# A Depth Function and a Scale Curve Based on Spatial Quantiles

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**Abstract.** Spatial quantiles, based on the $L_1$ norm in a certain sense, provide an appealing vector extension of univariate quantiles and generate a useful "volume" functional based on spatial "central regions" of increasing size. A plot of this functional as a "spatial scale curve" provides a convenient two-dimensional characterization of the spread of a multivariate distribution of any dimension. We discuss this curve and establish weak convergence of the empirical version. As a tool, we introduce and study a new statistical depth function which is naturally associated with spatial quantiles. Other depth functions that generate $L_1$-based multivariate quantiles are also noted.

## 1. Introduction

An effective way of working with probability distributions, especially when they are unspecified as in exploratory and nonparametric inference, is through "descriptive measures" that characterize features of particular interest. In the univariate case, based on the natural order of the real line, *quantiles* provide a popular approach. These have no definitive multivariate generalization, however, and a variety of *ad hoc* notions of multivariate quantiles have been formulated (see Serfling, 2002b, for a partial review). Here we focus on a particular form of multivariate quantiles, the *spatial quantiles*, introduced by Chaudhuri (1996) and Koltchinskii (1997) as a certain form of generalization of the univariate case based on the $L_1$ norm, and on a particular descriptive measure, *spread*. In particular, we treat the spatial "volume functional", whose values are noted by Chaudhuri (1996) to provide multivariate analogues of such univariate measures as interquantile ranges.

In the present development we consider the volume functional as a *spatial scale curve* that provides a convenient two-dimensional characterization of the spread of a multivariate distribution of any dimension. This is a "spatial" analogue of the scale curve introduced in the context of central regions based on

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statistical depth functions by Liu, Parelius and Singh (1999), who emphasize the appeal and importance of visualizing features of multivariate distributions by *one-dimensional curves*: "the very simplicity of such objects ... makes them powerful as a general tool for the practicing statistician". Thus, to provide an additional tool in the growing body of practical methods based on spatial quantiles, we explore the basic features of the spatial scale curve and derive weak convergence of the empirical version. We note that the basic idea of a depth-based scale curve for multivariate analysis was first mentioned in Liu (1990, p. 408).

The spatial quantiles are introduced in Section 2, along with a review of their attractive properties and applications. In terms of these quantiles, spatial "central regions" and the corresponding volume functional and scale curve are discussed.

Derived as the solution to an $L_1$ optimization problem, the spatial quantiles are formulated quite differently, for example, from those notions of multivariate quantiles which are defined in terms of *statistical depth functions*, as the boundary points of depth-based central regions of specified probability. Indeed, the spatial quantiles do not correspond in this sense to any of the many depth functions that have been proposed and studied thus far in the literature. In Section 3, however, we introduce and study a natural "spatial depth function", with respect to which the spatial quantiles are indeed "depth-based". We compare with the well-known halfspace depth in two examples, illustratrating a difference in how these two depth functions measure "centrality" and showing that the spatial depth is somewhat smoother than the halfspace depth. We also take note of several other "$L_1$-based" depth functions, whose corresponding multivariate quantiles differ from the "spatial" version.

The depth-based representation provides a new perspective on the use of spatial quantiles. Also, it is utilized to technical advantage in Section 4 in obtaining weak convergence of the associated "empirical spatial scale curve process" to a (rescaled) Brownian bridge. Such a result provides, for example, the foundation for placing confidence bands around an empirical spatial scale curve.

## 2. Spatial quantiles and a related volume functional

### 2.1. Spatial quantiles

For univariate $Z$ with $E|Z| < \infty$, and for $0 < p < 1$, the $L_1$-based definition of univariate quantiles characterizes the $p$th quantile as any value $\theta$ minimizing

$$E\{|Z-\theta| + (2p-1)(Z-\theta)\} \tag{1}$$

(Ferguson, 1967, p. 51). To extend to $\mathbb{R}^d$, Chaudhuri (1996) first rewrites (1) as

$$E\{|Z-\theta| + u(Z-\theta)\}, \tag{2}$$

where $u = 2p - 1$, thus re-indexing the univariate $p$th quantiles for $p \in (0,1)$ by $u$ in the open interval $(-1,1)$, and then formulates $d$-dimensional "spatial" or "geometric" quantiles by extending this index set to the open unit ball $\mathbb{B}^{d-1}(0)$ and minimizing a generalized form of (2),

$$E\{\Phi(u, X-\theta) - \Phi(u, X)\}, \tag{3}$$

where $X$ and $\theta$ are $\mathbb{R}^d$-valued and $\Phi(u,t) = \|t\| + \langle u, t\rangle$ with $\|\cdot\|$ the usual Euclidean norm and $\langle\cdot,\cdot\rangle$ the usual Euclidean inner product. (Subtraction of $\Phi(u, X)$ in (3) eliminates the need of a moment assumption.) This yields, corresponding to the underlying distribution function $F$ for $X$ on $\mathbb{R}^d$, and for $u \in \mathbb{B}^{d-1}(0)$, a "$u$th quantile" $Q_F(u)$ having both direction and magnitude. In particular, the well-known spatial median of $F$ is $Q_F(0)$, which we shall also denote by $M_F$. Unlike many notions of multivariate quantiles, it is relatively straightforward to extend spatial quantiles to the setting of Banach spaces, as discussed in Kemperman (1987) for the spatial median and by Chaudhuri (1996) for the general case.

The quantile $Q_F(u)$ always exists for any $u$, and it is unique if $d \geq 2$ and $F$ is not supported on a straight line (see Chaudhuri, 1996). Moreover, the spatial quantile function characterizes the associated distribution, in the sense that $Q_F = Q_G$ implies $F = G$ (see Koltchinskii, 1997, Cor. 2.9).

For each $u \in \mathbb{B}^{d-1}(0)$, the quantile $Q_F(u)$ may be represented as the solution $x = x_u$ of

$$-E\left\{\frac{X-x}{\|X-x\|}\right\} = u. \tag{4}$$

It follows that we may attach to each point $x$ in $\mathbb{R}^d$ a spatial quantile interpretation: namely, it is that spatial quantile $Q_F(u_x)$ indexed by the average unit vector $u_x$ pointing to $x$ from a random point having distribution $F$. Since $u_x$ is uniquely determined by (4) and satisfies $x = Q_F(u_x)$, we interpret $u_x$ as the inverse at $x$ of the spatial quantile function $Q_F$ and denote it by $Q_F^{-1}(x)$. When the solution $x$ of (4) is not unique, as illustrated for the univariate case in Section 2.3 below, multiple points $x$ can have a common value of $Q_F^{-1}(x)$.

Another inference from (4) is that "central" and "extreme" quantiles $Q_F(u)$ correspond to $\|u\|$ being close to 0 and 1, respectively. Thus we may think of the quantiles $Q_F(u)$ as indexed by a directional "outlyingness" parameter $u$ whose magnitude measures outlyingness quantitatively, and, accordingly, we measure the outlyingness of any point $x$ quantitatively by $\|Q_F^{-1}(x)\|$.

Also, a nice structural property for the spatial quantile function follows easily from (4) (or see Koltchinskii, 1997, p. 448). For $F$ *centrally symmetric* about $M_F$, that is, for $X - M_F$ and $M_F - X$ identically distributed, the corresponding median-centered spatial quantile function $Q_F$ is *skew-symmetric*:

$$Q_F(-u) - M_F = -(Q_F(u) - M_F),\ u \in \mathbb{B}^{d-1}(0). \tag{5}$$

Computation of the sample spatial quantile function $Q_n(u)$ for a data set $X_1, \dots, X_n$ via

$$-\sum_{i=1}^{n} \frac{X_i - x}{\|X_i - x\|} = u \tag{6}$$

is quite straightforward (Chaudhuri, 1996), whereas many depth-based notions of multivariate quantiles are computationally intensive. We note from (6) a *robustness* property of $Q_n(u)$: its value remains unchanged if the points $X_i$ are moved outward along the rays joining them with $Q_n(u)$. Moreover, it has favorable breakdown point (50% for the median – see Kemperman, 1987, and Lopuhaä and Rousseeuw, 1991) and bounded influence function (Koltchinskii, 1997, p. 459).

Also, spatial quantiles support a variety of useful methodological techniques. For example, the extension of the univariate *regression quantiles* of Koenker and Bassett (1978) to *multiresponse regresssion* is discussed in Chaudhuri (1996) and Koltchinskii (1997). Marden (1998) illustrates the use of *bivariate QQ-plots* based on spatial quantiles, along with some related devices, and similar methods based on a modified type of sample spatial quantile are developed in Chakraborty (2001). Notions of *multivariate ranks* may be based on spatial quantiles — see Jan and Randles (1994), Möttönen and Oja (1995), Chaudhuri (1996), Choi and Marden (1997), and Möttönen, Oja and Tienari (1997). And nonparametric multivariate *descriptive measures* based on spatial quantiles are treated in Serfling (2002c), with spread further treated in the present paper.

The primary weakness of spatial quantiles is lack of full affine equivariance. By (4), they are equivariant with respect to *shift*, *orthogonal*, and *homogeneous scale* transformations, and thus the outlyingness measure $\|Q_F^{-1}(x)\|$ associated with any fixed $x$ is invariant under such transformations. (See Serfling, 2002c, for detailed discussion.) In terms of a data cloud in $\mathbb{R}^d$, the sample spatial quantile function thus changes equivariantly if the cloud of observations is translated, homogeneously rescaled, rotated about the origin, and/or reflected about a $(d-1)$-dimensional hyperplane through the origin. For applications with coordinates measured in a common unit, such equivariance is quite sufficient. Full affine equivariance fails only if the coordinate variables are subject to *heterogeneous* scale transformations, a matter of possible concern only in applications with coordinates having differing measurement scales. The importance of this depends perhaps on the situation. Chakraborty (2001, p. 391) takes the point of view, for example, that outlyingness measures of data points which are potential “outliers” should not depend on the choice of coordinate system. On the other hand, Marden (1998) comments that in some cases it may be satisfactory to transform variables to have similar scales at the outset of data analysis. Likewise, as pointed out by Van Keilegom and Hettmansperger (2002), when the variables of interest have a special physical interpretation, there is no interest in affinely transforming them.

**2.2. Central regions and a volume functional**

Corresponding to the spatial quantile function $Q_F$, we call

$$C_F(r) = \{Q_F(u) : \|u\| \leq r\}$$

the $r$th *central region* and define the (real-valued) *volume functional* by

$$v_F(r) = \text{volume}(C_F(r)), \; 0 \leq r < 1.$$

When $F$ is centrally symmetric, the skew-symmetry of $Q_F - M_F$ given by (5) yields that the regions $C_F(r)$ have the nice property of being symmetric sets, in the sense that for each point $x$ in $C_F(r)$ its reflection about $M_F$ is also in $C_F(r)$. As an increasing function of $r$, $v_F(r)$ characterizes the spread of $F$ in terms of expansion of the central regions $C_F(r)$. For each $r$, $v_F(r)$ is invariant under shift and orthogonal transformations, and $v_F(r)^{1/d}$ is equivariant under homogeneous scale transformations.

Analogous to the scale curve introduced by Liu, Parelius and Singh (1999) in connection with *depth-based* central regions indexed by their *probability weight*, the spatial volume functional may likewise be plotted as a "scale curve" over $0 \leq r < 1$, thus providing a convenient two-dimensional device for the viewing or comparing of multivariate distributions of any dimension. Besides its role as a scale curve, the volume functional may be used in other ways. For example, we may compare two multivariate distributions $F$ and $G$ via the graph of $v_G \, v_F^{-1}$. This generalizes the "spread-spread plot" introduced for the univariate case in Balanda and MacGillivray (1990). Also, besides its intrinsic appeal for measuring spread, the volume functional plays key roles in defining spatial skewness and kurtosis measures in Serfling (2002c).

Since the central regions are ordered and increase with respect to the spatial "outlyingness" parameter $r$ that describes their boundaries, i.e., $r < r'$ implies $C_F(r) \subset C_F(r')$, their probability weights $p$ increase with $r$. Consequently, the central regions and associated volume functional and scale curve can equivalently be indexed by the probability weight of the central region. This relationship may be described by a mapping $\psi_F : r \mapsto p_r \in [0, 1)$, with inverse $\psi_F^{-1} : p \mapsto r_p$ (thus $p_r = \psi_F(r)$ and $r_p = \psi_F^{-1}(p)$), although characterization of this mapping is complicated.

The empirical depth-based scale curve of Liu, Parelius and Singh (1999) is shown in Serfling (2002a) to converge weakly to a Brownian bridge. In Section 4 below, a similar result is established for the spatial version.

An alternative notion of spatial dispersion function is developed by Avérous and Meste (1997), who extend the univariate interquantile intervals to multivariate "median balls" indexed by their radii, as a family of "central regions" which provide *optimal* summaries in a certain $L_1$ sense. Under regularity conditions on $F$, the probability weight of a median ball is a nondecreasing function of its radius, even in cases when the balls are not ordered by inclusion. This yields a "median balls" analogue of the scale curve described above, which, however, remains open for investigation.

### 2.3. A simple illustration

To illustrate the above definitions in familiar terms, note that for $d = 1$ and univariate $F$, we have $\mathbb{B}^0(0) = (-1, +1)$, $\mathbb{S}^0_r(0) = \{-r, r\}$, $M_F = F^{-1}(\frac{1}{2})$, and $Q_F(u) = F^{-1}(\frac{1}{2} + \frac{u}{2})$, $-1 < u < 1$. It is readily seen that $Q_F^{-1}(x) = 2F(x) - 1$, and, accordingly, $|2F(x) - 1|$ serves as a measure of the outlyingness of $x$ relative to the distribution $F$ on $\mathbb{R}$. Note that if $F$ is constant over an interval $[x_1, x_2]$, then $F(x)$ and thus also $Q_F^{-1}(x)$ are constant over this interval. The central regions $C_F(r)$ consist of (nested) "interquantile intervals" whose widths form the volume functional

$$v_F(r) = F^{-1}(\tfrac{1}{2} + \tfrac{r}{2}) - F^{-1}(\tfrac{1}{2} - \tfrac{r}{2}),\ 0 \le r < 1, \tag{7}$$

which increases with $r$ and shrinks to $M_F$ as $r \to 0$. This is recognized to be a classical nonparametric spread measure arising in many treatments of skewness and kurtosis in the univariate case (see, for example, Avérous and Meste, 1990, and Balanda and MacGillivray, 1990).

## 3. A "spatial" depth function

### 3.1. Depth functions

By "depth function" we mean a nonnegative real-valued function $D(x, F)$ adopted for the purpose of providing an $F$-based *center-outward ordering* of points $x$ in $\mathbb{R}^d$. A data set with empirical distribution $F_n$ may thus be ordered using $D(x, F_n)$. The role of "center" is played by the point(s) of maximal depth, and the depth-induced "contours" are interpreted as multivariate analogues of univariate rank and order statistics. The smallest inner region $\{x \in \mathbb{R}^d : D(x, F) \ge \alpha\}$ having probability $\ge p$ is called the "$p$th central region".

The use of such depth functions was initiated by Tukey (1975) with the *halfspace depth*, defined for each $x$ in $\mathbb{R}^d$ as the minimum probability mass carried by any closed halfspace containing $x$:

$$D_h(x, F) = \inf\{P_F(H) : H \text{ a closed halfspace},\ x \in H\},\ x \in \mathbb{R}^d.$$

Tukey pointed out the appeal of "beginning in the middle" for some purposes of data summarization. Clearly, lower halfspace depth is associated with greater outlyingness. For $d = 1$, we have $D_h(x, F) = \min\{F(x), 1 - F(x-)\}$, $x \in \mathbb{R}$. Donoho (1982) and Donoho and Gasko (1992) provided detailed development of the halfspace depth, which remains a leading example and continues to receive extensive investigation. Liu (1990) introduced an important new variety of depth function, the "simplicial depth", and emphasized the general role of a depth function as providing a center-outward ranking of data points. Additional depth functions have been introduced by Liu and Singh (1993), Liu, Parelius and Singh (1999), Zuo and Serfling (2000a,b) and others, and these references along with Zuo and Serfling (2000c,d) and Serfling (2002a) may be consulted for detailed background and further references.

### 3.2. A "spatial" notion of depth

Corresponding to the spatial quantile function $Q_F$ associated with $F$ on $\mathbb{R}^d$, and recalling the interpretation of $\|Q_F^{-1}(x)\|$ as a measure of outlyingness, a natural notion of "spatial depth" is thus given by taking

$$D_s(x, F) = 1 - \|Q_F^{-1}(x)\|.$$

Clearly, from our previous discussion, this depth function is invariant under shift, orthogonal and homogeneous scale transformations. The point of maximal depth is the spatial median, and decreasing depth corresponds to increasing outlyingness.

An important aspect of any depth function is whether its sample version converges to the population counterpart, as discussed in Zuo and Serfling (2000a, Remark A.3) and Zuo and Serfling (2000b, Appendix B), along with results for several depth functions. In particular, for the halfspace depth this convergence is given in Donoho and Gasko (1992). For the spatial depth function, the desired convergence follows from Theorem 5.5 of Koltchinskii (1997). Specifically, for any bounded set $S$ in $\mathbb{R}^d$, we obtain $\sup_S |D_s(x, F_n) - D_s(x, F)| \to 0$ a.s. In turn, by Theorem 4.1 of Zuo and Serfling (2000b), this yields almost sure convergence of the central regions, $C_{F_n}(r) \to C_F(r)$ a.s., provided that the boundary $\{x : \|Q_F^{-1}(x)\| = r\}$ of $C_F(r)$ has $F$-probability 0.

In the one-dimensional case the spatial depth is equivalent to the halfspace depth, as seen from

$$D_s(x, F) = 1 - |2F(x) - 1| = 2 \min\{F(x), 1 - F(x)\}.$$

In higher dimension, however, such an equivalence does not hold, these two depth functions differing in how they measure "centrality". This is illustrated in Table 3.1, which compares the *sample* halfspace and spatial depths for a (contrived) bivariate data set of size $n = 12$.

Further, in higher dimension, the spatial and halfspace depths differ with respect to smoothness. From comparison of the respective contours in Figure 3.1 in the case of $F$ the uniform distribution on the unit square, it is evident that surface and contours of the spatial depth are somewhat smoother than those of the halfspace depth. The halfspace and spatial depth functions used for this $F$ are straightforward to obtain. For the spatial depth we have

$$D_s((x, y), F) = 1 - \sqrt{a^2(x, y) + a^2(y, x)},$$

where

$$a(x, y) = \int_0^1 \left(\sqrt{(1 - x)^2 + (z - y)^2} - \sqrt{x^2 + (z - y)^2}\ \right) dz.$$

For the halfspace depth we have

$$D_h(x, y) = 2\ \min(x, 1 - x)\ \min(y, 1 - y),$$

as given by Rousseeuw and Ruts (1999).

Table 3.1. Halfspace and spatial depths, $D_{h,n}(x)$ and $D_{s,n}(x)$, for a bivariate data set.

| $x$ | $D_{s,n}(x)$ | $D_{h,n}(x)$ |
|---|---|---|
| $(0, 1)$ | 0.7480 | 4/12 |
| $(0, -1)$ | 0.7480 | 4/12 |
| $(-1, 0)$ | 0.7270 | 2/12 |
| $(1, 0)$ | 0.7270 | 2/12 |
| $(0, 3)$ | 0.4940 | 3/12 |
| $(1.7, 0)$ | 0.4923 | 1/12 |
| $(0, -3.1)$ | 0.4921 | 3/12 |
| $(0, 15)$ | 0.2640 | 2/12 |
| $(0, -15)$ | 0.2640 | 2/12 |
| $(-10, 0)$ | 0.2578 | 1/12 |
| $(0, 20)$ | 0.0921 | 1/12 |
| $(0, -20)$ | 0.0921 | 1/12 |

For the data set of Table 3.1, the spatial median is $(0, 0)$, while the halfspace "median" is the *line* joining $(0, 1)$ and $(0, -1)$. Both depth functions agree that $(0, 1)$ and $(0, -1)$ are the most central, and $(0, 20)$ and $(0, -20)$ the most outlying, points. Otherwise they differ somewhat. The halfspace depth selects $(0, 3)$ and $(0, -3.1)$ as the two next most central points (forming a configuration of *four points on a line* as the "middle third" of the data set), whereas the spatial depth selects $(-1, 0)$ and $(1, 0)$ (forming a *square* as the "middle third"). Neither depth function is strictly compatible with the Euclidean distance, of course. More or less the same findings are obtained if the data points are slightly perturbed so that no three lie on the same line.

In Figure 3.1 we see that for $F$ uniform on the unit square the contours of the spatial depth approximate circles, while those of the halfspace depth approximate squares which in the middle are rotated 90° relative to the boundary square. Since any depth function is equivalent to itself multiplied by any constant, we do not force agreement of vertical scales in these 3D plots. Rather, we let the vertical scales be chosen to accommodate ease of viewing. Note that in fact the maximum depth has value 1 for the spatial depth and $\frac{1}{2}$ for the halfspace depth.

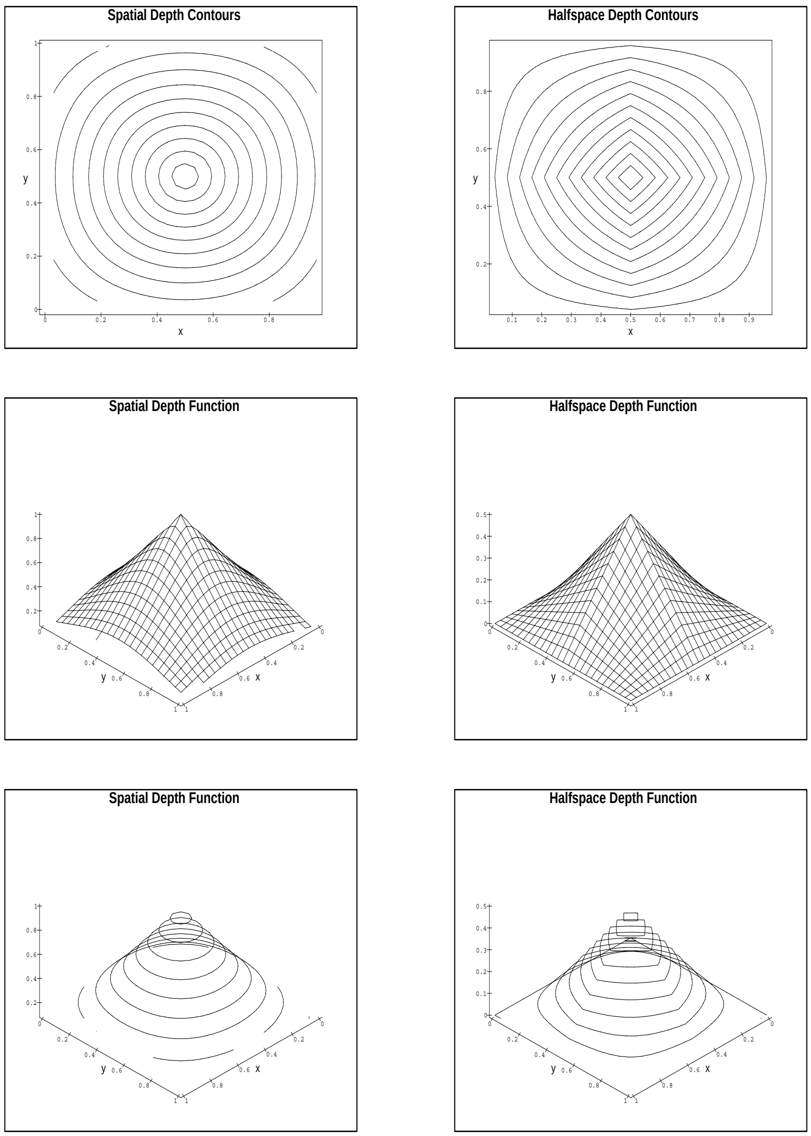


FIGURE 3.1. Views of Spatial and Halfspace Depth Functions, for $F$ Uniform on Unit Square.

### 3.3. Other quantile functions based on the $L_1$ norm

Besides the spatial quantiles, there are other versions of $L_1$-based multivariate quantiles. For example, Abdous and Theorodescu (1992) generalize (1) in a way quite different from Chaudhuri (1996). See Serfling (2002b) for comparison of the two approaches. Further, certain depth functions are based on the $L_1$ norm, and these yield corresponding quantile functions. Examples are

$$D_1(x, F) = \frac{1}{(1 + E_F\|x - X\|)},$$

$$D_2(x, F) = \frac{1}{(1 + E\{\|x - X\|_{\Sigma(F)^{-1}}\})},$$

with $\|x\|_M = \sqrt{x' M x}$ for $x \in \mathbb{R}^d$ and $\Sigma(F)$ the usual covariance matrix of $F$, and

$$D_3(x, F) = \frac{1}{(1 + \|x - M_F\|_{\widetilde{\Sigma}(F)^{-1}})},$$

with $M_F$ the spatial median as previously and $\widetilde{\Sigma}(F)$ a suitable version of covariance matrix for $F$. The motivation for $D_2(x, F)$ over $D_1(x, F)$ is its full affine invariance. Interest in $D_3(x, F)$ arises by comparison with the well-known (but *not* $L_1$-based) *Mahalanobis* depth,

$$D_4(x, F) = \frac{1}{(1 + \|x - \mu(F)\|^2_{\Sigma(F)^{-1}})},$$

with $\mu(F)$ the mean of $F$. The depth functions $D_1$, $D_2$ and $D_4$ are studied in Zuo and Serfling (2002a,b), for example, while $D_3$ is newly formulated here and will be investigated elsewhere, along with further study of the spatial depth defined above.

## 4. Asymptotic behavior of sample versions

Here we define an empirical process associated with the spatial volume functional and obtain its weak convergence to a Brownian bridge. Denote the sample version of $v_F$ by $v_n$, let $\tilde{v}_F(p)$ denote the volume functional of the spatial central regions as indexed by their probability content $p$, and let $\tilde{v}_n(p)$ denote the corresponding sample version. Recalling the mappings $\psi_F : r \mapsto p_r$ and $\psi_F^{-1} : p \mapsto r_p$ of Section 2.3, with $p_r = \psi_F(r)$ and $r_p = \psi_F^{-1}(p)$, it follows that the spatial central region having probability $p$ is $C_F(r_p)$ and thus

$$\tilde{v}_F(p) = v_F(r_p) = v_F(\psi_F^{-1}(p)) \tag{8}$$

and, assuming differentiability of $v_F(r)$ (see Condition C1 below) and of $\psi_F(r)$,

$$\tilde{v}'_F(p) = v'_F(\psi_F^{-1}(p))/\psi'_F(\psi_F^{-1}(p)). \tag{9}$$

Similar expressions relate the sample versions. In terms of these quantities, we define the *empirical spatial scale curve process* as

$$\xi_n(r) = \frac{\psi_F'(r)}{v_F'(r)}\, n^{1/2}\,(v_n(r) - v_F(r)),\ 0 < r < 1. \tag{10}$$

The relevance of the scaling factor in (10) derives from reexpression in terms of $\tilde{v}_n$ and $\tilde{v}_F$. Namely, using (8) and (9) and their sample versions with $r = \psi_F^{-1}(p)$, we may rewrite (10) as

$$\tilde{\xi}_n(p) = \frac{1}{\tilde{v}_F'(p)}\, n^{1/2}\,(\tilde{v}_n(p) - \tilde{v}_F(p)),\ 0 < p < 1. \tag{11}$$

This is precisely the empirical *depth-based* scale process in the form considered for general depth functions in Serfling (2002a), where weak convergence to a Brownian bridge is established in a result from which we derive the following lemma. Assume the conditions

**C1** The probability distribution $F$ possesses a density $f(x)$ positive for all $x \in \mathrm{supp}(F)$.

**C2** The volume functional $\tilde{v}_F(p)$ is finite, strictly decreasing and possesses a continuous derivative.

Let "$\stackrel{\mathrm{d}}{\longrightarrow}$" denote weak convergence in $(D[a,b], \mathcal{D}_a^b)$, where $D[a,b]$ is the space of left-continuous functions on $[a,b]$ and $\mathcal{D}_a^b$ is the class of corresponding Borel sets generated by the supremum norm on $[a,b]$. Also, let $B(\cdot)$ denote the usual Brownian bridge process.

**Lemma 4.1**. *Under C1 and C2, for any closed interval* $[a_0, b_0]$ *in* $(0,1)$ *we have*

$$\{\tilde{\xi}_n(p),\ a_0 < p < b_0\} \stackrel{\mathrm{d}}{\longrightarrow} \{B(p),\ a_0 < p < b_0\} \text{ on } (D[a_0,b_0], \mathcal{D}_{a_0}^{b_0}). \tag{12}$$

Proof. The validity of (12) follows under conditions A1–A3 of Theorem 3.1 of Serfling (2002a). Now, A1 and A3 are essentially C1 and C2, respectively. For A2, we need the spatial depth function $D_s(x,F)$ defined in Section 3.2 to satisfy (i) $D_s(x,F)$ is continuous in $x$, vanishes for $x \notin \mathrm{supp}(F)$, and converges to 0 as $\|x\| \to \infty$, and (ii) the set $\{x : D_s(x,F) = \alpha\}$ is nonempty for all $0 < \alpha < 1$. Now (i) follows if $\|Q_F^{-1}(x)\|$ is continuous in $x$, $= 1$ for $x \notin \mathrm{supp}(F)$, and $\to 1$ as $\|x\| \to \infty$. And (ii) is equivalent to nonemptiness of the set $\{x : \|Q_F^{-1}(x)\| = 1 - \alpha\}$ for all $0 < \alpha < 1$. These requirements for the spatial quantile function follow from Proposition 2.6 and Example 2.7 of Koltchinskii (1997), which give strict continuity and strict monotonicity of the "M-distribution" (which here we denote $Q_F^{-1}$) associated with $Q_F$. □

By the relation $\xi_n(r) = \tilde{\xi}_n(p)$ with $p = \psi_F(r)$, the preceding lemma converts to the following result for the empirical spatial scale process as indexed by the "outlyingness" parameter $r$.

**Proposition 4.1**. *Under C1 and C2, for any closed interval* $[a,b]$ *in* $(0,1)$ *we have*

$$\{\xi_n(r),\ a < r < b\} \xrightarrow{\mathrm{d}} \{B(\psi_F(r)),\ a < r < b\} \text{ on } (D[a,b], \mathcal{D}_a^b). \tag{13}$$

Since the mapping $\psi_F$ is not easy to determine, in practice the process $\tilde{\xi}_n(p)$ is more straightforward to use. Of course, practical implementation of (13) requires *studentization*, replacing $\tilde{v}_F'(p)$ in (11) by a suitable estimator based on the sample (see Serfling and Wang, 2002).

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