb-da)/c^2}\exp(\frac{ax}{c})\, I_0(x)\, dx}{x^2\,[(ax+b)\,I_0(x)+(cx+d)\,I_1(x)]^2} =$$

$$= \frac{(cx+d)^{1+(cb-da)/c^2}\exp(\frac{ax}{c})}{x\,[(ax+b)\,I_0(x)+(cx+d)\,I_1(x)]}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab+ac-2cd)x^2+(b^2+2ad-d^2)x+bd]\,(cx+d)^{(da-cb)/c^2}\exp(-\frac{ax}{c})\, K_0(x)\, dx}{x^2\,[(ax+b)\,K_0(x)+(cx+d)\,K_1(x)]^2} =$$

$$= -\,\frac{(cx+d)^{1+(da-cb)/c^2}\exp(-\frac{ax}{c})}{x\,[(ax+b)\,K_0(x)+(cx+d)\,K_1(x)]}$$

$$\int \frac{[(a^2+c^2)x^3+(2ab+ac+2cd)x^2+(b^2+2ad+d^2)x+bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,J_1(x)\,dx}{x\,[(ax+b)\,J_0(x)+(cx+d)\,J_1(x)]^2}=$$

$$=\frac{(ax+b)^{1+(da-cb)/a^2}\exp(\frac{cx}{a})}{(ax+b)\,J_0(x)+(cx+d)\,J_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab-ac-2cd)x^2+(b^2-2ad-d^2)x-bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,I_1(x)\,dx}{x\,[(ax+b)\,I_0(x)+(cx+d)\,I_1(x)]^2}=$$

$$=-\frac{(ax+b)^{1+(da-cb)/c^2}\exp(\frac{cx}{a})}{(ax+b)\,I_0(x)+(cx+d)\,I_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab+ac-2cd)x^2+(b^2+2ad-d^2)x+bd]\,(ax+b)^{(cb-da)/c^2}\exp(-\frac{cx}{a})\,K_1(x)\,dx}{x\,[(ax+b)\,K_0(x)+(cx+d)\,K_1(x)]^2}=$$

$$=\frac{(ax+b)^{1+(cb-da)/c^2}\exp(-\frac{cx}{a})}{(ax+b)\,K_0(x)+(cx+d)\,K_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}+2b^2\sqrt{x}+3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,J_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,J_0(x)+b\,J_1(x)]^2}=-\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}-3ab]\exp(\frac{2ax}{3b}\,x^{3/2})\,I_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,I_0(x)+b\,I_1(x)]^2}=\frac{2b\exp(\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}+3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,K_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,K_0(x)+b\,K_1(x)]^2}=-\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}+2b^2\sqrt{x}+3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,J_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,J_0(x)+b\,J_1(x)]^2}=\frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,J_0(x)+b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}-3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,I_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,I_0(x)+b\,I_1(x)]^2}=-\frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}+3ab]\exp(-\frac{2b}{a}\,\sqrt{x})\,K_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,K_0(x)+b\,K_1(x)]^2}=\frac{2a\,\sqrt{x}\exp(-\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}+2a^2\sqrt{x}+ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,J_0(x)\,dx}{x^{3/2}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]^2}=-\frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}+ab]\exp(\frac{2a}{b}\,\sqrt{x})\,I_0(x)\,dx}{x^{3/2}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]^2}=-\frac{2b\exp(\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}-ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,K_0(x)\,dx}{x^{3/2}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]^2}=\frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}+2a^2\sqrt{x}+ab]\exp(\frac{2b}{3a}\,x^{3/2})\,J_1(x)\,dx}{\sqrt{x}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]^2}=\frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,J_0(x)+b\sqrt{x}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}+ab]\exp(\frac{2b}{3a}\,x^{3/2})\,I_1(x)\,dx}{\sqrt{x}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]^2}=\frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,I_0(x)+b\sqrt{x}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}-ab]\exp(-\frac{2b}{3a}\,x^{3/2})\,K_1(x)\,dx}{\sqrt{x}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]^2}=-\frac{2a\exp(-\frac{2b}{3a}\,x^{3/2})}{a\,K_0(x)+b\sqrt{x}\,K_1(x)}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}+2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,J_0(x)\,dx}{x\,[ax^{3/2}\,J_0(x)+b\,J_1(x)]^2} = -\,\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^3-5ab\,\sqrt{x}-2b^2]\exp(\frac{2a}{5b}\,x^{5/2})\,I_0(x)\,dx}{x\,[ax^{3/2}\,I_0(x)+b\,I_1(x)]^2} = \frac{2b\exp(\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}-2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,K_0(x)\,dx}{x\,[ax^{3/2}\,K_0(x)+b\,K_1(x)]^2} = -\,\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}+2b^2]\exp(-2b/a\sqrt{x}\,)\,J_1(x)\,dx}{[ax^{3/2}\,J_0(x)+b\,J_1(x)]^2} = \frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,J_0(x)+b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^3-5ab\,\sqrt{x}-2b^2]\exp(-2b/a\sqrt{x}\,)\,I_1(x)\,dx}{[ax^{3/2}\,I_0(x)+b\,I_1(x)]^2} = -\frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}-2b^2]\exp(2b/a\sqrt{x}\,)\,K_1(x)\,dx}{[ax^{3/2}\,K_0(x)+b\,K_1(x)]^2} = \frac{2a\,x^{3/2}\exp(2b/a\sqrt{x}\,)}{ax^{3/2}\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}+2a^2]\exp(2a/b\sqrt{x}\,)\,J_0(x)\,dx}{x\,[a\,J_0(x)+bx^{3/2}\,J_1(x)]^2} = -\,\frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,J_0(x)+bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}-2a^2]\exp(-2a/b\sqrt{x}\,)\,I_0(x)\,dx}{x\,[a\,I_0(x)+bx^{3/2}\,I_1(x)]^2} = -\,\frac{2b\,\sqrt{x}\exp(-2a/b\sqrt{x}\,)}{a\,I_0(x)+bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3+ab\,\sqrt{x}-2a^2]\exp(2a/b\sqrt{x}\,)\,K_0(x)\,dx}{x\,[a\,K_0(x)+bx^{3/2}\,K_1(x)]^2} = \frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,K_0(x)+bx^{3/2}\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}+2a^2]\exp(2bx^{5/2}/5a)\,J_1(x)\,dx}{[a\,J_0(x)+bx^{3/2}\,J_1(x)]^2} = \frac{2a\exp(2bx^{5/2}/5a)}{a\,J_0(x)+bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}-2a^2]\exp(2bx^{5/2}/5a)\,I_1(x)\,dx}{[a\,I_0(x)+bx^{3/2}\,I_1(x)]^2} = \frac{2a\exp(2bx^{5/2}/5a)}{a\,I_0(x)+bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3+ab\,\sqrt{x}-2a^2]\exp(-2bx^{5/2}/5a)\,K_1(x)\,dx}{[a\,K_0(x)+bx^{3/2}\,K_1(x)]^2} = -\,\frac{2a\exp(-2bx^{5/2}/5a)}{a\,K_0(x)+bx^{3/2}\,K_1(x)}$$

b) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^2\,,\ n=0,1,2:$

$$\int \frac{[(a^2+b^2)x+ab]\cdot\exp\,(-\frac{2ax}{b})\cdot J_0^2(x)\,dx}{x^2\,[a\,J_0(x)\,+b\,J_1(x)]^2} = E_1\left(\frac{2ax}{b}\right) - \frac{b\,J_0(x)}{x\,[a\,J_0(x)\,+b\,J_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

with the exponential integral $E_1(x)$ (see page 443).

$$\int \frac{[(a^2-b^2)x-ab]\cdot\exp\,(\frac{2ax}{b})\cdot I_0^2(x)\,dx}{x^2\,[a\,I_0(x)\,+b\,I_1(x)]^2} = E_1\left(-\frac{2ax}{b}\right) + \frac{b\,I_0(x)}{x\,[a\,I_0(x)\,+b\,I_1(x)]}\cdot\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot\exp\,(-\frac{2ax}{b})\cdot K_0^2(x)\,dx}{x^2\,[a\,K_0(x)\,+b\,K_1(x)]^2} = E_1\left(\frac{2ax}{b}\right) - \frac{b\,K_0(x)}{x\,[a\,K_0(x)\,+b\,K_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\cdot\exp\,(-\frac{a^2-b^2}{ab}\,x)\cdot J_0(x)\cdot J_1(x)\,dx}{x\,[a\,J_0(x)\,+b\,J_1(x)]^2} = -\,\frac{ab\,[b\,J_0(x)+a\,J_1(x)]}{(a^2-b^2)\,[aJ_0(x)\,+\,bJ_1(x)]}\cdot\exp\left(-\frac{a^2-b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\cdot\exp\,(\frac{a^2+b^2}{ab}\,x)\cdot I_0(x)\cdot I_1(x)\,dx}{x\,[a\,I_0(x)\,+b\,I_1(x)]^2} = -\,\frac{ab\,[b\,J_0(x)-a\,J_1(x)]}{(a^2+b^2)\,[a\,I_0(x)\,+b\,I_1(x)]}\cdot\exp\left(\frac{a^2+b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot \exp\left(-\frac{a^2+b^2}{ab}\,x\right)\cdot K_0(x)\cdot K_1(x)\,dx}{x\,[a\,K_0(x)\;+b\,K_1(x)]^2} = -\,\frac{ab\,[b\,K_0(x)-a\,K_1(x)]}{(a^2+b^2)\,[a\,K_0(x)\;+b\,K_1(x)]}\cdot\exp\left(-\frac{a^2+b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\cdot \exp\left(\frac{2bx}{a}\right)\cdot J_1^2(x)\,dx}{[a\,J_0(x)\;+\;b\,J_1(x)]^2} = \frac{a\,[a(a-2bx)\,J_0(x)+b(a+2bx)\,J_1(x)]}{4b^2\,[a\,J_0(x)\;+b\,J_1(x)]}\cdot\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\cdot \exp\left(\frac{2bx}{a}\right)\cdot I_1^2(x)\,dx}{[a\,I_0(x)\;+b\,I_1(x)]^2} = -\,\frac{a\,[a(a-2bx)\,I_0(x)+b(a+2bx)\,I_1(x)]}{4b^2\,[a\,I_0(x)\;+b\,I_1(x)]}\cdot\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot \exp\left(-\frac{2bx}{a}\right)\cdot K_1^2(x)\,dx}{[a\,K_0(x)\;+b\,K_1(x)]^2} = -\,\frac{a\,[a(a+2bx)\,K_0(x)+b(a-2bx)\,K_1(x)]}{4b^2\,[a\,K_0(x)\;+b\,K_1(x)]}\cdot\exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\cdot \exp\left(-\frac{ax^2}{b}\right)\cdot J_0^2(x)\,dx}{x\,[ax\,J_0(x)\;+b\,J_1(x)]^2} = \frac{1}{2}\,E_1\left(\frac{ax^2}{b}\right)-\frac{b\,J_0(x)}{x\,[ax\,J_0(x)\;+b\,J_1(x)]}\cdot\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\cdot \exp\left(\frac{ax^2}{b}\right)\cdot I_0^2(x)\,dx}{x\,[ax\,I_0(x)\;+b\,I_1(x)]^2} = \frac{1}{2}\,E_1\left(-\frac{ax^2}{b}\right)+\frac{b\,I_0(x)}{x\,[ax\,I_0(x)\;+b\,I_1(x)]}\cdot\exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[(a^2x^2-b^2+2ab]\cdot \exp\left(-\frac{ax^2}{b}\right)\cdot K_0^2(x)\,dx}{x\,[ax\,K_0(x)\;+b\,K_1(x)]^2} = \frac{1}{2}\,E_1\left(\frac{ax^2}{b}\right)-\frac{b\,K_0(x)}{x\,[ax\,K_0(x)\;+b\,K_1(x)]}\cdot\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{(a^2(a+b)x^2+2a^2b+3ab^2+b^3)\,x^{1+2b/a}\;J_1^2(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^2} = -\,\frac{a\,x^{2+2b/a}\,[ax\,J_0(x)-(2a+b)\,J_1(x)]}{2\,[ax\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{(a^2(a+b)x^2-2a^2b-3ab^2-b^3)\,x^{1+2b/a}\;I_1^2(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^2} = \frac{a\,x^{2+2b/a}\,[ax\,I_0(x)-(2a+b)\,I_1(x)]}{2\,[ax\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{(a^2(a-b)x^2+2a^2b-3ab^2+b^3)\,x^{1+2b/a}\;K_1^2(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^2} = \frac{a\,x^{2-2b/a}\,[ax\,K_0(x)+(2a-b)\,K_1(x)]}{2\,[ax\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{(a^2+b^2x^2)\,x^{-1-2a/b}\;J_0^2(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,J_0(x)-bx\,J_1(x)]}{2a\,[a\,J_0(x)+bx\,J_1(x)]}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1+2a/b}\;I_0^2(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^2} = \frac{b\,x^{2a/b}\,[a\,I_0(x)-bx\,I_1(x)]}{2a\,[a\,I_0(x)+bx\,I_1(x)]}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1-2a/b}\;K_0^2(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,K_0(x)-bx\,K_1(x)]}{2a\,[a\,K_0(x)+bx\,K_1(x)]}$$

$$\int \frac{x(a^2+b^2x^2)\cdot \exp\left(\frac{bx^2}{a}\right)\cdot J_1^2(x)\,dx}{[a\,J_0(x)\;+bx\,J_1(x)]^2} = -\,\frac{a\,[a\,J_0(x)-bx\,J_1(x)]}{2b\,[a\,J_0(x)\;+bx\,J_1(x)]}\cdot\exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2-b^2x^2)\cdot \exp\left(\frac{bx^2}{a}\right)\cdot I_1^2(x)\,dx}{[a\,I_0(x)\;+bx\,I_1(x)]^2} = \frac{a\,[a\,I_0(x)-bx\,I_1(x)]}{2b\,[a\,I_0(x)\;+bx\,I_1(x)]}\cdot\exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2-b^2x^2)\cdot \exp\left(-\frac{bx^2}{a}\right)\cdot K_1^2(x)\,dx}{[a\,K_0(x)\;+bx\,K_1(x)]^2} = -\,\frac{a\,[a\,K_0(x)-bx\,K_1(x)]}{2b\,[a\,K_0(x)\;+bx\,K_1(x)]}\cdot\exp\left(-\frac{bx^2}{a}\right)$$

5.3. Denominator $[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^3$

b) Typ $f(x)\,Z_\nu/[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^3$, :

$$\int \frac{[(a^2+b^2)x+ab]\exp(-\frac{2ax}{b})\,J_0(x)\,dx}{x^3\,[a\,J_0(x)+b\,J_1(x)]^3} = -\,\frac{b}{2x^2\,[a\,J_0(x)+b\,J_1(x)]^2}\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{2ax}{b})\,I_0(x)\,dx}{x^3\,[a\,I_0(x)+b\,I_1(x)]^3} = \frac{b}{2x^2\,[a\,I_0(x)+b\,I_1(x)]^2}\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{2ax}{b})\,K_0(x)\,dx}{x^3\,[a\,K_0(x)+b\,K_1(x)]^3} = -\frac{b}{2x^2\,[a\,K_0(x)+b\,K_1(x)]^2}\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\exp(\frac{2bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x)+b\,J_1(x)]^3} = \frac{a}{2\,[a\,J_0(x)+b\,J_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{2bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x)+b\,I_1(x)]^3} = -\frac{a}{2\,[a\,I_0(x)+b\,I_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{2bx}{a})\,K_1(x)\,dx}{x\,[a\,K_0(x)+b\,K_1(x)]^3} = \frac{a}{2\,[a\,K_0(x)+b\,K_1(x)]^2}\exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{ax^2}{b})\,J_0(x)\,dx}{x^2\,[ax\,J_0(x)+b\,J_1(x)]^3} = -\frac{b}{2x^2\,[ax\,J_0(x)+b\,J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{ax^2}{b})\,I_0(x)\,dx}{x^2\,[ax\,I_0(x)+b\,I_1(x)]^3} = \frac{b}{2x^2\,[ax\,I_0(x)+b\,I_1(x)]^2}\exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{ax^2}{b})\,K_0(x)\,dx}{x^2\,[ax\,K_0(x)+b\,K_1(x)]^3} = -\frac{b}{2x^2\,[ax\,J_0(x)+b\,J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\,x^{1+2b/a}\,J_1(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^3} = \frac{ax^{2+2b/a}}{2\,[ax\,J_0(x)+b\,J_1(x)]^2}$$

$$\int \frac{[a^2x^2-b^2-2ab]\,x^{1+2b/a}\,I_1(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^3} = -\frac{ax^{2+2b/a}}{2\,[ax\,I_0(x)+b\,I_1(x)]^2}$$

$$\int \frac{[a^2x^2-b^2+2ab]\,x^{1-2b/a}\,K_1(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^3} = \frac{ax^{2-2b/a}}{2\,[ax\,K_0(x)+b\,K_1(x)]^2}$$

$$\int \frac{[a^2+b^2x^2]\,x^{-1-2a/b}\,J_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = -\frac{bx^{-2a/b}}{2\,[a\,J_0(x)+bx\,J_1(x)]^2}$$

$$\int \frac{[a^2-b^2x^2]\,x^{-1+2a/b}\,I_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = \frac{bx^{2a/b}}{2\,[a\,I_0(x)+bx\,I_1(x)]^2}$$

$$\int \frac{[a^2-b^2x^2]\,x^{-1-2a/b}\,K_0(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^3} = \frac{bx^{-2a/b}}{2\,[a\,K_0(x)+bx\,K_1(x)]^2}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{b}{a}\,x^2)\,J_1(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = \frac{a}{2\,[a\,J_0(x)+bx\,J_1(x)]^2}\exp\left(\frac{b\,x^2}{a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(\frac{b}{a}\,x^2)\,I_1(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^3} = -\frac{a}{2\,[a\,I_0(x)+bx\,I_1(x)]^2}\exp\left(\frac{b\,x^2}{a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(-\frac{b}{a}\,x^2)\,K_1(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^3} = \frac{a}{2\,[a\,K_0(x)+bx\,K_1(x)]^2}\exp\left(-\frac{b\,x^2}{a}\right)$$

5.4. Denominator $[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^4$

a) Typ $f(x)\,Z_\nu/[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^4$:

$$\int \frac{[(a^2+b^2)x+ab]\exp(-\frac{3ax}{b})\,J_0(x)\,dx}{x^4\,[a\,J_0(x)+b\,J_1(x)]^4} = -\frac{b}{3x^3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{3ax}{b})\,I_0(x)\,dx}{x^4\,[a\,I_0(x)+b\,I_1(x)]^4} = \frac{b}{3x^3\,[a\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{3ax}{b})\,K_0(x)\,dx}{x^4\,[a\,K_0(x)+b\,K_1(x)]^4} = -\frac{b}{3x^3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\exp(\frac{3bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x)+b\,J_1(x)]^4} = -\frac{a}{3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{3bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x)+b\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{3bx}{a})\,K_1(x)\,dx}{x\,[a\,K_0(x)+b\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x)+b\,K_1(x)]^3}\exp\left(-\frac{3bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{3ax^2}{2b})\,J_0(x)\,dx}{x^3\,[ax\,J_0(x)+b\,J_1(x)]^4} = -\frac{b}{3x^3\,[ax\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{3ax^2}{2b})\,I_0(x)\,dx}{x^3\,[ax\,I_0(x)+b\,I_1(x)]^4} = \frac{b}{3x^3\,[ax\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{3ax^2}{2b})\,K_0(x)\,dx}{x^3\,[ax\,K_0(x)+b\,K_1(x)]^4} = -\frac{b}{3x^3\,[ax\,K_0(x)+b\,K_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\,x^{2+3b/a}\ J_1(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^4} = \frac{a\,x^{3+3b/a}}{3\,[ax\,J_0(x)+b\,J_1(x)]^3}$$

$$\int \frac{[a^2x^2-b^2-2ab]\,x^{2+3b/a}\ I_1(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^4} = -\frac{a\,x^{3+3b/a}}{3\,[ax\,I_0(x)+b\,I_1(x)]^3}$$

$$\int \frac{[a^2x^2-b^2+2ab]\,x^{2-3b/a}\ K_1(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^4} = \frac{a\,x^{3-3b/a}}{3\,[ax\,K_0(x)+b\,K_1(x)]^3}$$

$$\int \frac{(a^2+b^2x^2)\,x^{-1-3a/b}\ J_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^4} = -\frac{b\,x^{-3a/b}}{3\,[a\,J_0(x)+bx\,J_1(x)]^3}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1+3a/b}\ I_0(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^4} = \frac{b\,x^{3a/b}}{3\,[a\,I_0(x)+bx\,I_1(x)]^3}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1-3a/b}\ K_0(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^4} = \frac{b\,x^{-3a/b}}{3\,[a\,K_0(x)+bx\,K_1(x)]^3}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{3bx^2}{2a})\,J_1(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^4} = \frac{a}{3\,[a\,J_0(x)+bx\,J_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(\frac{3bx^2}{2a})\,I_1(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x)+bx\,I_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(-\frac{3bx^2}{2a})\,K_1(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x)+bx\,K_1(x)]^3}\exp\left(-\frac{3bx^2}{2a}\right)$$

5.5. Denominator $p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)$

a) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p\,(x)\,Z_0^2(x) + q(x)\,Z_1^2(x)],\ n = 0,1,2:$

$$\int \frac{J_0^2(x)\,dx}{x\,[J_0^2(x)\ +J_1^2(x)]} = \frac{1}{2}\ \ln\{x^2\,[J_0^2(x)\ +J_1^2(x)]\}$$
$$\int \frac{I_0^2(x)\,dx}{x\,[I_0^2(x)\ -I_1^2(x)]} = \frac{1}{2}\ \ln\{x^2\,[I_0^2(x)\ -I_1^2(x)]\}$$
$$\int \frac{K_0^2(x)\,dx}{x\,[K_0^2(x)\ -K_1^2(x)]} = \frac{1}{2}\ \ln\{x^2\,[K_1^2(x)\ -K_0^2(x)]\}$$

$$\int \frac{(x^2-a)\,J_0(x)\,J_1(x)\,dx}{a\,J_0^2(x)\ +x^2\,J_1^2(x)} = \frac{1}{2}\ \ln[a\,J_0^2(x)\ +x^2\,J_1^2(x)]$$
$$\int \frac{(x^2+a)\,I_0(x)\,I_1(x)\,dx}{a\,I_0^2(x)\ +x^2\,I_1^2(x)} = \frac{1}{2}\ \ln[a\,I_0^2(x)\ +x^2\,I_1^2(x)]$$
$$\int \frac{(x^2+a)\,K_0(x)\,K_1(x)\,dx}{a\,K_0^2(x)\ +x^2\,K_1^2(x)} = -\,\frac{1}{2}\ \ln[a\,K_0^2(x)\ +x^2\,K_1^2(x)]$$

$$\int \frac{J_1^2(x)\,dx}{x\,[J_0^2(x)\ +J_1^2(x)]} = -\,\frac{1}{2}\ \ln[J_0^2(x)\ +J_1^2(x)]$$
$$\int \frac{I_1^2(x)\,dx}{x\,[I_0^2(x)\ -I_1^2(x)]} = -\,\frac{1}{2}\ \ln[I_0^2(x)\ -I_1^2(x)]$$
$$\int \frac{K_1^2(x)\,dx}{x\,[K_1^2(x)\ -K_0^2(x)]} = -\,\frac{1}{2}\ \ln[K_1^2(x)\ -K_0^2(x)]$$

b) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)],\ n = 0,1,2:$

$$\int \frac{x\,J_0^2(x)\,dx}{x^2\,J_0^2(x) + 2x\,J_0(x)\,J_1(x) + (x^2-2)\,J_1^2(x)} = \frac{1}{6}\ \ln\left|x^4\,J_0^2(x) + 2x^3\,J_0(x)\,J_1(x) + (x^4-2x^3)\,J_1^2(x)\right|$$
$$\int \frac{x\,I_0^2(x)\,dx}{x^2\,I_0^2(x) + 2x\,I_0(x)\,I_1(x) - (x^2+2)\,I_1^2(x)} = \frac{1}{6}\ \ln\left|x^4\,I_0^2(x) + 2x^3\,I_0(x)\,I_1(x) - (x^4+2x^2)\,I_1^2(x)\right|$$
$$\int \frac{x\,K_0^2(x)\,dx}{x^2\,K_0^2(x) - 2x\,K_0(x)\,K_1(x) - (x^2+2)\,K_1^2(x)} = \frac{1}{6}\ \ln\left|x^4\,K_0^2(x) - 2x^3\,K_0(x)\,K_1(x) - (x^2+2)\,K_1^2(x)\right|$$

$$\int \frac{J_0(x)J_1(x)\,dx}{x[xJ_0^2(x)x - J_0(x)J_1(x) + xJ_1^2(x)]} = \ln\left|xJ_0^2(x)x - J_0(x)J_1(x) + xJ_1^2(x)\right|$$
$$\int \frac{I_0(x)I_1(x)\,dx}{x[xI_0^2(x)x - I_0(x)I_1(x) - xI_1^2(x)]} = \ln\left|xI_0^2(x)x - I_0(x)I_1(x) - xI_1^2(x)\right|$$
$$\int \frac{K_0(x)K_1(x)\,dx}{x[xK_0^2(x)x + K_0(x)K_1(x) + xK_1^2(x)]} = -\ln\left|xK_0^2(x)x + K_0(x)K_1(x) - xK_1^2(x)\right|$$

$$\int \frac{(1+2\ln x)x\,J_0(x)\,J_1(x)\,dx}{xJ_0^2(x) - 2J_0(x)J_1(x) - 2x\ln xJ_1^2(x)} = -\frac{1}{2}\ln\left|x\left[xJ_0^2(x) - 2J_0(x)J_1(x) - 2x\ln xJ_1^2(x)\right]\right|$$
$$\int \frac{(1+2\ln x)x\,I_0(x)\,I_1(x)\,dx}{xI_0^2(x) - 2I_0(x)I_1(x) + 2x\ln xI_1^2(x)} = \frac{1}{2}\ln\left|x\left[xI_0^2(x) - 2I_0(x)I_1(x) + 2x\ln xI_1^2(x)\right]\right|$$
$$\int \frac{(1+2\ln x)x\,K_0(x)\,K_1(x)\,dx}{xK_0^2(x) + 2K_0(x)K_1(x) - 2x\ln xK_1^2(x)} = -\frac{1}{2}\ln\left|x\left[xK_0^2(x) + 2K_0(x)K_1(x) + 2x\ln xK_1^2(x)\right]\right|$$

$$\int \frac{(8x^2+3)\,J_0(x)J_1(x)\,dx}{x[xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)]} = -\ln\left[x^2\left(xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)\right)\right|$$

$$\int \frac{(8x^2-3)\,I_0(x)I_1(x)\,dx}{x[xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)]} = \ln\left[x^2\left(xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)\right)\right|$$

$$\int \frac{(8x^2-3)\,K_0(x)K_1(x)\,dx}{x[xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)]} = -\ln\left[x^2\left(xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)\right)\right|$$

$$\int \frac{x^2\,J_0(x)\,J_1(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)-(2x^2-4)\,J_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)-(x^4-2x^2)\,J_1^2(x)\right|$$

$$\int \frac{x^2\,I_0(x)\,I_1(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)+(2x^2+4)\,I_1^2(x)} = \frac{1}{6}\,\ln\left|x^4\,I_0^2(x)-2x^3\,I_0(x)\,I_1(x)+(2x^4+4x^2)\,I_1^2(x)\right|$$

$$\int \frac{x^2\,K_0(x)\,K_1(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)+(2x^2+4)\,K_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)+(2x^4+4x^2)\,K_1^2(x)\right|$$

$$\int \frac{(3x^3+4)\,J_0(x)\,J_1(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)-2x^2\,J_1^2(x)} = -\frac{1}{2}\,\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)-2x^4\,J_1^2(x)\right|$$

$$\int \frac{(3x^2-4)\,I_0(x)\,I_1(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)+2x^2\,I_1^2(x)} = \frac{1}{2}\,\ln\left|x^4\,I_0^2(x)-4x^3\,I_0(x)\,I_1(x)+2x^4\,I_1^2(x)\right|$$

$$\int \frac{(3x^2-4)\,K_0(x)\,K_1(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)+2x^2\,K_1^2(x)} = -\frac{1}{2}\,\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)+2x^4\,K_1^2(x)\right|$$

$$\int \frac{J_1^2(x)\,dx}{xJ_0^2(x)-2J_0(x)J_1(x)+xJ_1^2(x)} = \frac{1}{2}\ln\left|x^2\,J_0^2(x)-2x\,J_0(x)J_1(x)+x^2\,J_1^2(x)\right|$$

$$\int \frac{I_1^2(x)\,dx}{xI_0^2(x)-2I_0(x)I_1(x)-xI_1^2(x)} = -\frac{1}{2}\ln\left|x^2\,I_0^2(x)-2x\,I_0(x)I_1(x)-x^2\,I_1^2(x)\right|$$

$$\int \frac{K_1^2(x)\,dx}{xK_0^2(x)+2K_0(x)K_1(x)-xK_1^2(x)} = -\frac{1}{2}\ln\left|x^2\,K_0^2(x)+2x\,K_0(x)K_1(x)-x^2\,K_1^2(x)\right|$$

$$\int \frac{x\,J_1^2(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)+(x^2+4)\,J_1^2(x)} = \frac{1}{6}\,\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)+(x^4+4x^2)\,J_1^2(x)\right|$$

$$\int \frac{x\,I_1^2(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)-(x^2-4)\,I_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,I_0^2(x)-4x^3\,I_0(x)\,I_1(x)-(x^2-4)\,I_1^2(x)\right|$$

$$\int \frac{x\,K_1^2(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)-(x^2-4)\,K_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)-(x^4-4x^2)\,K_1^2(x)\right|$$

5.6. Denominator $\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}$

Generally: From

$$\int \varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx \quad = \quad a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)$$

follows

$$\int \frac{\varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx}{\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}} =$$

$$= 2\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}\,.$$

Therefore the formulas from 1. and 2. give a lot of integrals of this kind.
Some special cases:

$$\int \frac{J_1(x)\,dx}{\sqrt{J_0(x)}} = -2\sqrt{J_0(x)}\,, \qquad \int \frac{\sqrt{x}\,J_0(x)\,dx}{\sqrt{J_1(x)}} = 2\sqrt{x\,J_1(x)}$$

$$\int \frac{x^2\,J_1(x)\,dx}{\sqrt{2x\,J_1(x)-x^2\,J_0(x)}} = 2\sqrt{2x\,J_1(x)-x^2\,J_0(x)} \quad \text{(at least for } 0 < x < 5.1356\text{)}$$

$$\int \frac{x^3\,J_0(x)\,dx}{\sqrt{2x^2\,J_0(x)+(x^3-4x)\,J_1(x)}} = 2\sqrt{2x^2\,J_0(x)+(x^3-4x)\,J_1(x)} \quad \text{(at least for } 0 < x < 3.0542\text{)}$$

$$\int \frac{x\,\ln x\,J_0(x)\,dx}{\sqrt{J_0(x)+x\,\ln x\,J_1(x)}} = 2\sqrt{J_0(x)+x\,\ln x\,J_1(x)} \quad \text{(at least for } 0 < x < 3.6265\text{)}$$

$$\int \frac{\sin x\,J_0(x)\,dx}{\sqrt{x\,\sin x\,J_0(x)-x\,\cos x\,J_1(x)}} = 2\sqrt{x\,\sin x\,J_0(x)-x\,\cos x\,J_1(x)}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{\sqrt{x\,\cos x\,J_0(x)+x\,\sin x\,J_1(x)}} = 2\sqrt{x\,\cos x\,J_0(x)+x\,\sin x\,J_1(x)}$$

6. Miscellaneous:

$$\int x^n \cdot J_0(x) \cdot J_1^{n-1}(x)\, dx = \frac{x^n}{n}\, J_1^n(x)\ , \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{dx}{x^n \cdot J_0(x) \cdot J_1^{n+1}(x)} = -\, \frac{1}{n\, x^n\, J_1^n(x}\ , \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int x^n \cdot I_0(x) \cdot I_1^{n-1}(x)\, dx = \frac{x^n}{n}\, I_1^n(x)\ , \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{dx}{x^n \cdot I_0(x) \cdot I_1^{n+1}(x)} = -\, \frac{1}{n\, x^n\, I_1^n(x)}\ , \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{\Phi(x)}{x}\, dx = \Lambda_1(x) - x\, J_0(x) - \Phi(x)$$

$$\int \frac{x^5\, J_1(px^2+i)\, dx}{px^2+i} = \frac{1}{2p^3}\, \left[J_1(px^2+i) - (px^2-i) J_0(px^2+i)\right]$$

$$\int \frac{x^7\, J_1(px^2+\sqrt{3}\, i)\, dx}{px^2+\sqrt{3}\, i} = \frac{1}{2p^4}\, \left[(2px^2-\sqrt{3}\, i)\, J_1(px^2+\sqrt{3}\, i) - (p^2x^4-\sqrt{3}\, px^2\, i - 3) J_0(px^2+\sqrt{3}\, i)\right]$$

7. Used special functions and defined functions:

Used functions:

$J_\nu(x)$	Bessel function of the first kind	
$I_\nu(x)$	Modified Bessel function (of the first kind)	
$Y_\nu(x)$	Bessel function of the second kind, Neumann's function, Weber's function	
$K_\nu(x)$	Modified Bessel function (of the second [third] kind), MacDonald Function	
$H_\nu^{(p)}(x)$, $p = 1, 2$	Bessel function of the third kind, Hankel function	
$\Gamma(x)$	Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$	[1] 6.1, [2] 1.1, [4] II.1.,[5] V, [7]
$E_1(x)$	Exponential integral $E_1(x) = \displaystyle\int_x^\infty \frac{e^{-t}\,dt}{t} = \int_1^\infty \frac{e^{-xt}\,dt}{t}$	[1] 5.1, [4] II.5.
$\mathbf{H}_\nu(x)$	Struve functions	[1] 12.1, [2] 10.1, [5] XIII 2., [7] 8.55
$\mathbf{L}_\nu(x)$	Modified Struve functions	[1] 12.2, [2] 10.1, [7] 8.55
$Ji_0(x)$	$Ji_0(x) = \displaystyle\int_x^\infty \frac{J_0(t)\,dt}{t}$	[1] 11.1.19, [9]
$s_{\mu,\nu}(x)$	Lommel functions	[2] 10.1, [7] 8.57
$P_n^m(x)$	(Associated) Legendre functions of the first kind	[1] 8, [5] XII 3.1.
$P_n(x)$	Legendre polynom	$w(x) \equiv 1$
$T_n(x)$	Chebyshev polynom, first kind	$w(x) = 1/\sqrt{1-x^2}$
$U_n(x)$	Chebyshev polynom, second kind	$w(x) = \sqrt{1-x^2}$
$L_n(x)$	Laguerre polynom	$w(x) = e^{-x}$
$H_n(x)$	Hermite polynom	$w(x) = \exp(-x^2)$

Defined functions:

Function	Page
$\Phi(x) = \frac{\pi x}{2}[J_1(x) \cdot \mathbf{H}_0(x) - J_0(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_Y(x) = \frac{\pi x}{2}[Y_1(x) \cdot \mathbf{H}_0(x) - Y_0(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_H^{(1)} = \frac{\pi x}{2}[H_1^{(1)}(x) \cdot \mathbf{H}_0(x) - H_0^{(1)}(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_H^{(2)} = \frac{\pi x}{2}[H_1^{(2)}(x) \cdot \mathbf{H}_0(x) - H_0^{(2)}(x) \cdot \mathbf{H}_1(x)]$	7
$\Psi(x) = \frac{\pi x}{2}[I_0(x) \cdot \mathbf{L}_1(x) - I_1(x) \cdot \mathbf{L}_0(x)]$	7
$\Psi_K(x) = \frac{\pi x}{2}[K_0(x) \cdot \mathbf{L}_1(x) + K_1(x) \cdot \mathbf{L}_0(x)]$	7
$\Theta(x) = \int_0^x J_0^2(t)\, dt$	195
$\Omega(x) = \int_0^x I_0^2(t)\, dt$	195
$\Lambda_0(x) = \int_0^x J_0(t)\, dt$	110
$\Lambda_0^* = \int_0^x I_0(t)\, dt(x)$	112
$\Lambda_1(x) = \int_0^x t^{-1} \cdot \Lambda_0(t)\, dt$	112
$\Lambda_1^*(x) = \int_0^x t^{-1} \cdot \Lambda_0^*(t)\, dt$	114
$\Theta_0(x;\gamma) = \int_0^x J_0(t)\, J_0(\gamma t)\, dt$	264
$\Omega_0(x;\gamma) = \int_0^x I_0(t)\, I_0(\gamma t)\, dt$	264
$\Theta_1(x;\gamma) = \int_0^x J_1(t)\, J_1(\gamma t)\, dt$	266
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$\mathfrak{H}_p(x,a),\ p = 0,1$	68
$\mathfrak{H}_p^*(x,a),\ p = 0,1$	74

8. Errata

The formulas are checked once more. Misprints in some integrals were found and corrected. By now these formulas are marked with the sign *E* as a warning: 'There were errors in previous editions.'.

They are located on the following pages of the present text:

7, 9, 11, 13, 14, 16, 17, 19, 52, 55, 58, 61, 80, 187, 208, 357 .

In the previous editions the incorrect formulas may have different page numbers.

The check is not finished yet.