Gravitational Forces in
Dual-Porosity Models
of Single Phase Flow

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GRAVITATIONAL FORCES IN
DUAL-POROSITY MODELS OF SINGLE PHASE FLOW*

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Abstract—A dual porosity model is derived by the formal theory of homogenization. The model properly incorporates gravity in that it respects the equilibrium states of the medium.

1. INTRODUCTION

We consider flow in a naturally fractured reservoir which we idealize as a periodic medium as shown in Fig. 1. There are three distinct scales in this system, the pore scale, the scale of the average distance between fractures, and the scale of the entire reservoir. The concept of dual-porosity [4], [10] is used to average the two finer scales in such a way that the pore scale is recognized as being much smaller than the fracture spacing scale. The fracture system is modeled as a porous structure distinct from the porous structure of the rock (the *matrix*) itself.
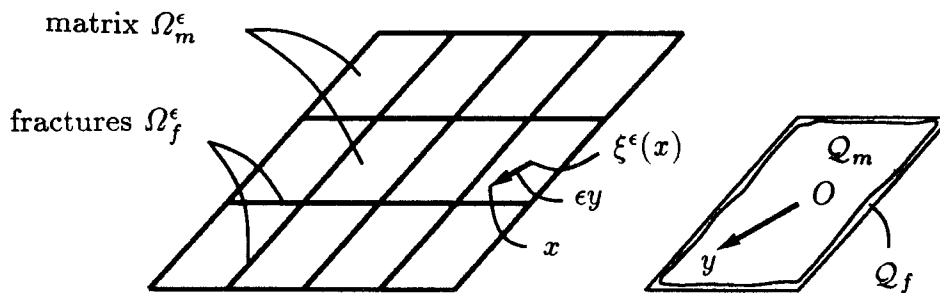

Fig. 1. The reservoir Ω.   Fig. 2. The unit cell $\mathcal{Q}$.

Dual-porosity models can be derived by the technique of homogenization [2], [3], [6] (see also the general references [5], [7], and [9]). Briefly, we pose the correct microscopic equations of the flow in the reservoir and then let the block size shrink to zero. The resulting macroscopic model is formulated in six space dimensions, three of them represent the entire reservoir over which the fracture system flow occurs. At each point of the reservoir, there exists a three dimensional, "infinitely small" matrix block (surrounded by fractures) in which matrix flow occurs.

For single phase, single component flow, it is recognized that diffusive, gravitational, and viscous forces affect the movement of fluids between the matrix and fracture systems; however, only diffusive forces are easily handled (see, e.g., [1], [4],

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[6], [8], [10], and the many multiphase models in the petroleum literature). Simply including gravity in the matrix of the standard model [2], [3], [6] creates an inconsistency in that when the fracture system is in gravitational equilibrium, the matrix system is not. In this paper we derive a consistent model.

2. The Microscopic and Macroscopic Models

Denote the reservoir by Ω. For a sequence of ϵ's decreasing to zero, we consider equivalent reservoirs with matrix blocks that are ϵ times the original size in any linear direction. Let Ω_f^ϵ and Ω_m^ϵ be the fracture and matrix parts of Ω, respectively. Each period of the ϵ-reservoir is congruent to the unit cell $\epsilon\mathcal{Q}$; the period at point $x \in \Omega$ is denoted by $\mathcal{Q}^\epsilon(x)$. For the fixed unit cell $\mathcal{Q}$ (see Fig. 2), we write $\mathcal{Q}_f$ and $\mathcal{Q}_m$ for the fracture and matrix parts, respectively. Let the centroid of $\mathcal{Q}$ be the origin, and the centroid of $\mathcal{Q}^\epsilon(x)$ be $\xi^\epsilon(x)$. Then $x = \xi^\epsilon(x) + \epsilon y \sim x + \epsilon y$. Asymptotically, $x \sim \xi^\epsilon(x)$ selects a period and y specifies a point in the enlarged, congruent period $\mathcal{Q}$. Let ν denote the unit normal vector to the matrix-fracture interface $\partial\Omega_m^\epsilon$ (or $\partial\mathcal{Q}_m$).

We use upper and lower case letters for fracture and matrix quantities, respectively. Let P (or p) be the fluid pressure, and Φ^* (or ϕ) and K^* (or k) be the porosity and permeability on the *pore* scale (so $\Phi^* \approx 1$ and K^* is very large). The fracture *system* porosity and permeability, Φ and K, are defined on the *fracture spacing* scale. Easily

$$\Phi = |\mathcal{Q}_f|\Phi^*/|\mathcal{Q}|, \tag{1}$$

where $|\cdot|$ denotes the volume of the set, while K is derived by homogenization. Finally, $\rho(P)$ (or $\rho(p)$) and μ are fluid density and viscosity, and g is the gravitational constant. Let $\mathbf{e}_j$ point in the jth Cartesian direction, where $\mathbf{e}_3$ points down.

Define the function $\psi(x_3)$ as the solution to

$$\psi' = \rho(\psi)g; \quad \text{i.e.,} \quad \int_{\psi_0}^{\psi} \frac{d\pi}{\rho(\pi)} = g(x_3 - x_{3,0}). \tag{2}$$

Then $\nabla p - \rho(p)g\mathbf{e}_3 = 0$ if and only if $p = \psi(x_3 + \bar{x}_3)$ for some constant $\bar{x}_3$, and so $\psi(x_3 + \bar{x}_3)$ is the gravitational equilibrium pressure distribution. We note that the pseudopotential of the flow is given by $\psi^{-1}(p) - x_3$.

We ignore boundary conditions on $\partial\Omega$, external sources/sinks, and initial conditions since we are interested in internal flow.

The microscopic model: For the fracture flow,

$$\Phi^* \frac{\partial}{\partial t}\rho(P^\epsilon) - \nabla \cdot \left[\mu^{-1}\rho(P^\epsilon)K^*\left(\nabla P^\epsilon - \rho(P^\epsilon)g\mathbf{e}_3\right)\right] = 0, \quad x \in \Omega_f^\epsilon, \tag{3a}$$

$$\mu^{-1}\rho(P^\epsilon)K^*\left(\nabla P^\epsilon - \rho(P^\epsilon)g\mathbf{e}_3\right) \cdot \nu$$
$$= \epsilon\mu^{-1}\rho(p^\epsilon)k\left(\epsilon\nabla p^\epsilon - \rho(p^\epsilon)g\mathbf{e}_3\right) \cdot \nu, \quad x \in \partial\Omega_m^\epsilon. \tag{3b}$$

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For the matrix,

$$\phi \frac{\partial}{\partial t}\rho(p^\epsilon) - \epsilon \nabla \cdot \left[\mu^{-1}\rho(p^\epsilon)k\left(\epsilon\nabla p^\epsilon - \rho(p^\epsilon)g\mathbf{e}_3\right)\right] = 0, \quad x \in \Omega_m^\epsilon, \tag{4a}$$

$$p^\epsilon = \psi\left(\psi^{-1}(P^\epsilon) + (\epsilon^{-1} - 1)(x_3 - \xi_3^\epsilon(x)) + \bar{\zeta}^\epsilon\right), \quad x \in \partial\Omega_m^\epsilon. \tag{4b}$$

On each $\mathcal{Q}^\epsilon(x)$, we need to define $\bar{\zeta}^\epsilon$. For a given P^ϵ, we can find for each constant $\bar{\zeta}^\epsilon$ the solution $\tilde{p}^\epsilon$ of the steady-state problem corresponding to (4). So, for the given fracture pressure P^ϵ, we take the $\bar{\zeta}^\epsilon$ which gives rise to the $\tilde{p}^\epsilon$ that satisfies

$$\int_{\mathcal{Q}_m^\epsilon(x)} \phi\rho(\tilde{p}^\epsilon)\,dx = \int_{\mathcal{Q}_m^\epsilon(x)} \phi\rho(\bar{p}^\epsilon)\,dx, \tag{5}$$

where $\bar{p}^\epsilon$ is the steady state solution of the unscaled problem corresponding to (4), given by removing the two ϵ's appearing as coefficients in (4a) and replacing (4b) by $\bar{p}^\epsilon = P^\epsilon$. (In the case of an incompressible fluid, simply take $\bar{\zeta}^\epsilon = 0$.)

This ϵ-family of microscopic models satisfies the following:

(i) Darcy flow governs the reservoir, and it does so in the standard way when $\epsilon = 1$ (since then $\bar{\zeta}^\epsilon = 0$);

(ii) For each ϵ, Darcy flow occurs in the fractures and within the *scaled* matrix blocks (i.e., if any matrix block $\mathcal{Q}_m^\epsilon$ is expanded to unit size $\mathcal{Q}_m$, the transformed equations indicate that Darcy flow results);

(iii) If the fracture system is in gravitational equilibrium in the vicinity of a block, then the boundary conditions on that block reflect this gravitational equilibrium;

(iv) For fixed fracture conditions around any matrix block, the steady state matrix solution gives rise to the same mass as calculated from the steady-state solution of the unscaled matrix problem.

We require (iv) so that mass is conserved, since when we scale the matrix problem with (ii)–(iii), we change the pressures which may change the total mass. Under steady-state conditions it is easy to account for any such spurious changes.

We remark that the standard microscopic model [2], [3], [6] replaces (4b) with $p^\epsilon = P^\epsilon$, omits (5), and to be consistent needs to have $\rho(p^\epsilon)g$ replaced by $\epsilon\rho(p^\epsilon)g$ in (3b) and (4a). The novel expression (4b) can be viewed as a scaled continuity of pseudopotential, since we can rewrite it as

$$\psi^{-1}(p^\epsilon) - \left(\xi_3^\epsilon(x) + \epsilon^{-1}(x_3 - \xi_3^\epsilon(x)) + \bar{\zeta}^\epsilon\right) = \psi^{-1}(P^\epsilon) - x_3.$$

The macroscopic model: For the fracture flow,

$$\Phi \frac{\partial}{\partial t}\rho(P^0) + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m} \phi\frac{\partial}{\partial t}\rho(p^0)\,dy$$
$$- \nabla_x \cdot \left[\mu^{-1}\rho(P^0)K\left(\nabla_x P^0 - \rho(P^0)g\mathbf{e}_3\right)\right] = 0, \quad x \in \Omega, \tag{6}$$

where (1), (9), and (10) define the new coefficients. For the matrix flow, for each $x \in \Omega$,

$$\phi \frac{\partial}{\partial t} \rho(p^0) - \nabla_y \cdot \left[\mu^{-1} \rho(p^0) k \left(\nabla_y p^0 - \rho(p^0) g e_3 \right) \right] = 0, \quad y \in \mathcal{Q}_m, \qquad \text{(7a)}$$

$$p^0 = \psi \left(\psi^{-1}(P^0) + y_3 + \bar{\zeta}^0 \right), \quad y \in \partial \mathcal{Q}_m, \qquad \text{(7b)}$$

where ψ is defined by (2) and $\bar{\zeta}^0$ is defined by

$$\frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} \phi \rho \left(\psi \left(\psi^{-1}(P^0) + y_3 + \bar{\zeta}^0 \right) \right) dy = \phi \rho(P^0). \qquad \text{(8)}$$

Note that no auxiliary steady-state problem need be solved.

The standard macroscopic model replaces (7b) by $p^0 = P^0$, omits (8), and should have $g = 0$ in (7a) to be consistent.

3. Formal Homogenization

We follow the homogenization of the standard model given in [2] and [6]. As usual, for some functions P^ℓ and p^ℓ, $\ell = 0, 1, 2, \ldots$, we assume the formal asymptotic expansions

$$x - \xi^\epsilon(x) \sim \epsilon y \quad \text{and} \quad \nabla \sim \epsilon^{-1} \nabla_y + \nabla_x,$$

$$P^\epsilon(x, t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell P^\ell(x, y, t), \quad x \in \Omega, \ y \in \mathcal{Q}_f,$$

$$p^\epsilon(x, t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell p^\ell(x, y, t), \quad x \in \Omega, \ y \in \mathcal{Q}_m,$$

where the P^ℓ are periodic in y with period $\mathcal{Q}_f$, reflecting the periodicity of the medium. We note that if some function F depends on $\pi^\epsilon \sim \sum_{\ell=0}^{\infty} \epsilon^\ell \pi^\ell$, then Taylor's Theorem shows that

$$F(\pi^\epsilon) \sim F \left(\sum_{\ell=0}^{\infty} \epsilon^\ell \pi^\ell \right) = F(\pi^0) + \sum_{\ell=1}^{\infty} \epsilon^\ell F^\ell,$$

for some F^ℓ that depend on the π^ℓ's.

Substituting the formal expansions into (3)–(5) and isolating the coefficients of powers of ϵ yield relations for the P^ℓ and p^ℓ.

We begin with two standard results which can be easily derived and appear in [2] and [6]. First, the ϵ^{-2} terms of (3a) and the ϵ^{-1} terms of (3b) imply that $P^0 = P^0(x, t)$ only. Second, the ϵ^{-1} terms of (3a) and the ϵ^0 terms of (3b) allow us to write

$$P^1 = \sum_{j=1}^{3} \frac{\partial P^0}{\partial x_j} \omega_j - \rho(P^0) g \omega_3 + \pi,$$

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for some $\pi(x,t)$, where the $\omega_j(y)$, $j = 1, 2, 3$, are periodic across $\partial \mathcal{Q}$ and satisfy

$$-\nabla_y \cdot (\nabla_y \omega_j) = 0, \quad y \in \mathcal{Q}_f, \tag{9a}$$

$$\nabla_y \omega_j \cdot \nu = -\mathbf{e}_j \cdot \nu, \quad y \in \partial \mathcal{Q}_m. \tag{9b}$$

Recognizing that $(\epsilon^{-1} - 1)(x_3 - \xi_3^\epsilon(x)) \sim (1 - \epsilon) y_3$, we have (7) from the ϵ^0 terms of (4).

We now consider (5). First, (4) or (7), without the time derivative term, implies $\tilde{p}^0 = \psi(\psi^{-1}(P^0) + y_3 + \bar{\zeta}^0)$. For $\bar{p}^\epsilon$, the ϵ^{-2} terms of its defining equation and the ϵ^0 terms of its boundary condition imply $\bar{p}^0 = P^0$. Now a rescaling shows that

$$\int_{\mathcal{Q}_m^\epsilon(x)} \phi \rho(\tilde{p}^\epsilon) \, dx \sim \int_{\mathcal{Q}_m} \phi \rho \left(\sum_{\ell=0}^\infty \epsilon^\ell \tilde{p}^\ell(x, y, t) \right) dy,$$

for some $\tilde{p}^\ell$ depending on the P^ℓ's and on $\bar{\zeta}^\epsilon$. A similar expression holds for the right side of (5), and so the ϵ^0 terms of (5) give the definition of $\bar{\zeta}^0$ as (8).

Finally, the ϵ^0 and ϵ^1 terms of (3a) and (3b) can be analyzed exactly as in the standard model [2], [6] to give (6), and the tensor K is seen to be given by

$$K_{ij} = \frac{K^*}{|\mathcal{Q}|} \left(\int_{\mathcal{Q}_f} \frac{\partial \omega_j}{\partial y_i} \, dy + |Q_f| \delta_{ij} \right); \tag{10}$$

K is symmetric and positive definite (see, e.g., [3]).

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1. INTRODUCTION

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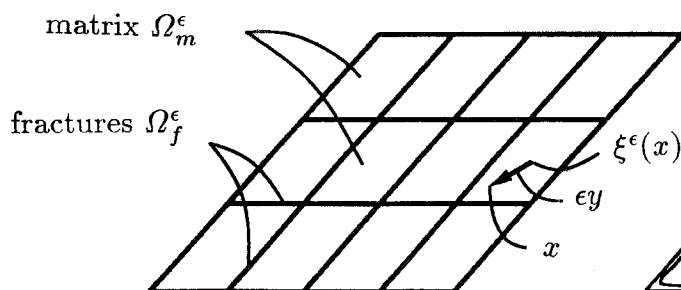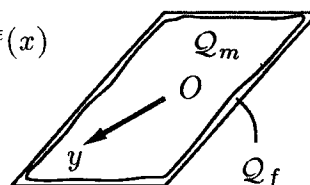

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