

# LARGE CARDINAL AXIOMS FROM TAMENESS IN AECS

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**ABSTRACT.** We show that various tameness assertions about abstract elementary classes imply the existence of large cardinals under mild cardinal arithmetic assumptions. For instance, we show:

**Theorem.** *Let  $\kappa$  be uncountable such that  $\mu^\omega < \kappa$  for every  $\mu < \kappa$ . If every AEC with Löwenheim-Skolem number less than  $\kappa$  is  $< \kappa$ -tame, then  $\kappa$  is almost strongly compact.*

This is done by isolating a class of AECs that exhibits tameness, etc. exactly when sufficiently complete ultrafilters exist.

## 1. INTRODUCTION

The birth of modern model theory is often said to be Morley’s proof of what was then called the Łoś Conjecture in [12]; of course, this is now called Morley’s Categoricity Theorem. Thus, it is only natural that this same question be an important test question when studying nonelementary model theory. In one of the most popular contexts for this study, Abstract Elementary Classes, this question is known as Shelah’s Categoricity Conjecture.

While still open, there are many partial results towards this conjecture that add various model-theoretic and set-theoretic assumptions. The most relevant for this discussion is the first author’s [5, Theorem 7.5], which shows that, if there are class many strongly compact cardinals, then any Abstract Elementary Class (AEC) that is categorical in *some* high enough successor cardinal is categorical in every high enough cardinal; here “high enough” means “above the second Hanf number of the first strongly compact cardinal above the Löwenheim-Skolem number.” The key advance in the first author’s work is the following<sup>1</sup>, which with a little more work allows the application of previous results of Shelah [14] and Grossberg and VanDieren [7] to obtain a version of Shelah’s categoricity conjecture.

**Fact 1.1** ([5].4.5). *If  $\mathbb{K}$  is an AEC with  $LS(\mathbb{K}) < \kappa$  and  $\kappa$  is strongly compact, then  $\mathbb{K}$  is  $< \kappa$ -tame.*

Sections 5 and 6 of [5] go on to give similar theorems for measurable and weakly compact cardinals. The main theorems of this paper give converses to these results under mild cardinal arithmetic assumptions; see Corollary 4.9 for proofs. We state the following theorem as a sample application of our methods. We note that by strengthening the tameness hypothesis we can drop the “almost” from the conclusion.

**Theorem.** *Let  $\kappa$  be uncountable such that  $\mu^\omega < \kappa$  for every  $\mu < \kappa$ .*

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<sup>1</sup>[5, Theorem 4.5] actually shows more than this, but this statement suffices for the present discussion.

- (1) If  $\kappa^{<\kappa} = \kappa$  and every AEC with Löwenheim-Skolem number less than  $\kappa$  is  $(<\kappa, \kappa)$ -tame, then  $\kappa$  is almost weakly compact.
- (2) If every AEC with Löwenheim-Skolem number less than  $\kappa$  is  $\kappa$ -local, then  $\kappa$  is almost measurable.
- (3) If every AEC with Löwenheim-Skolem number less than  $\kappa$  is  $<\kappa$ -tame, then  $\kappa$  is almost strongly compact.

Actually, the first step in this direction is Shelah [13], where the measurable version appears as Theorem 1.3. Indeed, the example constructed in this paper is a generalization of Shelah's to apply to more contexts. Note that Shelah's proof is essentially correct, but requires minor correction (see Remark 4.6 for a discussion).

The proof of the main theorems all follow the same plan, which we outline here. First, Section 2 codes large cardinals into a combinatorial statement  $\#(\mathcal{D}, \mathcal{F})$  (see Definition 2.4). Then Section 3 defines two structures  $H_1$  and  $H_2$  such that all small substructures of them are isomorphic, but  $H_1$  and  $H_2$  are only isomorphic if the relevant  $\#(\mathcal{D}, \mathcal{F})$  holds. Finally, Section 4 defines an AEC  $\mathbb{K}_\sigma$  that contains  $H_1$  and  $H_2$  and codes their isomorphism (and the isomorphism of their substructures) into equality of Galois types.

This gives rise, in  $\mathbb{K}_\sigma$ , to two Galois types  $p$  and  $q$  such that every small restriction of them are equal. Global tameness implies that  $p$  and  $q$  must be equal and this begins the chain of implications described in the last paragraph:  $p = q$  implies that  $H_1 \cong H_2$  implies  $\#(\mathcal{D}, \mathcal{F})$  implies  $\kappa$  has the appropriate large cardinal property. The full details appear in the proof of Theorem 4.8.

The main theorem has an immediate application to category theory. Makkai and Paré proved a theorem [11] about the accessibility of powerful images from the assumption of class many strongly compact cardinals and Lieberman and Rosicky [9] later applied this to AECs to give an alternate proof of Fact 1.1 above. Brooke-Taylor and Rosicky [6] have recently weakened the hypotheses of Makkai and Paré's result to almost strongly compact and our result completes the circle and shows that the conclusion of Makkai and Paré's result is actually a large cardinal statement in disguise. See Corollary 4.13 and the surrounding discussion.

Turning back to model theory, this shows that any attempt to prove that all AECs (even with the extra assumption of amalgamation) are eventually tame, and thus prove Shelah's Categoricity Conjecture, will fail. However, the AECs constructed are unstable and don't really fit into the picture of classification theory so far or the categoricity conjecture. This leaves open the possibility that eventual tameness can be proven in ZFC from *model-theoretic* assumptions, such as stability or categoricity. Partial work towards this goal has already been done by Shelah [14], which derives some tameness<sup>2</sup> from categoricity; see [2, Theorem 11.15] for an exposition. A related question of Grossberg asks if amalgamation can be derived from categoricity; even this conjecture is wide open.

In Section 5, we prove that even without our cardinal arithmetic assumption we can derive large cardinal strength from tameness assertions. Roughly speaking we show that if  $\kappa$  carries the tameness property corresponding to weak compactness, then  $\kappa$  is weakly compact in  $L$ .

The reader is advised to have some background in both set theory and model theory. The set-theoretic background is in large cardinals for which we recommend

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<sup>2</sup>Given amalgamation and categoricity in  $\lambda \geq H_1 := \beth_{(2^{LS(\mathbb{K})})^+}$  and  $\mu < \text{cf } \lambda$ , we can conclude  $(< H_1, \mu)$ -tameness for types whose domain is a saturated model

Kanamori's book [8]. For the model-theoretic background, see a standard reference on AECs such as Baldwin's book [2].

## 2. LARGE CARDINALS

We begin by recalling some relevant large cardinal definitions. These are slight tweaks on standard definitions in the spirit of  $\aleph_1$ -strongly compact cardinals (see for example [1]). The basic framework is to take a large cardinal property that has  $\kappa$  being large if there is a  $\kappa$ -complete object of some type and parameterizing the completeness by some  $\delta$ . Then  $\kappa$  is “almost large” if the  $\delta$ -complete object exists for all  $\delta < \kappa$  rather than at  $\kappa$ .

**Definition 2.1.** *Let  $\kappa$  be an uncountable cardinal.*

- (1)  $\kappa$  is  $\delta$ -weakly compact if for every field  $\mathcal{A}$  of subsets of  $\kappa$  with  $|\mathcal{A}| = \kappa$  there is a nonprincipal  $\delta$ -complete uniform filter measuring each set in  $\mathcal{A}$ .
- (2)  $\kappa$  is almost weakly compact if it is  $\delta$ -weakly compact for all  $\delta < \kappa$ .
- (3)  $\kappa$  is weakly compact if it is  $\kappa$ -weakly compact.
- (4)  $\kappa$  is  $\delta$ -measurable if there is a uniform,  $\delta$ -complete ultrafilter on  $\kappa$ .
- (5)  $\kappa$  is almost measurable if it is  $\delta$ -measurable for all  $\delta < \kappa$ .
- (6)  $\kappa$  is measurable if it is  $\kappa$ -measurable.
- (7)  $\kappa$  is  $(\delta, \lambda)$ -strongly compact for  $\delta \leq \kappa \leq \lambda$  if there is a  $\delta$ -complete, fine ultrafilter on  $\mathcal{P}_\kappa \lambda$ .
- (8)  $\kappa$  is  $(\delta, \infty)$ -strongly compact if it is  $(\delta, \lambda)$ -strongly compact for all  $\lambda \geq \kappa$ .
- (9)  $\kappa$  is  $\lambda$ -strongly compact if it is  $(\kappa, \lambda)$ -strongly compact.
- (10)  $\kappa$  is almost strongly compact if it is  $(\delta, \infty)$ -strongly compact for all  $\delta < \kappa$ .
- (11)  $\kappa$  is strongly compact if it is  $(\kappa, \infty)$ -strongly compact.

We note that the notions of  $\delta$ -weakly compact and  $\delta$ -measurable are not standard. From the definitions, it can be seen that being almost measurable implies being a limit of measurables. For almost weak and almost strong compactness, the relation is not so clear. For instance, the following seems open.

**Question 2.2.** *Is “there exists a proper class of almost strong compacts” equiconsistent with “there exists a proper class of strong compacts?”*

The motivation for this question are results on the consistency of Shelah's Categoricity Conjecture and Corollary 4.13.

For the section we fix a directed ordering  $(\mathcal{D}, \triangleleft)$  with  $\triangleleft$  strict. The intended applications are  $(\kappa, \in)$  and  $(\mathcal{P}_\kappa \lambda, \subset)$ .

**Definition 2.3.** *For  $d \in \mathcal{D}$  we define  $\lceil d \rceil = \{d' \in \mathcal{D} \mid d' \triangleleft d\}$  and  $\lfloor d \rfloor = \{d' \in \mathcal{D} \mid d \triangleleft d'\}$ .*

We also fix a collection  $\mathcal{F}$  of functions each of which has domain  $\mathcal{D}$ . For  $f_1, f_2 \in \mathcal{F}$  we set  $f_1 \leq f_2$  if and only if there is an  $e : \text{ran}(f_2) \rightarrow \text{ran}(f_1)$  such that  $f_1 = e \circ f_2$ . Obviously for each  $f \in \mathcal{F}$ , the set  $\{f^{-1}\{i\} \mid i \in \text{ran}(f)\}$  partitions  $\mathcal{D}$ . So  $f_1 \leq f_2$  is equivalent to saying that the partition from  $f_2$  refines the partition from  $f_1$ . We require that  $\mathcal{F}$  is directed under  $\leq$ .

We are interested in elements that appear cofinally often as values of  $f$  so we define

$$\text{ran}^*(f) = \bigcap_{d \in \mathcal{D}} \text{ran}(f \upharpoonright \lfloor d \rfloor).$$

With this notation in mind we formulate the following principle, which is implicit in [13].

**Definition 2.4.** Let  $\#(\mathcal{D}, \mathcal{F})$  be the assertion that there are  $f^* \in \mathcal{F}$  and a collection  $\{u_f \subseteq \text{ran}^*(f) \mid f \in \mathcal{F} \wedge f \geq f^*\}$  of nonempty finite sets such that whenever  $e$  witnesses that  $f \geq f^*$ ,  $e \restriction u_f : u_f \rightarrow u_{f^*}$  is a bijection.

This principle allows us to define<sup>3</sup> a filter on  $\mathcal{D}$ . It follows from the definition that  $e \restriction u_f$  is unique. So under  $\#(\mathcal{D}, \mathcal{F})$  we can choose  $i_f \in u_f$  such that  $e(i_f) = \min u_{f^*}$  where  $e$  witnesses  $f^* \leq f$ . Then we define  $U \subseteq P(\mathcal{D})$  by  $A \in U$  if and only if there are  $d \in \mathcal{D}$  and  $f \in \mathcal{F}$  with  $f \geq f^*$  such that  $f^{-1}\{i_f\} \cap [d] \subseteq A$ . Note that  $U$  depends on the many parameters we have defined so far:  $\mathcal{D}, \mathcal{F}, \{u_f\}$ , and  $i_f$ . Also, the choice of  $i_{f^*}$  as the minimum of  $u_{f^*}$  was arbitrary, any element would have done. Indeed, different elements generate different ultrafilters (Remark 2.9 hints at the reason), so  $\#(\mathcal{D}, \mathcal{F})$  is equivalent to the existence of  $|u_{f^*}|$ -many filters with the desired properties.

**Claim 2.5.**  $U$  is a proper filter and for all  $d \in \mathcal{D}$ ,  $[d] \in U$ .

*Proof.* It is not hard to see that  $f^{-1}\{i_f\} \cap [d]$  is nonempty for all  $d$  and  $f$ , so  $\emptyset \notin U$  provided that it forms a filter. The fact that  $[d] \in U$  for all  $d$  is immediate from the definition.

To see that  $U$  is a filter, let  $A, B \in U$  witnessed by  $f_1, d_1$  and  $f_2, d_2$  respectively, and let  $d_3 \in \mathcal{D}$  be above  $d_1$  and  $d_2$ . It is not hard to see that  $f^{-1}\{i_f\} \cap [d_3] \subseteq A \cap B$  for some  $f \geq f_1, f_2$ . Such an  $f$  exists, since  $\mathcal{F}$  is directed under  $\leq$ .  $\square$

We would like to generate highly complete filters. To do so we use the following ad-hoc definition.

**Definition 2.6.** We say that  $\mathcal{F}$  is  $\tau$ -replete if for every  $\mu < \tau$  and sequence  $\langle B_\epsilon \mid \epsilon < \mu \rangle$  of sets such that for each  $\epsilon$  there is a function  $f_\epsilon$  such that  $B_\epsilon = f_\epsilon^{-1}\{i\}$  for some  $i$ , there is a function  $f \in \mathcal{F}$  and  $\{i_\alpha : \alpha < \mu\} \subset \text{ran} f$  such that  $f^{-1}\{i_0\} = \bigcap_{\epsilon < \mu} B_\epsilon$  and for all  $\epsilon < \mu$  and  $d \in \mathcal{D}$ ,  $f(d) = i_{\epsilon+1}$  if and only if  $d \notin \bigcap_{\epsilon < \mu} B_\epsilon$  and  $\epsilon$  is least such that  $d \notin B_\epsilon$ .

**Claim 2.7.** If  $(\mathcal{D}, \triangleleft)$  is  $\tau$ -directed and  $(\mathcal{F}, \leq)$  is  $\tau$ -replete, then  $U$  is  $\tau$ -complete.

By  $\tau$ -directed we mean that sets of size less than  $\tau$  have an upperbound.

*Proof.* Let  $A_\epsilon$  for  $\epsilon < \mu$  be elements of  $U$  where  $\mu < \tau$ . By the definition of  $U$ , for each  $\epsilon < \mu$  we have  $f_\epsilon$  and  $d_\epsilon$  so that  $f_\epsilon^{-1}\{i_{f_\epsilon}\} \cap [d_\epsilon] \subseteq A_\epsilon$ . Let  $B_\epsilon = f_\epsilon^{-1}\{i_{f_\epsilon}\}$  for  $\epsilon < \mu$  and use the  $\tau$ -repleteness of  $\mathcal{F}$  to find  $f$ . Using the directedness of  $\mathcal{F}$  we can find  $\hat{f} \geq f, f^*$  (recall  $f^*$  is given by  $\#(\mathcal{D}, \mathcal{F})$ ). Using the  $\tau$ -directedness of  $\mathcal{D}$ , let  $d$  be above each  $d_\epsilon$  for  $\epsilon < \mu$ .

Let  $e$  witness that  $f \leq \hat{f}$ , ie  $f = e \circ \hat{f}$ . We want to show that  $e(i_{\hat{f}}) = 0$ , since then  $\hat{f}^{-1}\{i_{\hat{f}}\} \cap [d] \subseteq \bigcap_{\epsilon < \mu} (B_\epsilon \cap [d_\epsilon]) \subseteq \bigcap_{\epsilon < \mu} A_\epsilon$ .

Suppose that  $e(i_{\hat{f}}) = \epsilon$  is not zero. Then  $\hat{f}^{-1}\{i_{\hat{f}}\} \subseteq \mathcal{D} - B_\epsilon$  by the definition of  $f$ . This contradicts that  $U$  is filter containing all the sets  $[d]$  for  $d \in \mathcal{D}$ .  $\square$

**Claim 2.8.** If  $A \subseteq \mathcal{D}$  and there is an  $f$  in  $\mathcal{F}$  such that  $A = f^{-1}X$  for some  $X \subseteq \text{ran}(f)$ , then  $U$  measures  $A$ .

<sup>3</sup>We give the partition formulation of  $\#(\mathcal{D}, \mathcal{F})$  and the definition of this filter in Remark 2.9, which is more intuitive but harder to work with in later sections.

*Proof.* Let  $f$  and  $X$  witness the hypotheses of the claim. Since  $\mathcal{F}$  is directed, we can find  $\hat{f} \in \mathcal{F}$  such that  $f, f^* \leq \hat{f}$ . Let  $e$  be such that  $f = e \circ \hat{f}$ . Now it is not hard to see that if  $e(i_{\hat{f}}) \in X$ , then  $\hat{f}^{-1}\{i_{\hat{f}}\} \subseteq A$  and if  $e(i_{\hat{f}}) \notin X$ , then  $\hat{f}^{-1}\{i_{\hat{f}}\} \subseteq \mathcal{D} - A$ . In the first instance we have  $A \in U$  and in the second we have  $\mathcal{D} - A \in U$ .  $\square$

If  $\mathcal{F}$  satisfies the hypothesis of the previous claim for  $A$ , then we say that  $\mathcal{F}$  has a characteristic function for  $A$ .

**Remark 2.9.** The principle  $\#(\mathcal{D}, \mathcal{F})$  has a straightforward description in terms of the partitions of  $\mathcal{D}$  that the functions of  $\mathcal{F}$  generate. In that language,  $\#(\mathcal{D}, \mathcal{F})$  holds if and only if there is a special partition  $\mathcal{P}^*$  such that any finer partition  $\mathcal{P}$  has a distinguished piece  $X_{\mathcal{P}}$  that is chosen in a coherent way: if  $\mathcal{Q}$  is finer than  $\mathcal{P}$ , then  $X_{\mathcal{Q}} \subseteq X_{\mathcal{P}}$ . We also require that, for  $d \in \mathcal{D}$  and  $\mathcal{P}$  finer than  $\mathcal{P}^*$ , we have  $X_{\mathcal{P}} \cap [d] \neq \emptyset$ .

Then we can define the filter as follows, given  $A \subset \mathcal{D}$ , we form a partition  $\mathcal{P}_A$  that is finer than  $\mathcal{P}^*$  and  $\{A, \mathcal{D} - A\}$ . Then we set  $A \in U$  if and only if the distinguished piece  $X_{\mathcal{P}_A}$  is a subset of  $A$  rather than  $\mathcal{D} - A$ .

We can now reformulate many large cardinal notions that are witnessed by the existence of measures. Our first corollary is an equivalent formulation of weak compactness.

**Corollary 2.10.** Let  $\kappa$  be a regular cardinal.  $\kappa$  is weakly compact if and only if for all fields  $\mathcal{A}$  of subsets of  $\kappa$  with  $|\mathcal{A}| = \kappa$ ,  $\#(\kappa, \mathcal{F})$  holds for some set of functions  $\mathcal{F}$  on  $\kappa$  which is directed,  $\kappa$ -replete and contains characteristic functions for all elements of  $\mathcal{A}$ .

*Proof.* Assume that  $\kappa$  is weakly compact. Let  $\mathcal{A}$  be a field of subsets of  $\kappa$  with  $|\mathcal{A}| = \kappa$ . We can assume that  $\mathcal{A}$  is closed under intersections of size less than  $\kappa$ . Using the weak compactness of  $\kappa$ , we fix a  $\kappa$ -complete  $\mathcal{A}$ -ultrafilter  $U$ .

Let  $\mathcal{F}_{\mathcal{A}}$  be the collection of functions  $f : \kappa \rightarrow \kappa$  such that  $\text{ran}(f) \subseteq \alpha < \kappa$  for some  $\alpha$  and for all  $\beta \in \text{ran}(f)$ ,  $f^{-1}\{\beta\} \in \mathcal{A}$ . It is not difficult to show that  $\mathcal{F}_{\mathcal{A}}$  is  $\leq$ -directed and  $\kappa$ -replete.

For each  $f \in \mathcal{F}_{\mathcal{A}}$  let  $u_f = \{i_f\}$  where  $i_f$  is the unique element of  $\text{ran} f$  such that  $f^{-1}\{i_f\} \in U$ . If we take  $f^*$  to be the constantly zero function, then it is straightforward to see that  $f^*$  and  $\{u_f \mid f \in \mathcal{F}_{\mathcal{A}}\}$  satisfy  $\#(\mathcal{D}, \mathcal{F}_{\mathcal{A}})$ .

For the reverse direction, for each field  $\mathcal{A}$  we apply Claims 2.5, 2.7 and 2.8 to see that the filter  $U$  generated by  $\#(\mathcal{D}, \mathcal{F})$  is a nonprincipal  $\kappa$ -complete  $\mathcal{A}$ -ultrafilter.  $\square$

**Remark 2.11.** A similar proof gives a characterization of  $\sigma^+$ -weak compactness where we just replace  $\kappa$ -replete with  $\sigma^+$ -replete and consider functions with codomain  $\sigma$ .

We also have characterizations of  $\sigma^+$ -measurable,  $(\delta, \lambda)$ -strongly compact and  $\lambda$ -strongly compact.

**Corollary 2.12.**  $\kappa$  is  $\sigma^+$ -measurable if and only if  $\#(\kappa, \mathcal{F})$  holds for some set  $\mathcal{F}$  of functions from  $\kappa$  to  $\sigma$  such that  $\mathcal{F}$  is directed,  $\sigma^+$ -replete and has characteristic functions for all subsets of  $\kappa$ .

**Corollary 2.13.** Let  $\kappa \leq \lambda$  be cardinals.  $\kappa$  is  $\lambda$ -strongly compact if and only if  $\#(\mathcal{P}_{\kappa}(\lambda), \mathcal{F})$  holds for some set  $\mathcal{F}$  of functions with domain  $\mathcal{P}_{\kappa}(\lambda)$  and range

bounded in  $\kappa$ , such that  $\mathcal{F}$  is directed,  $\kappa$ -replete and has characteristic functions for all subsets of  $\mathcal{P}_\kappa(\lambda)$ .

**Remark 2.14.** *The previous corollary can be modified to give a natural characterization of “ $\kappa$  is  $(\delta, \lambda)$ -strongly compact”.*

The proofs of these corollaries are all similar to the proof of Corollary 2.10 and will be omitted.

### 3. MODEL CONSTRUCTIONS

In this section we describe a family of constructions of models which take  $\mathcal{D}$  and  $\mathcal{F}$  from the previous section as parameters. For this section we assume  $\mathcal{F}$  is countably closed in the sense that for all  $\leq$ -increasing sequences  $\langle f_n \mid n < \omega \rangle$  there is an  $f \in \mathcal{F}$  such that  $f \geq f_n$  for all  $n < \omega$ .

We begin by making a host of definitions. For ease of notation, we suppress the  $\mathcal{D}$  and  $\mathcal{F}$  parameters (writing for instance  $A_d$  instead of  $A_d^{(\mathcal{D}, \mathcal{F})}$ ).

- $X = \cup \{ \text{ran } f : f \in \mathcal{F} \}$ .
- $G = ([X]^{<\omega}, \Delta)$ ; an alternate characterization of this is as the direct sum of  $|X|$  many copies of  $\mathbb{Z}_2$ .
- $A = \mathcal{F} \times \mathcal{D} \times G$  and, for  $d \in \mathcal{D}$ ,  $A_d = \mathcal{F} \times [d] \times G$ , so that  $A$  is the direct (co)limit of the directed system  $\langle A_d \mid d \in \mathcal{D} \rangle$ .
- $I = \mathcal{F} \times \mathcal{D}$  and  $I_d = \mathcal{F} \times [d]$ .
- For  $d \in \mathcal{D}$ , set  $\pi_d : A_d \rightarrow I_d$  is given by  $\pi_d(f, d, u) = (f, d)$ .
- $E'$  is a binary relation on  $A$  given by  $(f, d, u)E'(f', d', u')$  if and only if  $(f, d) = (f', d')$ . Note that this is definable in terms of the  $\pi$ 's above.
- $E$  is a binary relation on  $A$  such that  $(f, d, u)E(f', d', u')$  if and only if  $(f, d, u)E'(f', d', u')$  and there are elements  $d_0, \dots, d_{2n-1} \in [d]$  such that  $u \Delta \{f(d_0)\} \Delta \dots \Delta \{f(d_{2n-1})\} = u'$ .
- $P(f, d, u)$  holds if and only if  $|u \cap \text{ran}(f \upharpoonright [d])|$  is odd. We call  $P$  the *parity predicate*.
- For each  $v \in G$  and  $(f, d, u) \in A$ , we say that  $D_v(f, d, u)$  holds if and only if  $u - \text{ran } f \upharpoonright [d] \subset v$  and  $v \cap \text{ran } f \upharpoonright [d] = \emptyset$ . We call  $D$  the *difference predicate*.
- For  $c \in G$ , we define  $F_c$  on  $A$  by  $F_c(f, d, u) = (f, d, u + c)$ . Notice that the  $F_c$  are the obvious transitive action of  $G$  on each  $E'$  class.
- We define binary  $R$  as follows  $(f, d, u)R(f', d', u')$  if and only if  $f \leq f'$  and, for every  $e$  that witnesses,  $u = \{i \in X \mid \exists^{\text{odd}} j \in u'. e(j) = i\}$ .
- Define a class of structures in a suitable language  $L_0$  by  $H_{1,d}$  is the structure with universe  $A_d$  and the functions and relations defined above.

First note that  $E$  is quantifier-free definable in terms of  $E'$ ,  $P$ , and the  $D_v(\cdot)$ .

**Claim 3.1.** *For all  $d^* \in \mathcal{D}$  and  $(f, d, u), (f', d', u') \in H_{1,d^*}$ , we have  $(f, d, u)E(f', d', u')$  if and only if the following hold:*

- $(f, d, u)E'(f', d', u')$ ;
- $P(f, d, u)$  if and only if  $P(f', d', u')$ ; and
- for all  $v \in G$ ,  $D_v(f, d, u)$  if and only if  $D_v(f', d', u')$ .

*In particular, within a particular  $E'$ -class,  $E$ -equivalence is determined by the quantifier-free type of the singletons.*

*Proof.* Clearly  $(f, d, u)E(f', d', u')$  if and only if

- $(f, d, u)E'(f', d', u')$  (hence  $f = f'$  and  $d = d'$ ),
- $u + v = u'$  for some  $v$  with an even number of elements from  $\text{ran}(f)$  and
- $u - \text{ran}(f \upharpoonright [d]) = u' - \text{ran}(f' \upharpoonright [d'])$

It is not hard to see that this is equivalent to the list from the claim.  $\square$

**Definition 3.2.** For  $d \in \mathcal{D}$ , set  $g_d$  to be a permutation of  $A_d$  of order two given by  $g_d(f, d', u) = (f, d', u + \{f(d)\})$ .

It will be important that  $g_d(f, d', u)$  is defined if and only if  $d' \triangleleft d$ .

**Remark 3.3.** For each  $d \in \mathcal{D}$ ,  $g_d$  preserves each difference predicate and exactly flips the parity predicate.

**Claim 3.4.** Given  $d_1, d_2 \in [d]$ ,  $g_{d_1} \circ g_{d_2}$  is an automorphism of  $H_{1,d}$ .

*Proof.* Most of this is clear from the definition of  $H_{1,d}$ . We show that  $g_{d_1}$  already preserves  $R$  on any pair for which the function is defined. Suppose  $(f, d, u)R(f', d', u')$  and let  $f = e \circ f'$ . Since  $u = \{i \in X \mid \exists^{\text{odd}} j \in u'.e(j) = i\}$ , we have that

$$u \Delta \{f(d_1)\} = u \Delta \{e \circ f'(d_1)\} = \{i \in X \mid \exists^{\text{odd}} j \in u' \Delta \{f'(d_1)\}.f'(j) = i\}$$

Since this holds for all  $e$ ,  $(f, \alpha, u \Delta \{f(d_1)\})R(f', \alpha', u' \Delta \{f'(d_1)\})$ .

We need the composition to prove that  $E$  and  $P$  are preserved. Recall that  $g_{d_1}$  and  $g_{d_2}$  both reverse the parity predicate, so their composition leaves it unchanged. To see that  $E$  is preserved note that the same elements of  $\mathcal{D}$  witnessing  $E$  relatedness in  $H_{1,d}$  will witness  $E$  relatedness of the  $g_{d_1} \circ g_{d_2}$ -images.  $\square$

**Claim 3.5.** If  $\langle (f, d, u_1), (f, d, u_2) \rangle$  have the same quantifier-free type in  $H_{1,d^*}$  as  $\langle (f, d, v_1), (f, d, v_2) \rangle$ , then  $u_1 - u_2 = v_1 - v_2$ .

*Proof.* Note that  $F_{u_1 - u_2}(x) = y$  is in the quantifier free type of the first and  $F_c(g, d, u) = (g, d, v)$  if and only if  $c = u - v$ .  $\square$

We know that  $g_d$  is a bijection from  $A_d$  to itself and, by Claim 3.4, if  $d \triangleleft d' \in \mathcal{D}$ , then  $H_{2,d} \subseteq H_{2,d'}$ . This justifies the following definition.

**Definition 3.6.** For  $d \in \mathcal{D}$ , set  $H_{2,d}$  to be the  $L_0$ -structure with universe  $A_d$  so that  $g_d : H_{1,d} \cong H_{2,d}$ .

For  $\ell = 1, 2$ , set  $H_{\ell, \mathcal{D}}$  be the direct union of the sequence  $\langle H_{\ell, d} \mid d \in \mathcal{D} \rangle$ .

Tameness type assumptions about our class of models will give an isomorphism from  $H_{1, \mathcal{D}}$  to  $H_{2, \mathcal{D}}$  respecting  $\pi$ . From this assumption we will prove  $\#(\mathcal{D}, \mathcal{F})$ . We pause briefly to point out that the relation  $R$  is the only real barrier to the existence of an isomorphism, in part because  $R$  is the only thing relating elements  $(f, d, u)$  and  $(f', d', u')$  for which  $f \neq f'$ .

**Claim 3.7.** Let  $L'_0$  be  $L_0$  without the relation  $R$ . Then there is  $h : H_{1, \mathcal{D}} \upharpoonright L'_0 \cong H_{2, \mathcal{D}} \upharpoonright L'_0$  such that  $\pi_d(a) = \pi_d(h(a))$  for all  $a \in H_{1, \mathcal{D}}$  and  $d \in \mathcal{D}$ .

*Proof.* Set  $h(f, d, u) := g_d(f, d, u) = (f, d, u \Delta \{f(d)\})$ . It is not difficult to see that this works.  $\square$

Note that the isomorphism above fails to respect  $R$  because different  $d$  give different  $f(d)$ . This could be remedied by picking a “generic” or “average” value to add, and we could use an ultrafilter to find such a value. The argument below shows that this is the only way to construct this isomorphism.

Once we move to the setting of abstract elementary classes, the assumption that our class is tame will be witnessed by an isomorphism. The heart of the derivation of large cardinals from tameness is the following lemma.

**Lemma 3.8.** *If there is an isomorphism  $h$  from  $H_{1,\mathcal{D}}$  to  $H_{2,\mathcal{D}}$  such that for all  $a \in H_{1,\mathcal{D}}$  and  $d \in \mathcal{D}$ ,  $\pi_d(a) = \pi_d(h(a))$ , then  $\#(\mathcal{D}, \mathcal{F})$  holds.*

For the remainder of the section we assume that there is an  $h$  as in the lemma and derive  $\#$ .

**Claim 3.9.** *The following are true of  $h$ :*

- (1) *If we let  $u_{f,d}$  be the unique element of  $G$  such that  $h(f, d, \emptyset) = (f, d, u_{f,d})$ , then  $u_{f,d}$  doesn't depend on  $d$ . Hence we denote the common value by  $u_f$ .*
- (2) *For all  $d \in \mathcal{D}$ ,  $f \in \mathcal{F}$  and  $u \in G$ ,  $h(f, d, u) = (f, d, u \Delta u_f)$ .*
- (3) *If  $f \leq f'$ , then  $|u_f| \leq |u_{f'}|$ .*
- (4) *For all  $f \in \mathcal{F}$ ,  $u_f \neq \emptyset$ .*

*Proof.* For (1) clearly

$$H_{1,\mathcal{D}} \models (f, d, \emptyset)R(f, d', \emptyset) \rightarrow H_{2,\mathcal{D}} \models (f, d, u_{f,d})R(f, d', u_{f,d'}).$$

Recall from the proof of Claim 3.4 that the  $H_{2,d}$ 's are correct about  $R$ , so we use the fact that the identity function is an injective witness (to  $f \leq f'$ ) to get

$$u_{f,d} = \{i \in X \mid \exists^{odd} j \in u_{f,d'} \cdot j = i\} = u_{f,d'}$$

For (2) we apply Claim 3.5 to  $\langle (f, d, \emptyset), (f, d, u) \rangle$  and  $\langle (f, d, u_f), (f, d, v) \rangle$  where  $h(f, d, u) = (f, d, v)$ .

For (3), we let  $e$  be any function such that  $f = e \circ f'$ . Then  $(f, d, \emptyset)R(f', d', \emptyset)$  implies  $(f, d, u_f)R(f', d', u_{f'})$ . So  $u_f \subseteq e \restriction u_{f'}$  and  $|u_f| \leq |u_{f'}|$ .

For (4), recall that  $H_{2,d}$  interprets the parity predicate to mean “ $|u|$  is even” and that  $H_{2,\mathcal{D}} \models \neg P(f, d, u_f)$ , since  $H_{1,\mathcal{D}} \models \neg P(f, d, \emptyset)$ . Since  $|u_f|$  is odd, it can't be empty.  $\square$

We are ready to produce the  $f^*$  for  $\#$ . It is here that we use for the first (and only) time the countable closure of the space of functions  $\mathcal{F}$  under the order  $\leq$ .

**Claim 3.10.** *There is  $f^* \in \mathcal{F}$  such that  $|u_f| = |u_{f^*}|$  for all  $f \geq f^*$  from  $\mathcal{F}$ . Moreover, if  $e$  witnesses  $f^* \leq f$ , then  $e \restriction u_f$  is a bijection from  $u_f$  to  $u_{f^*}$ .*

*Proof.* The moreover part follows from the first part because every member of  $u_{f^*}$  is in the image of  $u_f$  under an  $e$  witnessing  $f^* \leq f$  by the proof of Claim 3.9 part (3).

Suppose there is no such  $f^*$ . Then there is a  $\leq$ -increasing sequence  $\langle f_n \in \mathcal{F} \mid n < \omega \rangle$  such that  $|u_{f_n}| < |u_{f_{n+1}}|$  for all  $n < \omega$ . By our assumption that  $\mathcal{F}$  is countably closed we can find  $f \geq f_n$  for all  $n < \omega$ , but then  $|u_{f^*}|$  is a natural number above infinitely many natural numbers, a contradiction.  $\square$

The proof will be complete if we can show that  $u_f \subseteq \text{ran}^* f$  for all  $f \in \mathcal{F}$  with  $f \geq f^*$ . Recall that  $\text{ran}^* f := \bigcap_{d \in \mathcal{D}} \text{ran}(f \restriction [d])$ . Note that if  $(\mathcal{D}, \triangleleft)$  is  $\tau$ -directed and  $|\text{ran} f| < \tau$ , then  $\text{ran}^* f \neq \emptyset$ .

**Claim 3.11.** *For all  $f \in \mathcal{F}$ ,  $u_f \subseteq \text{ran}^* f$ .*

*Proof.* For all  $d \in \mathcal{D}$ ,  $H_{2,\mathcal{D}} \models D_\emptyset(f, d, u_f)$ , since  $H_{1,\mathcal{D}} \models D_\emptyset(f, d, \emptyset)$ . Moreover  $M_{2,\mathcal{D}}$  is correct about the meaning of this predicate. So  $u_f \subseteq \text{ran}(f \restriction [d])$ .  $\square$



So we have derived  $\#(\mathcal{D}, \mathcal{F})$  as witnessed by  $f^*$  and the finite sets  $u_f$ . This finishes the construction of our sequence of models. We will need further work to show that these models can be thought of as elements of some AEC and that tameness assumptions about that AEC give the hypothesis of Lemma 3.8.

#### 4. ABSTRACT ELEMENTARY CLASSES

The goal of this section is twofold. First, we put the algebraic constructions of Section 3 into the context of AECs. Second, we put the necessary pieces together to conclude large cardinal principles from global tameness and locality axioms.

The AEC is designed to precisely take in the algebraic examples constructed above with a single twist. As in Baldwin and Shelah [3], there is an extra predicate (called  $Q$  here) that is an index for the set  $A_\beta$ . This allows us to turn the “incompactness” results about the existence of isomorphisms above into the desired nonlocality results for Galois types.

In defining the language, we have tried to use similar letters for the language from Section 3. We fix an infinite cardinal  $\sigma$  and set  $(G, +) = ([\sigma]^{<\omega}, \Delta)$ ; recall that this is isomorphic to the direct sum of  $\sigma$  copies of  $\mathbb{Z}_2$ . The language  $\mathcal{L}_\sigma$  consists of the following:

- (1) disjoint unary predicates  $H, J, I$
- (2) unary functions  $\pi : H \rightarrow I$ ;  $F_u : H \rightarrow H$  for  $u \in G$ ; and  $Q : H \rightarrow J$
- (3) binary relations  $R, E, E' \subset H^2$  and unary relations  $D_u, P \subset H$  for  $u \in G$

We define an AEC parameterized by  $\sigma$  with very minimal structure. In applications we require  $\sigma^\omega = \sigma$ . The strong substructure relation is as weak as possible, leaving open the question of whether restricting to the case of stronger strong substructure relations carries the same large cardinal implications.

**Definition 4.1.** *We define  $\mathbb{K} = \mathbb{K}_\sigma$  as follows:  $M \in \mathbb{K}$  if and only if  $M$  is an  $\mathcal{L}_\sigma$ -structure satisfying:*

- (1)  $F$  is an action of  $G$  on  $H$  which respects  $\pi$  in this sense that it restricts to an action of  $G$  on  $\pi^{-1}\{i\}$  for every  $i \in I$ .
- (2)  $E'$  and  $E$  are equivalence relations and  $aE'b$  if and only if  $\pi(a) = \pi(b)$ .
- (3) For all  $i \in I$  and  $j \in J$ , there is an  $h \in H$  such that  $\pi(h) = i$  and  $Q(h) = j$ .

We let  $\prec_{\mathbb{K}}$  be the  $\mathcal{L}_\sigma$ -substructure relation.

Note that this is an AEC with  $LS(\mathbb{K}) = \sigma$ . The main difference between this definition and Shelah [13,  $\boxtimes_2$  in Proof of Theorem 1.3] is that we have encoded the entire group  $G$  into the language rather than adding a separate sort for it<sup>4</sup> (see Remark 4.6).

Now we realize the models constructed in the last section as members of this AEC. Fix a  $\mathcal{D}$  and  $\mathcal{F}$  as in Section 2 and set  $\sigma = |X|$ . For  $d \in \mathcal{D}$  and  $\ell = 1, 2$ , we define  $M_{\ell,d} \in \mathbb{K}_\sigma$  to be the structure corresponding to  $H_{\ell,d}$ , with the following additions:

- $I = I_d$
- $J = \{i_\ell\}$  is an index set not corresponding to anything previously defined
- $\pi = \pi_d$
- $Q^{M_{\ell,d}}$  is a function on  $A_d$  with constant value  $i_\ell$

<sup>4</sup>We note that the structures we call  $H_{\ell,d}$  and  $M_{\ell,d}$  are called  $M_{\ell,\alpha}$  and  $M_{\ell,\alpha}^+$  respectively in [13]

Note that the models arising from different  $\mathcal{D}$ 's and  $\mathcal{F}$ 's will live in the same AEC as long as the size of  $X = \bigcup_{f \in \mathcal{F}} \text{ran } f$  is the same. We also define base model  $M_{0,d}$  as follows:

- $H^{M_{0,d}} = \emptyset$
- $I^{M_{0,d}} = I_d$
- $J^{M_{0,d}} = \emptyset$
- All other functions and relations are empty.

Note that  $M_{0,d} \subseteq M_{\ell,d}$  for  $\ell = 1, 2$ . Thus we can define  $p_d := \text{gtp}(i_1/M_{0,d}; M_{1,d})$  and  $q_d := \text{gtp}(i_2/M_{0,d}; M_{2,d})$ ;  $p_{\mathcal{D}}$  and  $q_{\mathcal{D}}$  are defined similarly. The connection between this AEC and the previous work is the following proposition.

**Claim 4.2.**

- (1) For all  $d \in \mathcal{D}$ ,  $p_d = q_d$ .
- (2)  $p_{\mathcal{D}} = q_{\mathcal{D}}$  if and only if there is an isomorphism as in Lemma 3.8.

We begin by showing (1) and the reverse direction of (2). The type equality comes from the fact that, if  $f$  is an isomorphism from  $H_{1,d}$  to  $H_{2,d}$  that respects  $\pi$ , then

$$f^* = f \cup \text{id}_{I_d} \cup \{(i_1, i_2)\}$$

is a  $\mathcal{L}_{\sigma}$ -isomorphism from  $M_{1,d}$  to  $M_{2,d}$  that fixes  $M_{0,d}$  and sends  $i_1$  to  $i_2$ , which witnesses  $p_d = q_d$ . The same argument gives the reverse direction of (2).

To prove the forward direction of (2), we need the notion of *admitting intersection* coming from [3, Definition 1.2] in the AEC case.

**Definition 4.3.**  $\mathbb{K}$  admits intersections if and only if for all  $X \subseteq M \in \mathbb{K}$ ,  $cl_M(X) \prec M$ , where  $cl_M(X)$  is the substructure of  $M$  with universe  $\cap\{N : X \subseteq N \prec M\}$ .

The key consequence of closure under intersection is that it simplifies checking if two types are equal.

**Fact 4.4** ([3].1.3). Suppose  $\mathbb{K}$  admits intersections. Then  $\text{gtp}(a_1/M_0; M_1) = \text{gtp}(a_2/M_0; M_1)$  if and only if there is  $h : cl_{M_1}(M_0 a_1) \cong_{M_0} cl_{M_2}(M_0 a_2)$  with  $h(a_1) = a_2$ .

**Claim 4.5.**  $\mathbb{K}_{\sigma}$  admits intersections.

*Proof.* We define the closure on each of the predicates. Then  $cl_M(X)$  will be the substructure with the union of the  $cl_M^{\ell}(X)$  as the universe.

- $cl_M^1(X) = (X \cap J) \cup \{j \in J^M : \exists h \in X \cap H^M. Q^M(h) = j\}$ ;
- $cl_M^2(X) = (X \cap I) \cup \{i \in I^M : \exists h \in X \cap H^M. \pi^M(h) = i\}$ ;
- $cl_M^3(X) = \{h \in H^M : \exists (i, j) \in cl_M^1(X) \times cl_M^2(X). Q^M(h) = j \text{ and } \pi^M(h) = i\}$ .

It is routine to verify that  $cl_M$  satisfies Definition 4.3. □

**Remark 4.6.** The AEC as constructed in [13] is not closed under intersections. Shelah does not require that the entire group  $G$  be included in every model. This means that if there is an empty equivalence class of  $H$  that must be filled (due to it projecting into  $I$  and  $J$ ), then any proper subgroups  $G' < G$  allows a choice of orbits to fill the equivalence class. This choice is incompatible with closure under intersection.

We are now ready to prove the forward direction of (2) from Claim 4.2. Suppose  $p_{\mathcal{D}} = q_{\mathcal{D}}$ . It is easy to compute that  $cl_{M_{\ell, \mathcal{D}}}(M_{0, \mathcal{D}} i_{\ell}) = M_{\ell, \mathcal{D}}$ . So by Fact 4.4 we have an isomorphism  $h : M_{1, \mathcal{D}} \cong_{M_{0, \mathcal{D}}} M_{2, \mathcal{D}}$ . This restricts to an isomorphism from  $H_{1, \mathcal{D}}$  to  $H_{2, \mathcal{D}}$  that respects  $\pi$  as in Lemma 3.8 and so  $\#(\mathcal{D}, \mathcal{F})$  follows.

Much of the work on AECs takes place under the assumption of amalgamation. Although not necessary for this proof, we also point out that  $\mathbb{K}_{\sigma}$  has amalgamation. This means the use of the construction [3, Definition 4.5] in [13] is unnecessary.

**Claim 4.7.**  $\mathbb{K}_{\sigma}$  has amalgamation.

*Proof.* Suppose that  $M_0 \subset M_1, M_2 \in K$  and without loss of generality  $M_1 \cap M_2 = M_0$ . We define the universe of  $M^*$  as follows:

- $J^* = J_1 \cup J_2$ ;
- $I^* = I_1 \cup I_2$ ; and
- $H^* = I^* \times G \times J^*$ .

For each  $(i, j) \in I_{\ell} \times J_{\ell}$  and  $\ell = 1, 2$ , pick some  $x_{i, j}^{\ell} \in H_{\ell}$  such that

- $\pi^{M_{\ell}}(x_{i, j}^{\ell}) = i$ ;
- $Q^{M_{\ell}}(x_{i, j}^{\ell}) = j$ ; and
- if  $(i, j) \in I_0 \times J_0$ , then  $x_{i, j}^1 = x_{i, j}^2$ .

Now we are ready to define the maps  $f_{\ell} : M_{\ell} \rightarrow M^*$ . This map is the identity on  $J_{\ell}, I_{\ell}$ , and  $G$  and, given  $y \in H_{\ell}$ ,

$$f_{\ell}(y) = (i, g, j) \iff F^{M_{\ell}}(x_{i, j}^{\ell}, g) = y; \pi^{M_{\ell}}(y) = i; \text{ and } Q^{M_{\ell}}(y) = j$$

Then the relations and functions of  $\mathcal{L}$  are defined on  $M^*$  as follows:

- $\pi(i, g, j) = i$ ;
- $F_{g'}(i, g, j) = (i, g + g', j)$ ;
- $Q(i, g, j) = j$ ;
- $E'((i, g, j), (i', g', j'))$  iff  $i = i'$ ;
- $E((i, g, j), (i', g', j'))$  iff  $(i, g, j) = f_{\ell}(x)$ ,  $(i', g', j') = f_{\ell}(x')$ , and  $(E')^{M_{\ell}}(x, x')$ ; and
- $R((i, g, j), (i', g', j'))$  iff  $(i, g, j) = f_{\ell}(x)$ ,  $(i', g', j') = f_{\ell}(x')$ , and  $R^{M_{\ell}}(x, x')$

Then it is easy to check that we have constructed an amalgam.  $\square$

We are now able to put the pieces together and generate several equivalences between global tameness principles for AECs and large cardinal axioms.

**Theorem 4.8.** *Let  $\sigma < \kappa$  be infinite cardinals with  $\sigma^{\omega} = \sigma$ .*

- (1) *If  $\kappa^{\sigma} = \kappa$  and every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) = \sigma$  is  $(< \kappa, \kappa)$ -tame, then  $\kappa$  is  $\sigma^+$ -weakly compact.*
- (2) *If every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) = \sigma$  is  $\kappa$ -local, then  $\kappa$  is  $\sigma^+$ -measurable.*
- (3) *If every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) = \sigma$  is  $(< \kappa, \sigma^{(\lambda^{< \kappa})})$ -tame, then  $\kappa$  is  $(\sigma^+, \lambda)$ -strongly compact.*

So we have the following corollary.

**Corollary 4.9.** *Let  $\kappa$  be an infinite cardinal such that  $\mu^{\omega} < \kappa$  for all  $\mu < \kappa$ .*

- (1) *if  $\kappa^{< \kappa} = \kappa$  and every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) < \kappa$  is  $(< \kappa, \kappa)$ -tame, then  $\kappa$  is almost weakly compact.*
- (2) *if every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) < \kappa$  is  $< \kappa$ -tame, then  $\kappa$  is almost strongly compact.*

*Proof of Theorem 4.8.* We start with the proof of part (1). We wish to apply our AEC construction together with Remark 2.11. Let  $\mathcal{A}$  be a field of subsets of  $\kappa$  with  $|\mathcal{A}| = \kappa$ . Let  $\theta$  be a big regular cardinal and take  $X \prec H_\theta$  of size  $\kappa$  with  $\mathcal{A} \in X$ . By our cardinal arithmetic assumption we can take  $X$  to be closed under  $\sigma$ -sequences. Let  $\mathcal{F}$  be the collection of functions in  $X$  with domain  $\kappa$  and range a bounded subset of  $\kappa$  and  $(\mathcal{D}, \triangleleft) = (\kappa, \in)$ . A straightforward argument using the fact that  $X$  is closed under  $\sigma$ -sequences shows that  $\mathcal{F}$  is  $\sigma^+$ -replete. It is also clear that  $\mathcal{F}$  has a characteristic function for each  $A \in \mathcal{A}$ . By our tameness assumption, Claim 4.2 and Lemma 3.8, we have that  $\#(\mathcal{D}, \mathcal{F})$  holds. Note that Lemma 3.8 requires that  $\mathcal{F}$  is countably closed and this follows from the fact that  $\sigma^\omega = \sigma$ . So by Remark 2.11  $\kappa$  is  $\sigma^+$ -weakly compact.

Part (2) is essentially Shelah's theorem from [13], but with the required corrections. We let  $\mathcal{F} = {}^\kappa\sigma$  and  $(\mathcal{D}, \triangleleft) = (\kappa, \in)$ . By our locality assumption, Claim 4.2 and Lemma 3.8, we have  $\#(\mathcal{D}, \mathcal{F})$ . Hence by Corollary 2.12, we have that  $\kappa$  is  $\sigma^+$ -measurable.

For part (3) we let  $(\mathcal{D}, \triangleleft) = (\mathcal{P}_\kappa(\lambda), \subset)$  and  $\mathcal{F} = {}^\mathcal{D}\sigma$ . By our tameness assumption, Claim 4.2 and Lemma 3.8, we have  $\#(\mathcal{D}, \mathcal{F})$ . Hence by Remark 2.14,  $\kappa$  is  $(\sigma^+, \lambda)$ -strongly compact.  $\square$

By strengthening the hypotheses a little we can remove the ‘almost’ from the above theorem. To do so we need a definition.

**Definition 4.10.**  $(\mathbb{K}, \prec_{\mathbb{K}})$  is *quasi-essentially below  $\kappa$*  if and only if it is essentially below  $\kappa^5$  or there is a theory  $T$  in  $L_{\kappa, \omega}$  such that  $(\mathbb{K}, \prec_{\mathbb{K}}) = (\text{Mod } T, \subset)$ .

We have introduced quasi-essentially below instead of just essentially below from [5], because although the class of models in  $\mathbb{K}_\sigma$  are axiomatizable in  $L_{\sigma^+, \omega}$ , the strong substructure relation is even weaker than first-order elementary.

**Theorem 4.11.** *Let  $\kappa$  be an infinite cardinal with  $\kappa^{<\kappa} = \kappa$  and for every  $\mu < \kappa$ ,  $\mu^\omega < \kappa$ . If every AEC  $\mathbb{K}$  which is quasi-essentially below  $\kappa$  is  $(< \kappa, \kappa)$ -tame, then  $\kappa$  is weakly compact.*

*Proof.* We apply Corollary 2.10. Let  $\mathcal{A}$  be a collection of subsets of  $\kappa$  with  $|\mathcal{A}| = \kappa$ . Again we build  $\mathcal{A}$  into an elementary substructure  $X$  of  $H_\theta$  of size  $\kappa$ . By the cardinal arithmetic assumption we can take  $X$  to be closed under  $< \kappa$ -sequences. We let  $\mathcal{F}$  be the collection of functions in  $X$  with domain  $\kappa$  and range bounded in  $\kappa$ . Note that  $\mathcal{F}$  is countably closed since  $\kappa$  has uncountable cofinality and  ${}^\kappa\sigma$  is countably closed provided that  $\sigma^\omega = \sigma$ . By our tameness assumption, Claim 4.2 and Lemma 3.8, we have  $\#(\mathcal{D}, \mathcal{F})$ . Moreover  $\mathcal{F}$  is  $\kappa$ -replete by the closure of  $X$ , hence by Corollary 2.10,  $\kappa$  is weakly compact.  $\square$

**Theorem 4.12.** *Let  $\kappa$  be a cardinal with  $\text{cf}(\kappa) > \omega$  and for all  $\mu < \kappa$ ,  $\mu^\omega < \kappa$ . If every AEC  $\mathbb{K}$  that is quasi-essentially below  $\kappa$  is  $< \kappa$ -tame, then  $\kappa$  is strongly compact.*

This theorem has some level by level information. In particular  $(< \kappa, \sup_{\alpha < \kappa} \alpha^{(\lambda^{<\kappa})})$ -tameness will give that  $\kappa$  is  $\lambda$ -strongly compact.

*Proof.* The proof is similar to the other proofs above. We take  $(\mathcal{D}, \triangleleft) = (\mathcal{P}_\kappa(\lambda), \subset)$  and  $\mathcal{F}$  to be the set of functions with domain  $\mathcal{D}$  and range bounded in  $\kappa$ . A similar

<sup>5</sup>See [5, Definition 2.10]

argument to the one in the previous theorem shows that  $\mathcal{F}$  is countably complete. Our tameness assumption gives  $\#(\mathcal{D}, \mathcal{F})$  and hence  $\kappa$  is  $\lambda$ -strongly compact by Corollary 2.13.  $\square$

It is important to note that the converses of the main theorems (and corollaries) from this section are true. The bulk of the work is already done in the first author's paper [5]. In some cases the converses of the stated results are stronger than theorems appearing in the literature. With some small adjustment the proofs in the literature already give these stronger claims. We make a brief list of the improvements required.

- (1) Loś' Theorem for AECs [5, Theorem 4.3] holds for AECs with Löwenheim-Skolem number  $\sigma$  and any  $\sigma^+$ -complete ultrafilter. This allows us to prove for example that every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) < \kappa$  is  $< \kappa$ -tame from  $\kappa$  is almost strongly compact.
- (2) Loś' Theorem for AECs does not require an ultrafilter only that the filter measures enough sets. This allows us to prove every AEC  $\mathbb{K}$  with  $\text{LS}(\mathbb{K}) < \kappa$  is  $(< \kappa, \kappa)$ -tame from the assumption that  $\kappa$  is almost weakly compact. In particular we can build everything into a transitive model of set theory of size  $\kappa$  and the weak compactness assumption gives a filter measuring all subsets of  $\kappa$  in the model. This is enough to complete the proof.
- (3) Loś' Theorem for AECs applies to the class of AECs which are quasi-essentially below  $\kappa$ . This allows us to prove that every AEC  $\mathbb{K}$  which is quasi-essentially below  $\kappa$  is  $< \kappa$ -tame from  $\kappa$  is strongly compact. It could be that such AECs have no models of size less than  $\kappa$ , but existing arguments are enough to give tameness over sets<sup>6</sup> rather than models.

We collect a few remarks on our construction:

- (1) For a cardinal  $\delta > 2^\omega$ , the assumption that  $\delta^\omega > \delta^+$  gives a failure of the singular cardinals hypothesis. Such a failure of the singular cardinals hypotheses has consistency strength on the level of a (partially) strong cardinal. So in many instances the failure of our cardinal arithmetic assumption has large cardinal strength.
- (2) We do not know if the cardinal arithmetic assumptions are necessary in the main theorems of this section. For example, if we assume that  $\kappa$  is weakly compact and we force to add  $\kappa^+$  many subsets to some  $\sigma^+$  where  $\sigma < \kappa$ , then  $\kappa$  remains  $\sigma^+$ -weakly compact in the extension. It follows that every AEC with Löwenheim-Skolem number  $\sigma$  is  $(< \kappa, \kappa)$ -tame in the extension. We do not know if  $\kappa$  satisfies any stronger tameness properties in the extension.
- (3) Under our mild cardinal arithmetic assumptions, the global full tameness and type shortness and compactness results from [5] follow from the global tameness for 1-types, as this tameness are already enough to imply the necessary large cardinals.

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<sup>6</sup>Typically, Galois types are defined so that the domains are always models. The same definition works for defining Galois types over arbitrary sets. However, many model-theoretic arguments ([14, Claim 3.3] on local character of non-splitting from stability is an early example) only work for Galois types over models, explaining their prevalence. The set-theoretic nature of the arguments from large cardinals, on the other hand, mean that they carry through with little change.

We conclude this section with an application to category theory. There has been recent activity in exploring the connection between AECs and accessible categories; see Lieberman [10], Beke and Rosicky [4], and Lieberman Rosicky [9]. In [9, Theorem 5.2], the authors apply a result of Makkai and Pare to derive the result that class many strongly compact cardinals imply tameness everywhere from [5]. Here we show that this application is in fact equivalent to the whole result. Note that in the global version we do not need any cardinal arithmetic assumptions.

**Corollary 4.13.** *The following are equivalent:*

- (1) *The powerful image of any accessible functor is accessible.*
- (2) *Every AEC is tame.*
- (3) *There are class many almost strongly compact cardinals.*

*Fix and infinite cardinal  $\kappa$  with  $\mu^\omega < \kappa$  for all  $\mu < \kappa$ . The following are equivalent:*

- (1) *The powerful image of a  $< \kappa$ -accessible functor is  $< \kappa$ -accessible.*
- (2) *Every AEC with  $LS(K) < \kappa$  is  $< \kappa$ -tame.*
- (3)  *$\kappa$  is almost strongly compact.*

In saying that every AEC is tame, we allow AECs with no models of size  $\kappa$  or larger to be trivially tame by saying they are  $< \kappa$ -tame.

*Proof.* In the first set of equivalences, the first implies the second by [9, Theorem 5.2]. The third implies the first by Brooke-Taylor and Rosicky [6, Corollary 3.5], which is a modification of Makkai and Pare's original [11, 5.5.1].

To see the second implies the third, for all  $\sigma$ , we know that there is some  $\kappa_\sigma$  such that  $\mathbb{K}_\sigma$  is  $< \kappa_\sigma$ -tame. Let  $\mathbf{S}$  be the set of all limit points of the map that takes  $\sigma$  to  $\sigma^\omega + \kappa_\sigma$ . Clearly,  $\mathbf{S}$  is class sized.

We claim that each  $\kappa \in \mathbf{S}$  is almost strongly compact. First note that  $\sigma^\omega < \kappa$  for all  $\sigma < \kappa$ . Let  $\sigma < \kappa \leq \lambda$ . Then  $\mathbb{K}_{\sigma^\omega}$  is  $< \kappa$ -tame, so it's  $(< \kappa, (\sigma^\omega)^{(\lambda^{< \kappa})})$ -tame. The proof of Theorem 4.8.(3) only involves this AEC, so it implies that  $\kappa$  is  $((\sigma^\omega)^+, \lambda)$ -strongly compact. Of course, this means that it is  $(\sigma, \lambda)$ -strongly compact. Since  $\sigma$  and  $\lambda$  were arbitrary,  $\kappa$  is almost strongly compact, as desired.

The second set of equivalences is just the parameterized version of the first one, and follows by the parameterized versions of the relevant results. The cardinal arithmetic is only needed for (2) implies (3).  $\square$

## 5. THE CONSISTENCY STRENGTH OF $(< \kappa, \kappa)$ -TAMENESS

We have already remarked that we do not know if the cardinal arithmetic assumptions are necessary in the theorems of Section 4. In this section, we show that, in the absence of cardinal arithmetic assumptions, the degree of tameness we associate to weak compactness has the expected *consistency strength*.

**Theorem 5.1.** *Let  $\kappa$  be a regular cardinal greater than  $\aleph_1$ . If every AEC  $\mathbb{K}$  which is quasi-essentially below  $\kappa$  is  $(< \kappa, \kappa)$ -tame, then  $\kappa$  is weakly compact in  $L$ .*

*Proof.* We may assume that  $0^\#$  does not exist, since otherwise every uncountable cardinal is weakly compact in  $L$  (see [8, Theorems 9.17.(b) and 9.14.(b)]). Let  $\mathcal{A}$  in  $L$  be a collection of  $\kappa$  many subsets of  $\kappa$  which is closed under complements and intersections of size less than  $\kappa$ . Choose an ordinal  $\beta < (\kappa^+)^L$  with  $\mathcal{A} \in L_\beta$  and

such that  $L$  models  $[L_\beta \cap \mathcal{P}(\kappa)]^{<\kappa} \subseteq L_\beta$ . Let  $\mathcal{F}$  be the collection of functions in  $L_\beta$  from  $\kappa$  to  $\kappa$  whose ranges are bounded in  $\kappa$ .

We claim that  $\mathcal{F}$  is countably closed in  $V$  under the ordering on functions defined in Section 2. Suppose that  $X$  is a countable subset of  $\mathcal{F}$ . By the covering lemma and our assumption that  $0^\#$  doesn't exist, there is a set  $Y \in L$  of size  $\aleph_1$  with  $X \subseteq Y$ . By the choice of  $L_\beta$ ,  $Y \in L_\beta$  and hence it has an upperbound in  $L_\beta$ .

By our tameness assumption, Claim 4.2 and Lemma 3.8, we have  $\#(\kappa, \mathcal{F})$  from which we can derive a filter  $U$  on  $\kappa$ . From the way we chose  $\mathcal{F}$ ,  $U$  measures all subsets of  $\kappa$  in  $L_\beta$  and is  $\kappa$ -complete with respect to sequences in  $L_\beta$ . We are now ready to give a standard argument that  $U$  restricted to  $\mathcal{A}$  is in  $L$ .

Let  $j : L_\beta \rightarrow L_\gamma \simeq \text{Ult}(L_\beta, U)$  be the elementary embedding derived from the ultrapower by  $U$ . Standard arguments show that the critical point of  $j$  is  $\kappa$ . Let  $\langle A_\alpha \mid \alpha < \kappa \rangle$  be an enumeration of  $\mathcal{A}$  in  $L_\beta$ . The sequence  $\langle j(A_\alpha) \mid \alpha < \kappa \rangle$  is in  $L$ , since it is just  $j(\langle A_\alpha \mid \alpha < \kappa \rangle) \restriction \kappa$ . So the set  $\bar{U} = \{A_\alpha \mid \kappa \in j(A_\alpha)\}$  is in  $L$ . It is easy to see that  $\bar{U} \subseteq U$  and hence is the  $\kappa$ -complete  $\mathcal{A}$ -ultrafilter that we require.  $\square$

## REFERENCES

1. Joan Bagaria and Menachem Magidor, *Group radicals and strongly compact cardinals*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 1857–1877. MR 3152715
2. John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, Providence, RI, 2009.
3. John T Baldwin and Saharon Shelah, *Examples of non-locality*, The Journal of Symbolic Logic **73** (2008), no. 03, 765–782.
4. T. Beke and J. Rosick, *Abstract elementary classes and accessible categories*, Annals of Pure and Applied Logic **163** (2012), no. 12, 2008 – 2017.
5. Will Boney, *Tameness from large cardinal axioms*, J. Symb. Log. **79** (2014), no. 4, 1092–1119.
6. Andrew Brooke-Taylor and Jiri Rosický, *Accessible images revisited*, preprint.
7. Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, J. Math. Log. **6** (2006), no. 2, 181–201. MR 2317426 (2008k:03070)
8. Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Large cardinals in set theory from their beginnings.
9. Michael Lieberman and Jiri Rosický, *Classification theory for accessible categories*, preprint.
10. Michael J. Lieberman, *Category-theoretic aspects of abstract elementary classes*, Annals of Pure and Applied Logic **162** (2011), no. 11, 903 – 915.
11. Michael Makkai and Robert Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, RI, 1989.
12. Michael Morley, *Categoricity in power*, Trans. Amer. Math. Soc **114** (1965), no. 2, 514–538.
13. Saharon Shelah, *Maximal failures of sequence locality in aec*, Preprint.
14. ———, *Categoricity for abstract classes with amalgamation*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 261–294. MR 1696853 (2000m:03087)

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