Completeness and Total Boundedness of the Hausdorff Metric

Jeff Henrikson

Abstract. This paper defines and discusses the Hausdorff metric on the space of nonempty, closed, and bounded subsets of a given metric space. We consider two important topological properties, completeness and total boundedness. We prove that each of these properties is posessed by a Hausdorff metric space if the property is possesed by the underlying metric space. Finally, we explore applications of the Hausdorff metric, including fractal geometry.

1. Introduction. The Hausdorff metric is defined on the space of nonempty closed bounded subsets of a metric space. The resulting metric space will be referred to as the "induced Hausdorff metric space," or else simply as the "induced Hausdorff space." Normally the term "Hausdorff space" refers to a space satisfying a certain topological separation axiom. But since this paper refers only to metric spaces, which all satisfy the Hausdorff separation axiom, there will be no ambiguity.

We first explain why nonemptiness, closedness, and boundedness are the right restrictions to choose for sets on which we wish to put a metric. Many of the reasons that follow are based on the idea that we want our set metric to resemble our point metric. For example, nonemptiness isn't profound: the distance between two small blobs (picture the Euclidean plane if you like) seems like it should be approximately equal to the distance between a representative point in each. But what should we answer if we are asked the distance between a small blob and the empty set? Between a large blob and the empty set? Because there is no good answer, we throw out the empty set.

Next, why particularly closed and bounded? And why not compactness instead? To see why closedness is perhaps a good thing, let's consider the two sequences of subsets $a_n = [0, 2 - 1/n]$ and $b_n = [0, 2 + 1/n]$. If we choose our construction carefully, and our given metric space is sufficiently well behaved, then we will be able to take limits in our induced Hausdorff space. So should a_n converge to a set containing the endpoint 2 or a set not containing 2? What about b_n? What about the sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$? To prevent this apparent difficulty, we choose closed sets. This will turn out to be a good idea since it will not only allow us to take limits, it will insure that $d(A, B) = 0$ implies $A = B$, a necessary property of a true metric. (There is something called a *pseudometric* which is a metric without this property, but it turns out we don't need the concept.)

Finally, we consider reasons to choose between boundedness and compactness. One thought is that compactness is a topological property, whereas boundedness is metric-dependent, and maybe we should concentrate on the topological properties of the space. It turns out that we really have no reason to restrict in this way since boundedness is the only property we end up using. It guarantees that the Hausdorff metric gives a finite value. This restriction, for example, allows us to consider building a Hausdorff metric out of the standard bounded metric in the Euclidean plane $d(x, y) = min(1, d_{\text{eucl.}}(x, y))$.

If we choose boundedness instead of compactness, the elements $[a, b] \times \mathbf{R}$ will be elements of the induced Hausdorff space, since all subsets of the euclidean plane under the standard bounded metric are bounded.

Of course if we don't like being forced to use nontopological properties in exchange for the slightly increased generality of closed and boundedness over compactness, we can certainly choose compactness instead. In fact, many interesting restrictions of our space preserve the main result of this paper, that the Hausdorff distance is complete: limits of connected sets yield connected sets, and limits of sets of 1-dimensional Hausdorff measure x converge to sets of 1-dimensional Hausdorff measure x, etc.

In the next section, we define the Hausdorff distance function, and see that the properties above form a sufficient set of conditions for showing that the distance function is a metric. In Section 3, we demonstrate that, if in addition a metric space is complete, then the induced Hausdorff space is complete. We also show that total boundedness is inherited by our similar construction. In Section 4, we apply completeness in a discussion of iterated function systems. And finally, in Section 5, we discuss an idea that attempts to solve an open problem using both completeness and total boundedness.

2. Definitions. We now begin by defining the Hausdorff distance on nonempty sets. We present two definitions, and show that they are equal. Then we show that requiring elements to be closed and bounded is sufficient to show that the definition satisfies the axioms for a metric space.

Given a compact metric space S, we consider the space X of nonempty closed subsets of S:

$$X = \{A \subset S \mid \text{A is nonempty, closed, and bounded}\}.$$

Then the *Hausdorff metric* is defined on pairs of elements in X as follows:

$$d(A, B) = \max\{\sup_{e\in A} m(e, B),\ \sup_{e\in B} m(e, A)\}, \tag{2-1}$$

where $m(e, C)\colon S \times X \to \mathbf{R}$ is given by

$$m(e, C) = \inf_{c\in C} d(e, c).$$

The function m represents the "minimum" distance from a point e in S to a point in C in X (as a subset of S). Note that it is *not* the same as the Hausdorff distance between the single point set $\{e\}$ and C; indeed, we have

$$d(\{e\}, C) = \max\{m(e, C), \sup_{c\in C} d(c, e)\} = \max\{inf_{c\in C} d(e, c), \sup_{c\in C} d(c, e)\} = \sup_{c\in C} d(c, e).$$

From here on in this paper, we will write simply $d(e, C)$ in place of $d(\{e\}, C)$. This will comply with our convention of using lowercase letters for elements in S and uppercase letters for elements in X or other subsets of S.

This definition of the Hausdorff metric, while sometimes useful for symbolic manipulation, has a reformulation which is more visually appealing. Given $A \in X$, let the *ϵ-expansion* of A be the union of all ϵ-open balls around points in A. We denote it by $E_\epsilon(A)$; that is,

$$E_\epsilon(A) = \bigcup_{x\in A} B(x, \epsilon).$$

Then $d(A, B)$ is defined as the "smallest" ϵ that allows the expansion of A to cover B and vice versa:

$$d(A, B) = \inf\{\epsilon > 0 |\ E_\epsilon(A) \supset B \text{ and } E_\epsilon(B) \supset A\}. \tag{2-2}$$

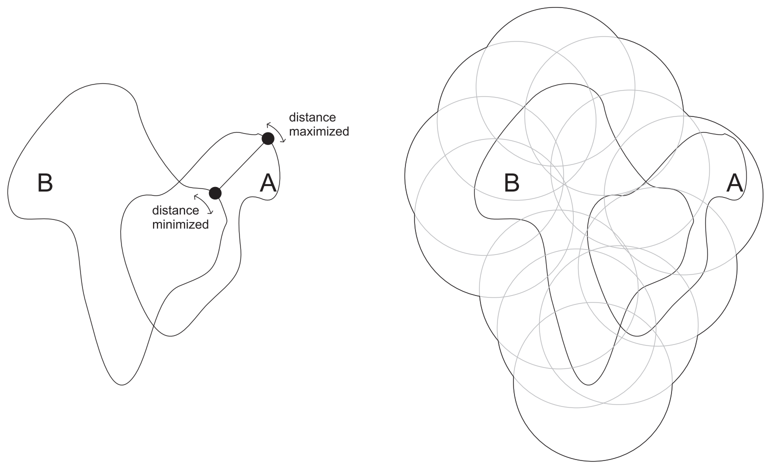

Figure 2-1. A visual depiction of the two definitions. Left: picking the largest $m(a, B)$. Right: picking the smallest ϵ-expansion of B to cover A.

Proposition 2-1. *The above two definitions of the Hausdorff metric are equivalent.*

Proof: We expand Equation (2-2), and reduce it to Equation (2-1). First, Equation (2-2) tells us that the distance from A to B is the larger of two infimums:

$$d(A, B) = \max\bigl\{\inf\{\epsilon > 0 \mid A \subset E_\epsilon(B)\},\ \inf\{\epsilon > 0 \mid B \subset E_\epsilon(A)\}\bigr\}.$$

The condition $A \subset E_\epsilon(B)$ simply means $A \subset \bigcup_{b\in B}\{x|d(x, b) < \epsilon\}$. To ask if a particular set A satisfies that containment is to ask whether or not, for every $a \in A$, we have $b \in B$ less than ϵ away. Equivalently we can ask if, for every a, the infimum of distances to $b \in B$ is small. Substituting for $A \subset E_\epsilon(B)$ and its symmetric counterpart, we get:

$$\begin{aligned}\max\bigl\{\inf\{\epsilon > 0|\ \forall\ a \in A,\ \inf_{b\in B} d(a, b) < \epsilon\}, \inf\{\epsilon > 0|\ \forall\ b \in B,\ \inf_{a\in A} d(a, b) < \epsilon\}\bigr\}\\ = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}\\ = \max\{\sup_{a\in A} m(a, B), \sup_{b\in B} m(b, A)\}.\end{aligned}$$

This is precisely the right-hand side of Equation (2-1). □

Proposition 2-2. *The function d is a metric on X.*

Proof: Recall the axioms for a metric space:

1. $d(A, B) = 0 \Leftrightarrow A = B$ and $d(A, B) > 0$;
2. $d(A, B) = d(B, A)$;
3. $d(A, C) \leq d(A, B) + d(B, C)$.

Proof of 1. We use Equation (2-1). If $A = B$, then $d(A, B) = 0$ because every $a \in A$ satisfies $m(a, B) = 0$. Conversely, if $d(A, B) = 0$, then both terms of the max expression are equal to zero, and thus $m(a, B) = 0$ for every a. Every such point a is a limit point of B since any neighborhood of a must contain a point of b if $m(a, B) = \inf_{b \in B} d(a, b)$ is to be equal to 0. So a is in B because B is by definition closed. Since $a \in A$ was arbitrary, $A \subset B$. By symmetry of our definition, $B \subset A$ also. Thus $B = A$. Also, the value $d(A, B)$ is always nonnegative because the $d(a, b)$ is always nonnegative.

Proof of 2. The max operation is symmetric, so d is symmetric.

Proof of 3. Let A, B, and C be elements of X. Let a be an arbitrariy element of A. There must exist $b \in B$ so that $d(a, b) < d(A, B)$. Given this b we can by the same logic choose some $c \in C$ so that $d(b, c) < d(B, C)$. Adding and applying the triangle inequality in S tells us that $d(a, c) < d(A, B) + d(B, C)$. Hence for every element $a \in A$ there is a $c \in C$ less than $d(A, B) + d(B, C)$ away; that is,

$$E_{d(A,B)+d(B,C)}(C) \supset A.$$

Since the ordering of A and C was arbitrary, we also know

$$E_{d(A,B)+d(B,C)}(A) \supset C.$$

Thus, $d(A, C) \leq d(A, B) + d(B, C)$. □

We now know that d is a metric on X. In the next section, we will give sufficient conditions for X to be compact.

3. Properties. Recall that a space S is called *totally bounded* if, for every $\epsilon > 0$, there exists an open covering of S by finitely many ϵ-balls. We now show that if S is totally bounded, then the induced Hausdorff metric space X is totally bounded.

Theorem 3-1. *If S is totally bounded, then the induced Hausdorff space X is totally bounded.*

Proof: Pick $\epsilon > 0$. Take a finite open cover of S by ϵ-balls. Denote their centers by $s_1, s_2, ..., s_n$. Define

$$C = \{C_i\}_{i=1}^{2^n-1} = \mathcal{P}(\{s_1, s_2, \dots, s_n\}) - \{\emptyset\}.$$

Note that the elements of C are points in X. We now show that the sets $B(C_i, \epsilon)$ form a finite open cover for X. Let A be an element of X; we will show it is in one of the $B(C_i, \epsilon)$. Take the set D of all s_n such that $B(s_n, \epsilon) \bigcap A$ is nonempty. Since A is nonempty and the $B(s_n, \epsilon)$ cover S, the set D is a nonempty subset of $\{s_1, s_2, \dots, s_n\}$. That is, it is one of the C_i. As depicted in Fig. 3-1, clearly

$$E_\epsilon(A) \supset D.$$

And $E_\epsilon(D) \supset A$ by construction. Thus, $d(A, D) < \epsilon$. □

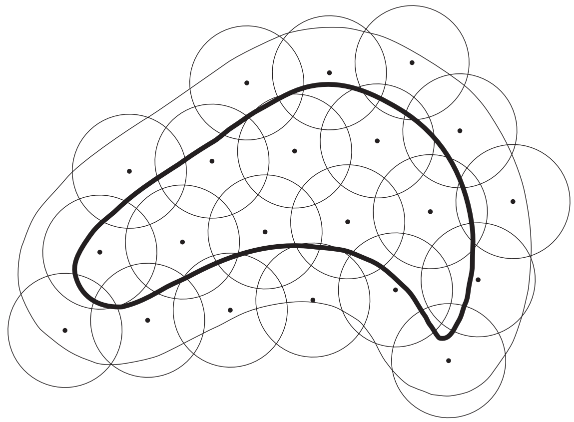

Figure 3-1. Theorem 3-1 is visually obvious. Balls from our finite open cover for S which intersect A are collected. The centers of these balls form the set D. Also depicted are A, in bold, and $E_\epsilon(A)$.

Recall that by a *Cauchy sequence* we mean a sequence of elements $\{a_n\}$ such that, for every $\epsilon > 0$, there is an integer N so that $m, n \geq N$ implies $d(a_m, a_n) < \epsilon$. We say that a space is *complete* when every Cauchy sequence converges to some element in the space. Theorem 3-1 tell us that if our metric space S is totally bounded, then its induced Hausdorff space X is totally bounded. Theorem 3-3 will tell us similarly that X is complete when S is complete. Now, a complete and totally bounded metric space is also compact. (Conversely, a compact metric space is complete and totally bounded, but this fact does not concern us.) Thus, Theorem 3-3 will also tell us that X is compact when S is compact. We now digress to confirm that complete, totally bounded metric spaces are compact.

The compactness criterion we use is the one called *sequential compactness*: every infinite sequence contains a convergent subsequence. It is equivalent to compactness in a metrizable space [**1**, p. 181]. We show only that every infinite sequence in a totally bounded metric space contains a Cauchy sequence, since the addition of completeness will automatically cause Cauchy sequences to converge. Note that unlike the rest of the theorems in this paper, which discuss a space S and its induced Hausdorff space X, this theorem holds in a totally arbitrary metric space.

Theorem 3-2. *If M is an arbitrary totally bounded metric space, then every sequence $\{a_i\}$ in M has a Cauchy subsequence.*

Proof: Cover M with finitely many balls $\{B_k\}$ of radius 1. Let n be the number of elements in $\{B_k\}$. Let $\phi\colon \mathbf{Z}^+ \to \{1, \ldots, n\}$ be given by

$$\phi(i) = \text{ the smallest } k \text{ so that } a_i \in B_k.$$

There is nothing important about k being the smallest such k. It is just a way to avoid ambiguity. Since ϕ maps from an infinite set into a finite set, at least one of the elements of the finite set must get hit infinitely many times. Find such an element $p \in \{1, \ldots, n\}$ and form a subsequence containing the elements that map to p. Call the subsequence $\{a_i^{(1)}\}$. It has diameter less than 2 since it is contained in a ball of radius 1.

Similarly construct $\{a_i^{(2)}\}$ with radius 1/2 as a subsequence of $\{a_i^{(1)}\}$, then $\{a_i^{(3)}\}$ with radius 1/4 as a subsequence of $\{a_i^{(2)}\}$, and so on. By taking the first elements of each sequence, we form a Cauchy sequence:

$$d_k = a_1^{(k)}.$$

By construction, the subsequence of d_k starting with the nth element will have diameter less than 2^{2-n}. Hence d_k is Cauchy since, for every $\epsilon > 0$, if $N = \max(0, -log_2(\epsilon)) + 2$, then

$$m, n > N \Longrightarrow d(d_m, d_n) < \epsilon.$$

The proof is now complete. □

We now return to talking about Hausdorff metric spaces, and prove our final result.

Theorem 3-3. *If S is complete, then its induced Hausdorff metric space X is complete.*

Proof: Let D_k be a Cauchy sequence in X. We need to show that D_k converges to some element in X.

Let the set F be the set of limit points of sequences $\{d_k\}$ with $d_k \in D_k$. We show that F is the limit point of the sequence D_k. To show that $d(F, D_k) < 2\epsilon$ for large enough k, we need to show two inclusions:

$$F \subset E_{2\epsilon}(D_k) \text{ and } E_{2\epsilon}(F) \supset D_k.$$

Pick $\epsilon > 0$. First we show that there aren't too many elements in F. Take N so that $m, n \geq N$ implies $d(D_m, D_n) < \epsilon$. Since $E_\epsilon(D_N) \supset D_n$, every Cauchy sequence d_k selected must have points in $E_\epsilon(D_N)$ for large enough k. So $\lim d_k$ must be inside $\overline{E_\epsilon(D_N)}$. Therefore $F \subset E_{2\epsilon}(D_N)$.

We now show that F has enough elements. Take $\epsilon > 0$ as before. Pick N_i strictly increasing so that $m, n \geq N_i$ implies $d(D_m, D_n) < \epsilon/2^i$. Our strategy will be to show that, for every x in some D_k with $k \geq N_1$, there is a sequence $\{f_i\}$ with $f_i \in D_i$ converging less than ϵ away. Specifically, we will construct f_i such that, for every $p \geq 2$ and $N_p \leq i, j \leq N_{p+1}$, we will have $d(f_i, f_j) < \epsilon/2^p$ and $d(x, f_{N_2}) < \epsilon/2$. Then for every $j \geq N_2$ we will have $q \geq 2$ so that $N_q \leq j \leq N_{q+1}$, and

$$\begin{aligned} d(x, f_j) &\leq d(x, f_{N_2}) + d(f_{N_2}, f_{N_3}) + \cdots + d(f_{N_q}, f_j) \\ &< \epsilon/2 + \epsilon/4 + \cdots + \epsilon/2^q < \epsilon. \end{aligned}$$

Therefore, our f_i will converge to a point less than 2ϵ away.

We will repeatedly use the following simple observation.

Claim 3-4. *For every pair $<x, k>$ with $x \in D_k$ and $k \geq N_i$, there is a nearby $y_j \in D_j$, less than $\epsilon/2^i$ away, so long as $j \geq k$.*

To prove the claim, recall that the N_i were constructed so that

$$d(D_j, D_k) < \epsilon/2^i \text{ for } j \geq k \geq N_i.$$

So we know that

$$E_{\epsilon/2^i}(D_j) \supset D_k \text{ for } j \geq k.$$

In particular, we know that, for every $j \geq k$, there is a $y_j \in D_j$ so that $B(y, \epsilon/2^i) \ni x$. This statement is the claim above.

To construct f_i, let the particular element f_k equal x. Also, pick the particular element $f_{N_2} \in D_{N_2}$ with $d(x, f_{N_2}) < \epsilon/2$ by applying Claim 3-4 to $<x, k>$. All other elements f_j with $j < N_2$ can be picked arbitrarily and not effect either the convergence of the sequence or our bound on its limit point.

Assume by induction on m that f_i is defined when $i \leq N_m$, and assume that, for $2 \leq p < m$ and $N_p \leq i, j \leq N_{p+1}$, we have $d(f_i, f_j) < \epsilon/2^p$. Then Claim 3-4 applied to $< f_{N_m}, N_m >$ tells us that, for every $j \geq N_m$, there is a $y_j \in D_j$ closer than $\epsilon/2^m$ away from f_{N_m}. Define f_i for $N_m < i \leq N_{m+1}$ to be these y_i. Since for $N_m \leq i, j \leq N_{m+1}$ we have $d(f_i, f_j) < \epsilon/2^m$, we have satisfied our induction hypothesis.

Thus by constructing a limit point less than 2ϵ away from an arbitrary $x \in D_k$ with $k \geq N_1$, we have shown that $E_{2\epsilon}(F) \supset D_k$ for $k \geq N_1$. We knew from before that $F \subset E_{2\epsilon}(D_k)$ for $k \geq N_1$, so we also know $d(F, D_k) < 2\epsilon$ for $k \geq N_1$. Since every $\epsilon > 0$ has such an N_1, we know F is our desired limit point of the sequence D_k. □

4. Applications of Completeness. Of the completeness and total boundedness properties, completeness is by far the more useful in applications of the Hausdorff metric space. For example, in fractal geometry, the Hausdorff metric space is frequently used to represent geometric entities. Closed bounded sets in $\mathbf{R}^2$, for example, yield enough variety to describe fractals, yet are restricted enough to allow the construction of the well-behaved Hausdorff metric. Its associated completeness property even lets us take limits of sequences in this space. To elaborate slightly, a typical construction is the "iterated function system," or IFS.

To allow definition of the iterated function system, consider a transformation w sending a metric space S into itself. Recall that $e \in \mathbf{R}$ is called a *contractivity factor* of w when $d(w(a), w(b)) < e \cdot d(a, b)$ for every $a, b \in S$. In other words, a contractivity factor is a bound on the expansion of distance that can be observed by application of w. We say w is a *contraction mapping* when it has a contractivity factor less than 1.

An *iterated function system* (S, w_n), then, is a complete metric space S along with a finite set of contraction mappings $w_n : S \longrightarrow S$ with respective contractivity factors e_n. If we take the Hausdorff metric space X induced by S, we define the *transformation* of the IFS to be a map $W : X \to X$ given by

$$W(A) = \bigcup_n w_n(A).$$

It is not difficult to show that $e = \max e_n$ is a contractivity factor for W. A good stepping stone for showing this bound is the inequality:

$$d(A \cup B, C \cup D) \leq \max\{d(A, C), d(B, D)\}.$$

When $e < 1$, it is an immediate consequence of completeness of X that any element $G \in X$ generates a convergent sequence G_i under repeated application of W:

$$G_0 = G,\ G_k = W(G_{k-1}).$$

We call the limit point A of our sequence G_i the "attractor" of the IFS with respect to G. Without much additional trouble, it can be shown that the attractor A is independent of the choice of G. Even a single point set G is enough to generate the same limit set A. (Conveniently, G must be nonempty to be in X.) Since A is independent of the choice of G, it makes sense to speak of the attractor of the IFS without any reference to G. In the Euclidean plane, IFS attractors exhibit a variety of visually interesting fractal shapes even when constructed from simple linear maps. The following four affine transformations completely determine a compact subset of $\mathbf{R}^2$ which resembles a fern, rendered in Figure 4-1:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ .16 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .2 & -.26 \\ .22 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .23 \\ 1.6 \end{pmatrix};$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -.15 & .28 \\ .24 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .26 \\ .44 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .75 & -.04 \\ .85 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -.04 \\ 1.6 \end{pmatrix}.$$

Figure 4-1. A rendering of the attractor of the linear IFS described by the matrices above.

The algorithm used to render the picture is stochastic, and actually requires four more numbers for producing the image, representing the weighting of each map required to make a visually well-balanced rendering of a fractal set that is not naturally suited to a pixelated grid. In this case, the fourth transform was favored heavily, with a weighting of 74%, while the other three transforms shared equally the remaining 26%. For a description of the algorithm, see Barnsley [**2**, p. 91]. The values for the fern transforms and rendering were taken from [**4**].

5. Applications of Compactness. Compactness and total boundedness are a bit more obscure to apply. Generally speaking, one way to use compactness is to use the maximum value theorem to show existence of solutions in optimization problems. Described in this section is a problem that attempts to find an optimal geometric shape satisfying a certain property. The author briefly conjectured that existence was demonstrable by describing the geometric shapes as elements of the Hausdorff metric space induced by the Euclidean metric on the unit square. Since then, he has thought of a counterexample which seems to indicate that attempting to use the Hausdorff metric in this way cannot yield an existence proof. Following the description of the problem, this reasoning is given.

The problem is the unsolved "opaque square" problem, which is discussed briefly in [**3**, p. 17]. As depicted in Figure 5-1, the opaque square problem is to find a subset of the unit square of minimal measure that intersects every line segment joining two boundary points of the square and crossing the interior. The shape shown is a conjectured minimum. It is shorter than an "X" crossing corners, and shorter than an "H," even with the sides bent in a bit.

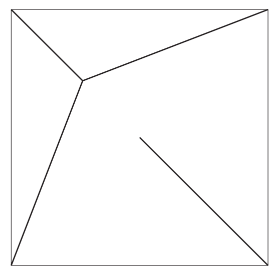

Figure 5-1. The shape strictly inside the square is the conjectured set of minimal length that intersects every line segment adjoining distinct sides of the square.

If we required our answer to be a closed and bounded subset of the unit square, it is conceivable that we would still have a meaningful question. And if we made this requirement, then we could topologize possible answers using the Hausdorff metric. Since the unit square is compact, its induced Hausdorff metric space is compact. The "opaque" sets would form a closed subset of this space, and thus would be compact. The maximum value theorem tells us that every continuous real function on a compact space has a minimum. So existence boils down to finding a continuous measure function on this space.

Many options are available for measure functions. One natural choice is the 1-dimensional Hausdorff measure, which in our case essentially counts the order of growth of the number of ϵ balls required to cover a certain set as ϵ goes to 0. The general definition can be found in [**2**, p. 200]. For the purposes of this discussion, it suffices to assume that we can pick a measure whose value on a line segment inside the unit square is the length of the line segment. We will for the moment call such a measure function *reasonable*.

Unfortunately, when we try to apply the maximum value theorem to the Hausdorff metric space with respect to a measure, we discover that any reasonable measure function is discontinuous with respect to the topology generated by the Hausdorff metric. That is, we cannot make a Hausdorff neighborhood around $A \in X$ small enough to keep the measure of points in this neighborhood within a certain interval. This is because, for any ϵ expansion of $A \in X$, we cannot prevent a dense scribble of arbitrary length inside the expansion from being added onto the side of A. See Figure 5-2.

To handle this difficulty, we might attempt to find a subspace of X that discards "pathological" shapes like the one depicted in Figure 5-2, while still preserving completeness. As was noted before, the Hausdorff limiting process is well behaved under some types of restrictions, since completeness still holds if we restrict X to contain only elements that are connected in S. Likewise, completeness holds if we restrict to elements of constant 1-dimensional Hausdorff measure.

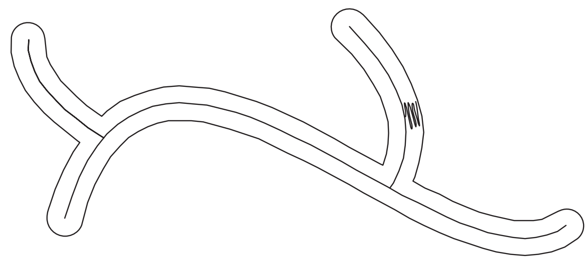

Figure 5-2. A depiction of the reason why measure functions are discontinuous with respect to the Hausdorff metric. Keeping a set within a certain ϵ expansion is not restrictive enough to put bounds on measure. A small shape of large 1-dimensional measure can be fit inside any epsilon expansion without changing the Hausdorff distance to another set by much.

There seem to be a variety of ways to impose "niceness" on shapes; for example we might require that a good approximation of a 1-dimensional shape in X be recoverable from its ϵ expansion for small enough ϵ. So we might surmise that we should think of various "niceness" conditions and check their completeness. A successful restriction might tell us what sort of set we were looking for to satisfy our opaque square problem. We can, after all, imagine that there might be closed bounded opaque sets that seem quite unsatisfactory as solutions to the opaque square problem. For example, if we found a pathological example of an opaque, closed set that was also totally disconnected and had zero measure, would we find that acceptable?

One necessary quality of a "niceness" property can be observed from the definition of continuity if we decide to use the 1-dimensional Hausdorff measure. To show that two sets are close in Hausdorff measure, we need, as ϵ goes to zero, a relative bound on the number of ϵ balls required to cover each of the two sets. We need to deduce such a bound from the Hausdorff distance between the two sets being small.

To spend time creating exotic "niceness" properties and checking their completeness, however, would be a mistake of optimism. The disappointment of our approach comes when considering extremely restrictive niceness properties. For example, since the conjectured minimal is the union of four line segments, it seems suspicious that we should reject any finite union of line segments. Indeed, if we accept only unions of N or fewer line segments for some constant N, we may as well throw away the Hausdorff metric completely and topologize via parameterization by $\mathbf{R}^{4N}$. (Which by the way gives us a trivial existence proof for optimality among line segment unions of bounded order.) But if we include all finite unions of line segments in our subspace of X, we already have enough elements to make any "reasonable" measure function discontinuous. Below is a counterexample demonstrating this fact. It is a convergent sequence of sets, each comprised of finitely many disjoint line segments, having the property that while each element has reasonable measure $1/2$, the limit has reasonable measure 1.

Define the subset of the real line I_n to be n equally spaced points between 0 and 1, excluding 0, but including 1:

$$I_n = \{1/n, 2/n, 3/n, \ldots, 1\}.$$

Consider the sequence $D_k = I_k \times [0, 1/2k]$. Clearly each element is closed and bounded, and must have "reasonable" measure 1/2, since each is the union of k disjoint line segments of length $1/2k$. But it is also easy to see that the limit of D_k in X is the line segment $[0, 1] \times 0$, which must have "reasonable" measure 1. We conclude that any "reasonable" measure defined on a subspace of X that includes all finite unions of line segments is necessarily discontinuous. And the set of all finite unions of line segments is not even closed.

Of course, a second and more basic problem of our approach is this: Although we might find a class of sets that had an optimum, it is not at all clear why it would directly yield any information about the shape of a minimal set. It seems very difficult even to come up with a method for approximating a particular minimal set since, with no notion of derivative or incremental displacement, it is unclear how to improve an estimate.

References

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