OPTIMAL REGULARITY RESULTS FOR PARABOLIC NONLINEAR INTEGRAL OPERATORS WITH OBSTACLES

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Abstract. We investigate the optimal regularity of the weak solution of an obstacle problem for a parabolic nonlinear integral equation.

1. Introduction

In this paper, we present our results about the optimal regularity of parabolic nonlocal obstacle problems with measurable kernels. Namely, we consider a parabolic obstacle problem with measurable kernels of the form

$$\begin{aligned} &\min\{w_t - \int [w(y,t) - w(x,t)]K(y-x)dy, w - \varphi\} = 0 \text{ in } \mathbb{R}^n \times [0,T], \\ &w(x,0) = w_0(x) \text{ in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where, $K(z)$ is a symmetric and "bounded measurable" kernel i.e.

$$\frac{\lambda}{|x-y|^{n+2s}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2s}},$$

$\lim_{|x|\to+\infty} w(x,t) = 0$ for every t. Non-local "heat equations" arise in material sciences in cases where there are long range interactions, like in polymers, or stochastic processes with jump (Levi) processes. Obstacle-like problems appear in both cases as an optimization change of strategy (optimal temperature control or American options in financial mathematics). In [1], the problem is analyzed for the weak solution without the obstacle. In [3, 4], some a priori estimates are obtained and in [2] a similar problem is analyzed for the fractional obstacle problem. We use the penalization method so that the solution of the equation (1.2) can be approximated by the sequence of solutions of the following equation for $\beta_\varepsilon(x) = e^{-x/\varepsilon}$:

$$\begin{aligned} &w_t - \beta_\varepsilon(w-\varphi) = \int [w(y,t) - w(x,t)]K(y-x)dy \\ &w(x,0) = w_0(x) \end{aligned} \tag{1.2}$$

Here, for simplifying the notation, solutions are denoted as w instead of w_ε.

At first we will consider $\mathbb{R}^n$ with some initial condition, then we will look at this problem on a subset Ω of $\mathbb{R}^n$. We can write the equation (1.2) as

$$(w-\varphi)_t - \beta_\varepsilon(w-\varphi) = \int [(w-\varphi)(y,t) - (w-\varphi)(x,t)]K(y-x)dy + \int [\varphi(y) - \varphi(x)]K(y-x)dy.$$

Theorem 1.1. *Let $w(x,t)$ be a weak solution of* (1.2) *with an initial data in $\mathcal{L}^2$. Then, w is bounded and Hölder continuous.*

Proof. We will use the test functions $w_\psi = (w-\varphi-\psi)^+$. First, multiply (1.2) with w_ψ, then integrate in x and t

$$\int_{T_1}^{T_2}\int w_\psi (w-\varphi)_t dxdt - \int_{T_1}^{T_2}\int w_\psi \beta_\varepsilon(w-\varphi)dxdt =$$
$$\int_{T_1}^{T_2}\int\int w_\psi[(w-\varphi)(y,t)-(w-\varphi)(x,t)]K(y-x)dydxdt+$$
$$\int_{T_1}^{T_2}\int\int w_\psi[\varphi(y)-\varphi(x)]K(y-x)dydxdt,$$

RHS can be symmetrized(exchange y and x),

$$\int\int w_\psi(x,t)[(w-\varphi)(y,t)-(w-\varphi)(x,t)]K(y-x)dydx+$$
$$\int\int w_\psi(x,t)[\varphi(y)-\varphi(x)]K(y-x)dydx =$$
$$-\frac{1}{2}\int\int [w_\psi(y,t)-w_\psi(x,t)]K(y-x)[(w-\varphi)(y,t)-(w-\varphi)(x,t)]dydx+$$
$$-\frac{1}{2}\int\int [w_\psi(y,t)-w_\psi(x,t)]K(y-x)[\varphi(y)-\varphi(x)]dydx :=$$
$$-\frac{1}{2}[B(w_\psi, w-\varphi)+B(w_\psi,\varphi)]$$

then we get

$$\int (w_\psi^2)_t dx\Big|_{T_1}^{T_2} - \int_{T_1}^{T_2}\int w_\psi\beta_\varepsilon(w-\varphi)dxdt + \int_{T_1}^{T_2}[B(w_\psi,w-\varphi)+B(w_\psi,\varphi)]dt = 0$$

$$\int w_\psi^2(x,T_2)dx - \int_{T_1}^{T_2}\int w_\psi\beta_\varepsilon(w-\varphi)dxdt + \int_{T_1}^{T_2} B(w_\psi,w_\psi)dt =$$
$$\int w_\psi^2(x,T_1)dx + \int_{T_1}^{T_2} B(w_\psi, w_\psi - w)dt$$

Here we have the following energy inequality:

$$\int w_\psi^2(x,T_2)dx + \int_{T_1}^{T_2} B(w_\psi,w_\psi)dt \geq$$
$$\sup_{T_1\leq t\leq T_2}\int w_\psi^2(x,t)dx + \int_{T_1}^{T_2}|w_\psi|^2_{\mathcal{L}^p}dt \geq \int_{T_1}^{T_2}|w_\psi|^q_{\mathcal{L}^q}dt, \text{for some } 2<q<p.$$

Let $\frac{1}{s}+\frac{1}{t}=1$ and $q=\frac{2}{s}+\frac{p}{t}$, then

$$\begin{aligned}\textstyle\int w^q dx = \int w^{\frac{2}{s}+\frac{p}{t}} &\leq (\int w^2 dx)^{\frac{1}{s}}(\int w^p dx)^{\frac{1}{t}}\\ (\textstyle\int w^q dx)^{\frac{2}{q}} &\leq \sup_{T_1\leq t\leq T_2}\|w\|_{\mathcal{L}_2}^{\frac{4}{qs}}(\int w^p dx)^{\frac{2}{qt}}\end{aligned}$$

We will apply the following lemma: □

Lemma 1.2. *Let $w\leq\phi=\max(2,|x|^\varepsilon)$ be a subsolution of* (1.2) *such that $|\{w>\psi_0\}|<\delta$ in $\Gamma_4 = B_4\times[-4,0]$, $\psi_0=\frac{1}{8}|x|^2$, then $w-\varphi<\min(\frac{1}{8}|x|^2+1,\phi)$ in $\mathbb{R}^n\times[-1,0]$.*

Proof. Define $\psi_k = \min(\frac{1}{8}|x|^2 + (1-2^{-k}), \phi)$ and $w_k = (w-\varphi-\psi_k)^+$, then we will show that "w_∞" = "$(w-\varphi-\psi_\infty)^+$" $\equiv 0$ in $\mathbb{R}^n \times [-1,0]$. Let

$$A_k = \int_{T_k}^0 \int w_k^2(x,t)dxdt, \text{with } T_k = -(1+2^{-k}),$$

we will show that $A_k \le C^k(A_{k-1})^{1+\varepsilon}$ that implies $A_k \to 0$, as $k \to \infty$. We have the following inequalities:

$$A_k = \int_{T_k}^0 \int w_k^2(x,t)dxdt \le \left(\int_{T_k}^0 \int w_k^p(x,t)dxdt\right)^{\frac{2}{p}} |\{w_k > 0\}|^{\frac{1}{q^*}}, \text{where } \frac{2}{p} + \frac{1}{q^*} = 1.$$

By the Sobolev Inequality,

$$A_k \le \int_{T_k}^0 B(w_k,w_k)dt\ |\{w_k>0\}|^\varepsilon \le \left\{\int w_k^2(x,0)dx + \int_{T_k}^0 B(w_k,w_k)dt\right\}\ |\{w_k>0\}|^\varepsilon,$$

now, we will find the bounds of the terms of RHS.

Since $\{w_k > 0\} \subseteq \{w_{k-1} > 2^{-k}\}$, we have

$$|\{w_k > 0\}| \le |\{w_{k-1} > 2^{-k}\}| \le 2^{2k} \int_{T_k}^0 \int w_{k-1}^2(x,t)dxdt.$$

$$\int w_k^2(x,0)dx + \int_{T_k}^0 B(w_k,w_k)dt \le \int w_k^2(x,T_k)dx + \int_{T_k}^0 \int w_k \beta_\varepsilon(w-\varphi)dxdt + \int_{T_2}^0 B(w_k, w_k - w)dt.$$

Let us analyze the RHS term by term:

We may replace the first term by $\inf_{T_{k-1}\le t\le T_k} \int w_k^2(x,t)dx$, since that will only increase A_k, then

$$\inf_{T_{k-1}\le t\le T_k} \int w_k^2(x,t)dx \le 2^k \int_{T_{k-1}}^{T_k} \int w_k^2(x,t)dxdt \le 2^k A_{k-1};$$

the second term has a sign which is < 0;

we can write the last term as

$$\int_{T_2}^0 B(w_k, w_k - w)dt = \int_{T_2}^0 B(w_k, (w-\varphi-\psi_k)^-)dt - \int_{T_2}^0 B(w_k, \varphi+\psi_k)dt,$$

since w_k and $(w-\varphi-\psi_k)^-$ have disjoint support, we have

$$B(w_k, (w-\varphi-\psi_k)^-) = \int\int w_k(x)K(x,y)(-(w-\varphi-\psi_k)^-(y))dydx \le 0,$$

lastly, $B(w_k, \varphi+\psi_k)$ equals to

$$\begin{aligned}
&\int\int (w_k(x)-w_k(y))(\chi_{\{w_k>0\}}(x)+\chi_{\{w_k>0\}}(y))K(x,y)((\varphi+\psi_k)(x)-(\varphi+\psi_k)(y))dydx \\
&= \int\int (w_k(x)-w_k(y))K^{1/2}K^{1/2}(\chi_{\{w_k>0\}}(x)+\chi_{\{w_k>0\}}(y))((\varphi+\psi_k)(x)-(\varphi+\psi_k)(y))dydx \\
&\le \frac{1}{2}B(w_k,w_k) + \frac{1}{2}\int\int K(x,y)(\chi_{\{w_k>0\}}(x)+\chi_{\{w_k>0\}}(y))((\varphi+\psi_k)(x)-(\varphi+\psi_k)(y))^2dydx
\end{aligned}$$

By the Lipschitz regularity and the slow decay at infinity of $K(x,y)(\varphi+\psi_k)$, when we fixed one variable the second term will be integrable on the other variable. So, let's say x is fixed on the support of w_k, then the second term of the RHS will be bounded by $|\{w_k > 0\}|$.

To sum up,

$$\begin{aligned}
A_k &\le \left\{\int w_k^2(x,0)dx + \int_{T_k}^0 B(w_k,w_k)dt\right\} |\{w_k>0\}|^\varepsilon,\\
&\le \left\{\int w_k^2(x,T_k)dx + \int_{T_k}^0 \int w_k\beta_\varepsilon(w-\varphi)dxdt + \int_{T_2}^0 B(w_k,w_k-w)dt\right\} 2^{2k\varepsilon}A_{k-1}^\varepsilon\\
&\le \left\{2^k A_{k-1} + \int_{T_2}^0 B(w_k,(w-\varphi-\psi_k)^-)dt - \int_{T_2}^0 B(w_k,\varphi+\psi_k)dt\right\} 2^{2k\varepsilon}A_{k-1}^\varepsilon\\
&\le \left\{2^k A_{k-1} - \int_{T_2}^0 B(w_k,\varphi+\psi_k)dt\right\} 2^{2k\varepsilon}A_{k-1}^\varepsilon\\
&\le \left\{2^k A_{k-1} - \frac{1}{2}B(w_k,w_k) - \frac{1}{2}|\{w_k>0\}|\right\} 2^{2k\varepsilon}A_{k-1}^\varepsilon
\end{aligned}$$

□

Lemma 1.3. *Let $|\{w<0\}| \ge \mu > 0$ in $\Gamma_4 = B_4 \times [-4,0]$, then for some (very tiny) value $\lambda^*(\mu)$, we have*

$$|\{w > -\lambda^*\}| \le \delta,$$

i.e. the measure goes below critical value, the first lemma applies and $w \le -\dfrac{\lambda^*}{8}$ in $\Gamma_{1/4}$.

Proof. Consider the following three consecutive cut offs:

$$\psi_0 = \min(\phi, \frac{1}{8}(|x|^2-4)),$$
$$\psi_1 = \min(\phi, \frac{\lambda}{8}(|x|^2-4)),$$
$$\psi_2 = \min(\phi, \frac{\lambda^2}{8}(|x|^2-4)),$$

where λ is much smaller than μ. We will show that some mass is lost going from the set $\{w > \psi_0\}$ to $\{w > \psi_2\}$, which implies after a finite number of cut offs we have $|\{w > -\psi_k\}| \le \delta$, and the first lemma will apply.

Consider the equality with the cut-off ψ_1:

$$\int (w_{\psi_1}^2)_t dx \Big|_{T_1}^{T_2} - \int_{T_1}^{T_2}\int w_{\psi_1}\beta_\varepsilon(w-\varphi)dxdt + \int_{T_1}^{T_2} B(w_{\psi_1},w_{\psi_1}) = \int_{T_1}^{T_2} B(w_{\psi_1},w_{\psi_1}-w)dt$$

$$\begin{aligned}
\int (w_{\psi_1}^2)_t dx \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} B(w_{\psi_1},w_{\psi_1}) \quad &= \quad \int_{T_1}^{T_2}\int w_{\psi_1}\beta_\varepsilon(w-\varphi)dxdt + \int_{T_1}^{T_2} B(w_{\psi_1},(w-\varphi-\psi_1)^-)dt\\
&\quad - \int_{T_1}^{T_2} B(w_{\psi_1},\varphi+\psi_1)dt,
\end{aligned}$$

we have

$$B(w_{\psi_1},\varphi+\psi_1) \le \frac{1}{2}B(w_{\psi_1},w_{\psi_1}) + \int\int K(y-x)(\chi_{\{w_{\psi_1}>0\}}(x)((\varphi+\psi_1)(x)-(\varphi+\psi_1)(y))^2dydx,$$

where $\displaystyle\int\int K(y-x)(\chi_{\{w_{\psi_1}>0\}}(x))((\varphi+\psi_1)(x)-(\varphi+\psi_1)(y))^2dydx \le C\lambda^2$

I NEED TO CHECK LAST INEQLTY!!!!!! then we have,

$$\begin{aligned}\int (w_{\psi_1}^2)_t dx \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} B(w_{\psi_1}, w_{\psi_1}) \quad &= \quad \int_{T_1}^{T_2} \int w_{\psi_1} \beta_\varepsilon (w-\varphi) dx dt + \int_{T_1}^{T_2} B(w_{\psi_1}, (w-\varphi-\psi_1)^-) dt \\ & \quad - \int_{T_1}^{T_2} B(w_{\psi_1}, \varphi + \psi_1) dt,\end{aligned}$$

□

Theorem 1.4. *(Comparison Principle) Let w, v be two weak solutions of* (**??**) *s.t. $w(x,0) \geq v(x,0)$, then $w(x,t) \geq v(x,t)$ for every t.*

Proof. (BWOC) Assume that w and v are two weak solutions of (**??**) s.t. $w(x,0) \geq v(x,0)$ and $w(x,t) < v(x,t)$ for some t. Let t_0 be the smallest t s.t. $w(x,t) < v(x,t)$. By the Fundamental Theorem of Calculus, we have

$$w(x,t_0) - w(x,0) = \int_0^{t_0} w_t(x,t) dt,$$

$$v(x,t_0) - v(x,0) = \int_0^{t_0} v_t(x,t) dt.$$

We will take $\eta(x,t) = (w-v)(x,t) > 0$ as a test function, since it is measurable, we can use the approximation to the identity as a smoothed version of η. For simplicity, we will take it as it is.

$$\begin{array}{lcl} 0 > w(x,t_0) - v(x,t_0) & = & w(x,0) - v(x,0) + \int_0^{t_0} [w_t(x,t) - v_t(x,t)] dt \\ \int \int \eta(x,t) [w(x,t_0) - v(x,t_0)] dx dt & = & \int \int \eta(x,t) [w(x,0) - v(x,0)] dx dt \\ & + & \int \int \int_0^{t_0} \eta(x,t) [w_t(x,t) - v_t(x,t)] dt dx dt \\ & \geq & \int_0^{t_0} \int \int [\int \eta(x,t) [(w-v)(y,t) - (w-v)(x,t))] K(y-x) dy] dt dx dt dy \\ & = & \frac{1}{2} \int_0^{t_0} \int \int \int [\eta(x,t) - \eta(y,t)] [(w-v)(y,t) - (w-v)(x,t))] K(y-x) dy] dt dx dt dy \\ & > 0 & \end{array}$$

by symmetrization, change the variables x and y and add them up. We get a contradiction. Hence, we have the comparison principle. □

Theorem 1.5. *Let $\varphi \in \mathcal{C}^{1,1}$, $w_0 \in \mathcal{C}^{1,1}$, and $C = \max(\sup -\partial_{ee}\varphi, \sup -\partial_{ee} w_0)$, then $\partial_{ee} w \geq -C$.*

Proof. Consider the second incremental quotients:

$$\frac{\varphi(x+se) - \varphi(x-se)}{2} + Cs^2 \geq \varphi,$$

$$\frac{w_0(x+se) - w_0(x-se)}{2} + Cs^2 \geq w_0.$$

Hence,

$$v(x,t) = \frac{w(x+se,t) - w(x-se,t)}{2} + Cs^2 \geq \varphi$$

and

$$\begin{aligned}
v_t &= \frac{w_t(x+se,t)-w_t(x-se,t)}{2}\\
&= \frac{\int[w(y,t)-w(x+se,t)]K(y-x-se)dy-\int[w(y,t)-w(x-se,t)]K(y-x+se)dy}{2}\\
&= \frac{\int[w(y+se,t)-w(x+se,t)]K(y-x)dy-\int[w(y-se,t)-w(x-se,t)]K(y-x)dy}{2}\\
&= \int(\frac{[w(y+se,t)-w(y-se,t)]}{2}-\frac{[w(x+se,t)-w(x-se,t)]}{2})K(y-x)dy\\
&= \int(v(y,t)-v(x,t))K(y-x)dy
\end{aligned}$$

i.e. it is a weak solution of (**??**), also

$$v(x,0)\geq w(x,0)=w_0(x).$$

By the comparison principle,

$$v(x,t)\geq w(x,t),$$

i.e.

$$\frac{w(x+se,t)-w(x-se,t)}{2}+Cs^2\geq w(x,t).$$

Hence, $\partial_{ee}w\geq -C$. Therefore, for every x,t there exists a paraboloid of opening C touching w from below, i.e. w is semiconvex. □

Remark 1.6. We know that w is bounded and Hölder continuous. We want to show that $w\in\mathcal{C}^{1,\alpha}$. Let's look at the equation satisfied by $v=w_e$:

$$\begin{aligned}
w_t-\beta_\varepsilon(w-\varphi) &= \int[w(y,t)-w(x,t)]K(y-x)dy\\
&= \int[w(x+z,t)-w(x,t)]K(z)dz, \text{by change of variable } y=x+z\\
v_t-\beta'_\varepsilon(w-\varphi).(v-\varphi_e) &= \int[v(x+z,t)-v(x,t)]K(z)dz.
\end{aligned}$$

Hence, $v=w_e$ satisfies

$$v_t-\beta'_\varepsilon(w-\varphi).(v-\varphi_e)=\int[v(y,t)-v(x,t)]K(y-x)dy.$$

If we can show that v is Hölder continuous, we will be done, up to the regularity of φ and the ellipticity of K.

Here $\beta'\leq 0$, and we I need to check the comparison principle for this equation, then hopefully I can show the result, I might use also DeGiorgi method again.

Let us try to prove the comparison principle for this equation

$$v_t-\beta'_\varepsilon(w-\varphi).(v-\varphi_e)=\int[v(y,t)-v(x,t)]K(y-x)dy. \tag{1.3}$$

Theorem 1.7. *(Comparison Principle) Let u and v be two weak solutions of* (1.3) *s.t.* $u(x,0) \geq v(x,0)$, *then* $u(x,t) \geq v(x,t)$ *for every* t.

Proof. (BWOC) Assume that u and v are two weak solutions of (1.3) s.t. $u(x,0) \geq v(x,0)$ and $u(x,t) < v(x,t)$ for some t. Let t_0 be the smallest t s.t. $u(x,t) < v(x,t)$. By the Fundamental Theorem of Calculus, we have

$$u(x,t_0) - u(x,0) = \int_0^{t_0} u_t(x,t)dt,$$
$$v(x,t_0) - v(x,0) = \int_0^{t_0} v_t(x,t)dt.$$

We can take $\eta(x,t) = \chi_{\{(u-v)(y,t) \geq (u-v)(x,t)\}}(x,t)$ as a test function, since it is measurable, we can use the approximation to the identity as a smoothed version of η. For simplicity, we will take it as it is.

$$\begin{array}{lcl} 0 > u(x,t_0) - v(x,t_0) & = & u(x,0) - v(x,0) + \int_0^{t_0} [u_t(x,t) - v_t(x,t)]dt \\ \int\int \eta[u(x,t_0) - v(x,t_0)]dxds & = & \int\int \eta[u(x,0) - v(x,0)]dxds + \int\int\int_0^{t_0} \eta[u_t(x,t) - v_t(x,t)]dtdxds \\ & \geq & \int_0^{t_0}\int\int \eta[u_t(x,t) - v_t(x,t)]dtdxds \\ & = & \int_0^{t_0}\int\int \eta\beta'_\varepsilon(w-\varphi).(u-v)dxdsdt \\ & + & \int_0^{t_0}\int\int\int \eta[(u-v)(y,t) - (u-v)(x,t)]K(y-x)dydxdsdt \\ & \geq & 0 \end{array}$$

contradiction. Hence, we have the comparison principle. □

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