# Combination Resonances of a Beam with Two-Mode Interaction 

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#### Abstract

We present the perturbation and numerical solutions of two-dimensional nonlinear differential equations that describe the oscillations of two modes of the beam under axial forces. The multiple scales and Rung-Kutta fourth order methods are utilized to investigate the system behavior and its stability. All possible resonance cases are extracted and effects of different parameters on system behavior at resonant condition are studied.


Keywords: oscillations, analytical solution, numerical solution, combination resonances
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## 1. Introduction

Problems involving nonlinear differential equations are extremely diverse, and methods of solutions or analysis are problem dependent. Nonlinear systems are interesting for engineers, physicists and mathematicians because most physical systems are nonlinear in nature. Resonance of oscillating systems may lead to their damage or destruction, so it is important to study the behavior of vibrating system under different resonance conditions [1]. The non-linear behavior of a single-link flexible viscoelastic Cartesian manipulator is investigated [2], and the responses obtained using method of multiple scales compared with those obtained by solving numerically the temporal equation of motion. A single-degree-of-freedom non-linear oscillating system subject to multi-parametric and external excitations was studied [3,4]. In addition the multiple time scale perturbation technique was applied to obtain solution up to the third order approximation to extract and investigate the available resonance cases. The occurrence of saturation phenomena at different parameters values is reported. The nonlinear instability problem of two superposed dielectric fluids is solved by using the method of multiple scales. Numerical solutions were presented graphically for the effects of the different parameters on the system stability, response and chaos [5]. The method of multiple time scales is applied to investigate the response of nonlinear mechanical systems with internal and external resonances [6, 7]. The stability of vibrating systems was studied applying both the frequency response equation and the phase plane methods. The numerical solutions were focused on both the effects of the different parameters and the behavior of the system at the considered resonance cases. The motion of a flexible cantilever beam carrying a moving spring-mass system was investigated [8]. The system was described by a set of two nonlinear coupled partial differential
equations where the coupling terms have to be evaluated at the position of the mass. The equations of motion are solved numerically using the Rayleigh-Ritz method and an automatic ODE solver. The analysis of the local and codimension-3 degenerate bifurcations in a simply supported flexible beam with quintic nonlinear terms subjected to a harmonic axial excitation is presented [9]. Moreover numerical method is used to compute the bifurcation response curves based on the averaged equations and the stability of trivial solution analyzed, where new jumping phenomena were discovered in amplitude modulated oscillations. Two types of resonances, which are fundamental and subharmonic, were considered in a post-buckled beam to harmonic excitation [10]. The regions of instability and chaotic response are shown for different damping levels. The measured data are illustrated through time histories and phase plots. The dynamics of two oscillators coupled with quadratic nonlinearities in the case of two-to-one internal resonance is investigated when the higher mode is subjected to a principal parametric excitation [11]. The method of multiple scales is used to obtain an approximate solution to the equations of motion and to investigate the stability of the system. Experimental work is conducted to validate the theoretical results. The theoretical and experimental results indicated that the system exhibits complicated responses, such as jumps, saturation phenomenon, types I and II intermittency, as well as periodically, and chaotically modulated motions. The asymptotic solutions and transition curves for the generalized form of the non-homogeneous Mathieu differential equation are studied [12]. The problem of suppressing the vibrations of a hinged-hinged flexible beam that is subjected to primary and principal parametric excitations is investigated [13]. Different control laws are proposed, and saturation phenomenon is studied to suppress the vibrations of the system.
In this paper, a two degree of freedom model of flexible beam system subjected to axial or tuned forces is studied.

The method of multiple scales perturbation technique is used to obtain a first order approximate solution. Some possible resonance cases were extracted and studied applying Runge-Kutta fourth order method. The stability of the model is studied using both frequency response function and the phase-plane method. The resonant frequency response curves indicated the phenomenon of multiple solutions, soft- and hardening-spring types.

## 2. A Two d.o.f Nonlinear Model of Flexible Beam

In this section, The nonlinear partial differential equation governing the flexural deflection $u(x, t)$ of the beam, shown in Figure 1, subject to harmonic axial excitation $p=p_{0}-p_{1} \cos \Omega t$ is given by [13]


Figure 1. The model of the beam

$$
\begin{align*}
& m \frac{\partial^{2} u}{\partial t^{2}}+c \frac{\partial u}{\partial t}+E I \frac{\partial^{4} u}{\partial x^{4}}+\left(p_{0}-p_{1} \cos \Omega t\right) \frac{\partial^{2} u}{\partial x^{2}} \\
& +\frac{3}{2}\left(p_{0}-p_{1} \cos \Omega t\right)\left(\frac{\partial u}{\partial x}\right)^{2} \frac{\partial^{2} u}{\partial x^{2}} \\
& +E I\left[\frac{27}{2}\left(\frac{\partial u}{\partial x}\right)^{2}\left(\frac{\partial^{2} u}{\partial x^{2}}\right)^{3}-3\left(\frac{\partial^{2} u}{\partial x^{2}}\right)^{3}\right.  \tag{1}\\
& \left.-3\left(\frac{\partial u}{\partial x}\right)^{2} \frac{\partial^{4} u}{\partial x^{4}}+\frac{9}{4}\left(\frac{\partial u}{\partial x}\right)^{4} \frac{\partial^{4} u}{\partial x^{4}}\right]=0 .
\end{align*}
$$

The model of the flexible beam is studied under the following boundary conditions:

$$
\begin{equation*}
u(x)=0 \text { and } \frac{\partial u}{\partial x}=0 \text { at } x=0, x=l \tag{2}
\end{equation*}
$$

where $m$ is the mass per unit length of the beam, $E$ is the young's modulus of the beam, $I$ is the moment of inertia of the beam cross section, and $c$ is the damping coefficient.

For the purpose of analysis of Eq. (1), the Galerkin's procedure is introduced. Substituting the expression $u(x, t)=q(t) \sin \left(\frac{\pi x}{l}\right)+R(t) \sin \left(\frac{2 \pi x}{l}\right)$, where $q(t), R(t)$ are the amplitudes of the two modes of vibration, performing integration and using the following nondimensional quantities $t *=\sqrt{\frac{E I}{m l^{4}} t} q^{*}=\frac{l}{r^{2}} q, R *=\frac{l}{r^{2}} R$, $\Omega *=\sqrt{\frac{m l^{4}}{E I}} \Omega$, where $r$ is the radius of gyration of the cross-section area of the beam. Then dropping the overbar, we obtain a two-degree-of-freedom differential equation
governing the motion of the beam in the horizontal and vertical directions

$$
\begin{align*}
& q^{\prime \prime}+\left(\pi^{4}-\frac{\pi^{2} l^{2} p_{0}}{E I}\right) q+c l^{2} \sqrt{\frac{1}{m E I}} q^{\prime} \\
& +\frac{\pi^{2} l^{2} p_{1}}{E I} q \cos (\Omega t)+\frac{3 \pi^{4} r^{4} p_{1}}{8 l^{2} E I} q^{3} \cos (\Omega t) \\
& +3 \frac{\pi^{4} r^{4} p_{1}}{l^{2} E I} q R^{2} \cos (\Omega t)+\left(\frac{3 \pi^{2}}{2 l^{2}}-\frac{3 p_{0}}{8 E I}\right) \frac{\pi^{4} r^{4}}{l^{2}} q^{3}  \tag{3}\\
& -\frac{45}{32}\left(\frac{r \pi}{l}\right)^{8} q^{5}+\left(\frac{66 \pi^{2}}{l^{2}}-\frac{3 p_{0}}{E I}\right) \frac{\pi^{4} r^{4}}{l^{2}} q R^{2} \\
& -\frac{621}{2}\left(\frac{r \pi}{l}\right)^{8} q R^{4}-\frac{333}{4}\left(\frac{r \pi}{l}\right)^{8} q^{3} R^{2}=0 . \\
& R^{\prime \prime}+\left(16 \pi^{4}-\frac{4 \pi^{2} l^{2} p_{0}}{E I}\right) R+c l^{2} \sqrt{\frac{1}{m E I} R^{\prime}} \\
& +\frac{4 \pi^{2} l^{2} p_{1}}{E I} R \cos (\Omega t)+\frac{6 r^{4} \pi^{4} p_{1}}{l^{2} E I} R^{3} \cos (\Omega t) \\
& +\frac{3 r^{4} \pi^{4} p_{1}}{l^{2} E I} q^{2} R \cos (\Omega t)+\left(\frac{96 \pi^{2}}{l^{2}}-\frac{6 p_{0}}{E I}\right) \frac{r^{4} \pi^{4}}{l^{2}} R^{3}  \tag{4}\\
& -360\left(\frac{r \pi}{l}\right)^{8} R^{5}+\left(\frac{-6 \pi^{2}}{l^{2}}-\frac{3 p_{0}}{E I}\right) \frac{\pi^{4} r^{4}}{l^{2}} q^{2} R \\
& -\frac{117}{8}\left(\frac{r \pi}{l}\right)^{8} q^{4} R-297\left(\frac{r \pi}{l}\right)^{8} q^{2} R^{3}=0 . \\
& \\
& +
\end{align*}
$$

Introducing the perturbation parameter $\varepsilon$, the above equations can be expressed in the following form:
$q^{\prime \prime}+\omega^{2} q+\varepsilon\left(\begin{array}{l}\mu q^{\prime}+f_{1} q \cos (\Omega t)+f_{2} q^{3} \cos (\Omega t) \\ +f_{3} q R^{2} \cos (\Omega t)+\alpha_{1} q^{3}-\alpha_{2} q^{5} \\ +\alpha_{3} q R^{2}-\alpha_{4} q R^{4}-\alpha_{5} q^{3} R^{2}\end{array}\right)=0 .(5)$
$R^{\prime \prime}+\omega_{1}^{2} R+\varepsilon\left(\begin{array}{l}\mu R^{\prime}+4 f_{1} R \cos (\Omega t) \\ +\frac{4}{9} f_{2} R^{3} \cos (\Omega t)+f_{3} q^{2} R \cos (\Omega t) \\ +\beta_{1} R^{3}-\beta_{2} R^{5}+\beta_{3} q^{2} R \\ -\beta_{4} q^{4} R-\beta_{5} q^{2} R^{3}\end{array}\right)=0$

## 3. Analytical and Numerical Approaches

The analytical solution of equations (5) and (6) is studied using the method of multiple scales by assuming $q$, $R$ in the form

$$
\begin{align*}
& q\left(T_{0}, T_{1}\right)=q_{0}\left(T_{0}, T_{1}\right)+\varepsilon q_{1}\left(T_{0}, T_{1}\right)+\ldots  \tag{7}\\
& R\left(T_{0}, T_{1}\right)=R_{0}\left(T_{0}, T_{1}\right)+\varepsilon R_{1}\left(T_{0}, T_{1}\right)+\ldots \tag{8}
\end{align*}
$$

where, $T_{n}=\varepsilon^{n} t, \mathrm{~T}_{0}$ is the fast time scale associated with changes occurring at the frequencies $\omega, \omega_{1}$ and $\Omega, T_{1}$ is the slow time scale associated with modulations in the
amplitudes and phases caused by the nonlinearity, damping, and resonances.

In terms of $T_{0}$ and $T_{1}$, the time derivatives become:

$$
\begin{equation*}
\frac{d}{d t}=D_{0}+\varepsilon D_{1}+\ldots, \frac{d^{2}}{d t^{2}}=D_{0}^{2}+2 \varepsilon D_{0} D_{1}+\ldots \tag{9}
\end{equation*}
$$

where $D_{j}=\frac{\partial}{\partial T_{j}}, j=0,1$.
Using equations (7-9), Substituting $q, q^{\prime}, q^{\prime \prime}, R, R^{\prime}, R^{\prime \prime}$ into equations (5), (6) and equating the coefficient of same powers of $\varepsilon$ yields:

$$
\begin{gather*}
O\left(\varepsilon^{0}\right):\left(D_{0}^{2}+\omega^{2}\right) q_{0}=0  \tag{10}\\
\left(D_{0}^{2}+\omega_{1}^{2}\right) R_{0}=0  \tag{11}\\
O\left(\varepsilon^{1}\right):\left(D_{0}^{2}+\omega^{2}\right) q_{1}=-2 D_{0} D_{1} q_{0}-\mu D_{0} q_{0} \\
-f_{1} q_{0} \cos (\Omega t)-f_{2} q_{0}^{3} \cos (\Omega t)-f_{3} q_{0} R_{0}^{2} \cos (\Omega t)  \tag{12}\\
-\alpha_{1} q_{0}^{3}+\alpha_{2} q_{0}^{5}-\alpha_{3} q_{0} R_{0}^{2}+\alpha_{4} q_{0} R_{0}^{4}+\alpha_{5} q_{0}^{3} R_{0}^{2} \\
\left(D_{0}^{2}+\omega_{1}^{2}\right) R_{1}=-2 D_{0} D_{1} R_{0}-\mu D_{0} R_{0} \\
-4 f_{1} R_{0} \cos (\Omega t)-\frac{4}{9} f_{2} R_{0}^{3} \cos (\Omega t)  \tag{13}\\
-f_{3} q_{0}^{2} R \cos (\Omega t)-\beta_{1} R_{0}^{3}+\beta_{2} R_{0}^{5} \\
-\beta_{3} q_{0}^{2} R_{0}+\beta_{4} q_{0}^{4} R_{0}+\beta_{5} q_{0}^{2} R_{0}^{3}
\end{gather*}
$$

The general solution of equations (10) and (11) is given by

$$
\begin{gather*}
q_{0}\left(T_{0}, T_{1}\right)=A_{0}\left(T_{1}\right) e^{i \omega T_{0}}+\bar{A}_{0}\left(T_{1}\right) e^{-i \omega T_{0}}  \tag{14}\\
R_{0}\left(T_{0}, T_{1}\right)=B_{0}\left(T_{1}\right) e^{i \omega_{1} T_{0}}+\bar{B}_{0}\left(T_{1}\right) e^{-i \omega_{1} T_{0}} \tag{15}
\end{gather*}
$$

where $A_{0}, B_{0}$ are complex functions in $T_{1}$.
Thus the general solution of equations (12) and (13) can be written in the form

$$
\begin{aligned}
& q_{1}\left(T_{0}, T_{1}\right) \\
& =A_{1} e^{i \omega T_{0}}-\frac{\left(-\alpha_{1} A_{0}^{3}+5 \alpha_{2} A_{0}^{4} \bar{A}_{0}+2 \alpha_{5} A_{0}^{3} B_{0} \bar{B}_{0}\right)}{8 \omega^{2}} e^{3 i \omega T_{0}} \\
& -\frac{\alpha_{2} A_{0}^{5}}{24 \omega^{2}} e^{5 i \omega T_{0}}-\frac{\binom{-\alpha_{3} A_{0} B_{0}^{2}+4 \alpha_{4} A_{0} B_{0}^{3} \bar{B}_{0}}{+3 \alpha_{5} A_{0}^{2} \bar{A}_{0} B_{0}^{2}}}{4 \omega_{1}\left(\omega_{1}+\omega\right)} e^{i\left(2 \omega_{1}+\omega\right) T_{0}} \\
& -\frac{\left(-\alpha_{3} \bar{A}_{0} B_{0}^{2}+4 \alpha_{4} \bar{A}_{0} B_{0}^{3} \bar{B}_{0}+3 \alpha_{5} A_{0} \bar{A}_{0}^{2} B_{0}^{2}\right)}{4 \omega_{1}\left(\omega_{1}-\omega\right)} e^{i\left(2 \omega_{1}-\omega\right) T_{0}} \\
& -\frac{\alpha_{4} A_{0} B_{0}^{4}}{8 \omega_{1}\left(2 \omega_{1}+\omega\right)} e^{i\left(4 \omega_{1}+\omega\right) T_{0}}-\frac{\alpha_{4} \bar{A}_{0} B_{0}^{4}}{8 \omega_{1}\left(2 \omega_{1}-\omega\right)} e^{i\left(4 \omega_{1}-\omega\right) T_{0}} \\
& -\frac{\alpha_{5} A_{0}^{3} B_{0}^{2}}{4\left(2 \omega+\omega_{1}\right)\left(\omega+\omega_{1}\right)} e^{i\left(3 \omega+2 \omega_{1}\right) T_{0}} \\
& -\frac{\alpha_{5} A_{0}^{3} \bar{B}_{0}^{2}}{4\left(2 \omega-\omega_{1}\right)} e^{i\left(3 \omega-2 \omega_{1}\right) T_{0}}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\frac{1}{2} f_{1} A_{0}+\frac{3}{2} f_{2} A_{0}^{2} \bar{A}_{0}+f_{3} A_{0} B_{0} \bar{B}_{0}}{\Omega(2 \omega+\Omega)} e^{i(\omega+\Omega) T_{0}} \\
& -\frac{\frac{1}{2} f_{1} A_{0}+\frac{3}{2} f_{2} A_{0}^{2} \bar{A}_{0}+f_{3} A_{0} B_{0} \bar{B}_{0}}{\Omega(2 \omega-\Omega)} e^{i(\omega-\Omega) T_{0}}
\end{aligned}
$$

$$
+\frac{\frac{1}{2} f_{2} A_{0}^{3}}{(4 \omega+\Omega)(2 \omega+\Omega)} e^{i(3 \omega+\Omega) T_{0}}
$$

$$
+\frac{\frac{1}{2} f_{2} A_{0}^{3}}{(4 \omega-\Omega)(2 \omega-\Omega)} e^{i(3 \omega-\Omega) T_{0}}
$$

$$
+\frac{\frac{1}{2} f_{3} A_{0} B_{0}^{2}}{\left(2 \omega_{1}+\Omega\right)\left(2 \omega+2 \omega_{1}+\Omega\right)} e^{i\left(\omega+2 \omega_{1}+\Omega\right) T_{0}}
$$

$$
+\frac{\frac{1}{2} f_{3} A_{0} B_{0}^{2}}{\left(2 \omega_{1}-\Omega\right)\left(2 \omega+2 \omega_{1}-\Omega\right)} e^{i\left(\omega+2 \omega_{1}-\Omega\right) T_{0}}
$$

$$
+\frac{\frac{1}{2} f_{3} \bar{A}_{0} B_{0}^{2}}{\left(2 \omega_{1}+\Omega\right)\left(2 \omega_{1}-2 \omega+\Omega\right)} e^{i\left(2 \omega_{1}-\omega+\Omega\right) T_{0}}
$$

$$
\begin{equation*}
+\frac{\frac{1}{2} f_{3} \bar{A}_{0} B_{0}^{2}}{\left(\Omega-2 \omega_{1}\right)\left(2 \omega-2 \omega_{1}+\Omega\right)} e^{i\left(2 \omega_{1}-\omega-\Omega\right) T_{0}} \tag{16}
\end{equation*}
$$

$$
\begin{aligned}
& R_{1}\left(T_{0}, T_{1}\right) \\
& =B_{1} e^{i \omega_{1} T_{0}}-\frac{-\beta_{1} B_{0}^{3}+5 \beta_{2} B_{0}^{4} \bar{B}_{0}+2 \beta_{5} A_{0} \bar{A}_{0} B_{0}^{3}}{8 \omega_{1}^{2}} e^{3 i \omega_{1} T_{0}} \\
& -\frac{\beta_{2} B_{0}^{5}}{24 \omega_{1}^{2}} e^{5 i \omega_{1} T_{0}}-\frac{\binom{-\beta_{3} A_{0}^{2} B_{0}+4 \beta_{4} A_{0}^{3} \bar{A}_{0} B_{0}}{+3 \beta_{5} A_{0}^{2} B_{0}^{2} \bar{B}_{0}}^{i \omega\left(\omega_{1}+\omega\right)} e^{i\left(2 \omega+\omega_{1}\right) T_{0}}}{-\frac{-\beta_{3} A_{0}^{2} \bar{B}_{0}+4 \beta_{4} A_{0}^{3} \bar{A}_{0} \bar{B}_{0}+3 \beta_{5} A_{0}^{2} B_{0} \bar{B}_{0}^{2}}{4 \omega\left(\omega-\omega_{1}\right)} e^{i\left(2 \omega-\omega_{1}\right) T_{0}}} \\
& -\frac{\beta_{4} A_{0}^{4} B_{0}}{8 \omega\left(2 \omega+\omega_{1}\right)} e^{i\left(4 \omega+\omega_{1}\right) T_{0}}-\frac{\beta_{4} A_{0}^{4} \bar{B}_{0}}{8 \omega\left(2 \omega-\omega_{1}\right)} e^{i\left(4 \omega-\omega_{1}\right) T_{0}} \\
& -\frac{\beta_{5} A_{0}^{2} B_{0}^{3}}{4\left(2 \omega_{1}+\omega\right)\left(\omega_{1}+\omega\right)} e^{i\left(2 \omega+3 \omega_{1}\right) T_{0}} \\
& -\frac{\beta_{5} \bar{A}_{0}^{2} B_{0}^{3}}{4\left(2 \omega_{1}-\omega\right)\left(\omega_{1}-\omega\right)} e^{i\left(3 \omega_{1}-2 \omega\right) T_{0}} \\
& +\frac{2 f_{1} B_{0}+\frac{2}{3} f_{2} B_{0}^{2} \bar{B}_{0}+f_{3} A_{0} \bar{A}_{0} B_{0}}{\Omega\left(2 \omega_{1}+\Omega\right)} e^{i\left(\omega_{1}+\Omega\right) T_{0}} \\
& +\frac{2 f_{1} B_{0}+\frac{2}{3} f_{2} B_{0}^{2} \bar{B}_{0}+f_{3} A_{0} \bar{A}_{0} B_{0}}{\Omega\left(2 \omega_{1}-\Omega\right)} e^{i\left(\omega_{1}-\Omega\right) T_{0}} \\
& -\frac{2}{9} f_{2} B_{0}^{3} \\
& +\frac{\left.4 \omega_{1}+\Omega\right)\left(2 \omega_{1}+\Omega\right)}{} e^{i\left(3 \omega_{1}+\Omega\right) T_{0}}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\frac{2}{9} f_{2} B_{0}^{3}}{\left(4 \omega_{1}-\Omega\right)\left(2 \omega_{1}-\Omega\right)} e^{i\left(3 \omega_{1}-\Omega\right) T_{0}} \\
& +\frac{\frac{1}{2} f_{3} A_{0}^{2} B_{0}}{(2 \omega+\Omega)\left(2 \omega+2 \omega_{1}+\Omega\right)} e^{i\left(2 \omega+\omega_{1}+\Omega\right) T_{0}} \\
& +\frac{\frac{1}{2} f_{3} A_{0}^{2} B_{0}}{(2 \omega-\Omega)\left(2 \omega+2 \omega_{1}-\Omega\right)} e^{i\left(2 \omega+\omega_{1}-\Omega\right) T_{0}}  \tag{17}\\
& +\frac{\frac{1}{2} f_{3} A_{0}^{2} \bar{B}_{0}}{(2 \omega+\Omega)\left(2 \omega-2 \omega_{1}+\Omega\right)} e^{i\left(2 \omega-\omega_{1}+\Omega\right) T_{0}} \\
& +\frac{\frac{1}{2} f_{3} A_{0}^{2} \bar{B}_{0}}{(2 \omega-\Omega)\left(2 \omega-2 \omega_{1}-\Omega\right)} e^{i\left(2 \omega-\omega_{1}-\Omega\right) T_{0}}+c c .
\end{align*}
$$

## 4. Stability Analysis

In this section, we consider the resonance case when $\Omega$ tends to $2 \omega+2 \omega_{1}$ where $\omega=\omega_{1}$, which can be expressed as

$$
\begin{equation*}
\Omega=2 \omega+2 \omega_{1}+\varepsilon \sigma_{1}, \omega_{1}=\omega+\varepsilon \sigma_{2} \tag{18}
\end{equation*}
$$

Substituting equations (14), (15) and (18) into equations (12) and (13) and eliminating the secular terms, yields

$$
\begin{align*}
& -2 i \omega A_{0}^{\prime}-i \omega \mu A_{0}-3 \alpha_{1} A_{0}^{2} \bar{A}_{0}+10 \alpha_{2} A_{0}^{3} \bar{A}_{0}^{2} \\
& -2 \alpha_{3} A_{0} B_{0} \bar{B}_{0}+6 \alpha_{4} A_{0} B_{0}^{2} \bar{B}_{0}^{2}+6 \alpha_{5} A_{0}^{2} \bar{A}_{0} B_{0} \bar{B}_{0} \\
& -\alpha_{3} \bar{A}_{0} B_{0}^{2} e^{2 i \sigma_{2} T_{1}}+4 \alpha_{4} \bar{A}_{0} B_{0}^{3} \bar{B}_{0} e^{2 i \sigma_{2} T_{1}}  \tag{19}\\
& +3 \alpha_{5} A_{0} \bar{A}_{0}^{2} B_{0}^{2} e^{2 i \sigma_{2} T_{1}}+\alpha_{5} A_{0}^{3} \bar{B}_{0}^{2} e^{-2 i \sigma_{2} T_{1}} \\
& -\frac{1}{2} f_{2} \bar{A}_{0}^{3} e^{i\left(2 \sigma_{2}+\sigma_{1}\right) T_{1}}-\frac{1}{2} f_{3} \bar{A}_{0} \bar{B}_{0}^{2} e^{i \sigma_{1} T_{1}}=0, \\
& -2 i \omega_{1} B_{0}^{\prime}-i \mu \omega_{1} B_{0}-3 \beta_{1} B_{0}^{2} \bar{B}_{0}+10 \beta_{2} B_{0}^{3} \bar{B}_{0}^{2} \\
& -2 \beta_{3} A_{0} \bar{A}_{0} B_{0}+6 \beta_{4} A_{0}^{2} \bar{A}_{0}^{2} B_{0}+6 \beta_{5} A_{0} \bar{A}_{0} B_{0}^{2} \bar{B}_{0} \\
& -\beta_{3} A_{0}^{2} \bar{B}_{0} e^{-2 i \sigma_{2} T_{1}}+4 \beta_{4} A_{0}^{3} \bar{A}_{0} \bar{B}_{0} e^{-2 i \sigma_{2} T_{1}}  \tag{20}\\
& +3 \beta_{5} A_{0}^{2} B_{0} \bar{B}_{0}^{2} e^{-2 i \sigma_{2} T_{1}}+\beta_{5} \bar{A}_{0}^{2} B_{0}^{3} e^{2 i \sigma_{2} T_{1}} \\
& -\frac{2}{9} f_{2} \bar{B}_{0}^{3 i} e^{i\left(\sigma_{1}-2 \sigma_{2}\right) T_{1}}-\frac{1}{2} f_{3} \bar{A}_{0}^{2} \bar{B}_{0} e^{i \sigma_{1} T_{1}}=0
\end{align*}
$$

Using the following forms for $A_{0}, B_{0}$

$$
\begin{equation*}
A_{0}=\frac{1}{2} a_{1} e^{i \theta_{1}}, B_{0}=\frac{1}{2} a_{2} e^{i \theta_{2}} \tag{21}
\end{equation*}
$$

where $a_{1}, a_{2}$ and $\theta_{1}, \theta_{2}$ are functions in $T_{1}$ called the steady-state amplitudes and the phases of motions in the horizontal and vertical directions.

Substituting $A_{0}, B_{0}$ from equation (21) into equations (19) and (20), then separating real and imaginary parts gives governing equations of the amplitudes $a_{\mathrm{i}}$ and phases $\theta_{\mathrm{i}}$

$$
\begin{align*}
& 2 a_{1}^{\prime}=-\mu a_{1}+\left(-\frac{1}{4 \omega} \alpha_{3}+\frac{1}{4 \omega} \alpha_{4} a_{2}^{2}+\frac{3}{16 \omega} \alpha_{5} a_{1}^{2}\right) a_{1} a_{2}^{2} \sin v_{1}  \tag{22}\\
& +\frac{1}{16 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \sin v_{2}-\frac{1}{8 \omega} f_{2} a_{1}^{3} \sin v_{4}-\frac{1}{8 \omega} f_{3} a_{1} a_{2}^{2} \sin v_{3} .
\end{align*}
$$

$$
\begin{align*}
& \frac{a_{1}}{2}\left(v_{1}^{\prime}+v_{3}^{\prime}\right)=\frac{a_{1}}{2}\left(\sigma_{1}+2 \sigma_{2}\right)-\frac{3}{4 \omega} \alpha_{1} a_{1}^{3}+\frac{5}{8 \omega} \alpha_{2} a_{1}^{5} \\
& -\frac{1}{2 \omega} \alpha_{3} a_{1} a_{2}^{2}+\frac{3}{8 \omega} \alpha_{4} a_{1} a_{2}^{4}+\frac{3}{8 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \\
& +\left(-\frac{1}{4 \omega} \alpha_{3}+\frac{1}{4 \omega} \alpha_{4} a_{2}^{2}+\frac{3}{16 \omega} \alpha_{5} a_{1}^{2}\right) a_{1} a_{2}^{2} \cos v_{1}  \tag{23}\\
& +\frac{1}{16 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \cos v_{2}-\frac{1}{8 \omega} f_{2} a_{1}^{3} \cos v_{4} \\
& -\frac{1}{8 \omega} f_{3} a_{1} a_{2}^{2} \cos v_{3}
\end{align*}
$$

and

$$
\begin{align*}
& 2 a_{2}^{\prime}=-\mu a_{2}+\binom{-\frac{1}{4 \omega_{1}} \beta_{3}+\frac{1}{4 \omega_{1}} \beta_{4} a_{1}^{2}}{+\frac{3}{16 \omega_{1}} \beta_{5} a_{2}^{2}} a_{1}^{2} a_{2} \sin v_{2} \\
& +\frac{1}{16 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \sin v_{1}-\frac{1}{8 \omega_{1}} f_{2} a_{2}^{3} \sin v_{5}  \tag{24}\\
& -\frac{1}{8 \omega_{1}} f_{3} a_{1}^{2} a_{2} \sin v_{3}, \\
& \frac{a_{2}}{2}\left(v_{3}^{\prime}-v_{1}^{\prime}\right)=\frac{a_{2}}{2}\left(\sigma_{1}-2 \sigma_{2}\right)-\frac{3}{4 \omega_{1}} \beta_{1} a_{2}^{3}+\frac{5}{8 \omega_{1}} \beta_{2} a_{2}^{5} \\
& -\frac{1}{2 \omega_{1}} \beta_{3} a_{1}^{2} a_{2}+\frac{3}{8 \omega_{1}} \beta_{4} a_{1}^{4} a_{2}+\frac{3}{8 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \\
& +\left(-\frac{1}{4 \omega_{1}} \beta_{3}+\frac{1}{4 \omega_{1}} \beta_{4} a_{1}^{2}+\frac{3}{16 \omega_{1}} \beta_{5} a_{2}^{2}\right) a_{1}^{2} a_{2} \cos v_{2}  \tag{25}\\
& +\frac{1}{16 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \cos v_{1}-\frac{1}{8 \omega_{1}} f_{2} a_{2}^{3} \cos v_{5} \\
& -\frac{1}{8 \omega_{1}} f_{3} a_{1}^{2} a_{2} \cos v_{3},
\end{align*}
$$

where

$$
\begin{aligned}
& v_{1}=2\left(\theta_{2}-\theta_{1}+\sigma_{2} T_{1}\right), v_{2}=-2\left(\theta_{2}-\theta_{1}+\sigma_{2} T_{1}\right)=-v_{1}, \\
& v_{3}=\sigma_{1} T_{1}-2 \theta_{1}-2 \theta_{2}, v_{4}=2 \sigma_{2} T_{1}+\sigma_{1} T_{1}-4 \theta_{1}, \\
& v_{5}=\sigma_{1} T_{1}-2 \sigma_{2} T_{1}-4 \theta_{2} .
\end{aligned}
$$

The steady-state solutions correspond to constant $a_{1}, a_{2}, v_{1}, v_{3}$ that is $a_{1}^{\prime}=a_{2}^{\prime}=v_{1}^{\prime}=v_{3}^{\prime}=0$. Thus, equations (22)- (25) can be reduced to the following nonlinear algebraic equations

$$
\begin{align*}
& \mu a_{1}=\left(-\frac{1}{4 \omega} \alpha_{3}+\frac{1}{4 \omega} \alpha_{4} a_{2}^{2}+\frac{3}{16 \omega} \alpha_{5} a_{1}^{2}\right) a_{1} a_{2}^{2} \sin v_{1}  \tag{26}\\
& +\frac{1}{16 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \sin v_{2}-\frac{1}{8 \omega} f_{2} a_{1}^{3} \sin v_{4}-\frac{1}{8 \omega} f_{3} a_{1} a_{2}^{2} \sin v_{3} \\
& -\frac{a_{1}}{2}\left(\sigma_{1}+2 \sigma_{2}\right)+\frac{3}{4 \omega} \alpha_{1} a_{1}^{3}-\frac{5}{8 \omega} \alpha_{2} a_{1}^{5}+\frac{1}{2 \omega} \alpha_{3} a_{1} a_{2}^{2} \\
& -\frac{3}{8 \omega} \alpha_{4} a_{1} a_{2}^{4}-\frac{3}{8 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \\
& =\left(-\frac{1}{4 \omega} \alpha_{3}+\frac{1}{4 \omega} \alpha_{4} a_{2}^{2}+\frac{3}{16 \omega} \alpha_{5} a_{1}^{2}\right) a_{1} a_{2}^{2} \cos v_{1}  \tag{27}\\
& +\frac{1}{16 \omega} \alpha_{5} a_{1}^{3} a_{2}^{2} \cos v_{2}-\frac{1}{8 \omega} f_{2} a_{1}^{3} \cos v_{4}-\frac{1}{8 \omega} f_{3} a_{1} a_{2}^{2} \cos v_{3}
\end{align*}
$$

and

$$
\begin{aligned}
& \mu a_{2}=\left(-\frac{1}{4 \omega_{1}} \beta_{3}+\frac{1}{4 \omega_{1}} \beta_{4} a_{1}^{2}+\frac{3}{16 \omega_{1}} \beta_{5} a_{2}^{2}\right) a_{1}^{2} a_{2} \sin v_{2} \\
& +\frac{1}{16 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \sin v_{1}-\frac{1}{8 \omega_{1}} f_{2} a_{2}^{3} \sin v_{5}-\frac{1}{8 \omega_{1}} f_{3} a_{1}^{2} a_{2} \sin v_{3} \\
& \frac{a_{2}}{2}\left(2 \sigma_{2}-\sigma_{1}\right)+\frac{3}{4 \omega_{1}} \beta_{1} a_{2}^{3}-\frac{5}{8 \omega_{1}} \beta_{2} a_{2}^{5}+\frac{1}{2 \omega_{1}} \beta_{3} a_{1}^{2} a_{2} \\
& \quad-\frac{3}{8 \omega_{1}} \beta_{4} a_{1}^{4} a_{2}-\frac{3}{8 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \\
& \quad=\left(-\frac{1}{4 \omega_{1}} \beta_{3}+\frac{1}{4 \omega_{1}} \beta_{4} a_{1}^{2}+\frac{3}{16 \omega_{1}} \beta_{5} a_{2}^{2}\right) a_{1}^{2} a_{2} \cos v_{2} \\
& \quad+\frac{1}{16 \omega_{1}} \beta_{5} a_{1}^{2} a_{2}^{3} \cos v_{1}-\frac{1}{8 \omega_{1}} f_{2} a_{2}^{3} \cos v_{5} \\
& \quad-\frac{1}{8 \omega_{1}} f_{3} a_{1}^{2} a_{2} \cos v_{3} .
\end{aligned}
$$

Squaring equations (26)-(29) then adding (25) to (26) and (27) to (29), and simplifying we obtain

$$
\begin{gather*}
\Gamma_{1} a_{1}^{10}+\Gamma_{2} a_{1}^{8}+\Gamma_{3} a_{1}^{6}+\Gamma_{4} a_{1}^{4}+\Gamma_{5} a_{1}^{2}=0  \tag{30}\\
\Gamma_{6} a_{2}^{10}+\Gamma_{7} a_{2}^{8}+\Gamma_{8} a_{2}^{6}+\Gamma_{9} a_{2}^{4}+\Gamma_{10} a_{2}^{2}=0 \tag{31}
\end{gather*}
$$

The coefficients $\Gamma_{i}, i=1,2, \ldots, 10$ are given by

$$
\begin{aligned}
& \Gamma_{1}=\frac{25}{64 \omega^{2}} \alpha_{2}^{2} \\
& \Gamma_{2}=\frac{-15}{16 \omega^{2}} \alpha_{1} \alpha_{2}+\frac{15}{32} \alpha_{2} \alpha_{5} a_{2}^{2} \\
& \Gamma_{3}=\frac{5}{8 \omega} \alpha_{2}\left(\sigma_{1}+2 \sigma_{2}\right)+\frac{9}{16 \omega^{2}} \alpha_{1}^{2}-\frac{9}{16 \omega^{2}} \alpha_{1} \alpha_{5} a_{2}^{2} \\
& -\frac{5}{8 \omega^{2}} \alpha_{2} \alpha_{3} a_{2}^{2}+\frac{15}{32 \omega^{2}} \alpha_{2} \alpha_{4} a_{2}^{4} \\
& +\frac{5}{16 \omega^{2}} \alpha_{5}^{2} a_{2}^{4}+\frac{1}{16 \omega^{2}} f_{2} \alpha_{5} a_{2}^{2}-\frac{1}{64 \omega^{2}} f_{2}^{2}, \\
& \Gamma_{4}=\frac{-3}{4 \omega} \alpha_{1}\left(\sigma_{1}+2 \sigma_{2}\right)+\frac{3}{8 \omega} \alpha_{5}\left(\sigma_{1}+2 \sigma_{2}\right) a_{2}^{2} \\
& +\frac{3}{4 \omega^{2}} \alpha_{1} \alpha_{3} a_{2}^{2}-\frac{9}{16 \omega^{2}} \alpha_{1} \alpha_{4} a_{2}^{4}-\frac{1}{4 \omega^{2}} \alpha_{3} \alpha_{5} \\
& +\frac{5}{32 \omega^{2}} \alpha_{4} \alpha_{5} a_{2}^{6}-\frac{1}{16 \omega^{2}} f_{2} \alpha_{3} a_{2}^{2}+\frac{1}{16 \omega^{2}} f_{2} \alpha_{4} a_{2}^{4} \\
& -\frac{1}{32 \omega^{2}} f_{2} f_{3} a_{2}^{2}+\frac{1}{16 \omega^{2}} f_{3} \alpha_{5} a_{2}^{4} \\
& \Gamma_{5}=\mu^{2}+\frac{1}{4}\left(\sigma_{1}+2 \sigma_{2}\right)^{2}-\frac{1}{2 \omega} \alpha_{3}\left(\sigma_{1}+2 \sigma_{2}\right) a_{2}^{2} \\
& +\frac{3}{8 \omega} \alpha_{4}\left(\sigma_{1}+2 \sigma_{2}\right) a_{2}^{4}+\frac{3}{16 \omega^{2}} \alpha_{3}^{2} a_{2}^{4}-\frac{1}{4 \omega^{2}} \alpha_{3} \alpha_{4} a_{2}^{6} \\
& +\frac{5}{64 \omega^{2}} \alpha_{4}^{2} a_{2}^{8}-\frac{1}{16 \omega^{2}} f_{3} \alpha_{3} a_{2}^{4} \\
& +\frac{1}{16 \omega^{2}} f_{3} \alpha_{4} a_{2}^{6}-\frac{1}{64 \omega^{2}} f_{3}^{2} a_{2}^{4} \\
& \Gamma_{6}=\frac{25}{64 \omega_{1}^{2}} \beta_{2}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \Gamma_{7}=-\frac{15}{16 \omega_{1}^{2}} \beta_{1} \beta_{2}+\frac{15}{32 \omega_{1}^{2}} \beta_{2} \beta_{5} a_{1}^{2}, \\
& \Gamma_{8}=-\frac{5}{8 \omega_{1}} \beta_{2}\left(2 \sigma_{2}-\sigma_{1}\right)+\frac{9}{16 \omega_{1}^{2}} \beta_{1}^{2}-\frac{9}{16 \omega_{1}^{2}} \beta_{1} \beta_{5} a_{1}^{2} \\
& -\frac{5}{8 \omega_{1}^{2}} \beta_{2} \beta_{3} a_{1}^{2}+\frac{15}{32 \omega_{1}^{2}} \beta_{2} \beta_{4} a_{1}^{4}+\frac{5}{64 \omega_{1}^{2}} \beta_{5}^{2} a_{1}^{4} \\
& +\frac{1}{16 \omega_{1}^{2}} f_{2} \beta_{5} a_{1}^{2}-\frac{1}{64 \omega_{1}^{2}} f_{2}^{2}, \\
& \Gamma_{9}=\frac{3}{4 \omega_{1}} \beta_{1}\left(2 \sigma_{2}-\sigma_{1}\right)-\frac{3}{8 \omega_{1}} \beta_{5}\left(2 \sigma_{2}-\sigma_{1}\right) a_{1}^{2} \\
& +\frac{3}{4 \omega_{1}^{2}} \beta_{1} \beta_{3} a_{1}^{2}-\frac{9}{16 \omega_{1}^{2}} \beta_{1} \beta_{4} a_{1}^{4}-\frac{1}{4 \omega_{1}^{2}} \beta_{3} \beta_{5} a_{1}^{4} \\
& +\frac{5}{32 \omega_{1}^{2}} \beta_{4} \beta_{5} a_{1}^{6}-\frac{1}{16 \omega_{1}^{2}} f_{2} \beta_{3} a_{1}^{2}+\frac{1}{16 \omega_{1}^{2}} f_{2} \beta_{4} a_{1}^{4} \\
& \quad-\frac{1}{32 \omega_{1}^{2}} f_{2} f_{3} a_{1}^{2}+\frac{1}{16 \omega_{1}^{2}} f_{3} \beta_{5} a_{1}^{4}, \\
& \Gamma_{10}=\mu^{2}+\frac{1}{4}\left(2 \sigma_{2}-\sigma_{1}\right)^{2}+\frac{1}{2 \omega_{1}} \beta_{3}\left(2 \sigma_{2}-\sigma_{1}\right) a_{1}^{2} \\
& -\frac{3}{8 \omega_{1}} \beta_{4}\left(2 \sigma_{2}-\sigma_{1}\right) a_{1}^{4}+\frac{3}{16 \omega_{1}^{2}} \beta_{3}^{2} a_{1}^{4} .
\end{aligned}
$$

## 5. Numerical Results and Discussions

In this section, the solution of the frequency response equations (30), (31) are obtained numerically. Results are presented graphically as the steady-state amplitude of both modes against the detuning parameters to give the frequency response curves. The stability of the steadystate solution is investigated using the phase plane method and frequency response function and the numerical results are focused on the effect of different parameters.

### 5.1. Time-response (Numerical) Solution

A non-resonant time response for both modes of the system is shown in Figure 2(a). Different resonance cases are listed and an approximate percentage of increase in maximum steady-state amplitude compared to that in the non-resonant case is indicated.
(a) Internal resonance
$\left(\omega=\omega_{1}\right),(150 \%$, None) Figure 2(b),
$\left(\omega=2 \omega_{1}\right)$, (None,150\%) Figure 2(c)
(b) Sub-harmonic resonance
( $\Omega=2 \omega$ ) , (None,125\%) Figure 2(e)
( $\Omega=2 \omega$ ) , (150\%,125\%) Figure 2(f)
( $\Omega=4 \omega$ ) , (None,125\%) Figure 2(m)
( $\Omega=4 \omega_{1}$ ) , (None,125\%) Figure 2(n)
(c) Simultaneous resonance
( $\left.\Omega=2 \omega=2 \omega_{1}\right),(150 \%$, None) Figure 2(i)
(d) Combination resonance
$\left(\Omega=2 \omega+2 \omega_{1}\right)$, (None,125\%) Figure 2(g)

$$
\begin{aligned}
& \left(\Omega=2 \omega+2 \omega_{1}, \omega=\omega_{1}\right),(150 \%, 125 \%) \text { Figure } 2(\mathrm{j}) \\
& \left(\Omega=2 \omega+2 \omega_{1}, \omega_{1}=2 \omega\right),(125 \%, 125 \%) \text { Figure } 2(\mathrm{k})
\end{aligned}
$$

It can be noticed that steady-state amplitudes have maximum peak at the simultaneous resonance case in Figure 2(j) and hence is considered as the worst resonance cases of the systems behavior.


Non-resonance case

resonance case: $\omega=\omega_{1}$

resonance case: $\omega=2 \omega_{1}$

resonance case: $2 \omega=\omega_{1}$

resonance case: $2 \omega=\Omega$

resonance case: $2 \omega_{1}=\Omega$

resonance case: $\Omega=2 \omega+2 \omega_{1}$


Figure 2. Nonresonant and different resonant time solution of the 2-D model to axial excitation $u_{1}, u_{2}$ denote the amplitudes in the horizontal and vertical directions $q$ and $R$, respectively.
$\alpha_{1}=3.5, \alpha_{2}=4.1, \alpha_{3}=7.9, \alpha_{4}=7.2, \alpha_{5}=2.1, \mu=0.0001, f_{1}=0.03, f_{2}=0.08, f_{3}=0.04$.
$\beta_{1}=0.2, \beta_{2}=1, \beta_{3}=2.55, \beta_{4}=3.07, \beta_{5}=12.29, \Omega=3, \omega=2.5, \omega_{1}=1.7$

### 5.2. Theoretical Frequency Response solution

The solution of equations (30) and (31) is plotted in Figure 3 and Figure 4, respectively, as the amplitudes $a_{1}$
and $a_{2}$ against the detuning parameters $\sigma_{1}$ and $\sigma_{2}$ for different values of system parameters. Each figure consists of two branches that are bent to the right, except for Figure 3(d).


Figure 3. Frequency response curves of the first mode of the system at resonance
$\omega=2.7, \mu=0.01, f_{2}=0.6, f_{3}=3.1, \alpha_{1}=7, \alpha_{2}=0.4, \alpha_{3}=90, \alpha_{4}=10.4, \alpha_{5}=7.3, a_{2}=0.08, \sigma_{2}=0.01$.

Considering Figure 3(a) as basic case to compare with, it can be seen from Figures 3(b), (d), (e) and (f) that the steady-state amplitude $a_{1}$ increases as each of the natural frequency $\omega$ and the nonlinear coefficients $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ are increased. But in Figures 3(c) and 3(h), the steadystate amplitude $a_{1}$ decreases as each of the force amplitude $f_{3}$ and the second mode amplitude $a_{2}$ are increased.

Moreover the frequency response curves are bent to the right and left if $\alpha_{1}>0$, and $\alpha_{1}<0$, respectively. Whereas the frequency response curves are shifted to the right and to the left if $\alpha_{3}>0$ and $\alpha_{3}<0$, respectively.

Considering Figure 4(a) as basic case to compare with, it can be seen from Figures 4(d), (e) and (f) that the steady-state amplitude $a_{2}$ increases as each of the force
amplitudes $f_{1}, f_{3}$ and the nonlinear coefficients $\beta_{1}$ are increasing. But in Figures. $4(\mathrm{~g})$ the steady- state amplitude $a_{2}$ decreases as the nonlinear coefficient $\beta_{2}$ increasing. The behavior of the frequency response curves with linear damping coefficient and natural frequency are illustrated
in Figures 4(b) and 4(c). On the other hand, the frequency response curves are shifted to the right or to the left, shown in Figures. 4(i), 4(k), and 4(l) as the non linear coefficient $\beta_{3}$, the first mode amplitude $a_{1}$, and the detuning parameter $\sigma_{1}$ are varied, respectively.


Figure 4. Frequency response curves of second mode of the system at resonance
$\omega=4.5, \mu=0.01, f_{2}=4.5, f_{3}=6, \beta_{1}=-7.1, \beta_{2}=1.4, \beta_{3}=20, \beta_{4}=2.5, \beta_{5}=2.3, a_{1}=0.2, \sigma_{1}=0.3$.

## 6. Conclusions

The analytic and numerical solutions of a second degree of freedom model of nonlinear dynamic beam system to
axial harmonic excitation force are investigated. The method of multiple scales is utilized to solve the nonlinear ordinary differential equations up to and including the first order approximation. Some possible resonance cases were extracted and studied applying Runge-Kutta fourth order method. The stability of the model is studied using both
frequency response function and the phase-plane method. The numerical solutions are focused on the effect of the system parameters and the behavior of the nonlinear system at resonant condition, where the variations of the response due to the change of different parameters are investigated and studied. We may conclude the following:

1. The resonant frequency response curves show the phenomenon of multiple solutions, soft-spring type and hardening-spring type.
2. The steady-state amplitude of the first mode of vibration is a monotonic increasing function in the second mode amplitude and in the axial and tuned force amplitudes $f_{1}, f_{2}, f_{3}$ and $F$, respectively.
3. The first mode amplitude is a monotonic decreasing function in the linear damping coefficient $\mu$ and the nonlinear coefficient $\alpha_{3}$.
4. The second mode amplitude is a monotonic increasing function in the first mode amplitude and in the axial force amplitude $f_{2}$.
5. The steady-state amplitude of the second mode of the beam is a monotonic decreasing function in the natural frequency of the first mode $\omega$.
6. The nonlinear parameters $\alpha_{1}$ and $\beta_{1}$ show the nonlinearity effect of hardening- and softeningspring type.

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