### PRINCIPLES OF PHASE LOCKED LOOPS (PLL)

### (TUTORIAL)

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### **Introduction**

In recent years, personal communications in high Megahertz and low Gigahertz frequency ranges are booming. Behind this achievements was the technological progress in integrated circuitry on one hand and application of frequency synthesis on the other hand.

### **I. Principles**

The task of the phase locked loops is to maintain coherence between input (reference) signal frequency,  $f_i$ , and the respective output frequency,  $f_o$ , via phase comparison. The theory is explained in many textbooks [e.g., 1, 2] and practically in all books on frequency synthesis. [3 through 10]. Here, we shall repeat, in short, all major features with some new achievements.

### A/ Basic equations

Each PLL loop works as a feedback system shown in Fig. 1.



Fig. 1 Basic feedback network of PLL

To get more insight into the PLL properties, we shall simplify, without any loss of generality, the block diagram to that shown in Fig. 2. and introduce the Laplace transfer functions of the individual building circuits - suitable for investigation of small signal properties.



## Fig. 2 Simplified block diagram of the PLL with individual transfer functions

Investigation of the above figure reveals that the input phase  $\varphi_i(t)$  is compared with the output phase  $\varphi_0(t)$  in phase detector (ring modulator, sampling circuit, etc.).

$$V_d(t) = \left[\varphi_i(t) - \varphi_o(t)\right] K_d \tag{1}$$

the proportionality factor,  $K_d$  [volt/2 $\pi$ ], is called the "phase detector gain."

Next,  $v_d(t)$  passes the loop filter, F(s)

$$v_2(t) = v_d(t) \otimes h_f(t)$$
<sup>(2)</sup>

where  $h_f(t)$  is the time response of the loop filter. After applying  $v_2(t)$  on the frequency control element of the voltage controlled oscillator (VCO) we get the output phase

$$\varphi_o(t) = \int \omega_o(t) dt = \omega_c t + \int K_o v_2(t) dt$$
(3)

the proportionality factor,  $K_0$  [2 $\pi$  Hz/volt], is the oscillator gain.

Since, in most cases, K<sub>d</sub> and K<sub>o</sub> are voltage dependent the general mathematical model of a PLL is a nonl+inear differential equation. Its linearization, justified in small signal cases ("steady state" working modes), provides a good insight into the problem.

the relation between input and output phase in the Laplace transform

$$[\Phi_i(s) - \Phi_o(s)F_M(s)]K_dK_oF(s) = \frac{\Phi_o(s)}{s}$$
(4)

The ratio,  $\Phi_0(s)/\Phi_i(s)$ , the PLL transfer function, is given by

KF(s)

$$H(s) = \frac{\frac{M(s)}{s}}{1 + \frac{KF(s)F_{M}(s)}{s}} = \frac{1}{F_{M}(s)} * \frac{G(s)}{1 + G(s)}$$
(5)

where we have introduced the forward loop gain  $K = K_d K_0$ and the open loop gain G(s)

$$G(s) = \frac{KF(s)F_{M}(s)}{s}$$
(6)

#### B/ Order of PLL

In the simplest case there are no filters both in forward and feedback paths.

$$H(s) = \frac{K}{s+K} \tag{7}$$

This phase lock loop is designated as the *first order loop* Generally the denominator in *H(s)* is of a higher order in *s* and we speak about PLL of the *second, third,* ect

### C/ Type of PLL

# The number of poles in the transfer function G(s), i.e. the number of integrators in the loop define the *type of the loop*

### **D**/ Phase error at the output of the phase detector (**PD**)

$$\Phi_e(s) = \Phi_i(s) - F_M(s)\Phi_o(s)$$
(8)

where

$$\Phi_o(s) = \Phi_e(s) \frac{KF_M(s)}{s}$$
(9)

After elimination of  $\boldsymbol{\varPhi}_{\mathbf{0}}(\mathbf{s})$ 

$$\Phi_{e}(s) = \Phi_{i}(s) \frac{1}{1 + G(s)}$$
(10)

By assuming the gain, G(s), as a ratio of two polynomials

$$G(s) = \frac{A(s)}{s^{n}B(s)}$$
(11)

where n is number of integrators in PLLwe get for the phase error

$$\Phi_e(s) = \Phi_i(s) \frac{s^n B(s)}{A(s) + s^n B(s)}$$
(12)

### E/ Transient and steady state errors

Due to input phase steps, frequency steps, and steady frequency changes

$$\frac{\Delta \phi_i}{s}; \qquad \frac{\Delta \omega_i}{s} = \frac{\Delta \phi_i}{s^2}; \qquad \frac{\Delta \dot{\omega}}{s} = \frac{\Delta \omega_i}{s^2} = \frac{\Delta \phi_i}{s^3}$$
(13)

After introducing any of the respective steps into (10 or 12) and performing the inverse Laplce transform we find the respective transients

With the assistance of the Laplace limit theorem we get for the final value of the phase error

$$\lim \Phi_{e}(t) = \lim [\Phi_{i}(s) \frac{s^{n+1}B(s)}{A(s) + s^{n}B(s)}]$$

$$t \to \infty \qquad s \to 0$$
(14)

### F/ Block diagram algebra

## Actual PLLs are often much more complicated than block diagrams in Fig. 1 or 2

For arriving at transfer functions,

 $|H(s)|^2$  and  $|1 - H(s)|^2$ 

we can apply the rules of the *Block diagram algebra*.

Investigation of the relation (5) reveals that the feedback block can be put outside of the basic loop. In this way we arrive at the effective transfer functions,

 $|H'(s)|^2$  and  $|1 - H'(s)|^2$ ,

$$H'(s) = \frac{1}{N}H(s) \tag{15}$$

or

$$H'(s) = MH(s) \tag{16}$$





Fig. 3 Simplification of the block diagrams of PLL: a/ series connection, b/ parallel connection, c/ and d/ feedback arrangement, e/ more complicated system.

## II. Phase locked loops of the 1<sup>st</sup> and 2<sup>nd</sup> order

The most common PLLs are those of the 2<sup>nd</sup> order. Their advantage is the absolute stability and simple theoretical and practical design.

### A/ PLL of the 1<sup>st</sup> order.

their open loop gain is

$$G(s) = \frac{K_d K_o K_A}{s} = \frac{K}{s}$$
(17)

with transfer functions

$$H(s) = \frac{K}{s+K}$$
;  $1 - H(s) = \frac{s}{s+K}$  (18)

Note that DC gain  $K_A$  can be used for changing the corner frequency, of this simple PLL, to any desired value - Fig. 4.

### Fig. 4. The block diagram of the 1<sup>st</sup> order PLL



Since the open loop gain K has dimension of the  $2\pi Hz$  normalization of the

input or reference frequency in respect to it provides nearly all information about the behaviour of the PLL.

$$\frac{s}{K} = \sigma = jx = \frac{j\omega}{K}$$
(19)

The transfer function H(jx) behaves as a low pass filter in respect to the noise and spurious signals accompanying the reference signal whereas 1 - H(jx) as a high pass filter in respect to the noise and spurious of the VCO.- see Fig. 5.



Fig. 5. Transfer functions Hi(jx) = 20log(|H(jx)|) and Ho(jx) = 20log(|1-H(jx)|)

### B/ PLLs of the 2<sup>nd</sup> order.

1<sup>st</sup> order PLL has only one degree of freedom, namely the DC gain

Other difficulties are rather modest attenuation in the respective stop bands

only 20 dB/ decade.

This last problem can be removed with introduction of a suitable low pass filter into the forward path.

### (1) A simple RC filter

In instances where we need to increase attenuation of the PLL for high frequencies application of the simple RC low pas filter, provides the desired effect. Note that the filter time constant T<sub>1</sub>

presents an additional degree of freedom





Fig. 6. 2<sup>nd</sup> order PLL loop filters: a simple RC filter.

### The open loop gain is

$$G(s) = K_{d}K_{A}\frac{1}{1+sT_{1}}\frac{K_{o}}{s} = \frac{K}{s(1+sT_{1})}$$
(20)

#### the transfer function *H(s)* of the PLL

$$H(s) = \frac{K/T_1}{s^2 + s/T_2 + K/T_2}$$
(21)

After introduction of the natural frequency  $\omega_n\,$  and the damping factor  $\zeta$ 

-

$$\omega_n = \sqrt{\frac{K}{T_1}}; \quad \zeta = \frac{\omega_n}{2K} = \frac{1}{2\sqrt{KT_1}} = \frac{1}{2\omega_n T_1}$$
(22)

#### we can rearrange the open loop gain into

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta s \omega_n}$$
(23)

#### and the PLL transfer function into its "characteristic form"

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$
(24)

After normalization of the frequency  $\omega$  in respect to the natural frequency

$$\sigma = \frac{s}{\omega_n} = jx = j\frac{\omega}{\omega_n}$$
(25)

we get for the open loop transfer function

$$G(jx) = \frac{1}{-x^2 + 2j\zeta x}$$
(26)



#### and for the PLL transfer functions

Fig.7(a) Transfer functions Hi(jx) = 20log(|H(jx)|) and Ho(jx) = 20log(|1-H(jx)|), (b) phase characteristic of the open loop gain G(jx) of the 2<sup>nd</sup> order PLL loop

with a simple RC filter. (2) Phase lag-lead or RRC filter (Fig. 8)





Transfer function of the RRC filter

$$F(s) = \frac{1+sT_2}{1+sT_1}$$
(28)

#### provides a further degree of freedom. The open loop gain is

$$G(s) = \frac{K_d K_A K_o (1 + sT_2)}{s(s + T_1)}$$
(29)

the respective transfer function

$$H(s) = \frac{K/T_1(1+sT_2)}{s^2+s(1+T_2)/T_1+K/T_1}$$
(30)

We can again introduce the natural frequency and the damping factor

$$\omega_n = \sqrt{K/T_1} \quad ; \qquad \zeta = \frac{\omega_n}{2} (T_2 + \frac{1}{K}) \tag{31}$$

and arrive to the characteristic form the transfer functions

$$G(s) = \frac{s(2\zeta\omega_n - \omega_n^2/K) + \omega_n^2}{s(s + \omega_n^2/K)} = \frac{\sigma(2\zeta - \omega_n/K) + 1}{\sigma(\sigma + \omega_n/K)}$$
(32)

and to

$$H(s) = \frac{s\omega_n(2\zeta - \omega_n/K) + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \ 1 - H(s) = \frac{s^2 + s\omega_n^2/K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
(33)

Note that the freedom for independent choice of  $\omega_n$  and  $\zeta$  resulted in reduced slope of the stop band of H(jx) on one hand and in a reduced phase margin on the other hand



Transfer functions Hi(jx) = 20log(|H(jx)|) and Ho(jx) = 20log(|1-H(jx)|), (b) phase characteristic of the open loop gain G(jx) of the 2<sup>nd</sup> order PLL loop with an RRC filter.

**Fig. 9(a)** 

### C/ PLLs of the 2<sup>nd</sup> order of the type 2.

The loop contains two integrators, the second one in the loop filter



Fig. 10 2<sup>nd</sup> order PLL loop filters: active phase-lag lead network (dashed is one of the 3<sup>rd</sup> order loop configuration).

#### Its transfer function is

$$F(s) = \frac{1 + sT_2}{sT_1 + 1/A}$$
(34)

For operation amplifier (A>>1) the time constants are

$$T_1 = (R_1 + \frac{R_1 + R_2}{A})C \approx R_1C; \quad T_2 = R_2C$$
 (35)

and the open loop gain

$$G(s) = \frac{K(1+sT_2)}{s[sT_1+(s(T_1+T_2)/A])} \approx \frac{K(1+sT_2)}{s[sT_1+1/A]}$$
(36)

Effective loop gain  $K = K_d K_A K_o$ , however, for DC the gain is  $K_{DC} = K_d K_A K_o F(\theta)$ 

 $=K_d K_A K_o A$  The PLL transfer function

$$H(s) = \frac{K(ST_2+1)}{s^2T_1 + s(KT_2+1/A) + K}$$
(37)

the natural frequency  $\omega_n$  and damping  $\zeta$ 

$$\omega_n = \sqrt{K/T_1} ; \quad 2\zeta \omega_n = \frac{KT_2 + 1/A}{T_1}$$
(38)

from which

$$\zeta = \omega_n \frac{T_2}{2} \tag{39}$$

#### Introduction of $\omega_n$ and damping $\zeta$ leads to the PLL transfer functions

$$H(s) = \frac{2\zeta s \omega_{n} + \omega_{n}^{2}}{s^{2} + 2\zeta s \omega_{n} + \omega_{n}^{2}}; \quad H(\sigma) = \frac{2\zeta \sigma + 1}{\sigma^{2} + 2\zeta \sigma + 1}$$

$$1 - H(s) = \frac{s^{2}}{s^{2} + 2\zeta s \omega_{n} + \omega_{n}^{2}}; \quad 1 - H(\sigma) = \frac{1}{\sigma^{2} + 2\zeta \sigma + 1}$$
(40)

After plotting the transfer functions Hi(x) and Ho(x) we find out that they coincide with those plotted in Fig. 9 for the PLL with the gain K (high gain loops). However, we find a substantial difference with the phase characteristic which starts, due to the two integrators in G(s), at nearly - 180 degrees. This is very important in instances with unintentionally introduced poles or delays, due to the use of sampled phase detectors, into the loop gain G(s) since the stability of the system deteriorates. The problem will be discussed in the next sections.



Fig. 11(a) Transfer functions Hi(jx) = 20log(|H(jx)|) and Ho(jx) = 20log(|1-H(jx)|); of the 2<sup>nd</sup> order PLL loop of the type 2;(b) phase characteristic of the open loop gain G(jx).

### III. Phase locked loops of the 3<sup>rd</sup> order type 2.

Investigation of Figs 7, 9, and 11 reveals PLL of the 2<sup>nd</sup> order with simple RC filter exhibits the slope of the transfer function **Hi(jx) in the stop band of** -40 dB/dec. But the high gain RRC loops have the slope of 40 dB/dec. in the stop band of the Ho(jx) transfer function.

The problem will be solved with introduction of an independent RC section in the loop filter F(s) in the type 2 systems

$$G(s) \approx \frac{K}{s} \cdot \frac{1 + sR_2(C + C_3)}{sR_1C(1 + sR_2C_3)} = \frac{K}{s} \cdot \frac{1 + sT_2}{sT_1(1 + \kappa sT_2)}$$
(41)

Note that even this  $3^{rd}$  order loop is unconditionally stable since G(s) exhibits a positive phase margin.

$$\Psi = -\frac{180^{\circ}}{\pi} (\pi + \arctan(\omega T_2) - \arctan(\omega \kappa T_2)) > -180^{\circ}$$
(42)

After introduction of the natural frequency  $\omega_n$  and the damping factor  $\zeta$  we get for the transfer function

$$H(jx) = \frac{jx^2\zeta + 1}{-jx^3 2\zeta \kappa - x^2 + jx^2\zeta + 1}$$
(43)

The transfer functions together with the phase margin are plotted in Fig. 12



Fig. 12 Transfer functions Hi(jx) =  $20\log(|H(jx)|)$ , Ho(jx) =  $20\log(|1-H(jx)|)$ , and open loop gain  $G_o(jx)$  of the 3rd order PLL loop of the type 2;  $\kappa$ =.3 and  $\zeta$ =1.5.



Fig. 13 Properties of the 3<sup>rd</sup> order PLL for different damping constants of the original 2<sup>nd</sup> order loop and for different  $\kappa$  of the additional RC section: a) phase of the open loop gain; b) magnitude of the overshoot M<sub>p</sub> of the transfer function 20log(|H(jx)|<sup>2</sup>).

### IV. Time delays in PLL.

### A/ Simple time delay

Simple time delay,  $\tau$ , is respected by multiplying the open loop gain by the factor

$$F_{dl}(s) = e^{-s\tau}; \quad F_{dl}(j\omega) = e^{-j\omega\tau}$$
 (44)

Evidently it only changes the phase margin. From Fig. 14 we see that its influence might be considerable [11].



Fig. 14 Phase shift introduced by a simple normalized time delay ωτ.

### **B**/ Sampling

In modern technology many analog processes are replaced with digital processing

#### .This is also true for PLLs

The proper approach would be the investigation with the assistance of the *z*-*transform*. The other possibility is to modify the original Laplace transform of *G(s)* 

$$\boldsymbol{G}_{\text{mod}}(s) = \boldsymbol{F}_{h}(s)\hat{\boldsymbol{G}}(s) \tag{45}$$

where

$$F_{h}(s) = \frac{1 - e^{-sT}}{s}$$
 (46)

and

$$\hat{G}(s) = \frac{1}{T} \sum_{n=\infty}^{\infty} G(s - jn\omega_s) ; \quad \omega_s = \frac{2\pi}{T}$$
(47)

Evidently

$$G_{\text{mod}}(s) = \frac{1 - e^{-sT}}{s} \cdot \frac{1}{T} \sum_{n=\infty}^{\infty} G(s - jn\omega_s) \approx \frac{1 - e^{-sT}}{s} \cdot \frac{1}{T} G(s)$$
(48)

and

$$G_{\text{mod}}(s) \approx \frac{\sinh(sT/2)}{sT/2} G(s) e^{-sT/2} \approx$$

$$G(s) e^{-sT/2} \approx \frac{G(s)}{1+sT/2}$$
(49)

The situation with the sampled PLL is illustrated in Fig. 15



Fig. 15 a) block diagram of the PLL with sampling phase detector; b) the simulating analog system

Finally we arrive at the often suggested approximation of the sampling process, with the assistance of an additional RC section.



Fig.16Properties of the transfer function  $H_{om} = 20log(|F_h(s)|)$  compared with that of a simple *RC* section  $H_{fm} = 20log[|1/(1+j\omega T/2)]$ .



Fig. 17 Transfer functions Hi(jx) = 20log(|H(jx)|), Ho(jx) = 20log(|1-H(jx)|) and 20log(|G(jx)|)of the sampled 3rd order PLL loop of the type 2 as in Fig. 12.

Note the reduced phase margin for the case where the ratio of the natural frequency  $\omega_n$  to sampling frequency  $\omega_s$  is  $1{:}10$ 

## V. Responses of PLL to the step and periodic phase and frequency changes.

The respective changes can be divided into three major groups:

Phase or frequency steps
 Periodic changes (spurious phase or frequency modulations, discrete spurious signals, etc.)
 Noises accompanying both reference and VCO signals

The information provides the phase difference at the output of the phase detector  $\Phi_e(s)$  or more exactly  $\varphi_e(t)$ . Since  $\Phi_e(s)/\Phi_i(s) = 1-H(s)$  we must investigate the following relation

$$\Phi_e(s) = \Delta \Phi_i(s) [1 - H(s)]$$
<sup>(50)</sup>

### A) Step changes

(1) Phase step  $\Delta \phi_i$  at the input of the phase detector  $(\Phi_i(s) = \Delta \Phi_i/s)$ .

In the normalized form we have

$$\Phi_{e}(\sigma) = \frac{\Delta \Phi_{i} / \omega_{n}}{\sigma} \cdot [1 - H(\sigma)] = \frac{\Delta \Phi_{i}}{\omega_{n}} \cdot \frac{\sigma + \omega_{n} / K_{v}}{\sigma^{2} + 2\zeta \sigma + 1}$$
(51)

Solution of the quadratic equation in the denominator reveals

$$\sigma_{1,2} = -\zeta \pm \sqrt{\zeta^2 - 1} \tag{52}$$

After application of the Laplace transform tables (e.g. [12]) and the above roots we get

$$\Phi_{el}(t) = \frac{\Delta \Phi_i}{\sigma_1 - \sigma_2} [(\sigma_1 + \omega_n / K_v) e^{(\sigma_1 \omega_n t)} - (\sigma_2 + \omega_n / K_v) e^{(\sigma_2 \omega_n t)}]$$
(53)



Fig. 18 Normalized transients  $\Delta \phi_{e1}(t)/\Delta \phi_i$  due to the phase step  $\Delta \phi_i$  for different damping factors  $\zeta$ ; a) for simple RC loop filter;

#### b) for high gain loop with lag lead RC filter.

(2)Frequency step  $\Delta \omega_i$  at the input of the phase detector ( $\omega_i(s) = \Delta \omega_i/s^2$ ).

After a step change of the division ratio N in the feedback path by  $\Delta N$  the effective change of the "feedback reference frequency" is  $\Delta f_r = f_r \Delta N/N$ . The consequence is the transient in the output phase  $\Phi_e(t)$ .

$$\Phi_{e}(\sigma) = \frac{\Delta \omega_{i} / \omega_{n}}{\sigma^{2}} [1 - H(\sigma)] = \frac{\Delta \omega_{i} / \omega_{n}}{\sigma^{2}} \cdot \frac{\sigma^{2}}{\sigma^{2} + 2\zeta \sigma + 1} = \frac{\Delta \omega_{i} / \omega_{n}}{\sigma^{2} + 2\zeta \sigma + 1}$$
(54)

Application of the roots from (52) and of the Laplace transform tables gives

$$\Phi_{e2}(t) = \frac{\Delta \omega_i / \omega_n}{\sigma_1 - \sigma_2} [(\sigma_1 + \omega_n / K_v) e^{(\sigma_1 \omega_n t)} - (\sigma_2 + \omega_n / K_v) e^{(\sigma_2 \omega_n t)}]$$
(55)

#### Which simplifies for very high gain and the type 2 loops

$$\phi_{e2}(t) = \frac{\Delta \omega_i}{\omega_n} \cdot \frac{e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} - e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}}{2\sqrt{\zeta^2 - 1}}$$
(56)





#### 

(3) A step of acceleration (frequency ramp)  $\Delta_{i}$  (radians /s^2)

#### In this case we get

$${}_{2}(\sigma) = \frac{\Delta \dot{\omega}_{i} / \omega_{n}^{2}}{\sigma^{2}} [1 - H(\sigma)] = \frac{\Delta \dot{\omega}_{i} / \omega_{n}^{2}}{\sigma^{3}} \cdot \frac{\sigma^{2}}{\sigma^{2} + 2\zeta \sigma + 1} = \frac{\Delta \dot{\omega}_{i} / \omega}{\sigma(\sigma^{2} + 2\zeta)}$$
(57)

#### After performing the inverse Laplace transform we arrive at

$$\phi_{e3}(t) = \frac{\Delta \dot{\omega}_i}{\omega_n^2} [1 + \frac{e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} - e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}}{2\sqrt{\zeta^2 - 1}}]$$
(58)



ed transients  $\Delta \phi_{e3}(t)/(\Delta_i/\omega_n^2)$  due to the frequency ramp  $\Delta_i$  for different damping factors  $\zeta$  for high gain loop (DC phase error is retained).

#### 2) Periodic changes.

In these instances we are interested in settled or steady states

(A) Phase modulation of the input signal.

For simplicity we shall consider modulation with a single sine wave

$$\phi_i(t) = \Delta \phi_i \sin \Omega t \tag{59}$$

The output modulation would remain sinusoidal,

$$\phi_o(t) = |H(j\Omega)| \Delta \phi_i \sin(\Omega t + \Psi_o)$$
(60)

#### however, shifted by the transfer function

In instances where PLL should be used as phase detector then the desired information must be recovered at the output of the loop detector, however, only for frequencies outside of the pass band ,i.e. for  $\Omega > \omega_n$ 

(B) Frequency modulation of the input signal.

By starting again with the sinusoidal modulation

$$\Delta \phi_i(t) = \Delta \omega_i \int_0^t \cos\Omega t dt = \frac{\Delta \omega_i}{\Omega} \sin\Omega t$$
(61)

which remains unaltered for modulation frequencies  $\Omega < \omega_n$ , however, only for PLLs of the *type 1*.

The amplitude of the normalized phase at the output of the loop detector in the instances of the PLLs of the *type 2* is peaking for  $\zeta < 1$ 

$$\frac{\Delta \phi_e}{\Delta \omega_i / \omega_n} = \frac{1 - H(jx)}{x}; \quad where \quad x = \frac{\Omega}{\omega_n}$$
(62)

### **VI. Stability of PLL**

Since PLL are feedback systems with the feedback transfer function G(s) they will oscillate whenever the gain G(s) is equal to minus 1, i.e.

$$1 + G(s) = 0 (63)$$

This condition is met in instances where

$$|G(j\omega)|e^{j\psi} = -1 \tag{64}$$

i.e. for

$$|G(j\omega)| = 1 \tag{65}$$

### and $\psi = (2k+1)\pi$ $(k = \pm 1, \pm 2,...)$ (66)

Investigation of the 1<sup>st</sup> and 2<sup>nd</sup> order loops reveals unconditionally stabile.

However, this need not be the case with higher order loops.

By taking into account that

$$G(s) = \frac{A(s)}{s^{m}B(s)}$$
(68)

#### condition (1) depends on character of the polynomial P(s)

$$P_{n}(s) = s^{m}B(s) + A(S) = a_{n}s^{n} + a_{n-1}s^{n-1} + \dots + a_{1}s + a_{0}$$
(69)

A) Hurwitz criterion of stability

We shall write a determinant  $\Delta_n$  from the coefficients of the polynomial  $P_n(s)$  in accordance with the following rules:

- 1) We start the first column with a<sub>n-1</sub> and proceeds with a<sub>n-3</sub>, etc. in rows below
- 2) We start the second column with a<sub>n</sub> and proceeds with a<sub>n-2</sub>, etc. in rows below
- 3) We start the 3<sup>rd</sup> and 4<sup>th</sup> column with zeros but further apply the 1<sup>st</sup> and 2<sup>nd</sup> columns
- 4) We start the 5<sup>th</sup> and 6<sup>th</sup> column with two zeros but further apply the 1<sup>st</sup> and 2<sup>nd</sup> columns, etc
- 5) We finish as soon as the determinant has *n columns and n rows*.

We evaluate all principle minor subdeterminants (minors)  $\Delta_i$ ; if they all are larger than zero the feedback system is a stable one.

**B)** Computation of the roots of the polynomial P(s).

If real parts of all roots are negative the loop is stable.

C) Expansion of the function 1/[1+G(s)] into a sum of simple fractions

Investigation of the function 1/[1+G(s)] reveals that that it is equal to the ratio of two polynomials R(s)/S(s)

$$\frac{1}{1+G(s)} = \frac{R(s)}{S(s)} = \frac{R(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$
(70)

where  $s_1, s_2, ...s_n$ , are roots of the polynomial S(s).

Application of the tables with Laplace transform pairs provides solution in the time domain. Another procedure is in changing the above relation into a sum of simple fractions with constants in the nominators, i.e.

$$\frac{R(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} = \frac{K_1}{s-s_1} + \frac{K_2}{s-s_2} + \dots + \frac{K_n}{s-s_n}$$
(71)

D/ The root-locus method

Root-locus method of the function 1+G(s) is intended to find location of the respective roots in the complex plain.

At present computer solution of the polynomial of  $P_n(s)$ , with the changing parameter K or any other, provides us with a set of roots which can be thereafter plotted in the complex plain.

Example: We will plot roots of the 2<sup>nd</sup> order PLL with the open loop gain

$$G(s) = \frac{K_d K_A K_o (1 + sT_2)}{s(s + T_1)}$$
(72)

The polynomial for computation of roots is of the 2<sup>nd</sup> order

$$s^{2}T_{1} + s(1+KT_{2}) + K = 0$$
(73)

The above equation is that of the circle with the center,  $-1/T_2$ ,  $\theta$ , and the radius  $r^2 = 1/T_2^2 - 1/T_1T_2$ . After introducing the loop parameters  $\omega_n$  and  $\zeta$  the root locus is their function.



Fig. 25 Root for the 2<sup>nd</sup> order PLL type 2 with the RRC loop filter

locus of 1+G(s)

#### E/ Frequency analysis of the transfer functions - Bode plots

Transfer functions of individual PLL blocks provide information about all important properties of

Phase-lock loops enclosing stability.

1/ Frequency independent gain	K=KdKaKo
2/ Factor with one zero in the origin	jω
<b>3/ Factor with one pole in the origin</b>	1/ <b>jω</b>
4/ Factor with one zero	1+jωT <sub>o</sub>
5/ Factor with one pole	1/(1+jωT <sub>0</sub> )
6/ Time delay	exp(-jωτ)

7/ A quadratic transfer function which can be encountered both in the nominator and denominator

 $[(j\omega)^2+2j\zeta\omega_n+\omega_n{}^2\ ]^{\pm 1}$ 

In the earlier and often in the contemporary literature stability of the PLL systems is investigated with the simple Bode plots in accordance with the old tradition of servo systems.

However, application of modern computers provides more insight and more precision solutions.

Nevertheless, for the sake of completeness we shall repeat here some basic rules for construction of the Bode plots. After computing logarithm of the open loop gain we get



Fig. 26 Bode plots for 1/ Frequency independent gain K=KdKAKo , 2/ Factor with one zero in the origin  $j\omega$ , 3/ Factor with one pole in the origin  $1/j\omega$ : A/ Decibel gain, B/ phase



Fiug. 27 Bode plot of the function  $1+j\omega T_0$ 



Fiug. 28 Bode plot of the function  $1/(1+j\omega T_0)$ 



#### the computer drawing.





Fig. 29 Bode plot of the 3<sup>rd</sup> order type 2 PLL

VII. Phase locked loops of the 4<sup>th</sup> and higher orders.

We have seen that the 2<sup>nd</sup> and 3<sup>rd</sup> order loops were unconditionally stable

However, we often introduce intentionally additional filtering sections to improve properties of PLL's but the stability is endangered.

A/ Twin-T RC filter

In instances where we need large attenuation at a specific frequency addition of the Twin-T RC filter, shown in Fig. 30 may solve the problem.



Fig. 17 Twin-T RC filter

This network exhibits "infinite attenuation" for the following arrangement

$$\omega^{2} = \frac{1}{2R_{1}R_{2}C_{1}^{2}}$$

$$\omega^{2} = \frac{2}{C_{1}C_{2}R_{2}^{2}}$$
(74)

After introducing following relations

 $R = R_1/n = R_2$  and  $C = C_1 = C_2/4n$  (75)

we get for the "resonant" frequency,  $\omega_{rf}$ ,

$$\omega_{rf} = \frac{1}{RC\sqrt{2}n} \tag{76}$$

For the input resistance R<sub>i</sub> << R and the output resistance R<sub>out</sub> >> R

#### The transfer function of the *Twin-T* is

$$F(jy) = \frac{1-y^2}{1+4jy-y^2}$$
where  $y = \frac{\omega}{\omega_{rf}} = \frac{\omega}{\omega_n} \cdot \frac{\omega}{\omega_{rf}} = x\alpha$ 
(77)





Investigation of the properties of these PLL will be started with the normalized open loop gain of the second order loop-type two,  $G_2(jx)$ ,

$$G_2(jx) = \frac{1+j2\zeta x}{(jx)^2}$$
 (78)

and thereafter by adding additional gains as that of Twin-T,

### G<sub>T</sub>(jx), and sampling Ge(jx)

$$G_{T}(j) = \frac{1 + (jx\alpha)^{2}}{1 + 4jx\alpha + (jx\alpha)^{2}}; \quad Ge(jx) = \frac{\sin(\pi x\delta)}{\pi x\delta} e^{-j\pi x\delta}$$
(79)

where we have introduced the "resonant" frequency  $f_{rf}$  and the sampling frequency  $f_s$ 

$$\alpha = \frac{\omega_n}{\omega_{rf}} \quad and \quad \delta = \frac{\omega_n}{\omega_s}$$
(80)

The overall open loop gain

$$G_{4,T}(jx) = G_2(jx)G_T(jx)Ge(jx)$$
(81)

The transfer functions Hi(x) and Ho(x) are plotted in Fig. 32 together with the open loop gain G(jx) and the phase margin  $\psi(jx)$  for  $\alpha = .1$  and  $\delta = .05$ ;

Note that the phase margin is small, *20 deg.*, only. In addition both transfer functions have peaks of about 10 dB which indicates under damping.



Fig. 32 Transfer functions Hi(x) = 20log(|H(jx)|), Ho(jx) = 20log(|1-H(jx)|) and 20log(|G(jx)|) of the sampled 4rd order PLL loop of the type 2 with additional Twin-T filter with parameters  $\alpha = .1$  and  $\delta = .05$ 

### B/ Active 2<sup>nd</sup> order low pass filter

From different configurations we shall investigated the only one shown



Fig. 33 Active 2<sup>nd</sup> order low pass filter

Its transfer function with a very large gain of the operation amplifier is

$$F(j\omega) = \frac{1}{1 + j\omega 2RC_2 - \omega^2 R^2 C_1 C_2} = \frac{1}{1 + j\omega 2T_2 - \omega^2 T_1 T_2}$$
(82)

After introduction of the natural frequency  $\omega_{nf}$ 

$$\omega_{nf}^2 = 1/T_1 T_2 = 1/R^2 C_1 C_2$$
(83)

and damping d

$$d = \omega_{nf}T_2 = \sqrt{T_2/T_1} = \sqrt{C_2/C_1}$$
(84)

we get for the transfer function (in the normalized form)

$$F(jy) = \frac{1}{1 + jy^2 d - y^2}; \text{ where } y = \omega/\omega_{nf} = x\alpha$$
(85)

which is plotted in Fig. 34 for different damping constants together with the respective phase characteristics.



Fig. 34 a/ Transfer functions of the active 2<sup>nd</sup> order low pass filter; b/ its phase characteristics.



Fig. 35 Transfer functions Hi(x) =  $20\log(|H(jx)|)$ , Ho(jx) =  $20\log(|1-H(jx)|)$  and  $20\log(|G(jx)|)$  of the sampled: a) 4<sup>th</sup> order PLL loop of the type 2 with additional 2<sup>nd</sup> order low pass filter with parameters  $\alpha$  =.1 and d = .6; b) of the 5<sup>th</sup> order PLL with parameters  $\alpha$  =.1, d = .6 and  $\kappa$ =.2.

### C/Phase lock loop of type 3

Loop of the type 3 are encountered rarely for special services only. For the sake of

simplicity we will consider two active RRC filters (see Fig. 10) in series.

$$G(s) = \frac{K(sT_2+1)^2}{s(sT_1+1/A)^2}$$
(86)

After introducing the natural loop frequency  $\omega_n$  and the damping factor  $\zeta$  we can change the above relation into

$$G(jx) = \gamma \frac{(2\zeta jx+1)^2}{(jx)^3} \qquad \text{where} \qquad \gamma = \frac{\omega_n}{K}$$
(87)

![](_page_44_Figure_4.jpeg)

#### A typical transfer functions with the respective phase characteristic are

Fig. 36 Transfer functions Hi(x) = 20log(|H(jx)|), Ho(jx) = 20log(|1-H(jx)|2) and 20log(|G(jx)|) of the 3<sup>rd</sup> order PLL loop of the type 3 with two additional 2<sup>nd</sup> order low pass filters and with parameters  $\gamma = .5$  and  $\zeta = .7$ 

### **VIII.** Noise properties of PLL

Random fluctuations of phase and amplitudes (generally designated as noise) of frequency

generators are often limiting factors for many applications even in PLL's.

$$v(t) = V_o[1 + A(t)]\cos[\omega_o + \varphi(t)]$$
(88)

Due to the limiting processes we can consider only

$$v(t) = V_o \cos[\omega_o t + \varphi(t)] = V_o \cos[\omega_o t + \int \dot{\varphi}(t) dt]$$
(89)

where

 $\dot{\varphi}(t) \approx \Delta v(t) + discrete \ sp. \ sig. + secular \ terms$  (90)

#### A/ Basic frequency instability measures in the frequency domain

#### 1) Phase measures

#### The autocorrelation of the random phase departures $\phi(t)$ is defined

$$R_{\phi}(\tau) = \langle \phi(t+\tau/2)\phi(t-\tau/2) \rangle =$$

$$[\lim \frac{1}{T} \int_{-t/2}^{T/2} \phi(t+\tau/2)\phi(t-\tau/2)d\tau]_{T\to\infty}$$
(91)

and the respective *Power Spectral Density* (PSD)  $S'_{\phi}(\omega)$  (primed indicate two sided spectra) is

$$S'_{\phi}(\omega) = \int_{-\infty}^{\infty} R_{\phi}(\tau) e^{-j\omega\tau} d\tau \quad [radian^{2}/Hz]$$
(92)

We often encounter another definition, i.e.  $\mathcal{L}(f)$ , defining ration of the phase power at frequencies  $f_0 \pm f$  in the 1 Hz bandwidth (where f is the so called Fourier frequency) in respect to the whole power of the investigated signal

$$\mathscr{L}(f) = S'_{\phi}(f) = \frac{1}{2}S_{\phi}(f) \tag{93}$$

where  $S_{\phi}(f)$  is the so called one sided PSD.

### 2) Frequency measures

In contradistinction to the uncertainty about the first moment of phase fluctuations, the first moment of frequency fluctuations can be put to zero

$$\boldsymbol{m}_1 = \langle \Delta v(t) \rangle \approx \mathbf{0} \tag{94}$$

However, this is not the case with the 2<sup>nd</sup> moment which can be defined as

$$m_2 = \langle [\Delta v(t)]^2 \rangle = \int_0^\infty S_{\Delta v}(f) df$$
(95)

A further simplification will be achieved by normalizing frequency fluctuations in respect to the carrier frequency  $v_0 = f_0$ 

$$y(t) = \frac{\Delta v(t)}{v_o}$$
(96)

Relation between the Power Spectral Density (PSD)  $S_V(f)$  and  $S_{\phi}(f)$ 

$$S_{\phi}(f) = \left(\frac{f_o}{f}\right)^2 S_y(f) \tag{97}$$

### B/ Basic frequency instability measures in the time domain

At very low frequencies direct evaluation of phase PSD is difficult. The

problem is solved with sample variances which provide other and very effective frequency stability measures. Nevertheless, in actual practice we encounter the Allan variance (two sample variance) defined as

$$\sigma_{y}^{2}(\tau) = \frac{1}{2} \langle (\bar{y}_{k+1} - \bar{y}_{k})^{2} \rangle$$
(98)

or the modified Allan variance

$$Mod \sigma_{y}^{2} = \frac{1}{2n^{2}} < [\sum_{i=1}^{n} (\overline{y}_{i+n} - \overline{y}_{i})]^{2} >$$
(99)

where

$$\overline{y}_{k} = \frac{1}{\tau} \int_{t_{k}}^{t_{k}+\tau} y(t) dt ; \quad t_{k}+\tau = t_{k+1}$$
(100)

Frequency stability defined in the frequency and time domain measures are related with the assistance of a transfer function

$$\sigma_y^2(\tau) \approx \int_0^\infty S_y(f) |H_A(jf)|^2 df$$
(101)

The difficulty is that we can evaluate the integral in (102), in the closed form, only for a very particular form of  $S_y(f)$ , namely a piece-wise linearized

$$S_{y}(f) \approx \frac{h_{-2}}{f^{2}} + \frac{h_{-1}}{f} + h_{o} + h_{1} \cdot f + h_{2} \cdot f^{2}$$
(102)

![](_page_48_Figure_0.jpeg)

Note two dB measures on the vertical axis: the on the r.h. side are values of  $S\varphi$  (f), however, that on the l.h. side retains slopes of the  $S\varphi$  (f), but it is invariant in respect to the carrier frequency as  $S_y$  (f). Consequently we can compare noise characteristics of different generators in one and the same figure.

### All important noise processes, generally encountered by evaluating the frequency instability, are

the random walk of frequency with the noise constant h<sub>-2</sub>

the flicker frequency noise with the noise constant h<sub>-1</sub>

h

the white frequency noise with the noise constant

the flicker phase noise with the noise constant h<sub>1</sub>

the white phase noise with the noise constant h <sub>2</sub>		
S <sub>v</sub> (f)	$\sigma_v^2(\tau)$	Mod $\sigma_v^2(\tau)$
$h_{-2}/f^2$	$(2\pi)^2 \tau h_{-2}/6$	$\approx$ 5.4n $\tau_{o}h_{-2}$
<b>h</b> <sub>-1</sub> / <b>f</b>	$2h_{-1}ln(2 h_{0}/2\tau)$	≈ <b>0.94h</b> <sub>-1</sub>
<b>h</b> <sub>o</sub>	$h_o/2\tau$	≉ho/4nτ₀
h <sub>1</sub> f	$h_1(2\pi\tau)^{-2}[1.38+3\ln(\omega_H\tau)]$	$\approx .084h_1/(n\tau_o)^2$
$h_2 f2$	$3h_2f_H(2\pi\tau)^{-2}$ $\approx .07$	$f_{\rm H}/n(2\pi n\tau)^2$ $6\tau_{\rm o}h_2f_{\rm H}/(n\tau_{\rm o})^3$

### C/ Noise in oscillators

#### 1/ Crystal oscillators

The resonator circuit exhibits the flicker and white noise

$$S_{y}(f) \approx \frac{a_{r}}{f} + \frac{2kT}{P_{r}}$$
(104)

where  $P_r$  is the dissipated power and  $a_r$  the flicker noise constant.

$$S_{\psi}(f) = \frac{a_e}{f} + a_o \qquad (a_o = \frac{2kTFG}{P_o})$$
 (105)

The noise of the maintaining circuit is

$$S_{\varphi,out}(f) \approx \left(\frac{a_r}{f} + \frac{2kT}{P_r} + \frac{a_e}{f} + a_o\right) \frac{fo^2}{f^2 Q_U^2} + \frac{a_e}{f} + a_o$$
(106)

Finally, we arrive at the PSD of the oscillator phase noise where we have introduced the unloaded  $Q_U$  by putting  $2Q_L \approx Q_U$ .

The magnitude of  $a_r$  can be appreciated from noise measurements performed on quartz resonators. Its value was found approximately to be  $a_r \approx 10^{-12.75}$ . After introducing this value, together with the quartz material constant,  $f_o Q_U \approx 1.3 * 10^{13}$ , we get

$$S_y(1) = h_{-1} \approx \frac{10^{-12.75} + a_e}{Q_U^2}$$
 (107)

## and the plateau in the Allan variance is approximately for all quartz crystal resonators (since generally $a_r > a_e$ )

$$\sigma(\tau) \approx 10^{-6.4} Q_U^{-1} \approx 10^{-19\pm 1} f_o$$
(108)

#### 2/ LC oscillators

The relation (106) is also valid for LC oscillators. For the mean values we can write

$$\frac{S_{\phi}(f)}{f_{o}^{2}} \approx \frac{1}{f^{3}} \frac{10^{-11.6}}{Q_{L}^{2}} + \frac{1}{f^{2}} \frac{10^{-15.6}}{Q_{L}^{2}} + \frac{1}{f} \frac{10^{-15}}{f_{o}^{2}} + 10^{-15}$$
(109)

Note that the coefficients **h**<sub>i</sub> (see 103) are mean values form experimental measurements. Actual noise coefficients can differ by -2 to +1 order

For a preliminary estimation of the oscillator noise both crystal and LC we can use the following diagram

![](_page_51_Figure_5.jpeg)

Fig. 37 Noise characteristic of oscillators with parameters  $Q_L$  and  $f_0$ .

### D/ Noise in digital frequeny dividers

## We provide practical formulae for a preliminary estimation of the output noise of digital dividers.

For TTL and ECL divider family

$$S_{\phi,D}(f) \approx \frac{10^{-14\pm l} + 10^{-27\pm l} f_o^2}{f} + 10^{-16\pm l} + 10^{-22\pm l} f_o$$
(110)

#### For GaAs divider family the above relation requires only a small correction in the first term

$$S_{\phi,D}(f) \approx \frac{10^{-10\pm 1} + 10^{-27\pm 1} f_o^2}{f} + 10^{-16\pm 1} + 10^{-22\pm 1} f_o$$
(111)

## We expect that these formulae can be also used for appreciation of the noise quality of actual devices.

![](_page_52_Figure_7.jpeg)

Fig. 38 a) Flicker phase noise of TTL and ELC digital dividers b) white phase noise of TTL and ELC digital dividers.

### E/ Noise in Phase detectors and amplifiers

For a preliminary estimation we can apply an experimentally found relation

$$S_{\phi,PD} = \frac{S_{\nu,PD}}{K_d^2} \approx \frac{10^{-14\pm 1}}{f} + 10^{-15.5\pm 2}$$
(112)

### F/ Noise in loop filters

After comparing actual PLL PSD  $S_{\phi,L}$  (in the white noise region) with magnitudes added by dividers  $S_{\phi,D}$  and  $S_{\phi,PD}$  we find out that its level is

$$S_{\phi,L}(f) \approx \frac{S_{\nu,F}(f)}{K_d^2} = \frac{4kTR}{K_d^2} \approx \frac{4kT\zeta}{\pi f_n K_d^2 C} \cdot \frac{T_1}{T_2}$$
(113)

orders higher; the reason is Johnson noise generated in the filter resistors. Consequently

![](_page_53_Figure_7.jpeg)

![](_page_53_Figure_8.jpeg)

Fig. 39 PSD  $S_{\phi,L}$  of the additive noise of different PLL's together with practical (full line) and theoretical limits (R).

### **G**/ Noise in PLL

#### We shall start from a rather general PLL arrangement.

![](_page_54_Figure_2.jpeg)

Fig. 40 Block diagram of a general PLL with additive noise sources.

#### By assuming a locked loop we can write with the assistance of the Laplace transform for the linearized arrangement

$$\Phi_{o,n} = \int (\frac{\Phi_{i,n}}{Q} - \Phi'_{o,n}) K_d + \Phi_{L,n} J F_L(s) \frac{K_o}{s} + \Phi_{osc,n}$$
(114)

where

$$\Phi'_{o,n} = (\Phi_{o,n} - \Phi_{m,n} + \Phi_{MI,n}) \frac{F_M(s)}{N} + \Phi_{DN,n}$$
(115)

#### Since most of the noise components are random by nature and uncorrelated the PSD of the PLL output phase is

$$S_{\phi,o}(f) = \{S_{\phi,i}(f)[M + \frac{N}{Q}]^2 + [S_{\phi,DQ}(f) + S_{\phi,DN}(f) + \frac{S_{v,PD}(f) + S_{v,F}(f)}{K_d^2}]N^2 + S_{\phi,MU}(f) + S_{\phi,MI}(f)\} |H'(jf)|^2 + S_{\phi,osc}(f)|1 - H'(jf)|^2$$
(116)

## All the additive noises, due to the phase detector, loop frequency dividers, loop amplifiers, and loop filters can be summarized into a PSD $\rm S_{L}(f)$

$$S_{\phi,L} \approx S_{\phi,DQ}(f) + S_{\phi,DN}(f) + \frac{S_{\nu,PD}(f) + S_{\nu,F}(f)}{K_d^2}$$
(117)

![](_page_56_Figure_0.jpeg)

Fig. 41 Output noise of 100MHz VCO locked to a 10 MHz crystal oscillator via a 5<sup>th</sup> order loop investigated in Fig. 35.

10

 $1 \cdot 10^4$ 

 $1.10^{5}$ 

 $1.10^{6}$ 

 $1.10^{7}$ 

 $1.10^{3}$ 

100

fm

-110

-150

-170

0.01

0.1

1

 $\overline{20 \cdot \log} \left( 1 - H5_{m} \right)_{-130}$ 

### **IX.** Acquisition

#### Working ranges of PLL

1) Hold–in range  $\Delta \omega_{\rm H}$ 

$$\Delta \omega_H = K_o v_{2,\text{max}} \leq K_{DC} \tag{115}$$

2) Pull-in range  $\Delta \omega_{\mathbf{P}}$ 

Let us assume that the difference between the reference frequency  $\omega_i$  and the free running VCO frequency  $\omega_c$  is larger than  $\Delta\omega_P$  the result is a beat

$$\mathbf{v} = |\omega_i - \omega_o| = |\omega_i - \omega_c| - \Delta \omega$$
(116)

evidently

$$\mathbf{v} = \mathbf{v}_c - \Delta \boldsymbol{\omega}(\mathbf{v}) \tag{117}$$

After taking into account the feedback properties of PLL and the principle of the harmonic balance we get for the 2<sup>nd</sup> order type 2 loops

$$\mathbf{v}_{c} = \mathbf{v} + \frac{AK^{2}}{2\mathbf{v}} \Phi(\mathbf{v}) \; ; \; \Phi(\mathbf{v}) = Re \left| F(j\mathbf{v}) F_{M}(j\mathbf{v}) \right|$$
(118)

Minimum of the above relation reveals

$$v_{\min} \approx K_{\sqrt{A\Phi(v_{\min})/2}}$$
 (119)

and for the pull-in range we get

$$\Delta \omega_{P} \approx K_{\sqrt{2A\Phi(\Delta \omega_{P}/2)}}$$
(120)

a) 2<sup>nd</sup> order simple DC filter

$$\frac{\Delta \omega_p}{\omega_n} = x_p \approx 1.475 \tag{121}$$

b) Lag lead (RRC) filter (PLL type 1)

$$\Delta \omega_p \approx K \sqrt{2T_2/T_1} \approx 2 \sqrt{\zeta \omega_n K}$$
(122)

c) Lag lead (RRC) filter (PLL type 2)

$$\Delta \omega_p \approx K_{\sqrt{2AT_2/T_1}} \approx 2\sqrt{\zeta \omega_n K_v}$$
(123)

#### d) Lag lead (RRC) filter (PLL type 2) with time delay

$$\Delta \omega_{p,\tau} \approx \Delta \omega_p \sqrt{1 - A \frac{T_2}{T_1} (\frac{K\tau}{2})^2}$$

Note that it exists certain delay for which the pull-in range is zero. This is illustrated with Fig. 42 and 43. The oscillating branch in Fig. 42 indicates the possibility of false locks.

![](_page_59_Figure_0.jpeg)

Fig. 42 Normalized detuning  $x_c = v_c / \omega_n$  as function of  $x = v / \omega_n$  for PLL of the 2<sup>nd</sup> order type 2 for two amplifier gains and different delays

![](_page_59_Figure_2.jpeg)

#### Fig. 43 Normalized pull-im range $x_P = \Delta \omega_P / \omega_n$ for PLL of the 2<sup>nd</sup> order type 2 for two amplifier gains and normalized delay (points were found by computers).

3) Lock-in range  $\Delta \omega_{\rm L}$ 

Acquisition is expected without cycle slipping. This condition is met with zero beat note at the output of the PD

$$\frac{d\phi_{e1}(t)}{dt} + \frac{d\phi_{e2}(t)}{dt} = 0$$
(125)

a) PLL of the 1<sup>st</sup> order

$$\Delta \omega_L \approx K, \quad K \frac{\pi}{2}, \quad \pi K \tag{126}$$

#### b) PLL of the 2nd order with RC filter

$$\Delta \omega_L \approx \omega_n (2\zeta - \omega_n / K) \tag{127}$$

#### c) PLL of the 2nd order with RRC filter (high gain loops)

$$\Delta \omega_L \approx \omega_n 2\zeta \approx K \frac{T_1}{T_2}$$
(128)

#### 4) Pull-out frequency $\Delta \omega_{PO}$

From investigation of the transients due to the frquency step we get for its maximum

$$\Phi_{e2,\max} \approx \frac{\Delta\omega}{\omega_n} f(\zeta)$$
(129)

and finally

$$\Delta \omega_{PO} \approx 1.8(\zeta + 1)\omega_n \tag{130}$$

![](_page_61_Figure_2.jpeg)

5) False locks

#### In some instances the pull-in process may result in The principle can be explained with the assistance of the following figure Fig. 44 Block diagram of the PLL with the beat note a bit smaller than $\Delta \omega_P$

For the slowly varying detuning  $\Delta \omega$  we have

$$\Delta \omega \approx \frac{K^2}{2\nu} |F(j\nu)| \cos \psi$$
(131)

Additional filtering or time delays may cause  $\Psi > \pi/2$  which will change the sign of the slowly varying tuning voltage  $u_2(t)$  and starts to push the loop out of lock and in some instances lock the VCO on a false frequency cf. Fig.45

![](_page_62_Figure_0.jpeg)

Fig. 45 The DC component  $\Delta\omega/K$  in the pull-in process: a)PLL of the 4<sup>th</sup> order Type 2 with two additional sections RC; b) PLL of the 3<sup>rd</sup> order Type 2 with additional time delay  $(\zeta = .7, \kappa = .3)$ .

6) Pull-in time

Solution will start with the simplified block diagram in Fig. 44. Note that the AC path is responsible for the magnitude of the beat note  $v_c$ . Furthermore we will assume the 2<sup>nd</sup> order loop with RRC filter with the reduced gain

$$K_r \approx K \frac{T_1}{T_2} \tag{132}$$

Finally we arrive at an approximate pull-in time for the sine wave PD

$$T_P \approx \frac{1}{2\zeta\omega_n} (\frac{\Delta\omega}{\omega_n})^2$$
 (133)

![](_page_63_Figure_2.jpeg)

#### Fig. 46 Asymptotic approximations of the pull-in time for PLL of the 2<sup>nd</sup> order : a) for a simple phase detector; b) for a phase-frequency detector for two different damping constants

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