is minimized where W is an arbitrary positive definite weighting matrix and where the expectation, $\mathcal{E}[\cdot]$, is conditioned on all previous measurements. As is well known, the solution to this optimization problem is obtained from [3]

$$\dot{\hat{x}} = A\hat{x} + \Sigma(t)C'P^{-1}\{y - C\hat{x}\} \tag{16}$$

$$\hat{x}(0) = 0 \tag{17}$$

where

$$\Sigma(t) = [\Phi_{21}(t) + \Phi_{22}R_0][\Phi_{11}(t) + \Phi_{12}(t)R_0]^{-1} \tag{18}$$

and where the Φ_{ij} are partitions of the exponential matrix of

$$\left[\begin{array}{c|c} -A' & C'P^{-1}C \\ \hline Q & A \end{array}\right]t. \tag{19}$$

Furthermore, the solution is independent of W and

$$\Sigma = \mathcal{E}[(x - \hat{x})(x - \hat{x})']. \tag{20}$$

It may be difficult to obtain Q, R_0 accurately. The sensitivity of the performance to R_0 is studied here. Define the sensitivity matrix

$$S \triangleq \frac{\partial I}{\partial R_0}. \tag{21}$$

Use basic properties of the trace function [1] to show

$$S = \frac{\partial}{\partial R_0}\operatorname{tr}\{W\Sigma\}. \tag{22}$$

Use (18) to show that the differential of $\operatorname{tr}\{W\Sigma\}$ is given by

$$\operatorname{tr}\left\{W\Phi_{22}dR_0[\Phi_{11} + \Phi_{12}R_0]^{-1}\right\} + \operatorname{tr}\left\{W[\Phi_{21} + \Phi_{22}R_0]d[\Phi_{11} + \Phi_{12}R_0]^{-1}\right\}. \tag{23}$$

Use $MM^{-1} = I$ to show that for any nonsingular M

$$dM^{-1} = -M^{-1}dMM^{-1}$$

so that

$$d[\Phi_{11} + \Phi_{12}R_0]^{-1} = -[\Phi_{11} + \Phi_{12}R_0]^{-1}\Phi_{12}dR_0[\Phi_{11} + \Phi_{12}R_0]^{-1}. \tag{24}$$

Combine (18), (23), and (24) and invoke (5) to show

$$d(\operatorname{tr}[W\Sigma]) = \operatorname{tr}\left\{[\Phi_{11} + \Phi_{12}R_0]^{-1}W\Phi_{22}dR_0\right\} - \operatorname{tr}\left\{[\Phi_{11} + \Phi_{12}R_0]^{-1}W\Sigma\Phi_{12}dR_0\right\}. \tag{25}$$

Now use Theorem 1 to show that the sensitivity

$$S = S_0' + S_0 - \operatorname{diag} S_0 \tag{26}$$

where

$$S_0 \triangleq [\Phi_{11} + \Phi_{12}R_0]^{-1}W[\Phi_{22} - \Sigma\Phi_{12}]. \tag{27}$$

Notice that the sensitivity is *not* independent of the weighting matrix W.

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On the Instability of Large-Scale Systems

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Abstract—**A set of sufficient conditions for the instability of large-scale systems is presented. These conditions are much less restrictive than similar conditions derived in [1].**

In a recent paper [1], Grujic and Šiljak derive a set of sufficient conditions for a large-scale interconnected system to be unstable in the sense of Lyapunov. Reference [1, theorem 5] appears to be of limited application, since it requires all subsystems to be completely unstable. In this note, we study conditions under which an interconnection of some stable and some unstable subsystems is itself unstable. The set of assumptions on the interconnections, as well as the method of proof, are very closely related to the corresponding items in [1]. However, by a slight modification of the technique of [1], we come up with results having a significantly larger scope of application.

The class of large-scale systems under study is described by a collection of equations of the form

$$\dot{x}_i(t) = f_i(t, x_i(t)) + \sum_{j=1}^{m} h_{ij}(t, x_j(t)), \qquad i = 1, \cdots, m \tag{1}$$

where $x_i \in R^{n_i}$, $f_i : R \times R^{n_i} \to R^{n_i}$, and $h_{ij} : R \times R^{n_j} \to R^{n_i}$ are continuous functions. We assume that $f_i(t,0) = 0$, $h_{ij}(t,0) = 0$, $\forall i,j$. Since we are interested in large-scale systems that consist of an interconnection of some stable and some unstable subsystems, we assume without loss of generality that for $i = 1, \cdots, k$, the equilibrium point $x_i = 0$ of the system

$$\dot{x}_i(t) = f_i(t, x_i(t)) \tag{2}$$

is asymptotically stable, while for $i = k+1, \cdots, m$, the equilibrium point $x_i = 0$ of (2) is unstable. We further assume that there exist Lyapunov functions $V_i : R \times R^{n_i} \to R$ satisfying the following conditions.

(A1) For $i = 1, \cdots, k$, there exist ϕ_{i1}, ϕ_{i2}, ϕ_{i3}, and $\phi_{i4} : R \to R$ which are functions of class K (see [1]), such that

$$\phi_{i1}(\|x_i\|) \le V_i(t, x_i) \le \phi_{i2}(\|x_i\|), \qquad \forall t, \forall x_i \in S_i \tag{3}$$

$$-\phi_{i3}(\|x_i\|) \le \dot{V}_i(t, x_i) \le -\phi_{i4}(\|x_i\|), \qquad \forall t, \forall x_i \in S_i \tag{4}$$

where, as usual, $\dot{V}_i$ is defined by

$$\dot{V}_i(t, x_i) = \frac{\partial}{\partial t}V_i(t, x_i) + (\nabla V_i)^T f_i(t, x_i) \tag{5}$$

and S_i is an open neighborhood of the origin.

(A2) For $i = k+1, \cdots, m$, there exist ϕ_{i3}, $\phi_{i4} : R \to R$, which are functions of class K, such that

$$-\phi_{i3}(\|x_i\|) \le \dot{V}_i(t, x_i) \le -\phi_{i4}(\|x_i\|), \qquad \forall t, \forall x_i \in S_i. \tag{6}$$

Moreover, $V_i(0, \cdot)$ assumes negative values arbitrarily close to the origin; i.e., given any $\epsilon > 0$, there is an x_i with $\|x_i\| < \epsilon$, such that $V_i(0, x_i) < 0$.

It can be easily verified [2] that (A1) implies that $x_i = 0$ is an asymptotically stable equilibrium point of (2) for $i = 1, \cdots, k$, while (A2) implies that $x_i = 0$ is an unstable equilibrium point of (2) for $i = k+1, \cdots, m$.

We now state the assumptions on the "interconnection" terms h_{ij}.

(A3) For $i,j = 1, \cdots, m$, there exist real constants ξ_{ij} such that

$$[\nabla V_i(t, x_i)]^T h_{ij}(t, x_j) \le \xi_{ij}\phi_{j4}(\|x_j\|), \qquad \forall x_j \in S_j. \tag{7}$$

With these definitions, we can state the first instability result.

Theorem 1: Assume (A1)–(A3) hold, and define the "test matrix" P

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by

$$P_{ij} = \delta_{ij} - \max\{0, \xi_{ij}\} \tag{8}$$

where δ_{ij} denotes the Kronecker delta. Then the equilibrium point $x_i = 0$, $\forall i$ of the interconnected system (1) is unstable if the leading principal minors of P are all positive.

Proof: Consider the function

$$V(t, x_1, \cdots, x_m) = \sum_{i=1}^{m} \alpha_i V_i(t, x_i) \tag{9}$$

as a Lyapunov function candidate where the α_i's are positive constants to be selected later. Calculating the derivative of V, we see that, provided $x_i \in S_i$, $\forall i$, we have

$$\begin{aligned} \dot{V}(t, x_1, \cdots, x_m) &= \sum_{i=1}^{m} \alpha_i \dot{V}_i(t, x_i) + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i [\nabla V_i(t, x_i)]^T h_{ij}(t, x_j) \\ &\leq \sum_{i=1}^{m} -\alpha_i \phi_{i4}(\|x_i\|) + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \xi_{ij} \phi_{j4}(\|x_j\|) \\ &\leq \sum_{i=1}^{m} \sum_{j=1}^{m} -\alpha_i p_{ij} \phi_{j4}(\|x_j\|). \end{aligned} \tag{10}$$

Now, if the leading principal minors of P are all positive, then exactly as in [1], we have that for every set of positive numbers $\alpha_1, \cdots, \alpha_m$, the corresponding numbers $\beta_1, \cdots, \beta_m$ defined by

$$\beta_j = \sum_{i=1}^{m} \alpha_i p_{ij} \tag{11}$$

are also positive. Since

$$\dot{V}(t, x_1, \cdots, x_m) \leq -\sum_{j=1}^{m} \beta_j \phi_{j4}(\|x_j\|), \tag{12}$$

it follows that $\dot{V}$ is negative definite. Moreover, it is clear that $V(0, \cdots)$ assumes negative values arbitrarily close to the origin. Hence, the equilibrium point at the origin is unstable [2]. ■

In the above theorem and proof, it should be noted that the formulation of the test matrix P, the conditions on P, as well as the method of proof, exactly follow the development in [1]. Thus, the major difference between [1, theorem 5] and the present note is in the assumed nature of the uncoupled subsystems. Whereas the assumptions in [1] [particularly the inequalities (43)] require *each* of the subsystems to be *completely* unstable, the present setup permits some of the subsystems to be stable, and requires a more natural type of instability on the part of the remaining subsystems. It is also clear that Theorem 1 can be applied to systems of the form

$$\dot{x}_i(t) = f_i(t, x_i(t)) + \sum_{j=1}^{m} g_i(t, x_i(t), \cdots, x_m(t)) \tag{13}$$

in which case the inequality (7) should be modified to

$$[(\nabla V_i(t, x_i))]^T g_i(t, x_1, \cdots, x_m) \leq \sum_{j=1}^{m} \xi_{ij} \phi_{j4}(\|x_j\|). \tag{14}$$

The existence of a Lyapunov function V_i satisfying (A2) is only one of several known sufficient conditions for instability. By replacing (A2) by other conditions, it is possible to generate alternate instability criteria for large-scale systems.

Example 1: Consider two subsystems described by

$$\dot{x}_1 = -x_1$$

and

$$\begin{aligned} \dot{x}_{21} &= -x_{21} + x_{21} x_{22}^2 \\ \dot{x}_{22} &= x_{22} + x_{21}^2 x_{22}. \end{aligned}$$

Assume that the interconnection terms g_1 and g_2 in (13) are of the form

$$\begin{aligned} g_1 &= \alpha_1 x_1 x_{21} \sin x_{22} + \alpha_2 x_{22}^2 \sin x_{21} \\ g_2 &= [\gamma_1 x_{21} \operatorname{sat}(x_1 + x_{22}) + \gamma_2 x_{22} \operatorname{sat}(x_1 + x_{21})]^T \end{aligned}$$

where α_1, α_2, γ_1, γ_2 are nonnegative, and "sat" denotes the saturation function defined by

$$\operatorname{sat}\sigma = \begin{cases} -1, & \text{if } \sigma < -1 \\ \sigma, & \text{if } |\sigma| \leq 1 \\ 1, & \text{if } \sigma > 1. \end{cases}$$

Let the neighborhoods S_1 and S_2 be defined by

$$\begin{aligned} S_1 &= \{x_1 : |x_1| \leq 1\} \\ S_2 &= \{x_2 : x_{21}^2 + x_{22}^2 \leq 1\}. \end{aligned}$$

Finally, choose the Lyapunov functions

$$\begin{aligned} V_1 &= x_1^2 \\ V_2 &= x_{21}^2 - x_{22}^2. \end{aligned}$$

Then

$$\begin{aligned} \nabla V_1 &= 2x_1 \\ \nabla V_2 &= [2x_{21} \quad -2x_{22}]^T. \end{aligned}$$

With no interactions, we have

$$\begin{aligned} \dot{V}_1 &= -2x_1^2 \triangleq -\phi_{14}(x_1) \\ \dot{V}_2 &= -2x_{21}^2 - 2x_{22}^2 \triangleq -\phi_{24}(x_2). \end{aligned}$$

With the interaction terms present, we have

$$\begin{aligned} \nabla V_1^T g_1 &= 2\alpha_1 x_1^2 x_{21} \sin x_{22} + 2\alpha_2 x_1 x_{22}^2 \sin x_{21} \\ &\leq 2\alpha_1 x_1^2 + 2\alpha_2 (x_{22}^2 + x_{22})^2, \qquad \text{for } x_1 \in S_1,\ x_2 \in S_2 \\ &= \alpha_1 \phi_{14}(x_1) + \alpha_2 \phi_{24}(x_2). \end{aligned}$$

Hence, (14) is satisfied with

$$\xi_{11} = \alpha_1, \quad \xi_{12} = \alpha_2.$$

Similarly,

$$\begin{aligned} \nabla V_2^T g_2 &= 2\gamma_1 x_{21}^2 \operatorname{sat}(x_1 + x_{22}) + 2\gamma_2 x_{22}^2 \operatorname{sat}(x_1 + x_{21}) \\ &\leq 2\gamma_0 (x_{21}^2 + x_{22}^2) = \gamma_0 \phi_{24}(x_2) \end{aligned}$$

where

$$\gamma_0 \triangleq \max(\gamma_1, \gamma_2).$$

Hence, (14) is satisfied with

$$\xi_{21} = 0, \quad \xi_{22} = \gamma_0.$$

Thus, the test matrix P is given by

$$P = \begin{bmatrix} 1 - \alpha_1 & -\alpha_2 \\ 0 & 1 - \gamma_0 \end{bmatrix}.$$

It now follows, by applying Theorem 1, that instability is assured whenever $\alpha_1 < 1$, $\gamma_0 < 1$, α_2 arbitrary.

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Stabilizability of the System $\dot{x}(t)=Fx(t)+Gu(t-h)$ by a Discrete Feedback Control

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Abstract—**Stabilizability problem for the system $\dot{x}(t)=Fx(t)+Gu(t-h)$ is considered. For appropriate discrete model $x_{k+1}=Ax_k+Bu_{k-1}$ the feedback controller which has the form $u_k=\sum_{i=0}^{l}F_ix_{k-i}$ is proposed. It is proven that controllability of the pair (A,B) and cyclicity of the A matrix imply stabilizability. Some extensions and applications are also mentioned.**

I. Introduction and Problem Statement

In this technical note, a stabilizability problem for the linear system

$$\dot{x}(t)=Fx(t)+Gu(t-h) \tag{1}$$

where $t\geqslant 0$, $x(t)\in R^n$, $u(t)\in R^m$, $F\in R^{n\times n}$, $G\in R^{n\times m}$ is considered. The positive number h represents time delay in control action. For analysis of the problem it is assumed that the control $u(t)$ is constant on the intervals $[kh,(k+1)h)$, $k\in N_0$. Thus, it follows that

$$x_{k+1}=Ax_k+Bu_{k-1} \tag{2}$$

where $x_k=x(kh)$, $u_k=u(kh)$, $A\in R^{n\times n}$, $B\in R^{n\times m}$ are appropriate matrices. It is assumed that the A matrix is cyclic and the pair (A,B) is controllable. This implies that there exists a vector $b=Bq$ such that (A,b) is controllable (see [2 p. 42]). The well-known relations between controllability of the pairs $(A,B),(F,G)$ and the systems (1),(2) are recalled briefly in the Appendix.

The stated problem has the following form. For the system

$$x_{k+1}=Ax_k+bu_{k-1} \tag{3}$$

$k\in N_0$, find l and $\{f_i\}_{i=0}^{i=l}$, $f_i\in R^{1\times n}$ such that a linear feedback

$$u_k=\sum_{i=0}^{i=l}f_ix_{k-i} \tag{4}$$

gives asymptotically stable closed system

$$x_{k+1}=Ax_k+b\sum_{i=0}^{i=l}f_ix_{k-i-1} \tag{5}$$

$k\in N_0$.

II. Main Result

Theorem: Let (A,b) be controllable. Then system (3) is stabilizable by feedback (4).

Proof: From the controllability assumption it follows that there exists a (unique) nonsingular matrice $S\in R^{n\times n}$ such that

$$\hat{A}=SAS^{-1}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\\a_1&a_2&a_3&\cdots&a_n\end{bmatrix} \tag{6}$$

$$\hat{b}=Sb=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}. \tag{7}$$

Let

$$\hat{f}_j=f_jS^{-1}=\left[\hat{f}_{j1},\hat{f}_{j2},\cdots,\hat{f}_{jn}\right]\in R^{1\times n} \tag{8}$$

and

$$\bar{a}_i(\lambda)=a_i\lambda^{1+l}+\sum_{j=0}^{j=l}\hat{f}_{ji}\lambda^{l-j}. \tag{9}$$

After standard manipulations it follows that (5) is stable iff every zero of the monic polynomial

$$\mathcal{X}(\lambda)=\lambda^{n+l+1}-\left\{\bar{a}_1(\lambda)+\bar{a}_2(\lambda)\cdot\lambda+\cdots+\bar{a}_n(\lambda)\cdot\lambda^{n-1}\right\} \tag{10}$$

lies in the open disc of the unit radius in the complex plane. It is easy to check [from (9) and (10)] that

$$\begin{aligned}\mathcal{X}(\lambda)=\lambda^{n+l+1}-\Bigg\{&\sum_{k=1}^{l+1}\lambda^{k-1}\cdot\left(\sum_{j=0}^{k-1}\hat{f}_{l-j,k-j}\right)+\sum_{k=l+2}^{n}\lambda^{k-1}\\&\cdot\left(\sum_{j=0}^{l}\hat{f}_{l-j,k-j}+a_{k-(2+l)+1}\right)+\sum_{k=n+1}^{l+n+1}\lambda^{k-1}\\&\cdot\left(\sum_{j=k-n}^{l}\hat{f}_{l-j,k-j}+a_{k-(l+2)+1}\right)\Bigg\}.\end{aligned} \tag{11}$$

It is obvious that exactly one (unique) coefficient of $\mathcal{X}(\lambda)$ which is not effected by feedback parameters $\hat{f}_i$, $i=0,1,\cdots,l$ is equal to a_n, the others may by arbitrary chosen numbers. It is known (from Vietà formulas) that a_n must be equal to the sum of all zeros of $\mathcal{X}(\lambda)$. Let

$$p=\inf_{r\in N}\{r>|a_n|\} \tag{12}$$

and

$$z_0=a_n/p. \tag{13}$$

Note that $|z_0|<1$ and let

$$l=\max\{p-(n+1),0\}. \tag{14}$$

The monic polynomial $\psi(\lambda)$ defined by

$$\psi(\lambda)=\begin{cases}(\lambda-z_0)^p, & \text{if } l>0\\ \lambda^{(n+1)-p}(\lambda-z_0)^p, & \text{if } l=0\end{cases} \tag{15}$$

has a representation

$$\psi(\lambda)=\lambda^{n+l+1}-\{a_n\lambda^{n+l}+\text{lower powers terms}\} \tag{16}$$

where $1\geqslant 0$ is given by (14) and all zeros of (16) lie in the open disc of the unit radius in the complex plane; hence ψ is stable. Now it is possible to choose feedback parameters $\{\hat{f}_i\}_{i=0}^{i=l}$ so that

$$\mathcal{X}(\lambda)=\psi(\lambda) \tag{17}$$

and the last relation implies stability of (5) with $f_j=\hat{f}_jS$.

Corollary: If (A,b) is not a controllable but an uncontrollable part of A is stable then (3) is stabilizable by feedback (4)

Remark: From proof of the theorem it follows that in order to achieve smaller zeros in (17) l should be increased.

III. Concluding Remarks

The theorem presented above has the following extensions:
1) for essentially multiinput (i.e., noncyclic) controllable systems;
2) for systems

$$x_{k+1}=Ax_k+Bu_{k-M} \tag{18}$$

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