Poisson homotopy algebra
An idiosyncratic survey of homotopy algebraic topics related to Alan's interests

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Dedicated to Alan Weinstein on his 60th birthday

Abstract

Homotopy algebra is playing an increasing role in mathematical physics. Especially in the Hamiltonian and Lagrangian settings, it is intimately related to some of Alan's interests, e.g. Courant and Lie algebroids. There is a comparatively long history of such structure in cohomological physics in terms of equations that hold mod exact terms (typically, divergences) or only 'on shell', meaning modulo the Euler-Lagrange equations of 'motion'; more recently, higher homotopies have come into prominence. Higher homotopies were developed first within algebraic topology and may not yet be commonly available tools for symplectic geometers and mathematical physicists.

This is an expanded version of my talk at the Alanfest, which was planned as a gentle introduction to the basic point of view with a variety of applications to substantiate its relevance. Most technical details are supplied by references to the original work or to [MSS02].

Contents

1 Introduction

Cohomological physics is a phrase I introduced some time ago in the context of anomalies in gauge theory, but it all began with Gauss in 1833, if not sooner (cf. Kirchoff's laws, as Nash's "Topology AND physics - a historical survey" [Nas98] reminded me). The cohomology referred to in Gauss was that of differential forms, div, grad, curl and especially Stoke's Theorem (the de Rham complex).

2 Basic concepts: DG Space of states, maps and homotopies

Whatever 'states' are physically, mathematically it is crucial that they form a vector space, in fact, usually a Hilbert or pre-Hilbert space. In cohomological physics, the *physical* space of states H is the (co)homology of a dg vector space, $V = \oplus V^n, d : V^n \to V^{n+1}$ with $d^2 = 0$. (As is more common in physics, we have adopted the *co*homological conventions with the grading as a superscript and the differential of degree 1. Of course there is a corresponding theory with differentials of degree -1 for which we would indicate the grading with a subscript. These conventions are equivalent just by raising/lowering: $V^n \Leftrightarrow V_{-n}$.) The space H is often considered as a subspace of the dg vector space by some (implicit) choice of representatives. In physical language, this might be referred to as *gauge fixing.*

Although much of physics is phrased in terms of manifolds and even analysis, my point of view is almost entirely (differential graded) algebraic, e.g. think of an *algebra of observables* without considering them as functions.

Maps (morphisms) of dg vector spaces $f : (V, d_V) \to (W, d_W)$ are linear maps of degree 0 which respect the differentials: $d_W f = f d_V$. For cochains/differential forms on topological spaces/manifolds, maps of spaces induce morphisms of the dg vector spaces. Assuming the differentials are of degree 1, a homotopy between two such maps is a linear map $h : (V, d_V) \to (W, d_W)$ of degree -1 such that

$$f - g = d_W h + h d_V.$$

Homotopies in the topological or smooth sense induce such dg homotopies. Notice, when applied to cocycles (representatives of physical states), $f = g$ mod exact terms.

Higher homotopies refer to homotopies of homotopies, and so on. For example, for given f and g with two such homotopies h and k as above, a second level homotopy is a linear map $h_2 : (V, d_V) \to (W, d_W)$ of degree -2 such that

$$h - k = d_W h_2 - h_2 d_V.$$

In full generality, higher homotopies refer to a family of linear maps $h_n : (V, d_V) \to (W, d_W)$ of degree $-n$ such that $d_W h_n - (-1)^n h_n d_V$ satisfies some relation among the h_i for $i < n$.

It is time to look at examples.

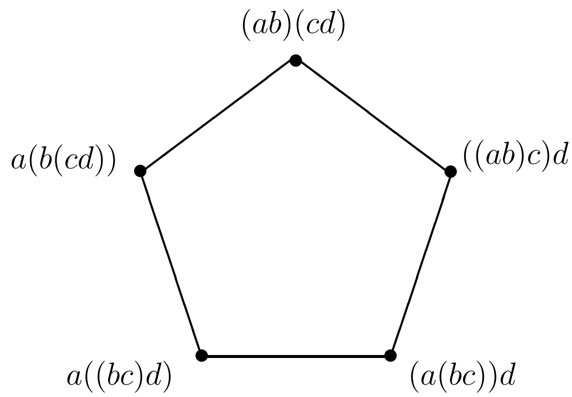

Figure 1: The pentagon K_4

3 A_∞-structure

The earliest example of higher homotopies occurs in Steenrod's $\smile_i$ products which underlie his operations. The examples currently most relevant to physics and Poisson structure are those of A_∞- and L_∞-structures [MSS02]. The topological version of A_∞-structure came first and is the easiest to visualize. It also has an obvious relevance to open string field theory (OSFT) [Kaj03] as L_∞-structure has to closed string field theory (CSFT) [Zwi93].

Consider the space of *based* loops ΩX on a space X with base point $*$. That is, a based loop is a map λ of the unit interval I into X such that $\lambda(0) = * = \lambda(1)$. Because we define the 'product' of two loops by reparameterizing the result of following one loop by another, this product is only homotopy associative.

Consider a specific associating homotopy $h(a,b,c)$ from $a(bc)$ to $(ab)c$. There are 5 ways of parenthesizing the product of 4 loops, which results in a pentagon of loops, where the sides represent a single application of a specific associating homotopy $h(a,b,c)$ from $a(bc)$ to $(ab)c$. For example, the bottom edge from left to right of Figure 1 is given by $h(a,bc,d)$. By looking at the parameterizations in more detail, it can be seen that the pentagon can be filled in by a 2-parameter family of loops.

Now there are 14 ways of parenthesizing the product of 5 loops and so on. The combinatorics, in general for n-loops, can be realized in terms of a polyhedron, called an *associahedron* and denoted K_n, described as a convex polytope with one vertex for each way of associating n ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of n letters.

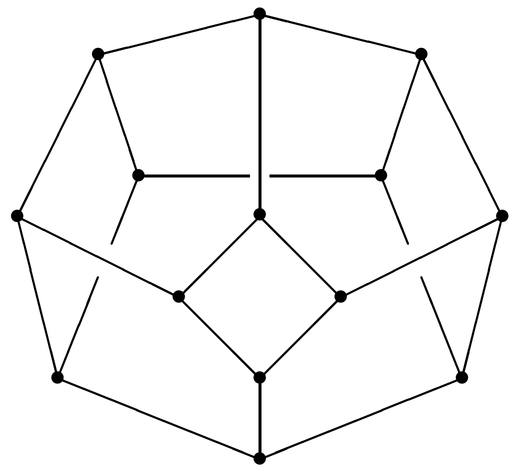

Figure 2: Saito's portrait of K_5.

For $n = 5$, a portrait due to Masahico Saito is in Figure 2; a rotatable image is available at `www.labmath.uqam.ca/~chapoton/stasheff.html`.

An A_∞*-space* Y is a topological space Y together with a family of maps

$$m_n : K_n \times Y^n \to Y$$

which fit together in a way suggested by the pentagon. (In technical language, they make Y an algebra over the pre-operad $\mathcal{K} = \{K_n\}_{n\geq 1}$.)

The main result is the following theorem.

Theorem 3.1 *A connected space* Y *(of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based loop space* ΩX *for some space* X *if and only if* Y *admits the structure of an* A_∞*-space.*

Now consider a *chain complex functor* from topological spaces to dg modules (over a commutative ground ring) which respects products. Applied to an A_∞-space, such a functor reveals an algebraic structure generalizing that of a dg associative algebra.

Definition 3.2 *An* A_∞-algebra *(or strongly homotopy associative algebra) consists of a graded module* V *with maps*

$$m_n : V^{\otimes n} \to V \text{ of degree } 2 - n$$

satisfying suitable compatibility conditions $(A_n)_{n\geq 1}$.

In particular,

(A_1) $m_1 = d$ is a differential,

(A_2) $m = m_2 : V \otimes V \to V$ is a chain map, that is, d is a derivation with respect to $m = m_2$,

(A_3) $m_3 : V^{\otimes 3} \to V$ is a chain homotopy for associativity of the multiplication m, i.e.

$$m_3 d^{\otimes 3} + dm_3 = m(m \otimes 1) - m(1 \otimes m),$$

where $d^{\otimes 3}$ denotes $d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d$,

(A_4) m_4 is a 'higher homotopy' such that $m_4 d^{\otimes 4} - dm_4$ has five terms, corresponding to the edges of the pentagon K_4:

$$m_4 d^{\otimes 4} - dm_4 = m_3(m_2 \otimes 1 \otimes 1 - 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2) - m_2(m_3 \otimes 1 + 1 \otimes m_3).$$

An alternate formulation generalizes the bar construction on an associative differential graded algebra. Define the suspension sA of a graded vector space A by shifting the grading down: $(sA)^n = A^{n+1}$.

Alternate Definition 3.3 *An* A_∞-algebra *structure on a positively graded vector space* A *is equivalent to a coderivation differential* δ *of degree* 1 *with respect to the total grading on the tensor coalgebra* $\mathbf{T}^c(sA)$ *on the suspension of the graded vector space* A*. As a coderivation,* δ *is determined by the formula* $\delta = \delta_1 + \delta_2 + \cdots$*, where*

$$\delta_n(sa_1 \otimes \cdots \otimes sa_n) := \epsilon \cdot sm_n(a_1 \otimes \cdots \otimes a_n), \text{ for } a_1, \ldots, a_n \in A,$$

and ϵ *is an appropriate sign.*

4 L_∞-structure

Although there was no topological predecessor, the notion of an L_∞-algebra follows a similar pattern algebraically.

4.1 A Lie algebra is...

A grad student in mathematics is likely to encounter the following.

Definition 4.1 *A Lie algebra is a vector space* $\mathfrak{g}$ *with a skew-bilinear operation* $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *satisfying the Jacobi identity.*

Physicists are more likely to assume a basis $\{X_i\}$ for $\mathfrak{g}$ and write

$$[X_i, X_j] = C_{ij}^k X_k$$

with the structure *constants* C_{ij}^k being skew-symmetric and satisfying the corresponding Jacobi identity.

There is another alternative that plays a key role in the homological study of Lie algebras.

Consider the familiar exterior (=Grassmann) algebra $E\mathfrak{g}$, except think of it as a *co*algebra. That is, define

$$\Delta : E\mathfrak{g} \to E\mathfrak{g} \otimes E\mathfrak{g}$$

by

$$\Delta(X_1 \wedge \cdots X_n) = \Sigma\ X_{i_1} \wedge \cdots \wedge X_{i_p} \otimes X_{i_{p+1}} \wedge \cdots \wedge X_{i_n}),$$

where the summation is over all $0 \leq p \leq n$ and all permutations $(i_1, \cdots, i_n)$. Grade $E\mathfrak{g}$ by wedge degree and define a coderivation

$$\delta : E\mathfrak{g} \to E\mathfrak{g}$$

by

$$\delta(X \wedge Y) = [X, Y]$$

extended as a coderivation. The Jacobi identity is equivalent to the condition $\delta^2 = 0$.

This dg coalgebra is the Chevalley-Eilenberg chain complex for Lie algebra homology. Many will be more familiar with the dual algebra and dual derivation, which is the standard Chevalley-Eilenberg coboundary and a special example of a BRST operator.

Now I want to up the ante and consider dg (differential graded) Lie algebras. For a dg Lie algebra, the classical Chevalley-Eilenberg complex needs a slight adjustment to account for the grading.

Definition 4.2 *A* dg Lie algebra *is a graded vector space* $\mathfrak{g} = \oplus \mathfrak{g}_n$ *with a differential* $d : \mathfrak{g}^n \to \mathfrak{g}^{n+1}$ *with* $d^2 = 0$ *and a graded-skew bilinear operation* $[\ ,\ ] : \mathfrak{g}^p \otimes \mathfrak{g}^q \to \mathfrak{g}^{p+q}$ *which is a chain map satisfying the graded Jacobi identity.*

Following our example of having associativity satisfied only up to homotopy, we can do the same for the Jacobi identity and then consider higher homotopies. One can give the definition in terms of a family of maps

$$l_n = [-, \cdots, -] : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$$

which are suitably 'skew symmetric' and compatible, but it is definitely a lot simpler to give the definition in terms of $\Lambda s\mathfrak{g}$.

Definition 4.3 *An* L_∞-algebra *structure on a positively graded vector space* $\mathfrak{g}$ *is equivalent to a coderivation differential* δ *of degree* 1 *with respect to the total grading on the graded symmetric coalgebra* $\Lambda^c(s\mathfrak{g})$ *on the suspension of the graded vector space* $\mathfrak{g}$*.*

The notation Λ^c for the graded symmetric coalgebra is well established in rational homotopy theory; the alternative S^c is used elsewhere.

Notice that if we write $d := -sl_1$ and $B := \delta - d$, then the condition $\delta^2 = 0$ can be rewritten in (generalized) Maurer-Cartan form:

$$dB + 1/2[B, B] = 0. \tag{1}$$

If you'd like some hands-on examples, consider very small finite dimensional L_∞-algebras. There are two versions of the classification, depending on whether we consider L_∞-algebras in the original Z-graded sense or the super, i.e. $= Z/2Z$-graded, sense. There are classifications here by, respectively, Daily [Dai02] or Fialowski-Penkava [FP03b, FP03a] for very small dimensional examples.

Now what does this have to do with physics or symplectic geometry or Alan's interests?

The answers include moment maps, symplectic reduction and Courant algebroids.

4.2 Courant algebroids

The most straightforward connection with Alan's interests appears in his paper with Roytenberg [RW98] on Courant algebroids. There is no point in repeating the very clear exposition in their original paper, so I will mention only the salient facts in re: higher homotopies.

Definition 4.4 *A* Courant algebroid *is a vector bundle $E \to M$ equipped with a non-degenerate bilinear form $< \cdot, \cdot >$ on the bundle, a skew symmetric bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(E)$ and a bundle map $a : E \to TM$ satisfying 5 properties.*

Kosmann-Schwarzbach has simplified the definition to 2 properties, but in the non-skew-symmetric form [KS03].

The skew-bracket does not in general satisfy the Jacobi identity, but property number 5 addresses the defect in terms of a map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ related to the deRham differential on M. Roytenberg and Weinstein consider the following small complex:

$$X^{-1} = C^\infty(M) \to X^0 = \Gamma(E) \tag{2}$$

with the differential l_1 given by $\mathcal{D}$. They define a very specific L_∞-structure as follows:

l_2 is given by the bracket on $X_0 \otimes X_0$ and by $< \cdot, \mathcal{D} \cdot >$ on $X_0 \otimes X_1$ and 0 in higher degrees,

l_3 is given by one third of the sum of the cyclic permutations of $< [\cdot, \cdot], \cdot >$ on $X_0 \otimes X_0 \otimes X_0$ and 0 otherwise,

while l_n for $n > 3$ is 0.

Of course, many of these 0's follow just from the fact that $X_n = 0$ for $n \geq 2$, in contrast to the L_∞-structures that appear in the work of Fulp, Lada and myself concerning higher spin algebras [FLS02b, FLS02a], where the complex is of the form $X^0 \to X^1$ (see section 6.2 below).

5 Homological reduction of constrained Poisson algebras

Cohomological physics had a major breakthrough with the 'ghosts' introduced by Fade'ev and Popov [FP67]. These were incorporated into what came to be known as *BRST cohomology* (Becchi-Rouet-Stora [BRS75] and Tyutin [Tyu75]) and which was applied to a variety of problems in mathematical physics. There the ghosts were reinterpreted by Stora [Sto77] and others in terms of the Maurer-Cartan forms in the case of a finite dimensional Lie group and more generally as generators of the Chevalley-Eilenberg cochain complex [CE48] for Lie algebra cohomology.

If, as geometers, you feel more comfortable with manifolds, one can make the following algebra seem more palatable as functions on 'supermanifolds', but most (all?) of the work is just algebraic (homological).

WARNING! The term 'BRST cohomology' has a variety of meanings in the existing literature. From time to time, it threatens to be used for any cohomology in physics, at least if the coboundary operator is called 'Q'. At other times, it refers (only implicitly) to the case in which the Lie algebra is the Virasoro algebra. I prefer to reserve the term for situations in which the coboundary operator has at least some part corresponding to that of Chevalley-Eilenberg.

Such is the case for the ghost technology for the cohomological reduction of constrained Poisson algebras, introduced by Batalin, Fradkin and Vilkovisky [BF83, FF78, FV75], which extended the complex of BRST by reinventing the Koszul-Tate [Tat57] resolution of the ideal of constraints and producing a synergistic combination of both Chevalley-Eilenberg and resolution cohomology. Here it was that I saw the essential features of a strong homotopy Lie algebra (L_∞-algebra).

5.1 Moment maps, momaps and symplectic reduction

The setting is one of Alan's favorites [Wei02], that of a moment map, though generalized in an important way. Alan considers:

> a phase space with a symmetry group consists of a manifold P equipped with a symplectic structure ω and a hamiltonian action of a Lie group G. By the latter, we mean a symplectic action of G on P together with an equivariant moment map J from P to the dual $\mathfrak{g}^*$ of the Lie algebra of G such that, *for each $v \in \mathfrak{g}$, the 1-parameter group of transformations of P generated by v is the flow of the hamiltonian vector field with hamilitonian $x \rightarrow< J(x), v >$.* The map J is called the *momentum map* (or, by many authors, *moment map*) of the hamiltonian action. If one is simply given a symplectic action of G on P, any map J satisfying the condition in italics above, even if it is not equivariant, is called a momentum map for the action.

By contrast, Batalin-Fradkin-Vilkovisky consider *constraints* on the symplectic manifold P to be primary.

A *Hamiltonian system with constraints* means we have functions $\phi_\alpha : P \to \mathbf{R}, 1 \leq \alpha \leq r$, the constraints. Solutions of the system are constrained to lie in a subspace $V \subset P$ given as the zero set of a smooth *momap* $\phi : P \to W = \mathbf{R}^r$ with components ϕ_α. In contrast to the more restrictive case in which $W = \mathbf{R}^r$ has the structure of a Lie algebra $\mathfrak{g}$ and ϕ is assumed to be equivariant with respect to the action of the corresponding Lie group on P, here we do not assume any Lie group G action. To emphaisze this, I refer to ϕ as a *momap*. (This also avoids the moment versus momentum controversy.) When V is a smooth submanifold of P, the algebra $C^\infty(V)$ can be expressed as $C^\infty(P)/I$ where I is the ideal generated by the ϕ_α. Dirac calls the constraints *first class* if I is closed under the Poisson bracket. In terms of the constraints, the condition is then

$$\{\phi_\alpha, \phi_\beta\} = f^\gamma_{\alpha\beta}\phi_\gamma,$$

where we have *structure functions* $f^\gamma_{\alpha\beta}$ on P, not structure constants. In other words, we have a Lie algebroid with anchor map $a : C^\infty(P) \to \Gamma(TP)$ given by the Hamiltonian vector field associatied to a function.

If we let W denote the *vector space* spanned by the ϕ_α, physicists speak of W as an *open algebra* since the bracket defined on W does not close in W. Compare this with Lie's notion of *function group* [Lie90] as discussed by Alan [Wei02].

In this first class case, the Hamiltonian vector fields X_{ϕ_α} determined by the constraints are tangent to V (where V is smooth) and give a foliation $\mathcal{F}$ of V. Similarly, $C^\infty(P)/I$ is an I-module with respect to the bracket. (In symplectic geometry, the corresponding space is called *coisotropic*. The passage from P to $V/\mathcal{F}$ is known as *symplectic reduction*.) The true physics of the system is the induced system on the space of leaves $V/\mathcal{F}$. If that space is a smooth manifold, $C^\infty(V/\mathcal{F})$ is the true algebra of observables. When smoothness is lacking and $C^\infty(V/\mathcal{F})$ makes no sense, the Batalin-Fradkin-Vilkovisky construction provides a replacement, as described below (see [HT92] for a comprehensive treatment).

In this context, the classical BRST construction, at least as developed by Batalin-Fradkin-Vilkovisky in the case of regular constraints, is a *homological model* for $C^\infty(V/\mathcal{F})$ or rather for the full de Rham complex $\Omega(V, \mathcal{F})$ consisting of forms on vertical vector fields, those tangent to the leaves. If we think of $\mathcal{F}$ as an involutive sub-bundle of the tangent bundle to V, then $\Omega(V, \mathcal{F})$ consists of sections of $\Lambda^*\mathcal{F}$, the exterior algebra bundle on the dual of $\mathcal{F}$. The ordinary exterior derivative of differential forms restricts to determine the vertical exterior derivative because $\mathcal{F}$ is involutive. The meaning of being a homological model is provided by the next Theorem.

The model is constructed as follows: First, consider the most common case of an equivariant moment map $\phi : P \to W = \mathfrak{g}^*$ with respect to a Lie group action of G on P where $\mathfrak{g}$ is the Lie algebra of G. Let A denote $C^\infty(P)$ considered as a Poisson algebra. Extend A as a graded commutative algebra to

$$\mathcal{BFV} = A \otimes E\mathfrak{g}^* \otimes E\mathfrak{g} \tag{3}$$

and extend the Poisson bracket $\{\cdot, \cdot\}$ (still of degree 0) as determined by the fundamental pairing $\mathfrak{g}^* \otimes \mathfrak{g} \to \mathbf{R}$. Note: $\mathfrak{g}$ here has degree -1 and $\mathfrak{g}^*$ has degree 1. Now make $\mathcal{BFV}$ a dg Poisson algebra by defining

$$d_{\mathcal{BFV}} = d_K + \delta^* \tag{4}$$

where δ^* is the Chevalley-Eilenberg coboundary and d_K is the Koszul differential on $A \otimes E\mathfrak{g}$ regarded as a resolution of the ideal of constraints. In terms of a basis $\{e_\alpha\}$ for $\mathfrak{g}$ so that $\phi_\alpha = e_\alpha \circ \phi$, this means that d_K is the graded derivation determined by

$$d_K(e_\alpha) = \phi_\alpha.$$

If we denote e_α as $\mathcal{P}_\alpha$ and similarly define η^α in terms of a dual basis, then $d_{\mathcal{BFV}} = \{Q_0 + Q_1, \cdot\}$ for

$$Q_0 = \eta^\alpha \phi_\alpha \textit{ and } Q_1 = 1/2\ \eta^\alpha \eta^\beta C^\gamma_{\alpha\beta} \mathcal{P}_\gamma, \tag{5}$$

the formula that often appears in the physics literature.

Because we have a strict Lie group action and, hence, structure constants, it is straightforward to verify $d^2_{\mathcal{BFV}} = 0$, but this is not the case for our momaps. The definition of the algebra is no problem:

$$\mathcal{BFV} = A \otimes EW \otimes EW^* \tag{6}$$

and $d_K + \delta^*$ is defined as before but fails to square to 0, essentially because we now have structure *functions*. In the regular case, the brilliance of Batalin-Fradkin-Vilkovisky was to define $d_{\mathcal{BFV}}$ by adding terms of higher order to $d_K + \delta^*$ so that $(d_{\mathcal{BFV}})^2 = 0$. With hindsight, the existence of such terms of higher order was due to the fact that $A \otimes E\mathfrak{g}$ provided a resolution of the ideal of constraints, thus permitting the techniques of Homological Perturbation

Theory (see Section 6.3 and [Gug82, GL89, GLS90, GS86, Hue84, HK91]). However, the proof crucially involves keeping $d_{\mathcal{BFV}}$ as an inner derivation $\{Q, \cdot\}$ by adding terms of higher order to $Q_0 + Q_1$. If we write $Q = Q_0 + Q_\infty$, then we see we are seeking a solution of the Master Equation (cf. Section 6.3):

$$\{Q_0, Q_\infty\} + 1/2\ \{Q_\infty, Q_\infty\} = 0 \tag{7}$$

or, equivalently, that we seek to deform Q_0 in the 'direction' of Q_1 (cf. Section 7).

The point of doing this is:

Theorem 5.1 *Under suitable regularity conditions, the cohomology of $\mathcal{BFV}$ is isomorphic to the cohomology of $\Omega(V, \mathcal{F})$ with respect to the leaf-wise exterior differential. In particular, $H^0(\mathcal{BFV})$ is isomorphic to $H^0(\Omega(V/\mathcal{F}))$, the algebra of 'observables' on the reduced phase space.*

In the more general non-regular case, the Koszul complex can be extended to the Koszul-Tate resolution by adding the polynomial algebra generated by 'anti-ghosts of anti-ghosts' (given degree -2), etc. To preserve the crucial Poisson algebra structure, one also adds 'ghosts of ghosts' (given degree 2), etc.

In general, the quotient space is not a manifold, often not even Hausdorff, then $H^0(\mathcal{BFV})$ provides a suitable candidate for the algebra of observables on the 'reduced phase space'.

Since $\mathcal{BFV}$ is a free graded commutative algebra over A, assuming sufficient finiteness, the differential derivation $d_{\mathcal{BFV}}$ is graded dual to a differential coderivation on a free graded cocommutative coalgebra over A and hence is equivalent to an L_∞-algebra. This is spelled out in considerable detail by Kjeseth [Kje96, Kje01a, Kje01b]subsequent to some relevant observations of Huebschmann [Hue90] and myself.

6 Lagrangians with symmetries

Lagrangian physics derives 'equations of motion' from a variational principle of least action. Here an action refers to an integral

$$S(\phi) = \int_M L((j^n\phi)(x))vol_M$$

over some manifold M where ϕ is a (possibly vector valued) function on M or section of a bundle E over M and L is a 'local function' on E, meaning a function on some finite jet space $J^n E$. The action may have symmetries, i.e. variations in ϕ which do not change the value of S and hence are physically irrelevant in the sense that ϕ and its transformed value encode the same physical information.

Emmy Noether had two major theorems regarding the variational calculus. The first, much better known and often referred to as *Noether's theorem*, asserts a correspondence between symmetries and conserved quantities. Noether's second variational theorem establishes a correspondence between symmetries, notably gauge symmetries, and differential algebraic relations among the Euler-Lagrange equations. It is this second theorem that has an important role in the Batalin-Vilkovisky construction for Lagrangians with symmetries.

These symmetries create difficulties for quantization of such physical theories. The method of Batalin and Vilkovisky [BV84, BV83] was invented to handle these difficulties, but turns out to be of interest also in a classical context. The construction is quite parallel to that of Batalin-Fradkin-Vilkovisky in the constrained Hamiltonian case, but with one crucial difference: instead of a grading preserving bracket, they use an 'anti-bracket' (independently due to Zinn-Justin [ZJ75, ZJ76]) which is of degree 1. Therefore it is also known as an odd Poisson or Gerstenhaber bracket. In this Lagrangian setting, Batalin and Vilkovisky extend the BRST cohomological approach by introducing anti-fields (independently and previously due to Zinn-Justin) dual to the original fields and anti-ghosts which (with hindsight) correspond to the Noether relations and are dual to the ghosts which generate the BRST complex for the Lie algebra of symmetries.

The original version of Noether in 'Invariante variationsprobleme' [Noe18], was written in terms of an infinite continuous group, $G_{\infty\rho}$, 'understood to be a group whose most general transformations depend on ρ essential arbitrary functions and their derivatives'. Noether's Theorem II refers to an integral I ($= S$ in our notation) and reads:

> If the integral I is invariant with respect to a $G_{\infty\rho}$ in which the arbitrary functions occur up to the σ-th derivative, there subsist ρ identity relationships between the Lagrange expressions and their derivatives up to the σ-th order. ... the converse holds.

Later in that paper these relations are called *dependencies*.

The relevance of Noether's theorem is not emphasized in most of the literature using the BV approach. As with $\mathcal{BFV}$, part of the differential of the Batalin-Vilkovisky complex $\mathcal{BV}$ is that of the Koszul-Tate resolution, in this case of the differential ideal generated by the Euler-Lagrange equations. The anti-fields generate the Koszul complex, which is not a resolution; the anti-ghosts provide the next level of generators, as described by Tate [Tat57], corresponding to the relations among the Euler-Lagrange equations. It is the full acyclicity of the Koszul-Tate resolution that permits the application of Homological Perturbation Theory [Gug82, GL89, GLS90, GS86, Hue84, HK91](among others) and thus guarantees the existence of the terms of higher order in the full differential $d_{\mathcal{BV}}$. As in the concluding remark in Section 5, the graded dual to $d_{\mathcal{BV}}$ is equivalent to an L_∞-algebra. We comment on this further in Section 6.2.

Rather than carrying out this analysis in the abstract, we indicate two particularly striking realizations of this structure: the Poisson sigma models of Cattaneo and Felder [CF99] and our analysis with Fulp and Lada [FLS02a, FLS02b] of Lagrangians with field dependent symmetries as in the case of higher spin particles.

6.1 The Poisson sigma model

The fields of the Cattaneo-Felder Poisson σ-model are ordered pairs (X, η) such that X is a mapping from a 2-dimensional manifold Σ into a Poisson manifold M and η is a section of the bundle $Hom(T\Sigma, X^*T^*M) \longrightarrow \Sigma$. These fields are subject to boundary conditions, namely they should satisfy the conditions: $X(u) = 0$ and $\eta(u)(v) = 0$ for arbitrary u in the boundary of Σ and for v tangent to the boundary of Σ at u. In terms of local coordinates $\{x^i\}$ on M, the Poisson structure is given by a *Poisson tensor* α which is a skew-symmetric tensor on M

$$\alpha = \alpha^{ij}(x)(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}) \tag{8}$$

which satisfies a Jacobi condition:

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0. \tag{9}$$

The action S of the model is defined in such local coordinates by

$$S(X, \eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij} \circ X)(\eta_i \wedge \eta_j). \tag{10}$$

The Euler-Lagrange equations for this action are:

$$E_{X^i} := d\eta_i + \frac{1}{2}\partial_i \alpha^{jk}(\eta_j \wedge \eta_k) = 0 \tag{11}$$

and

$$E_{\eta_i} := -dX^i - \alpha^{ij}\eta_j = 0. \tag{12}$$

The gauge symmetries of the action are parameterized by all sections β of the bundle $X^*T^*M \longrightarrow \Sigma$ which vanish on the boundary of Σ. For each such β, define δ_β acting on the fields by

$$(\delta_\beta X)^i = (\alpha \circ X)(dx^i, \beta) \tag{13}$$

$$(\delta_\beta \eta)(W \circ X) = -(d\beta)(U \circ X) - ((\mathcal{L}_U \alpha) \circ X)(\eta, \beta) \tag{14}$$

where U is a vector field on M, and $\mathcal{L}_U \alpha$ is the Lie derivative of α with respect to U.

In terms of the components of the fields, we write

$$E_{X^i} = (\partial_\mu \eta_{i,\nu} + \frac{1}{2}\partial_i \alpha^{jk}\eta_{j,\mu}\eta_{k,\nu})\epsilon^{\mu\nu} \tag{15}$$

and

$$E_{\eta_{i,\nu}} = -(\partial_\mu X^i + \alpha^{ij}\eta_{j,\mu})\epsilon^{\mu\nu}. \tag{16}$$

It follows from Noether's theorem that

$$\alpha^{ik}E_{X^i} + \partial_\mu E_{\eta_{k,\mu}} - \partial_i \alpha^{jk}\eta_{j,\mu}E_{\eta_{i,\mu}} = 0 \tag{17}$$

are the Noether identities corresponding to the gauge symmetry δ_β defined above.

Applied to this Poisson sigma model, the Batalin-Vilkovisky graded algebra $\mathcal{BV}$ is a graded commutative algebra over Loc_E, (the algebra of local functions on E) with generators X_i^+ and η^{+i}, called 'anti-fields', γ_i called 'ghosts' and γ^{+i}, called 'anti-ghosts. (If only the ghosts were used as generators, this would be just a BRST algebra.) These generators are bigraded. The graded commutativity is with respect to the sum of the ghost degree and the form degree (which we call the total degree). The pairing between symmetries and identities is now expressed as the pairing between ghosts and anti-ghosts, which plays a crucial role in the Batalin-Vilkovisky anti-bracket $(\cdot, \cdot)$.

Now the initial differential on $\mathcal{BV}$ can be expressed as $(S^0 + S^1, \cdot)$ where

$$S^0 = (X, \eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij} \circ X)(\eta_i \wedge \eta_j),$$

our original action, and S^1 is

$$\int_\Sigma X_i^+ \alpha^{ij}(X)\gamma_j - \eta^{+i} \wedge (d\gamma_i + \partial_i \alpha^{kj}(X)\eta_k \gamma_j) - \frac{1}{2}\gamma^{+i}\partial_i \alpha^{jk}(X)\gamma_j\gamma_k. \quad (18)$$

Corresponding to the fact that $(d_{KT} + \delta^*)^2 \neq 0$, we have

$$(S^0 + S^1, S^0 + S^1) \neq 0.$$

The additional terms in the differential D are found by extending $S^0 + S^1$ by terms of higher order to achieve the full BV action S_{BV}, but in the Cattaneo-Felder model, only one more term is needed:

$$S^2 = \int_\Sigma -\frac{1}{4}\eta^{+i} \wedge \eta^{+j}\partial_i\partial_j\alpha^{kl}(X)\gamma_k\gamma_l. \quad (19)$$

Thus the total Batalin-Vilkovisky generator is

$$\begin{aligned} S_{BV} &= \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i \wedge \eta_j \quad (20) \\ &\quad + X_i^+\alpha^{ij}(X)\gamma_j - \eta^{+i} \wedge (d\gamma_i + \partial_i\alpha^{kl}(X)\eta_k\gamma_l) - \frac{1}{2}\gamma_+^i\partial_i\alpha^{jk}(X)\gamma_j\gamma_k \\ &\quad -\frac{1}{4}\eta^{+i} \wedge \eta^{+j}\partial_i\partial_j\alpha^{kl}(X)\gamma_k\gamma_l. \end{aligned}$$

6.2 Field dependent gauge symmetries

Field dependent gauge symmetries appear in several field theories, most notably in a class due to Ikeda [Ike94] and Schaller and Strobl [SS94], including the Poisson sigma model of Cattaneo and Felder [CF99] above. A significant generalization occurs in the Berends, Burgers and van Dam [Bur85, BBvD86, BBvD85] approach to "particles of spin ≥ 2". The physics of "particles of spin ≤ 2" leads to representations of a Lie algebra Ξ of gauge parameters on a vector space Φ of fields. Berends, Burgers and van Dam attacked particles of higher spin by letting the gauge parameters act in a *field dependent* way. By a field dependent action of Ξ on Φ, Berends, Burgers and van Dam mean a polynomial (or power series) map $\delta(\xi)(\phi) = \Sigma_{i\geq 0}T_i(\xi,\phi)$ where T_i is linear in ξ and polynomial of homogeneous degree i in ϕ. Berends, Burgers and van Dam consider arbitrary field theories, subject only to the requirement that the commutator of two gauge symmetries be another gauge symmetry whose gauge parameter is possibly field dependent. Thus they do not require an a priori given Lie structure to induce the algebraic structure of the gauge symmetry "algebra".

Let Φ denote the vector space of fields and Ξ the vector space of gauge parameters. Let $\Lambda^*\Phi$ denote the free graded cocommutative coalgebra cogenerated by Φ. Although the space Ξ of gauge parameters has no natural Lie structure, the space of linear maps from $\Lambda^*\Phi$ into Ξ is a Lie algebra under certain mild assumptions along with a hypothesis which we refer to as the *BBvD hypothesis.* Under these assumptions, the gauge algebra gives rise to an L_∞-algebra.

The vector space $\mathrm{Coder}(\Lambda^*\Phi)$ of *graded coderivations* on $\Lambda^*\Phi$ is a Lie algebra with bracket given by the commutator with respect to composition. The vector space $\mathrm{Hom}(\Lambda^*\Phi,\Phi)$ is isomorphic to $\mathrm{Coder}(\Lambda^*\Phi)$ and hence inherits a Lie algebra structure; the bracket on $\mathrm{Hom}(\Lambda^*\Phi,\Phi)$ is known as the Gerstenhaber bracket [Ger63, Sta93]. Suppose that we are given a linear map $\delta : \Xi \to \mathrm{Hom}(\Lambda^*\Phi,\Phi)$, a 'field dependent action' of Ξ on Φ. We can write $\delta(\xi) = \Sigma_{i=0}T_i(\xi)$ where T_i is 0 except on $\Lambda^i\Phi$. (This T_i is adjoint to the polarization of the Berends, Burgers and van Dam T_i.) We extend δ to a map

$$\hat{\delta} : \mathrm{Hom}(\Lambda^*\Phi,\Xi) \to \mathrm{Hom}(\Lambda^*\Phi,\Phi)$$

by

$$\hat{\delta}(\pi) = ev \circ (\delta \circ \pi \otimes 1) \circ \Delta$$

where ev is the evaluation map. Consider the possibility of inducing a Lie-type bracket on $\mathrm{Hom}(\Lambda^*\Phi,\Xi)$ via the mapping $\hat{\delta}$. Under certain conditions, e.g δ is injective, such a bracket may then be used to obtain a bracket on the parameter space defined by restricting the induced bracket on $\mathrm{Hom}(\Lambda^*\Phi,\Xi)$ to the parameter space Ξ, where we think of the space Ξ as being contained in the space $\mathrm{Hom}(\Lambda^*\Phi,\Xi)$ by identifying $\xi \in \Xi$ with the map, also denoted ξ, in $\mathrm{Hom}(\Lambda^*\Phi,\Xi)$ which is 0 except on the scalars where $\xi(1) = \xi$. In order to assure the Jacobi property of the bracket on $\mathrm{Hom}(\Lambda^*\Phi,\Xi)$, we introduce a *correction term,* following Berends, Burgers and van Dam, by assuming that there is a map

$$C : \Xi \otimes \Xi \to \mathrm{Hom}(\Lambda^*\Phi,\Xi)$$

such that

$$[\delta(\xi), \delta(\eta)] = \delta C(\xi, \eta) \in \mathrm{Hom}(\Lambda^*\Phi, \Phi)$$

for all $\xi, \eta \in \Xi$. We will refer to this as the *BBvD hypothesis.* We then extend C to a mapping

$$\hat{C} : \mathrm{Hom}(\Lambda^*\Phi, \Xi) \otimes \mathrm{Hom}(\Lambda^*\Phi, \Xi) \to \mathrm{Hom}(\Lambda^*\Phi, \Xi)$$

and then we redefine the bracket on $\mathrm{Hom}(\Lambda^*\Phi, \Xi)$ given above by including the correction term C :

$$[\pi_1, \pi_2] := \pi_1 \odot \hat{\delta}(\pi_2) - \pi_2 \odot \hat{\delta}(\pi_1) + \hat{C}(\pi_1, \pi_2).$$

Theorem 6.1 *The mapping $\hat{\delta}$ preserves brackets: $\hat{\delta}[\pi_1, \pi_2] = [\hat{\delta}(\pi_1), \hat{\delta}(\pi_2)]$. Moreover, if $\hat{\delta} : \mathrm{Hom}(\Lambda^*\Phi, \Xi) \to \mathrm{Hom}(\Lambda^*\Phi, \Phi)$ is injective, then $[\pi_1, \pi_2]$ satisfies the Jacobi identity.*

This result suggests that the parameter space should be enlarged to include all of $Hom(\Lambda^*\Phi, \Xi)$, but this is apparently unacceptable to physicists since the number of independent parameters is linked to the number of independent Noether identities. However, the polynomial equations of physical relevance define an L_∞- structure on an appropriate graded vector space. We restrict our attention to the constant maps in $\mathrm{Hom}(\Lambda^*\Phi, \Xi)$ and show that the algebraic structure of $\mathrm{Hom}(\Lambda^*\Phi, \Xi)$ induces an L_∞- structure on $\Xi \oplus \Phi$. We assume the BBvD hypothesis and that $\hat{\delta}$ is injective, so Theorem 6.1 holds.

Define a differential graded vector space V with Ξ in degree 0, Φ in degree 1 and 0 in all other degrees. Take $\partial : \Xi \to \Phi$, given by $\partial(\xi) = \delta(\xi)(1) \in \Phi$, as the only non-trivial differential. Define

$$D : \Lambda^*(sV) \to sV$$

by

$$\begin{aligned} D(\xi) =& \partial(\xi) \\ D(\xi \wedge \phi_1 \wedge \cdots \wedge \phi_n) =& \delta(\xi)(\phi_1 \wedge \cdots \wedge \phi_n) \text{ for } n \geq 1 \\ D(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n) =& C(\xi_1, \xi_2)(\phi_1 \wedge \cdots \wedge \phi_n) \end{aligned}$$

and $D = 0$ on elements of $\Lambda^*(sV)$ with more than two entries from Ξ or with no entry from Ξ.

Notice this is essentially *not* of the same form as that of Roytenberg and Weinstein in section 4.2, although both have just two components. The crucial difference is in the grading: $0, 1$ here versus $-1, 0$ for them.

Theorem 6.2 *[FLS02a] $D : \Lambda^*(sV) \to sV$ as defined above gives V the structure of an L_∞-algebra*

6.3 The Master Equation and Homological Perturbation Theory

The name 'Master Equation' derives from the physics literature, especially of the Batalin-Vilkovisky approach considered above, but equations of this form are well known in mathematics, though under various names:

- the defining equation for a *twisting cochain*,
- the *integrability equation* of deformation theory,
- the *Maurer-Cartan equation*,

the latter being perhaps the most famous. The equation makes sense as applied to elements of a dg algebra, associative or Lie or Gerstenhaber, etc., as well as to higher homotopy versions thereof. In [HS02], Huebschmann and I give an extensive comparison of these various intrpretations. We show how to construct solutions using the tools of Homological Perturbation Theory, working in characteristic zero. In particular, we endow the homology $H(\mathfrak{g})$ of a strict dg Lie algebra $\mathfrak{g}$ with the structure of an L_∞-algebra such that $\mathfrak{g}$ and $H(\mathfrak{g})$ are equivalent as L_∞-algebras, i.e. via L_∞-maps (see Section 7). The much older analogous result for A_∞-algebras is due to Kadeishvili [Kad80]. Note that $H(\mathfrak{g})$ is a strict dg Lie algebra with $d = 0$, but the higher order operations l_i are often non-trivial. If $\mathfrak{g}$ is equivalent as L_∞-algebra to $H(\mathfrak{g})$ with all $l_i = 0$ for $i \geq 3$, then $\mathfrak{g}$ is called *formal*.

7 L_∞-maps, deformation quantization, String Field Theory (SFT) and more

Definition 7.1 *An L_∞-*map *$f : \mathfrak{h} \to \mathfrak{g}$ of L_∞-algebras (or dg Lie algebras) is a dg coalgebra map $\Lambda^c(s\mathfrak{h}) \to \Lambda^c(s\mathfrak{g})$.*

The Cattaneo and Felder Poisson sigma model was developed to provide an alternative, 'path integral', proof of Kontsevich's theroem that any Poisson *manifold* can be deformation quantized. In both proofs, the key issue is the *formality* of a certain dg Lie algebra $\mathfrak{g}$. The L_∞-equivalence of this $\mathfrak{g}$ and $H(\mathfrak{g})$ implies that all the obstructions to deformation quantization vanish.

For this important application, Σ was a disk so the maps $\Sigma \to M$ could be considered as world sheets as in SFT.

The relevance of A_∞- and L_∞-structure to (respectively) OSFT and CSFT has a particularly 'physical' interpretation. The higher order operations describe multiple string interactions, *not* obtained from 3-string interactions (multiplication, respectively bracketing of 2 strings) or the equivalent *correlation* functions [Zwi93]. (Here too there is contact with Alan and his student Tang in their recent paper [TW03].)

Because BBvD give an explicit expansion T_i, C_i, the corresponding multibrackets l_i are visible or at least easy to extract. In contrast, in the Cattaneo and Felder Poisson sigma model, they are hidden in the single *function* α and it's derivatives.

8 Homological Legendre transform

In their concluding remarks in [RW98], Dmitry and Alan muse:

> L_∞-algebras occur in physics in the framework of the Batalin-Vilkovisky procedure for quantizing gauge theories. On the other hand, the Courant bracket seems to provide a geometric framework for constrained Hamiltonian systems. It is known [HT92] that gauge Lagrangians lead to constrained theories in the Hamiltonian formalism. This suggests that homotopy Lie algebras arising in the Batalin-Vilkovisky formalism and those in the Courant formalism might be somehow related.

In response to my paper for the Alanfestschrift, Dmitry pointed out to me the paper of Grigoriev and Damgaard [GD00] which establishes an analog of the Legendre transform in terms of the BFV and BV constructions. At Alanfest, we discussed their transforms in further detail. This bears further investigation, but for now let me mention only that in either direction, Hamiltonian to Lagrangian of vice versa, the essential idea is the 'oddification' of all the fields, ghosts, etc., then substituting these into the respective formulas for the Hamiltonian, the Lagrangian, etc. and keeping only the parts of the appropriate total degree. Dmitry can interpret this further by looking at the algebra as that of a graded path space.

9 Coda

There are still other examples of A_∞- and L_∞-structures with potential physical relevance, for example, the notion of an A_∞-*category*. This can be thought of as a many object version of an A_∞-algebra, that is, satisfying the usual axioms of a category, except that composition of morphisms is associative only up to homotopy and higher homotopies of all orders. A_∞-categories have been used by Fukaya for remarkable applications to Morse theory and Floer homology and by Batanin and by May in higher category theory. More recently, they play a role in string and D-brane theory and homological mirror symmetry.

But this takes us further afield from today's topic: the L_∞-structures directly involved in some of Alan's work and closely related to his foundational work on symplectic reduction. There are further relations to be discovered, as he has indicated. Leaving that for the future, let me conclude with best wishes for the continuation of a long, happy and inspiring career to Alan on his 60th birthday.

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